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ROBUSTNESS OF INSTABILITY OF TWO–LAYER QUASI–GEOSTROPHIC EQUATIONS

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Abstract. In this article, we investigate the instability of two-layer quasi-geostrophic equations, which is a prototypical geophysical fluid model. It is proved that any equilibrium which is sufficiently close to an unstable equilibrium is also unstable.

1. Introduction

One of the simplest useful models in geophysical fluid dynamics that takes into consideration rotation and stratification is the two layer model for quasi-geostrophic flows without external forces (cf.[5] and [13]):

$$\frac{\partial P_i}{\partial t} + J(\Phi_i, P_i) = 0, \tag{1.1}$$

$$P_i = \Delta\Phi_i + (-1)^{i+1} F_i(\Phi_2 - \Phi_1) + \beta x_2, \quad i = 1,2, \tag{1.2}$$

where Φ_1 and Φ_2 are stream functions for the upper and lower layer of fluids respectively, $J(f,g) = \nabla^{\perp} f \cdot \nabla g$ is the Jacobi operator, where $\nabla^{\perp}$ is a normal vector of ∇. And

$$F_i = \frac{f_0^2 L^2}{g(\frac{\rho_2 - \rho_1}{\rho_0}) D_i}, \quad i = 1,2,$$

where f_0 and β are the Coriolis parameters, L is a typical horizontal length scale, g is the gravity acceleration, D_i is the depth of the i-th layer, ρ_0 is the characteristic fluid density, the densities ρ_1 and ρ_2 are different constants.

The domain Ω under consideration is a compact $2D$ Riemannian manifold with C^2 boundary. And the boundary condition are the usual ones of no normal flow in each layer, i.e.

$$\nabla^{\perp}\Phi_i \cdot \vec{n}|_{\partial\Omega} = U_i \cdot \vec{n}|_{\partial\Omega} = 0, \quad i = 1,2, \tag{1.3}$$

where $U_i = \nabla^{\perp}\Phi_i$ is the velocity in i-th layer (cf. [13]). For instance, the channel geometry $\mathbb{R}/l\mathbb{Z} \times [0,1]$, where all unknowns are periodic in the zonal (longitude x_1)

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direction with periodic l and no penetration/ no flow boundary condition in the meridional (x_2) direction, belongs to that kind of domain. Such a set of boundary conditions is more appealing than the usual doubly periodic boundary conditions. Physically the no-penetration in the latitudinal direction is closer to physics for flows on mid-latitude beta planes than the periodic boundary conditions, and the channel geometry allows us to derive conservation in time of the maximum modulus of the potential vorticity in the undamped/unforced environment (cf. [9]).

The stability and instability of quasi-geostrophic equations have been studied by many authors. In [10] and [11], Mu et al. investigated and derived the nonlinear stability criteria of quasi-geostrophic equations, which is analogous to Arnol'd's second theorem. The results establish rigorous upper bounds on both the energy and potential enstrophy of finite-amplitude disturbance to steady basic states, which are expressed in terms of the initial disturbance fields. These bounds hold uniformly in time, and tend to zero uniformly as the initial disturbance amplitude decreases to zero. In [7], Lin proved one-layer quasi-geostrophic flow is nonlinearly unstable if the linearized equation has an exponentially growing solution.

In this paper, we devote to the investigation of instability by operator theory. In general, it is a complicated but important job to deal with the linear instability. Only for some special cases, we can use normal modes or variational methods to get the unstable eigenvalue. This short paper demonstrates that an equilibrium sufficiently close to an unstable equilibrium is also unstable, which is indirect to partially do with the linear instability problem.

This paper is divided into four sections. After this introduction, which constitutes Section 1, we give some notations in Section 2. In Section 3, some properties of operators are stated. In the last section, we give the proof of the main result.

2. Preliminaries

In this section, we state some notations.

