ON POSITIVE SOLUTION FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS WITH SIGN–CHANGING WEIGHTS

DIANPENG HE AND ZUODONG YANG

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Abstract. In this paper, we consider the problem for the existence of positive solutions of quasi-linear elliptic system

\[
\begin{align*}
-\Delta_p u &= \lambda a(x)u^\alpha v^\gamma, & x \in \Omega, \\
-\Delta_q v &= \lambda b(x)u^\eta v^\beta, & x \in \Omega, \\
u &= v = 0, & x \in \partial \Omega,
\end{align*}
\]

where the \( \lambda > 0 \) is a parameter, \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N > 1) \) with smooth boundary \( \partial \Omega \), and the \( \Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z) \) is the \( p \)-Laplacian operator. Here \( a(x) \) and \( b(x) \) are \( C^1 \) sign-changing functions that maybe are negative near the boundary. Using the method of sub-super solutions and comparison principle, which studied the existence of positive solutions for quasilinear elliptic system. The main results of the present paper are new and extend the previously known results.

1. Introduction

In this note we consider the existence of positive solutions for the system

\[
\begin{align*}
-\Delta_p u &= \lambda a(x)u^\alpha v^\gamma, & x \in \Omega, \\
-\Delta_q v &= \lambda b(x)u^\eta v^\beta, & x \in \Omega, \\
u &= v = 0, & x \in \partial \Omega,
\end{align*}
\]

where \( \lambda > 0 \) is a parameter, \( 1 < p, q < N \), and \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N > 1) \) with smooth boundary \( \partial \Omega \), and the \( \Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z) \) is the \( p \)-Laplacian operator. Here \( a(x) \) and \( b(x) \) are \( C^1 \) sign-changing functions that maybe are negative near the boundary.

Problem (1.1) arises in the theory of quasi-regular and quasi-conformal mappings as well as in the study of non-Newtonian fluids. In the latter case, the pair \( (p, q) \) is a characteristic of the medium. Media with \( (p, q) > (2, 2) \) are called dilatant fluids and those with \( (p, q) < (2, 2) \) are called pseudo-plastics. If \( (p, q) = (2, 2) \), they are Newtonian fluids.


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When \( p = q = 2 \), the following system
\[
\begin{align*}
\Delta u &= a(|x|)v^\alpha, \quad x \in \mathbb{R}^N, \\
\Delta v &= b(|x|)u^\beta, \quad x \in \mathbb{R}^N,
\end{align*}
\]
for which existence results for boundary blow-up positive solutions can be found in a recent paper by Lair and Wood [12]. The authors established that all positive entire radial solutions of systems above are boundary blow-up provided that
\[
\int_0^\infty ta(t)dt = \infty, \quad \int_0^\infty tb(t)dt = \infty.
\]
On the other hand, if
\[
\int_0^\infty ta(t)dt < \infty, \quad \int_0^\infty tb(t)dt < \infty,
\]
then all positive entire radial solutions of this system are bounded.

F. Cîrstea and V. Rădulescu [5] extended the above results to a larger class of systems
\[
\begin{align*}
\Delta u &= a(|x|)g(v), \quad x \in \mathbb{R}^N, \\
\Delta v &= b(|x|)f(u), \quad x \in \mathbb{R}^N.
\end{align*}
\]
Z.D. Yang [15] extended the above results to a class of systems
\[
\begin{align*}
\text{div}(|\nabla u|^{p-2}\nabla u) &= a(|x|)g(v), \quad x \in \mathbb{R}^N, \\
\text{div}(|\nabla v|^{q-2}\nabla v) &= b(|x|)f(u), \quad x \in \mathbb{R}^N.
\end{align*}
\]
Caisheng Chen [3] discussed the existence and non-existence of positive weak solution to the following system
\[
\begin{align*}
-\Delta u &= \lambda u^\alpha v^\gamma, \quad x \in \Omega, \\
-\Delta v &= \lambda u^\delta v^\beta, \quad x \in \Omega, \\
u(x) = v(x) &= 0, \quad x \in \partial\Omega.
\end{align*}
\]
D.D. Hai [11] studied the existence and nonexistence of positive solutions for the quasilinear system
\[
\begin{align*}
-\Delta_{p} u &= \lambda a(x)f(u,v), \quad x \in \Omega, \\
-\Delta_{q} v &= \mu b(x)g(u,v), \quad x \in \Omega, \\
u = v &= 0, \quad x \in \partial\Omega, \quad (E)
\end{align*}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial\Omega \), \( p, q > 1, \lambda, \mu \) are positive parameters, \( a(x), b(x) \) are bounded functions that can change sign, which obtained existence results for the quasilinear system \( (E) \) when \( f(t,t) \) is \( p \)-sublinear in 0 and \( g(t,t) \) is \( q \)-sublinear at 0, and \( \lambda, \mu \) are small. Nonexistence results are also obtained.
Motivated by the above results, we focus on further extending the study in [3] to the system (1.1) and supplementary the results in [11]. In fact, we study the existence of positive solution to the system (1.1) with sign-changing weight functions \(a(x)\) and \(b(x)\). Due to this weight functions, the existence are challenging and nontrivial. Our approach is based on the method of sub-super solutions, see [3, 14].

