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BLOWUP OF NONLINEAR SCHRÖDINGER EQUATIONS WITH INVERSE–SQUARE POTENTIALS

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Abstract. Blowup in finite time for nonlinear Schrödinger equations (NLS) with inverse-square potential $a|x|^{-2}$ and nonlocal nonlinearities described by integral operators $\textbf{(HE)}_a$ is considered. The local and global existence for $\textbf{(HE)}_a$ is studied in Suzuki [10]. To show the blowup for (NLS) the virial identity is important role. But the identity for $\textbf{(HE)}_a$ has not proved in consequence of the strongly singular potential. Thus we give a strict proof of the virial identity for $\textbf{(HE)}_a$.

1. Introduction and main results

In this paper we consider the blowup in finite time of weak solutions to the following Hartree type equations (nonlinear Schrödinger equations with nonlocal nonlinearities) with inverse-square potentials

$$\begin{cases} i\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + uK(|u|^2) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \qquad \textbf{(HE)}_a$$

where $i=\sqrt{-1}$, $N \geqslant 3$, $a > -(N-2)^2/4$. Here the operator K is defined as

$$Kf(x) = K(f)(x) := \int_{\mathbb{R}^N} k(x,y)f(y)\,dy, \tag{1.1}$$

where $k \in L^1_{\mathrm{loc}}(\mathbb{R}^N\times\mathbb{R}^N)$ satisfies several conditions. One of the features of $\textbf{(HE)}_a$ is the presence of the strongly singular potential $a|x|^{-2}$. In fact, $-\Delta$ and $a|x|^{-2}$ are the same scaling. Moreover, if $N=1$, then the potential $a|x|^{-2}$ is appeared in the Calogero-Moser dynamical system as many-body problem. Thus the inverse-square potential is worth observing in both mathematics and quantum physics. The other is the nonlocal nonlinearity described by the integral operator (1.1). The typical examples of the problem are so-called Hartree equations:

$$\begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta u + \lambda\, u\,(|x|^{-1} * |u|^2) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

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where $|x|^{-1} * |u|^2$ is the convolution of $|x|^{-1}$ and $|u|^2$. In fact, (1.2) is the special case of **(HE)**$_a$ with $a=0$ and $k(x,y)=\lambda|x-y|^{-1}$ ($\lambda\in\mathbb{R}$). In quantum physics, (1.2) is considered very well (see Chadam-Glassey [4] and Lieb-Simon [7]).

In this paper the (local) weak solution is called in the following sense:

DEFINITION 1.1. Given an open interval $I\subset\mathbb{R}$ containing 0, a function u is said to be a *local weak solution* to **(HE)**$_a$ on I if u belongs to $L^\infty(I;H^1(\mathbb{R}^N))\cap W^{1,\infty}(I;H^{-1}(\mathbb{R}^N))$ and satisfies **(HE)**$_a$ in the sense of $L^\infty(I;H^{-1}(\mathbb{R}^N))$. If I coincides with $\mathbb{R}$, then the local weak solution is said to be a *global weak solution* to **(HE)**$_a$.

In Suzuki [10] he showed the unique existence of global weak solution to **(HE)**$_a$ (see Theorem 1.1). In fact, he solved under the following three conditions:

(K1) k is a symmetric real-valued function, that is, $k(x,y)=k(y,x)\in\mathbb{R}$ a.a. $x,y\in\mathbb{R}^N$;

(K2) $k\in L^\infty_x(L^\infty_y)+L^\beta_x(L^\alpha_y)$ for some $\alpha,\beta\in[1,\infty]$ such that $\alpha\leqslant\beta$, $\alpha^{-1}+\beta^{-1}\leqslant 4/N$;

(K3) $k_-:=-\min\{k,0\}\in L^\infty_x(L^\infty_y)+L^{\tilde\beta}_x(L^{\tilde\alpha}_y)$ for some $\tilde\alpha,\tilde\beta\in[1,\infty]$ such that $\tilde\alpha\leqslant\tilde\beta$, $\tilde\alpha^{-1}+\tilde\beta^{-1}\leqslant 2/N$.

