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EXISTENCE AND GLOBAL ATTRACTIVITY OF PERIODIC SOLUTIONS FOR CHEMOSTAT MODEL WITH DELAYED NUTRIENTS RECYCLING

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Abstract. In this paper, a chemostat model involving distributed delays with two-microorganism and nutrients recycling is considered. Some sufficient conditions ensuring the existence and global attractivity of periodic solutions for the chemostat model are derived by employing the theory of coincidence degree and differential inequality technique.

1. Introduction

The chemostat, a laboratory apparatus used for the continuous culture of microorganisms, has played an important role in microbiology and population biology[8]. They have a wide range of applications, for example, waste water treatment, production by genetically altered organisms (like production of insulin), etc. The growth in a chemostat is described by a system of ordinary differential equations and is studied by many researchers. Generally speaking,the loss or death of biomass in a chemostat attributing to the washout rate of the system is very high. But, when we try to model a natural lake system the washout rate tends to be low. As the washout rate is low, the dead biomass(naturally die or unnaturally die) remains in the system and it is possible that the bacterial decomposition of dead biomass resulting in the regeneration of nutrient. Consequently, we will introduce a recycling of dead biomass as nutrient.

However, when one tries to model a natural lake system, the system of ordinary differential equations is not suitable. At present, considerable work has been done to develop the dynamics for chemostat systems with delays. Sree Hari Rao and Raja Sekhara Rao, He and Ruan, Yuan et al.,[1, 4, 6, 8] studied the stability of the chemostat models with delays and nutrient recycling. Beretta and Takeuchi and Sree Hari Rao and Raja Sekhara Rao and He et al.,[2, 5, 7] described the mechanism in a chemostat by a system of integro-differential equations involving distributed time lags both in growth response and nutrient recycling. They obtained some sufficient conditions ensuring the stability of the system by employing Lyapunov functional method.However, to the best of our knowledge, few study the periodic oscillatory behavior of chemostat system involving distributed delay.

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Motivated by the above discussion, in this paper, we study the nonautonomous chemostat system involving distributed delays and nutrient recycling. We can write the following system:

$$\begin{cases} \dot{x}(t) = D(t)(x_0 - x(t)) - U(x(t))[a_1(t)y_1(t) + a_2(t)y_2(t)] \\ \qquad + \int_{-\infty}^{t} f(t-s)[b_1(s)\gamma_1(s)y_1(s) + b_2(s)\gamma_2(s)y_2(s)]ds, \\ \dot{y}_1(t) = -(\gamma_1(t) + D(t))y_1(t) + c_1(t)y_1(t)\int_{-\infty}^{t} g(t-s)U(x(s))ds, \\ \dot{y}_2(t) = -(\gamma_2(t) + D(t))y_2(t) + c_2(t)y_2(t)\int_{-\infty}^{t} g(t-s)U(x(s))ds, \\ x(t) = \phi(t), y_i(t) = \psi_i(t), -\infty < t \leqslant 0, i = 1,2, \end{cases} \tag{1.1}$$

where $x(t), y_1(t), y_2(t)$ represent the concentration of limiting substrate X, microorganism Y_1 and Y_2, respectively. x_0 denotes the nutrient input concentration which is assumed to be constant here. $D(t)$ is the washout rate at time $t, \gamma_1(t), \gamma_2(t) > 0$ denote the death rate coefficient of the microorganism Y_1 and Y_2, respectively. $b_1(t), b_2(t) > 0$ is the fraction of the nutrient recycled by the dead microorganisms, $a_1(t), a_2(t)$ denote the maximum uptake rate of the species, $c_1(t))(< a_1(t)), c_2(t)(< a_2(t))$ denote the maximum specific growth rate of the species, respectively. $D(t)$, $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$, $c_1(t)$, $c_2(t)$ are positive $\omega(\omega > 0)$-periodic functions. $U(x(t))$ denotes the uptake function and $U(0) = 0$. The kernel f describes the contribution of the dead biomass from the past to the nutrient recycled at time t where as g tells that the growth is not immediate to consumption and there is a time delay.

This paper is organized as follows. In Section 2, we introduce notations and lemmas which will be useful for our main results. In Section 3, we show the existence of periodic solution. Attractivity of periodic solution is discussed in Section 4. Finally, we give a brief discussion in Section 5.

2. Preliminary

For convenience, we will give some notations and lemmas which will be useful for our main results. Let $X = Z = \{z \in C(\mathbb{R}, \mathbb{R}^3) | z(t+\omega) = z(t)\}$, $z(t) = (x(t), y_1(t), y_2(t))^T$ and

$$\|z(t)\| = d_1^{-1}|x(t)| + d_2^{-1}|y_1(t)| + d_3^{-1}|y_2(t)|, \quad |||z(t)||| = \max_{t\in[0,\omega]} \|z(t)\|,$$

then $(X, |||\cdot|||)$ is a Banach space.