To precisely state our existence result we need the eigenvalue problem

\[-\Delta_p \phi_1 = \lambda_1 |\phi_1|^{p-2}\phi_1, \quad x \in \Omega, \quad \phi_1 = 0, \quad x \in \partial \Omega, \tag{1.2}\]

\[-\Delta_q \phi_1^* = \lambda_1^* |\phi_1^*|^{q-2}\phi_1^*, \quad x \in \Omega, \quad \phi_1^* = 0, \quad x \in \partial \Omega. \tag{1.3}\]

Let \(\lambda_1 > 0\) be the principal eigenvalue and \(\phi_1 > 0\) with \(\|\phi_1\|_\infty = 1\) the corresponding eigenfunction of \(-\Delta_p\) and \(\lambda_1^* > 0\) be the principal eigenvalue and \(\phi_1^* > 0\) with \(\|\phi_1^*\|_\infty = 1\) the corresponding eigenfunction of \(-\Delta_q\), with the Dirichlet boundary condition. It is well known that

\[\frac{\partial \phi_1}{\partial \nu} < 0 \quad \text{and} \quad \frac{\partial \phi_1^*}{\partial \nu} < 0 \quad \text{on} \quad \partial \Omega, \]

where \(\nu\) is the unit outward normal, while \(\phi_1, \phi_1^* = 0\) on \(\partial \Omega\). This result is well known and hence, depending on \(\Omega\), there exist \(\sigma, \sigma^* \in (0, 1], \delta > 0\) and \(m > 0\) such that (see [14])

\[\lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -m, \quad \text{on} \quad \overline{\Omega_\delta}, \tag{1.4}\]

\[\phi_1 \geq \sigma, \quad \text{on} \quad \Omega_0 = \Omega \setminus \overline{\Omega_\delta}, \tag{1.5}\]

and

\[\lambda_1 \phi_1^{*q} - |\nabla \phi_1^*|^q \leq -m, \quad \text{on} \quad \overline{\Omega_\delta}, \tag{1.6}\]

\[\phi_1^* \geq \sigma^*, \quad \text{on} \quad \Omega_0 = \Omega \setminus \overline{\Omega_\delta}, \tag{1.7}\]

where \(\Omega_\delta = \{x \in \Omega | d(x, \partial \Omega) < \delta\}\). We will also consider the unique solution, \(e_1(x)\), \(e_2(x) \in C^1(\overline{\Omega})\), of the boundary value problem

\[-\Delta_p e_1 = 1, \quad x \in \Omega, \quad e_1 = 0, \quad x \in \partial \Omega, \tag{1.8}\]

\[-\Delta_q e_2 = 1, \quad x \in \Omega, \quad e_2 = 0, \quad x \in \partial \Omega, \tag{1.9}\]

to discuss our existence result. It is known that \(e_i(x) > 0 (i = 1, 2)\) in \(\Omega\) and

\[\frac{\partial e_i(x)}{\partial \nu} < 0 \quad \text{on} \quad \partial \Omega (i = 1, 2) \quad \text{(see [8, 9, 10])}. \]