Initially, let us follow [2] and [4] to derive the definition of the classical Lyapunov exponent in Ω. Let $X_0(x,t)$ be the flow induced by the steady velocity field u_0, that is

$$\frac{\partial X_0}{\partial t} = u_0(X_0), \qquad X_0(x,0) = x, \tag{2.1}$$

and $\frac{\partial X_0}{\partial x}$ denotes the 2×2 matrix $(\partial X_0^i/\partial x^j)$ with

$$\left|\frac{\partial X_0}{\partial x}\right| = \Big(\sum_{i,j=1,2}\left|\frac{\partial X_0^i}{\partial x^j}\right|^2\Big)^{\frac{1}{2}}. \tag{2.2}$$

Thus, the classical Lyapunov exponent is defined by

$$\sigma = \sup_{x\in\Omega}\lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{\partial X_0}{\partial x}\right|(x,t) = \lim_{t\to\infty}\frac{1}{t}\sup_{x\in\Omega}\ln\left|\frac{\partial X_0}{\partial x}\right|(x,t). \tag{2.3}$$

We consider the equilibrium of (1.1) and (1.2), $(\Psi_i(x), Q_i(x))$, independent on t, where

$$Q_i = \Delta\Psi_i + (-1)^{i+1} F_i(\Psi_2 - \Psi_1) + \beta x_2, \quad u_i = \nabla^{\perp}\Psi_i. \tag{2.4}$$

A finite-amplitude disturbance (ψ_i, q_i) to the steady basic state is defined according to

$$\Phi_i = \Psi_i + \psi_i, \quad P_i = Q_i + q_i, \tag{2.5}$$

with

$$\begin{cases} q_1 = \Delta\psi_1 + F_1(\psi_2 - \psi_1), \\ q_2 = \Delta\psi_2 - F_2(\psi_2 - \psi_1), \\ \psi_1|_{\partial\Omega} = \psi_2|_{\partial\Omega} = 0. \end{cases} \tag{2.6}$$

Here, for purposes of simplification, we only consider the case that the boundary values are zero, since by (1.3), ψ_i are some constants on the boundary of Ω and the nonzero boundary problem can be transformed into the zero boundary problem with a C^2 function (cf. [3]). According to the L^2 theory of elliptic equations (cf. [1] and [3]), the operator S can be defined by

$$S\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.7}$$

Linearizing the system (1.1) and (1.2) at the equilibrium (Ψ_i, Q_i), we get

$$\frac{\partial}{\partial t}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = -\begin{pmatrix} u_1\cdot\nabla & 0 \\ 0 & u_2\cdot\nabla \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \begin{pmatrix} \nabla Q_1\cdot\nabla^{\perp} & 0 \\ 0 & \nabla Q_2\cdot\nabla^{\perp} \end{pmatrix} S\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{2.8}$$

It is convenient to define the operator

$$A = -\begin{pmatrix} u_1\cdot\nabla & 0 \\ 0 & u_2\cdot\nabla \end{pmatrix}, \quad K = \begin{pmatrix} \nabla Q_1\cdot\nabla^{\perp} & 0 \\ 0 & \nabla Q_2\cdot\nabla^{\perp} \end{pmatrix} S. \tag{2.9}$$

For the sake of simplification, let $L = A - K$, and $\mathbf{q} = (q_1, q_2)^{\top}$.

At last, we recall the Sobolev space of vector-valued functions. The H^k norm of any vector $(u_1, u_2)^{\top}$ is defined by $\|(u_1,u_2)^{\top}\|_{H^k} = (\|u_1\|_{H^k}^2 + \|u_2\|_{H^k}^2)^{\frac{1}{2}}$, for $k \geqslant 0$. And we will use the same notation H^k for the norm of in the vector-value case without any confusion.

3. Some properties of operators

In order to study the properties of operators in §2, we give the following lemma, see [8].

LEMMA 1. *For a smooth flow which is incompressible the following three conditions are equivalent:*

(a) a flow is volume preserving,

(b) $\nabla\cdot u_0 = 0$,

(c) $\det(\frac{\partial X_0}{\partial x}) = 1$.

Next, we state the following lemmas about the transport PDE.