Let X and Y be normed vector spaces. Let $L : \mathrm{Dom}L \subset X \to Y$ be a Fredholm mapping of index zero and let $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dim(\mathrm{Ker}L) = condim(\mathrm{Im}L) < +\infty$ and $\mathrm{Im}L$ is closed in Y.

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$, then $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P}$: $(I-P)X \to \mathrm{Im}L$ is invertible, and its inverse is denoted by K_P. If Ω is a bounded open subset of X, the mapping N is called L-compact on N if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to \mathrm{Im}L$ is compact. Because $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

LEMMA 1. *(Continuation Theorem, Gaines and Mawhin [3]) Let Ω is a bounded open set of X, L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose that:*

(i) *for each $\lambda \in (0,1)$, every solution x of $Lx \neq \lambda Nx$ is such that $x \in \mathrm{Ker} L \cap \partial\Omega$;*
(ii) *$QNx \neq 0$ for each $x \in \mathrm{Ker} L \cap \partial\Omega$;*
(iii) *$deg\{JQN, \Omega \cap \mathrm{Ker} L, 0\} \neq 0$.*

Then the equation $Lx = \lambda Nx$ has at least one solution lying in $\mathrm{Dom} L \cap \partial\Omega$.

For the sake of convenience and simplicity, we introduce some notation as follows:

$$\hat{k} = \frac{1}{\omega}\int_0^\omega k(t)dt, \quad \overline{k} = \sup_{t\in R}|k(t)|, \quad \underline{k} = \inf_{t\in R}|k(t)|,$$

where k is a positive continuous ω-periodic function with $\omega > 0$.

From now on, we always assume that:

(H_1) There exists positive constant L, such that $|U(x) - U(\bar{x})| \leqslant L|x - \bar{x}|$, for all $x, \bar{x} \in \mathbb{R}$.
(H_2) $\int_0^\infty f(s)ds = 1, \int_0^\infty g(s)ds = 1$.
(H_3) $$M_1 \triangleq \frac{(\underline{D} - (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}) + \sqrt{N}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L} > 0,$$
$$M_2 \triangleq \frac{(\underline{D} - (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}) - \sqrt{N}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L}$$
where
$$N \triangleq [(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \underline{D}]^2 - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0 > 0.$$

3. Existence of periodic solution

In this section, we shall consider the existence of periodic solution of system (1.1). Corresponding equation $Lz = \lambda Nz, \lambda \in (0,1)$, we have

$$\begin{cases} \dot{x}(t) = \lambda[D(t)(x_0 - x(t)) - U(x(t))[a_1(t)y_1(t) + a_2(t)y_2(t)] \\ \qquad + \int_{-\infty}^t f(t-s)[b_1(s)\gamma_1(s)y_1(s) + b_2(s)\gamma_2(s)y_2(s)]ds], \\ \dot{y}_1(t) = \lambda[-(\gamma_1(t) + D(t))y_1(t) + c_1(t)y_1(t)\int_{-\infty}^t g(t-s)U(x(s))ds], \\ \dot{y}_2(t) = \lambda[-(\gamma_2(t) + D(t))y_2(t) + c_2(t)y_2(t)\int_{-\infty}^t g(t-s)U(x(s))ds], \\ x(t) = \phi(t), y_i(t) = \psi_i(t), -\infty < t \leqslant 0, i = 1,2. \end{cases} \tag{3.1}$$

THEOREM 1. *If conditions (H_1)-(H_3) hold, then for any $\phi, \psi_i \in C$, there exists a positive constant $B \in [M_2, M_1]$ which is independent of λ, such that*

$$\|z(t)\| < B, \quad t \geqslant 0. \tag{3.2}$$

Proof. For any $\phi, \psi_i \in C$, we will prove that (3.2) holds. If (3.2) is not true, then there must be $t_1 > 0$, such that