2. Existence results

In this section, we shall establish our existence result via the method of sub and supersolutions. A pair of nonnegative functions \((\psi_1, \psi_2), (z_1, z_2)\), are called a subsolution and supersolution of (1.1) if they satisfy \((\psi_1, \psi_2) = (0, 0) = (z_1, z_2)\), on \(\partial \Omega\)

\[\int_{\Omega} |\nabla \psi_1|^{p-2} |\nabla \psi_1| \cdot \nabla f_1 dx \leq \lambda \int_{\Omega} a(x) \psi_1^\alpha \psi_2^\gamma f_1 dx, \]

\[\int_{\Omega} |\nabla \psi_2|^{q-2} |\nabla \psi_2| \cdot \nabla f_2 dx \leq \lambda \int_{\Omega} b(x) \psi_1^\eta \psi_2^\beta f_2 dx, \]
\[
\int_\Omega |\nabla z_1|^{p-2} |\nabla f_1| \cdot \nabla f_1 dx \geq \lambda \int_\Omega a(x) z_1^\alpha z_2^\gamma f_1 dx,
\]

\[
\int_\Omega |\nabla z_2|^{p-2} |\nabla f_2| \cdot \nabla f_2 dx \geq \lambda \int_\Omega a(x) z_1^\eta z_2^\beta f_2 dx,
\]

for all test functions \( f_1(x) \in W_{0}^{1,p}(\Omega) \) and \( f_2(x) \in W_{0}^{1,q}(\Omega) \) with \( f_1, f_2 \geq 0 \). Then the following result holds:

**Lemma 1.** (See [17]) Suppose there exist sub and super-solutions \((\psi_1, \psi_2)\) and \((z_1, z_2)\) respectively of (1.1) such that \((\psi_1, \psi_2) \leq (z_1, z_2)\). Then (1.1) has a solution \((u, v)\) such that \((u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]\).

We make the following assumptions:

(i) \(\alpha, \beta \geq 0, \gamma, \eta > 0\) and \((p-1-\alpha)(q-1-\beta) > \gamma \eta\);

(ii) Assume that there exist positive constants \(a_0, a_1, b_0\) and \(b_1\), such that

\[
a(x) \geq -a_0, b(x) \geq -b_0 \text{ on } \overline{\Omega_\delta}
\]

and

\[
a(x) \geq a_1, b(x) \geq b_1 \text{ on } \Omega \setminus \overline{\Omega_\delta};
\]

(iii) Suppose that there exists \(\varepsilon > 0\) such that:

\[
\frac{\lambda_1}{m} a_0 < \min \left\{ c_1, c_2 \varepsilon^{d_2-d_1} \right\}, \quad \frac{\lambda_1}{m} a_0 < \min \left\{ c_1 \varepsilon^{-(d_2-d_1)}, c_2 \right\}
\]

and

\[
\max \left\{ \frac{\lambda_1}{c_1} \varepsilon^{1-d_1}, \frac{\lambda^*_1}{c_2} \varepsilon^{1-d_2} \right\} \leq \min \left\{ \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty} \right\},
\]

where

\[
c_1 = a_1 \left( \frac{p-1}{p} \sigma_{p-1}^p \right)^\alpha \left( \frac{q-1}{q} \sigma_{q-1}^q \right)^\gamma, \quad c_2 = b_1 \left( \frac{p-1}{p} \sigma_{p-1}^p \right)^\eta \left( \frac{q-1}{q} \sigma_{q-1}^q \right)^\beta,
\]

\[
d_1 = \frac{\alpha}{p-1} + \frac{\gamma}{q-1}, \quad d_2 = \frac{\eta}{p-1} + \frac{\beta}{q-1}.
\]

Now we are ready to state our existence results.
THEOREM 1. Let (i) – (iii) hold. Then there exists a positive solution of (1.1) for every \( \lambda \in [\overline{\lambda}(\varepsilon), \overline{\lambda}(\varepsilon)] \), where

\[
\overline{\lambda} = \min \left\{ \frac{m}{a_0\varepsilon^{d_1-1}}, \frac{m}{b_0\varepsilon^{d_1-1}}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty} \right\},
\]

and since

\[
\int |\nabla \psi|^p dx = \varepsilon \int |\nabla \phi_1|^p dx,
\]

we shall verify that \((\psi_1, \psi_2)\) is a sub-solution of (1.1). Let \( f_1 \in W^1,p(\Omega) \), then a calculation shows that

\[
\int_{\Omega} |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla f_1 \, dx = \varepsilon \left\{ \int_{\Omega} |\nabla \phi_1|^{p-2} \nabla \phi_1 \cdot \nabla f_1 \, dx - \int_{\Omega} |\nabla \phi_1|^p f_1 \, dx \right\} \quad \text{(2.3)}
\]