LEMMA 2. *The operator e^{At} is an isometry on $L^2(\Omega)$ and the generator A has a purely imaginary spectrum on $L^2(\Omega)$.*

Proof. Let X_i be the flows on Ω corresponding to the vector field $-u_i$(i = 1, 2), $\nabla\cdot u_i = 0$:

$$\frac{\partial X_i}{\partial t} = -u_i(X_i), \qquad X_i(x,0) = x, \qquad x\in\Omega, \quad t\in\mathbb{R}.$$

Thus, the group e^{At} is given by the following formula for $f_i \in L^2(\Omega)$(i =1, 2),

$$e^{At}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} f_1(X_1(x,t)) \\ f_2(X_2(x,t)) \end{pmatrix}. \tag{3.1}$$

Due to Lemma 1, we can obtain

$$\|f_i(X_i(x,t))\|_{L^2}^2 = \|f_i(x)\|_{L^2}^2.$$

So, by spectral mapping theorems (cf. [12]), we can complete the proof.

LEMMA 3. *Suppose that $\sigma(-u_1) = \sigma(-u_2) = 0$. Then*

$$\lim_{t\to\infty}\frac{1}{t}\ln\|e^{At}\|_{\mathscr{L}(H^1)} = 0,$$

and the spectrum of its generator A on $H^1(\Omega)$ can't appear on the right half-plane .

Proof. For any $\mathbf{f} = (f_1, f_2)^\top \in H^1$ and any $t \geqslant 0$, by (3.1) and Lemma 1,

$$\begin{aligned} \|e^{At}\mathbf{f}\|_{H^1}^2 &\leqslant \sum_{i=1,2}\int_\Omega |f_i(X_i(x,t))|^2 + |\nabla f_i(X_i(x,t))|^2\left|\frac{\partial X_i}{\partial x}\right|^2 dx \\ &\leqslant \sum_{i=1,2}\max\left\{1, \sup_{x\in\Omega}\left|\frac{\partial X_i}{\partial x}\right|^2\right\}\int_\Omega |f_i(X_i(x,t))|^2 + |\nabla f_i(X_i(x,t))|^2 dx \\ &= \sum_{i=1,2}\|f_i\|_{H^1}^2 \max\left\{1, \sup_{x\in\Omega}\left|\frac{\partial X_i}{\partial x}\right|^2\right\}. \end{aligned} \tag{3.2}$$

By the definition of classical Lyapunov exponent (2.3) and (3.2), for all $\varepsilon > 0$, there exists a $T > 0$ such that

$$\|e^{At}\|_{\mathscr{L}(H^1)} \leqslant e^{\varepsilon t}, \qquad t \geqslant T.$$

Since ε is arbitrary, this implies $\lim_{t\to\infty}\frac{1}{t}\ln\|e^{At}\|_{\mathscr{L}(H^1)} = 0$, and the spectrum of e^{At} on $H^1(\Omega)$ is contained in the unit circle. By spectral mapping theorems, the real part of the spectrum of its generator will be less than 0 or equal to 0. Thus, the proof is completed.

Let λ be an eigenvalue of L in $L^2(\Omega)$ with $\mathrm{Re}\lambda > 0$, i.e. there exists a nonzero vector $\mathbf{q} \in L^2(\Omega)$ such that

$$\lambda \mathbf{q} = L\mathbf{q}. \tag{3.3}$$

After knowing the distribution of the the spectrum of the operator A, we rewrite (3.3) in terms of the resolvent of A. We define

$$M\mathbf{q} := M(\lambda, u_i)\mathbf{q} := -(\lambda - A)^{-1}K\mathbf{q}. \tag{3.4}$$

LEMMA 4. *M is a compact operator from L^2 to L^2, and depends analytically on λ for* $\mathrm{Re}\lambda > 0$.

Proof. According to elliptic regularity theory (cf. [3]) and the smoothness of Q_i, K is a compact operator from L^2 to L^2 since the domain Ω is compact (cf. [1]). By Lemma 3.2, the operator $-(\lambda - A)^{-1}$ is a bounded linear operator from L^2 to L^2. M is the composition of the above continuous linear operators. Hence, M is a compact operator from L^2 to L^2. We rewrite $-(\lambda - A)^{-1}$ in the integral form:

$$-(\lambda - A)^{-1} = -\int_0^\infty e^{-(\lambda - A)t}\,dt,$$

which converges in the operator norm and is analytic in the half-plane $\mathrm{Re}\lambda > 0$, where M is also analytic.