$$\|z(t_1)\| = B, \|z(t)\| < B, 0 \leqslant t < t_1, \tag{3.3}$$

and

$$\|z(t)\| \leqslant B, 0 \leqslant t \leqslant t_1.$$

Then, we have

$$\begin{aligned}
\|z(t_1)\| \leqslant\; & e^{-\lambda \int_0^{t_1} D(t)dt} d_1^{-1}|x(0)| + \int_0^{t_1} e^{-\lambda \int_t^{t_1} D(u)du} d_1^{-1} \lambda \\
& \times [D(t)x_0 + |U(x(t))|(a_1(t)|y_1(t)| + a_2(t)|y_2(t)|) \\
& \quad + \int_{-\infty}^{t} f(t-s)[b_1(s)\gamma_1(s)|y_1(s)| + b_2(s)\gamma_2(s)|y_2(s)|]ds]dt \\
& \quad + e^{-\lambda \int_0^{t_1} (\gamma_1(t)+D(t))dt} d_2^{-1}|y_1(0)| + \int_0^{t_1} e^{-\lambda \int_t^{t_1} (\gamma_1(u)+D(u))du} \\
& \quad \times d_2^{-1} \lambda [c_1(t)|y_1(t)| \int_{-\infty}^{t} g(t-s)|U(x(s))|ds]dt \\
& \quad + e^{-\lambda \int_0^{t_1} (\gamma_2(t)+D(t))dt} d_3^{-1}|y_2(0)| + \int_0^{t_1} e^{-\lambda \int_t^{t_1} (\gamma_2(u)+D(u))du} \\
& \quad \times d_3^{-1} \lambda [c_2(t)|y_2(t)| \int_{-\infty}^{t} g(t-s)|U(x(s))|ds]dt \\
\leqslant\; & e^{-\lambda \underline{D} t_1}(d_1^{-1}|x(0)| + d_2^{-1}|y_1(0)| + d_3^{-1}|y_2(0)|) + \int_0^{t_1} e^{-\lambda \underline{D}(t_1 - t)} \lambda \\
& \quad \times [d_1^{-1}\bar{D}x_0 + d_1^{-1}|U(x(t))|(a_1(t)|y_1(t)| + a_2(t)|y_2(t)|) \\
& \quad + d_1^{-1} \int_{-\infty}^{t} f(t-s)[\bar{b_1}\bar{\gamma_1}|y_1(s)| + \bar{b_2}\bar{\gamma_2}|y_2(s)|]ds \\
& \quad + \bar{c_1} d_2^{-1}|y_1(t)| \int_{-\infty}^{t} g(t-s)|U(x(s))|ds \\
& \quad + \bar{c_2} d_3^{-1}|y_2(t)| \int_{-\infty}^{t} g(t-s)|U(x(s))|ds]dt \\
\leqslant\; & e^{-\lambda \underline{D} t_1}(d_1^{-1}|x(0)| + d_2^{-1}|y_1(0)| + d_3^{-1}|y_2(0)|) + \int_0^{t_1} e^{-\lambda \underline{D}(t_1 - t)} \lambda \\
& \quad \times \Big[(\bar{a_1} d_2 + \bar{a_2} d_3)LB^2 + d_1^{-1}(\bar{b_1}\bar{\gamma_1} d_2 + \bar{b_2}\bar{\gamma_2} d_3)B + \bar{D} d_1^{-1} x_0 \\
& \qquad\qquad + (\bar{c_1} + \bar{c_2})L d_1 B^2\Big] dt \\
\leqslant\; & e^{-\lambda \underline{D} t_1} B + \frac{1}{\underline{D}}[(\bar{a_1} d_2 + \bar{a_2} d_3 + \bar{c_1} d_1 + \bar{c_2} d_1)LB^2 \\
& \quad + d_1^{-1}(\bar{b_1}\bar{\gamma_1} d_2 + \bar{b_2}\bar{\gamma_2} d_3)B + \bar{D} d_1^{-1} x_0][1 - e^{-\lambda \underline{D} t_1}] \\
<\; & B.
\end{aligned}$$

This contradicts the first equality of (3.3), so (3.2) holds. Thus, the proof is completed. □

THEOREM 2. *If* $(H1)$ - $(H3)$ *hold, then there exists at least one* ω*-periodic solution of system* (1.1).

Proof. Define

$$X = Y = \Big\{ (x(t), y_1(t), y_2(t))^T \in C(\mathbb{R}, \mathbb{R}^3) : \\ x(t+\omega) = x(t), y_1(t+\omega) = y_1(t), y_2(t+\omega) = y_2(t) \Big\},$$

then X, Y be normed vector spaces, with norm

$$\|z(t)\| = \|(x(t), y_1(t), y_2(t))^T\| = \max_{t \in [0,\omega]} (d_1^{-1}|x(t)| + d_2^{-1}|y_1(t)| + d_3^{-1}|y_2(t)|).$$

where $d_i^{-1}|\cdot|$ denote Euclidean norm. Let

$$L : \operatorname{Dom} L \cap X \to Y, L(x(t), y_1(t), y_2(t))^T = (x'(t), y_1'(t), y_2'(t))^T$$