A similarly calculation shows that

\[
\int_{\Omega} |\nabla \psi_2|^{q-2} \nabla \psi_2 \cdot \nabla f_2 \, dx = \varepsilon \left\{ \int_{\Omega} |\nabla \phi_1|^{q} - |\nabla \phi_1|^q f_2 \, dx \right\}. \quad \text{(2.4)}
\]

First, we consider the case when \( x \in \overline{\Omega} \backslash \delta \). We have \( \lambda_1^* - |\nabla \phi_1|^p \leq -m \) on \( \overline{\Omega} \backslash \delta \) and since \( \lambda \leq \overline{\lambda} \), we have \( \lambda \leq \frac{m}{a_0\varepsilon^{d_1-1}} \). Then

\[
-\varepsilon m \leq -\lambda a_0 \varepsilon^{\frac{\alpha}{p-1} + \frac{\gamma}{q-1}}. \quad \text{(2.5)}
\]

Hence

\[
\varepsilon (\lambda_1^* - |\nabla \phi_1|^p) \leq -m \varepsilon \leq -\lambda a_0 \varepsilon^{\frac{\alpha}{p-1} + \frac{\gamma}{q-1}} \leq -\lambda a_0 \left( \frac{p-1}{p} \varepsilon^{\frac{\alpha}{p-1}} \| \phi_1 \|_{\infty}^{p-1} \right)^{\alpha} \left( \frac{q-1}{q} \varepsilon^{\frac{\alpha}{q-1}} \| \phi_1 \|_{\infty}^{q-1} \right)^{\gamma} \quad \text{(2.6)}
\]

A similar argument shows that:

\[
\varepsilon (\lambda_1^* - |\nabla \phi_1|^q) \leq \lambda b(x) \psi_1^\eta \psi_2^\beta. \quad \text{(2.7)}
\]
Then we obtain from (2.3), (2.4) and (2.6), (2.7) that
\[
\int_{\Omega_\delta} |\nabla \psi_1|^{p-2}|\nabla \psi_1| \cdot \nabla f_1 \, dx \leq \lambda \int_{\Omega_\delta} a(x) \psi_1^\alpha \psi_2^\gamma f_1 \, dx, \tag{2.8}
\]
\[
\int_{\Omega_\delta} |\nabla \psi_2|^{q-2}|\nabla \psi_2| \cdot \nabla f_2 \, dx \leq \lambda \int_{\Omega_\delta} b(x) \psi_1^\eta \psi_2^\beta f_2 \, dx. \tag{2.9}
\]
On the other hand, on $\Omega \setminus \Omega_\delta$, we note that
\[
\phi_1 \geq \sigma > 0, \quad \phi_1^* \geq \sigma^* > 0, \quad a(x) \geq a_1, \quad b(x) \geq b_1
\]
and since $\lambda \geq \lambda$, we have $\lambda \geq \frac{\lambda}{c_1 \epsilon^{1-r}}$. Then
\[
\epsilon \lambda_1 \leq \lambda a_1 \left( \frac{p-1}{p} \epsilon^{\frac{1}{p-1}} \sigma^{\frac{p}{p-1}} \right)^\alpha \left( \frac{q-1}{q} \epsilon^{\frac{1}{q-1}} \sigma^{* \frac{q}{q-1}} \right)^\gamma. \tag{2.10}
\]
Hence
\[
\epsilon (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq \epsilon \lambda_1 \phi_1^p \leq \epsilon \lambda_1 \|\phi_1\|_\infty^p \\
\leq \epsilon \lambda_1 \leq a_1 \left( \frac{p-1}{p} \epsilon^{\frac{1}{p-1}} \sigma^{\frac{p}{p-1}} \right)^\alpha \left( \frac{q-1}{q} \epsilon^{\frac{1}{q-1}} \sigma^{* \frac{q}{q-1}} \right)^\gamma \\
\leq \lambda a(x) \psi_1^\alpha \psi_2^\gamma. \tag{2.11}
\]
A similar argument shows that:
\[
\epsilon (\lambda_1 \phi_1^* q - |\nabla \phi_1^*|^q) \leq \lambda b(x) \psi_1^\eta \psi_2^\beta. \tag{2.12}
\]
Then we obtain from (2.3), (2.4) and (2.11), (2.12) that
\[
\int_{\Omega \setminus \Omega_\delta} |\nabla \psi_1|^{p-2}|\nabla \psi_1| \cdot \nabla f_1 \, dx \leq \lambda \int_{\Omega \setminus \Omega_\delta} a(x) \psi_1^\alpha \psi_2^\gamma f_1 \, dx, \tag{2.13}
\]
\[
\int_{\Omega \setminus \Omega_\delta} |\nabla \psi_2|^{q-2}|\nabla \psi_2| \cdot \nabla f_2 \, dx \leq \lambda \int_{\Omega \setminus \Omega_\delta} b(x) \psi_1^\eta \psi_2^\beta f_2 \, dx. \tag{2.14}
\]
Since $\Omega = \Omega_\delta \cup (\Omega \setminus \Omega_\delta)$, We obtain from (2.8), (2.9) and (2.13), (2.14) that
\[
\int_{\Omega} |\nabla \psi_1|^{p-2}|\nabla \psi_1| \cdot \nabla f_1 \, dx \leq \lambda \int_{\Omega} a(x) \psi_1^\alpha \psi_2^\gamma f_1 \, dx, \tag{2.15}
\]
\[
\int_{\Omega} |\nabla \psi_2|^{q-2}|\nabla \psi_2| \cdot \nabla f_2 \, dx \leq \lambda \int_{\Omega} b(x) \psi_1^\eta \psi_2^\beta f_2 \, dx, \tag{2.16}
\]
we have shown that $(\psi_1, \psi_2)$ is sub-solution.