LEMMA 5. *Let $u_i, v_i \in C^3(\Omega)$ (i = 1, 2) satisfy the same conditions as Lemma 3. For $\forall \varepsilon > 0$, there exists a constant $C > 0$, which depends on ε, β, and $\|u_i\|_{C^3}$, such that*

$$\sup_{Re\lambda \geqslant \varepsilon} \|M(\lambda, u_i) - M(\lambda, v_i)\|_{\mathscr{L}(L^2)} \leqslant C\|u_i - v_i\|_{C^2}. \tag{3.5}$$

Proof. For $\mathbf{q} \in L^2(\Omega)$,

$$\begin{aligned}
& M(\lambda, u_i)\mathbf{q} - M(\lambda, v_i)\mathbf{q} \\
= & -(\lambda - A(u_i))^{-1}K(u_i)\mathbf{q} + (\lambda - A(v_i))^{-1}K(v_i)\mathbf{q} \\
= & -(\lambda - A(v_i))^{-1}(K(u_i) - K(v_i))\mathbf{q} \\
& + (\lambda - A(v_i))^{-1}\mathrm{diag}((u_1 - v_1)\cdot\nabla, (u_2 - v_2)\cdot\nabla)(\lambda - A(u_i))^{-1}K(u_i)\mathbf{q} \\
= & \; I + II.
\end{aligned} \tag{3.6}$$

Now, we estimate I and II. By Lemma 2,

$$\begin{aligned}
\|I\|_{L^2} &\leqslant C(\varepsilon)\|(K(u_i) - K(v_i))\mathbf{q}\|_{L^2} \\
&= C(\varepsilon)\|(\nabla(Q_1^1 - Q_1^2)\cdot\nabla^\perp\psi_1, \nabla(Q_2^1 - Q_2^2)\cdot\nabla^\perp\psi_2)^\top\|_{L^2} \\
&\leqslant C(\varepsilon)\max_{i=1,2}\|u_i - v_i\|_{C^2}\|(\psi_1, \psi_2)^\top\|_{H^1}
\end{aligned}$$

$$\leqslant C(\varepsilon)\max_{i=1,2}\|u_i-v_i\|_{C^2}\|\mathbf{q}\|_{L^2}, \tag{3.7}$$

where

$$Q_i^j=\Delta\Psi_i^j+(-1)^iF(\Psi_2^j-\Psi_1^j)+\beta x_2,$$

and

$$u_i=\nabla^{\perp}\Psi_i^1,\quad v_i=\nabla^{\perp}\Psi_i^2.$$

Due to Lemma 2 and Lemma 3, we get $(\lambda-A)^{-1}$ is a bounded linear operator from L^2 to L^2 and from H^1 to H^1, for $\mathrm{Re}\lambda>0$. Thus,

$$\begin{aligned}\|II\|_{L^2}&\leqslant C(\varepsilon)\|\mathrm{diag}((u_1-v_1)\cdot\nabla,(u_2-v_2)\cdot\nabla)(\lambda-A(u_i))^{-1}K(u_i)\mathbf{q}\|_{L^2}\\&\leqslant C(\varepsilon)\max_{i=1,2}\|u_i-v_i\|_{C^0}\|(\lambda-A(u_i))^{-1}K(u_i)\mathbf{q}\|_{H^1}\\&\leqslant C(\varepsilon)\max_{i=1,2}\|u_i-v_i\|_{C^0}\|K(u_i)\mathbf{q}\|_{H^1}\\&\leqslant C(\varepsilon)\max_{i=1,2}\|u_i-v_i\|_{C^0}(\max_{i=1,2}\|u_i\|_{C^3}+\beta)\|\mathbf{q}\|_{L^2}.\end{aligned}\tag{3.8}$$

This completes the proof.

4. The main result

Before the proof of main theorem, we state some propositions about the family of compact operators, which plays an important role in the proof of our theorem. First, we state the following proposition (cf. [6]).

PROPOSITION 1. *Suppose $T(z,x)$ is a family of compact operators analytic in z and jointly continuous in (z,x) for each $(z,x)\in\Lambda\times B$, where Λ is an open set in $\mathbb{C}$ and B is an interval in $\mathbb{R}$. If $I-T(z,x)$ is somewhere invertible for each x, then $(I-T(z,x))^{-1}$ is meromorphic in Λ for each x. If z_0 is not a pole of $(I-T(z,x_0))^{-1}$, then $(I-T(z,x))^{-1}$ is jointly continuous in (z,x) at (z_0,x_0). Moreover, the poles of $(I-T(z,x))^{-1}$ depend continuously on x and can appear or disappear only at the boundary of Λ (including ∞).*

As a special case of the above proposition, the analytic Fredholm theorem asserts that the set of $I-T(z,x)$ is not invertible in a discrete subset of $\mathbb{C}$ and each such z is a pole of finite multiplicity (cf. [14]).