and $\operatorname{Dom} L = \{(x(t), y_1(t), y_2(t))^T \in C'(\mathbb{R}, \mathbb{R}^3)\}$. Define $N : X \to X$,

$$N \begin{pmatrix} x(t) \\ y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} D(t)(x_0 - x(t)) - U(x(t))[a_1(t)y_1(t) + a_2(t)y_2(t)] \\ + \int_{-\infty}^{t} f(t-s)[b_1(s)\gamma_1(s)y_1(s) + b_2(s)\gamma_2(s)y_2(s)]ds \\ -(\gamma_1(t) + D(t))y_1(t) + c_1(t)y_1(t) \int_{-\infty}^{t} g(t-s)U(x(s))ds \\ -(\gamma_2(t) + D(t))y_2(t) + c_2(t)y_2(t) \int_{-\infty}^{t} g(t-s)U(x(s))ds \end{pmatrix}.$$

It is easy to see that $\operatorname{Ker} L = \mathbb{R}^3$,

$$\operatorname{Im} L = \left\{ (x(t), y_1(t), y_2(t)) \in Y : \int_0^{\omega} (x(t), y_1(t), y_2(t))^T dt = (0,0,0)^T \right\}$$

is closed in X and $dim(\operatorname{Ker} L) = condim(\operatorname{Im} L) = 3$. Therefore, L is a Fredholm mapping of index zero.

Define $P : X \to X$ and $Q : Y \to Y$:

$$P \begin{pmatrix} x(t) \\ y_1(t) \\ y_2(t) \end{pmatrix} = Q \begin{pmatrix} x(t) \\ y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} x\hat{(t)} \\ y_1\hat{(t)} \\ y_1\hat{(t)} \end{pmatrix}, \qquad \begin{pmatrix} x(t) \\ y_1(t) \\ y_2(t) \end{pmatrix} \in X = Y.$$

Clearly, P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L = \mathbb{R}^3, \ \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Thus, there exists L_P which is the converse projectors of L, and

$$K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P, \ K_P(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)dsdt,$$

then $QN: X \to Y, K_P(I-Q)N: X \to X$

$$QNz = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega (D(t)(x_0 - x(t)) - U(x(t))[a_1(t)y_1(t) + a_2(t)y_2(t)] \\ + \int_{-\infty}^t f(t-s)[b_1(s)\gamma_1(s)y_1(s) + b_2(s)\gamma_2(s)y_2(s)]ds)dt \\ \frac{1}{\omega}\int_0^\omega (-(\gamma_1(t)+D(t))y_1(t) + c_1(t)y_1(t)\int_{-\infty}^t g(t-s)U(x(s))ds)dt \\ \frac{1}{\omega}\int_0^\omega (-(\gamma_2(t)+D(t))y_2(t) + c_2(t)y_2(t)\int_{-\infty}^t g(t-s)U(x(s))ds)dt \end{pmatrix},$$

$$K_P(I-Q)Nz = \int_0^t Nz(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t Nz(s)dsdt - (\frac{t}{\omega} - \frac{1}{2})\int_0^\omega Nz(s)ds.$$

Clearly, QN and $K_P(I-Q)N$ are continuous, By Arezera-Ascoli's theorem, we can obtain that $QN(\bar{\Omega})$ and $K_P(I-Q)N : \bar{\Omega} \to \mathrm{Im}L$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L-compact on Ω.

Corresponding equation $Lz = \lambda Nz, \lambda \in (0,1)$, denote $z(t) \in X$ be a solution of system (3.1). And take $B_0 \in [M_{20}, M_{10}]$, where

$$M_{10} = \min\{M_1, \widetilde{M}_1\}, \qquad M_{20} = \max\{M_2, \widetilde{M}_2\},$$

$$\widetilde{M}_1 \triangleq \frac{(\bar{D} - (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}) + \sqrt{\widetilde{N}}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L} > 0,$$

$$\widetilde{M}_2 \triangleq \frac{(\bar{D} - (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}) - \sqrt{\widetilde{N}}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L}$$

and

$$\widetilde{N} \triangleq [(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \bar{D}]^2 - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0 > 0.$$