Now, we will construct a super-solution $(z_1, z_2)$ of (1.1). It is clear that:
\[
- \text{div}(|\nabla z_1|^{p-2} \nabla z_1) = A, \quad x \in \Omega, \tag{2.17}
\]
\[- \text{div}(|\nabla z_2|^{q-2}\nabla z_2) = B, \quad x \in \Omega. \quad (2.18)\]

We denote
\[
z_1(x) = Ae_1(x), \quad z_2(x) = Be_2(x), \quad (2.19)
\]
where the constants \( A, B > 0 \) are large and to be chosen later. We shall verify that is a super-solution of (1.1).

Next, since \( \bar{\lambda} \leq \lambda \) we have \( \lambda \leq \|b\|_{\infty}, \lambda \leq \|a\|_{\infty} \). Let \( l_1 = \|e_1\|_{\infty}, \ l_2 = \|e_2\|_{\infty} \), since (i) hold, it is easy to prove that there exist positive large constants \( A,B \) such that(\[3\]):
\[
A^{p-1-\alpha} \geq B^{q-1-\alpha} \|a\|_{\infty} t_1^{\gamma} t_2^{\alpha}, \quad (2.20)
\]
and
\[
B^{q-1-\beta} \geq A^{\eta} t_1^{\eta} t_2^{\beta}, \quad (2.21)
\]
These imply that:
\[
A^{p-1} \geq \lambda a(x) z_1^{\alpha} z_2^{\gamma}, \quad B^{q-1} \geq \lambda b(x) z_1^{\delta} z_2^{\beta} \quad (2.22)
\]

Let \( f_1 \in W^{1,p}_0(\Omega), \ f_2 \in W^{1,q}_0(\Omega) \) with \( f_1, f_2 > 0 \). Then we obtain from (2.12), (2.13) and (2.17) that
\[
\int_{\Omega} |\nabla z_1|^{p-2} |\nabla z_1| \cdot \nabla f_1 \, dx = A^{p-1} \int_{\Omega} f_1(x) \, dx \geq \lambda \int_{\Omega} a(x) z_1^{\alpha} z_2^{\gamma} f_1 \, dx, \quad (2.23)
\]
\[
\int_{\Omega} |\nabla z_2|^{p-2} |\nabla z_2| \cdot \nabla f_2 \, dx = B^{q-1} \int_{\Omega} f_2(x) \, dx \geq \lambda \int_{\Omega} a(x) z_1^{\eta} z_2^{\beta} f_2 \, dx. \quad (2.24)
\]
a.e. in \( \Omega \). Thus, \((z_1, z_2)\) is a super-solution of (1.1). Obviously, we have \( z_i(x) \geq \psi_i(x) \) in \( \Omega \) with large \( A,B \) for \( i = 1, 2 \). Thus, by Lemma 1, there exists a positive solution \((u, v)\) of (1.1) such that \((\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)\). This completes the proof of Theorem 1. \( \square \)

REFERENCES


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