PROPOSITION 2. *Let D be an open connected subset of $\mathbb{C}$. Let $f:D\to\mathscr{L}(H)$ be an analytic operator-valued function such that $f(z)$ is compact for each $z\in D$. Then, either*

(a) $(I-f(z))^{-1}$ exists for no $z\in D$,

or

(b) $(I-f(z))^{-1}$ exists for all $z\in D-G$ where G is a discrete subset of D (i.e. a set which has no limit points in D). In this case, $(I-f(z))^{-1}$ is meromorphic in D, analytic in $D-G$, the residues at the poles are finite rank operators, and if $z\in G$ then $f(z)\psi=\psi$ has a nonzero solution in $\mathscr{H}$.

We are now in a position to prove the main theorem.

THEOREM 1. *Let $u_i, v_i \in C^3(\Omega)$ (i =1, 2) be the velocities of an steady basic state of (1.1), (1.2) and (1.3), where $\sigma(-u_i) = \sigma(-v_i) = 0$ and $\max_{i=1,2} \|u_i - v_i\|_{C^2} \leqslant \varepsilon$. If ε is sufficiently small, then the two steady basic states have the same property of linear instability.*

Proof. First, we define the family of operators

$$T(\lambda, s) = (1-s)M(\lambda, u_i) + sM(\lambda, v_i) \tag{4.1}$$

for $\mathrm{Re}\lambda > 0$ and $s \in [0,1]$. By (3.3) and (3.4), λ is the eigenvalue of L if and only if 1 is the eigenvalue of M. According to Lemma 3.4 and Proposition 4.2, the set of such λ in the right half-plane is discrete. The operators $T(\lambda, s)$ are also compact on L^2, analytic in λ by Lemma 3.4. Due to Lemma 3.5, $T(\lambda, s)$ satisfy the following estimate

$$\|T(\lambda, s) - T(\lambda, 0)\|_{\mathscr{L}(L^2)} = |s| \|M(\lambda, u_i) - M(\lambda, v_i)\|_{\mathscr{L}(L^2)} \leqslant C\|u_i - v_i\|_{C^2} = \delta,$$

where C depends on β, $\|u_i\|_{C^3}$ and sufficiently small $\varepsilon_0 > 0$. Suppose $\mathrm{Re}\lambda_0 > 0$ is a pole of $(I - T(\cdot, 0))^{-1}$. Let ε be so small that the operator $(I - T(\cdot, 0))^{-1}$ exists on the circle $\Gamma = \{|\lambda - \lambda_0| = \varepsilon\}$. For all $s \in [0,1]$ and sufficiently small δ, $(I - T(\cdot, s))^{-1}$ also exists on the circle Γ and somewhere within the disk $\{|\lambda - \lambda_0| < \varepsilon\}$. Due to Proposition 4.1, there is a pole λ_1 of $(I - T(\cdot, 1))^{-1}$ within the disk $\{|\lambda - \lambda_0| < \varepsilon\}$. So, λ_1 is an eigenvalue for the perturbed problem with the equilibrium v_i. Hence, the proof of the theorem completes.

REMARK 1. In [13], by normal mode, the necessary and sufficient condition of linear instability of the Phillips model was obtain:

$$|U_1 - U_2|^2 > \frac{4\beta^2 F^2}{K^4(4F^2 - K^4)}, \quad \text{and } K^2 < 2F,$$

where, $u_1 = (U_1, 0)$, and $u_2 = (U_2, 0)$, U_1 and U_2 are constants, $F_1 = F_2 = F$, K is the total wave number. By virtue of the above analysis, if u_i have small disturbances, for example, $(\varepsilon \sin(\pi x_2), 0)$, for sufficiently small ε, the linear instability of the equilibrium doesn't vary. By our recent result [15], we know the nonlinear instability for Phillips model is robust as well.

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