We will prove $[M_{20}, M_{10}]$ is not empty. Obviously, $M_2 < \widetilde{M}_1$. We next prove $\widetilde{M}_2 < M_1$. From $(H_3), M_1 > 0$,and because of $\underline{D} - (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}) > 0$, we have

$$\begin{aligned} & M_1 - \widetilde{M}_2 \\ & = \frac{\underline{D} - (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L} \\ & \quad + \frac{\sqrt{[(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \underline{D}]^2 - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L} \\ & \quad - \frac{\bar{D} - (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L} \\ & \quad - \frac{\sqrt{[(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \bar{D}]^2 - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0}}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L}\Big[-\Delta D \\
&\quad + \sqrt{[(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \underline{D}]^2 - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0} \\
&\quad + \sqrt{[(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \bar{D}]^2 - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0}\Big] \\
&= \frac{1}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L}\Big[-\Delta D \\
&\quad + \Big([(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \underline{D}]^2 - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0\Big) \\
&\quad + \Big(-\Delta D + [(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \bar{D}])^2 \\
&\qquad\qquad - 4(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L\bar{D}d_1^{-1}x_0\Big)^{1/2}\Big] \\
&> \frac{1}{2(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)L}\Big[-\Delta D \\
&\qquad\qquad + \sqrt{\Delta D^2 - 2\Delta D[(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1} - \bar{D}]}\Big] \\
&> 0,
\end{aligned}
$$

where $\Delta D = \bar{D} - \underline{D}$.so $\widetilde{M}_2 < M_1$. hence $[M_2, m_1]\bigcap[\widetilde{M}_2, \widetilde{m}_1]$ is not empty, which implies $[M_{20}, m_{10}]$ is not empty.

Define $\Omega = \{z \in X|\ |||z||| < B_0\}$, according to the proof of Theorem 1, when $x \in \partial\Omega$, we have $Lz \neq \lambda Nz, \lambda \in (0,1)$. Then condition (i) of Lemma 1 holds. Since $z \in \partial\Omega \bigcap \operatorname{Ker} L = \partial\Omega \bigcap R^3, z$ is a constant vector, and $|||z||| = B_0$. Thus we have

$$
\begin{aligned}
(d_1^{-2}x, d_2^{-2}y_1, d_3^{-2}y_2)QNz \leqslant \frac{1}{\omega}\int_0^{\omega} &[\bar{D}x_0 d_1^{-2}x - \bar{D}d_1^{-2}x^2 + Ld_1^{-2}x^2(\bar{a}_1 y_1 + \bar{a}_2 y_2) \\
&+ d_1^{-2}x\int_{-\infty}^t f(t-s)[\bar{b}_1\bar{\gamma}_1 y_1 + \bar{b}_2\bar{\gamma}_2 y_2]ds \\
&- (\bar{\gamma}_1 + \bar{D})d_2^{-2}y_1^2 + \bar{c}_1 L d_2^{-2}y_1^2\int_{-\infty}^t g(t-s)xds \\
&- (\bar{\gamma}_2 + \bar{D})d_3^{-2}y_1^2 + \bar{c}_2 L d_3^{-2}y_2^2\int_{-\infty}^t g(t-s)xds]dt \\
\leqslant [&\bar{D}x_0 d_1^{-1} - \bar{D}B_0 + LB_0^2(\bar{a}_1 d_2 + \bar{a}_2 d_3) \\
&+ (\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3)d_1^{-1}B_0 + \bar{c}_1 L d_1^{-1}B_0^2 + \bar{c}_2 L d_1^{-1}B_0^2]B_0 \\
= [&(\bar{a}_1 d_2 + \bar{a}_2 d_3 + \bar{c}_1 d_1 + \bar{c}_2 d_1)LB_0^2 \\
&+ [(\bar{b}_1\bar{\gamma}_1 d_2 + \bar{b}_2\bar{\gamma}_2 d_3]d_1^{-1} - \bar{D}]B_0 + \bar{D}x_0 d_1^{-1}]B_0 < 0.
\end{aligned}
$$

Therefore $QNz \neq 0$,, which means condition (ii) of Lemma 1 holds. Notice that

$$
\operatorname{Im} Q = \operatorname{Ker} L \quad \text{and} \quad J = -I : \operatorname{Im} Q \to \operatorname{Ker} L,\ Jz = -z.
$$

From homotopy invariant, we obtain

$$deg(JQN, \Omega \cap \mathrm{Ker} L, (0,0,0)^T) = deg(-z^T, \Omega \cap \mathrm{Ker} L, (0,0,0)^T) = 1 \neq 0,$$

which means condition (iii) of Lemma 1 holds. By Lemma 1, the proof is completed. □

4. Attractivity of periodic solution

Let $(x^*(t), y_1^*(t), y_2^*(t))^T$ be the periodic solution of system (1), $(x(t), y_1(t), y_2(t))^T$ be any solution of system (1.1), and denote

$$u(t) = x(t) - x^*(t),\ v(t) = y_1(t) - y_1^*(t),\ w(t) = y_2(t) - y_2^*(t),$$

then system (1.1) becomes

$$\begin{cases} \dot{u}(t) &= -D(t)u(t) - a_1(t)[U(x(t))y_1(t) - U(x^*(t))y_1^*(t)] \\ &\quad -a_2(t)[U(x(t))y_2(t) - U(x^*(t))y_2^*(t)] \\ &\quad + \displaystyle\int_{-\infty}^{t} f(t-s)[b_1(s)\gamma_1(s)v(s) + b_2(s)\gamma_2(s)w(s)]ds, \\ \dot{v}(t) &= -(\gamma_1(t) + D(t))v(t) + c_1(t)[y_1(t)\int_{-\infty}^{t} g(t-s)U(x(s))ds \\ &\quad -y_1^*(t)\displaystyle\int_{-\infty}^{t} g(t-s)U(x^*(s))ds], \\ \dot{w}(t) &= -(\gamma_2(t) + D(t))w(t) + c_2(t)[y_2(t)\int_{-\infty}^{t} g(t-s)U(x(s))ds \\ &\quad -y_2^*(t)\displaystyle\int_{-\infty}^{t} g(t-s)U(x^*(s))ds], \\ u(t) &= \Phi(t), v(t) = \Psi_1(t), w(t) = \Psi_2(t), -\infty < t \leqslant 0, \end{cases} \tag{4.1}$$

where $\Phi(t) = \phi(t) - x^*(t), \Psi_i(t) = \psi_i(t) - y_i^*(t), i = 1,2$. Clearly, $(x^*(t), y_1^*(t), y_2^*(t))$ is global attractive for system (1.1) if and only if the zero solution of (4.1) is global attractive.

THEOREM 3. (Semilinear equations) *If (H_1)-(H_3) hold,then the zero solution of system (4.1) is global attractive.*

Proof. For any $\Phi, \Psi_i \in C, (i = 1,2)$, we first prove

$$\lim_{t\to+\infty} \sup \|z(t)\| = 0, \tag{4.2}$$

where $z(t) = (u(t), v(t), w(t))^T$. In view of Theorem 1, for any given $\Phi, \Psi_i \in C$, we have $\|z(t)\| < B$, for $t \geqslant 0$. Thus, there is a nonnegative constant σ, such that

$$\lim_{t\to+\infty} \sup \|z(t)\| = \sigma. \tag{4.3}$$

According to definition of superior limit and (4.3), for sufficient small constant $\varepsilon > 0$, there is $t_2 > 0$, such that

$$\|z(t)\| < (1+\varepsilon)\sigma, \forall t \geqslant t_2. \tag{4.4}$$

Since $\int_0^\infty f(s)ds = \int_0^\infty g(s)ds = 1$, for the above ε and B, there must exist $T > 0$, such that

$$\int_0^\infty ((\bar{b_1}\bar{\gamma_1}d_2 + \bar{b_2}\bar{\gamma_2}d_3)d_1^{-1}Bf(s) + (\bar{c_1}+\bar{c_2})Ld_1B^2g(s))ds \leqslant \frac{1}{3}\varepsilon. \tag{4.5}$$

From (4.1), (4.4), (4.5) and Taylor's theorem, when $t \geqslant t_2 + T$, we obtain

$$\begin{aligned}
\dot{u}(t) + D(t)u(t) &\leqslant |a_1(t)[U(x(t))y_1(t) - U(x^*(t))y_1^*(t)]| \\
&\quad + |a_2(t)[U(x(t))y_2(t) - U(x^*(t))y_2^*(t)]| \\
&\quad + \left| \int_{-\infty}^t f(t-s)[b_1(s)\gamma_1(s)v(s) + b_2(s)\gamma_2(s)w(s)]ds \right| \\
&= a_1(t)[|U(x(t)) - U(x^*(t))|\,|y_1(\xi)| + |y_1(t) - y_1^*(t)|\,|U(x(\xi))|] \\
&\quad + a_2(t)[|U(x(t)) - U(x^*(t))|\,|y_2(\xi)| + |y_2(t) - y_2^*(t)|\,|U(x(\xi))|] \\
&\quad + \int_{-\infty}^t f(t-s)[b_1(s)\gamma_1(s)|v(s)| + b_2(s)\gamma_2(s)|w(s)|]ds \\
&\leqslant \bar{a}_1[L|u(t)|\,|y_1(\xi)| + |v(t)|L|x(\xi)|] + \bar{a}_2[L|u(t)|\,|y_2(\xi)| \\
&\quad + |w(t)|L|x(\xi)|] + \int_{-\infty}^t f(t-s)[\bar{b}_1\bar{\gamma}_1|v(s)| + \bar{b}_2\bar{\gamma}_2|w(s)|]ds \\
&\leqslant \bar{a}_1Ld_1d_2B[d_1^{-1}|u(t)| + d_2^{-1}|v(t)|] + \bar{a}_2Ld_1d_3B[d_1^{-1}|u(t)| \\
&\quad + d_3^{-1}|w(t)|] + \left(\int_{-\infty}^{t-T} + \int_{t-T}^t \right)(\bar{b}_1\bar{\gamma}_1|v(s)| + \bar{b}_2\bar{\gamma}_2|w(s)|) \\
&\leqslant (\bar{a}_1d_2 + \bar{a}_2d_3)Ld_1B(1+\varepsilon)\sigma + \int_T^\infty (\bar{b}_1\bar{\gamma}_1d_2 + \bar{b}_2\bar{\gamma}_2d_3)f(s)Bds \\
&\quad + (\bar{b}_1\bar{\gamma}_1d_2 + \bar{b}_2\bar{\gamma}_2d_3)(1+\varepsilon)\sigma \\
&\leqslant [(\bar{a}_1d_2 + \bar{a}_2d_3)Ld_1B + (\bar{b}_1\bar{\gamma}_1d_2 + \bar{b}_2\bar{\gamma}_2d_3)(1+\varepsilon)\sigma] + d_1\frac{\varepsilon}{3}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\dot{v}(t) + (\gamma_1(t) + D(t))v(t) &\leqslant c_1(t)\left| y_1(t)\int_{-\infty}^t g(t-s)U(x(t))ds \right. \\
&\qquad \left. - y_1^*(t)\int_{-\infty}^t g(t-s)U(x^*(t))ds \right| \\
&\leqslant c_1(t)\int_{-\infty}^t g(t-s)|y_1(t)U(x(s)) - y_1^*(t)U(x^*(s))|ds \\
&= c_1(t)\int_{-\infty}^t g(t-s)[|U(x(t)) - U(x^*(t))|\,|y_1(\xi)|
\end{aligned}$$

$$
\begin{aligned}
&\quad + |y_1(t) - y_1^*(t)|\,|U(x(\xi))|]ds \\
&\leqslant \bar{c}_1 \int_{-\infty}^{t} g(t-s)[L|u(s)|\,|y_1(\xi)| + |v(s)|L|x(\xi)|]ds \\
&\leqslant \bar{c}_1 \left(\int_{-\infty}^{t-T} + \int_{t-T}^{t} \right) g(t-s)[L|u(s)|\,|y_1(\xi)| \\
&\quad + |v(s)|L|x(\xi))|]ds \\
&\leqslant \bar{c}_1 \int_{-\infty}^{t} g(t-s) L d_1 d_2 B(1+\varepsilon)\sigma + d_2 \frac{\varepsilon}{3} \\
&\leqslant \bar{c}_1 L d_1 d_2 B(1+\varepsilon)\sigma + d_2 \frac{\varepsilon}{3},
\end{aligned}
$$

and

$$
\dot{w}(t) + (\gamma_2(t) + D(t))w(t) \leqslant \bar{c}_2 L d_1 d_3 B(1+\varepsilon)\sigma + d_3 \frac{\varepsilon}{3}.
$$

Thus, we obtain

$$
\begin{aligned}
|u(t)| &\leqslant e^{-\int_0^t D(s)ds}|u(0)| + \int_0^t e^{-\int_s^t D(u)du} \Big\{ [(\bar{a}_1 d_2 + \bar{a}_2 d_3) L d_1 B \\
&\quad + (\bar{b}_1 \bar{\gamma}_1 d_2 + \bar{b}_2 \bar{\gamma}_2 d_3)](1+\varepsilon)\sigma + d_1 \frac{\varepsilon}{3} \Big\} ds \\
&\leqslant e^{-\underline{D}t}|u(0)| + \int_0^t e^{-\underline{D}(t-s)} \Big\{ [(\bar{a}_1 d_2 + \bar{a}_2 d_3) L d_1 B \\
&\quad + (\bar{b}_1 \bar{\gamma}_1 d_2 + \bar{b}_2 \bar{\gamma}_2 d_3)](1+\varepsilon)\sigma + d_1 \frac{\varepsilon}{3} \Big\} ds, \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
|v(t)| &\leqslant e^{-\int_0^t (\gamma_1(s) + D(s))ds}|v(0)| \\
&\quad + \int_0^t e^{-\int_s^{t_1} (\gamma_1(u) + D(u))du} \left[\bar{c}_1 d_1 d_2 L B(1+\varepsilon)\sigma + d_2 \frac{\varepsilon}{3} \right] ds \\
&\leqslant e^{-\underline{D}t}|v(0)| + \int_0^t e^{-(\underline{D} + \gamma_1(t))(t-s)} \left[\bar{c}_1 d_1 d_2 L B(1+\varepsilon)\sigma + d_2 \frac{\varepsilon}{3} \right] ds, \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
|w(t)| &\leqslant e^{-\int_0^t (\gamma_2(s) + D(s))ds}|w(0)| \\
&\quad + \int_0^t e^{-\int_s^{t_1} (\gamma_2(u) + D(u))du} \left[\bar{c}_2 d_1 d_3 L B(1+\varepsilon)\sigma + d_2 \frac{\varepsilon}{3} d_3 \right] ds \\
&\leqslant e^{-\underline{D}t}|v(0)| + \int_0^t e^{-(\underline{D} + \gamma_2(t))(t-s)} \left[\bar{c}_2 d_1 d_3 L B(1+\varepsilon)\sigma + d_3 \frac{\varepsilon}{3} \right] ds. \qquad (4.8)
\end{aligned}
$$

Combining (4.6), (4.7), (4.8), we have

$$\begin{aligned}\|z(t)\| &\leqslant e^{-\underline{D}t}(d_1^{-1}|u(0)|+d_2^{-1}|v(0)|+d_3^{-1}|w(0)|)\\ &\quad +(1-e^{-\underline{D}t})\frac{1}{\underline{D}}\Big[(\bar{a}_1d_2+\bar{a}_2d_3)LB+(\bar{b}_1\bar{\gamma}_1d_2+\bar{b}_2\bar{\gamma}_2d_3)d_1^{-1}(1+\varepsilon)\sigma]\\ &\quad +\frac{\varepsilon}{3}+\bar{c}_1d_1LB(1+\varepsilon)\sigma+\frac{\varepsilon}{3}+\bar{c}_2d_1LB(1+\varepsilon)\sigma+\frac{\varepsilon}{3}\Big]\\ &\leqslant e^{-\underline{D}t}B+\frac{\varepsilon}{\underline{D}}+\frac{1}{\underline{D}}\Big[[(\bar{a}_1d_2+\bar{a}_2d_3+\bar{c}_1d_1+\bar{c}_2d_1)LB\\ &\quad +d_1^{-1}(\bar{b}_1\bar{\gamma}_1d_2+\bar{b}_2\bar{\gamma}_2d_3)\Big](1+\varepsilon)\sigma.\end{aligned}$$

Letting $t\to+\infty,\varepsilon\to 0,$ then we have

$$\sigma\leqslant\frac{1}{\underline{D}}[[(\bar{a}_1d_2+\bar{a}_2d_3+\bar{c}_1d_1+\bar{c}_2d_1)LB+d_1^{-1}(\bar{b}_1\bar{\gamma}_1d_2+\bar{b}_2\bar{\gamma}_2d_3)]\sigma.$$

If $\sigma\neq 0$, we have

$$\frac{1}{\underline{D}}[(\bar{a}_1d_2+\bar{a}_2d_3+\bar{c}_1d_1+\bar{c}_2d_1)LB+(\bar{b}_1\bar{\gamma}_1d_2+\bar{b}_2\bar{\gamma}_2d_3)d_1^{-1}]\geqslant 1,$$

which means

$$(\bar{a}_1d_2+\bar{a}_2d_3+\bar{c}_1d_1+\bar{c}_2d_1)LB^2+[(\bar{b}_1\bar{\gamma}_1d_2+\bar{b}_2\bar{\gamma}_2d_3)d_1^{-1}-\underline{D}]B\geqslant 0.$$

However, by the definition of B in Theorem 1, we have

$$(\bar{a}_1d_2+\bar{a}_2d_3+\bar{c}_1d_1+\bar{c}_2d_1)LB^2+[(\bar{b}_1\bar{\gamma}_1d_2+\bar{b}_2\bar{\gamma}_2d_3)d_1^{-1}-\underline{D}]B< 0.$$

Hence, σ must be zero. Therefore, $\lim_{t\to+\infty}\|z(t)\|=0$, which implies

$$\lim_{t\to+\infty}\|u(t)\|=0,\ \lim_{t\to+\infty}\|v(t)\|=0\quad\text{and}\quad\lim_{t\to+\infty}\|w(t)\|=0.$$

Thus, the proof is completed. □

5. Conclusions

In this paper, we have analyzed a nonautonomous chemostat model with distributed delays nutrient recycling. By employing the theory of coincidence degree and differential inequality technique, we have derived several easily verifiable sufficient conditions ensuring the existence and global attractivity of periodic solution.

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