Here the class $L^\beta_x(L^\alpha_y)$ is the family of $k:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ such that

$$\|k\|_{L^\beta_x(L^\alpha_y)} := \left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}|k(x,y)|^\alpha\,dy\right)^{\beta/\alpha}dx\right)^{1/\beta}<\infty. \tag{1.3}$$

THEOREM 1.1. ([10, Theorem 1.1]) *Let $N\geqslant 3$ and $a>-(N-2)^2/4$. Assume that k satisfies* **(K1)–(K3)**. *Then for every $u_0\in H^1(\mathbb{R}^N)$ there exists a unique global weak solution u to* **(HE)**$_a$. *Moreover, u belongs to $C(\mathbb{R};H^1(\mathbb{R}^N))\cap C^1(\mathbb{R};H^{-1}(\mathbb{R}^N))$ and satisfies conservation laws*

$$\|u(t)\|_{L^2}=\|u_0\|_{L^2},\quad E(u(t))=E(u_0)\quad \forall\, t\in\mathbb{R}, \tag{1.4}$$

where the "energy" is defined as

$$\begin{aligned} E(\varphi) := \frac{1}{2}\|\nabla\varphi\|_{L^2}^2+\frac{a}{2}\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 \\ +\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}k(x,y)|\varphi(x)|^2|\varphi(y)|^2\,dxdy,\quad \varphi\in H^1(\mathbb{R}^N). \end{aligned} \tag{1.5}$$

Here note that to show the local existence of weak solutions to **(HE)**$_a$ **(K1)** and **(K2)** are assumed; **(K3)** is used only for global existence. **(K3)** implies that the energy functional E is almost positive. Thus we are interested in the case where **(K3)** is not satisfied and $E(u_0)$ is negative.

Blowup in finite time of weak solutions to nonlinear Schrödinger equations (NLS) is important. In particular, the finite blowup for (NLS) is related to the instability of standing waves (see Berestycki-Cazenave [1] and Weinstein [11]). First Glassey [6] proved the blowup for (NLS) with initial value $u_0\in H^1(\mathbb{R}^N)$ with $|x|u_0\in L^2(\mathbb{R}^N)$

over 30 years ago. Later, Ogawa-Tsutsumi [8] proved the blowup for (NLS) without $|x|u_0 \in L^2(\mathbb{R}^N)$ but u_0 being radial. The key of showing the finite blowup in [6] is the *virial identity*. For example, if u is a local weak solution to

$$\begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta u + \lambda\,|u|^{p-1}u & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \qquad \textbf{(NLS)}_0$$

then u satisfies the virial identity for **(NLS)**$_0$:

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\,\|\nabla u(t)\|_{L^2}^2 + \frac{4N(p-1)\lambda}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}.$$

On the one hand, formal calculation ensures the virial identity for **(HE)**$_a$:

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\,\|\nabla u(t)\|_{L^2}^2 + 8a\left\|\frac{u(t)}{|x|}\right\|_{L^2}^2 - 4\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x\cdot(\nabla_x k)(x,y)|u(t,x)|^2|u(t,y)|^2\,dxdy.$$

But the justification seems not to be simple and hence the proof has not been carried out. One of the typical reasons is the strong singularity of the potential $a|x|^{-2}$. If $N \geqslant 5$, then we can use the Rellich inequality:

$$\left\|\frac{\varphi}{|x|^2}\right\|_{L^2} \leqslant \frac{4}{N(N-4)}\|\Delta\varphi\|_{L^2} \quad \forall\,\varphi\in H^2(\mathbb{R}^N).$$

This implies that if $u\in C([-T,T];H^2(\mathbb{R}^N))$ is a local weak solution to **(HE)**$_a$, then we see that $u\in C^1([-T,T];L^2(\mathbb{R}^N))$. But the Rellich inequality does not hold if $N=3$ and $N=4$. Moreover, if $k(x,y)=U(x)U(y)$ $(U\in L^{N/2}(\mathbb{R}^N))$ with $N=3$ and $N=4$, then $K(|u|^2)u \notin L^2(\mathbb{R}^N)$ for $u\in H^2(\mathbb{R}^N)$ in general. Thus we can not construct a (strong) solution $u\in C([-T,T];H^2(\mathbb{R}^N))\cap C^1([-T,T];L^2(\mathbb{R}^N))$ to **(HE)**$_a$.

Our purpose of this paper is the blowup in finite time for weak solutions to **(HE)**$_a$ by applying the virial identity (see Section 3). To show this we add the following condition:

(K4) $\widetilde{k}\in L_y^\infty(L_x^\infty)+L_y^{\tilde\beta}(L_x^{\tilde\alpha})$ for some $\tilde\alpha,\tilde\beta\in[1,\infty]$ such that $\tilde\alpha\leqslant\tilde\beta$, $\tilde\alpha^{-1}+\tilde\beta^{-1}\leqslant 4/N$ and

$$k(x,y)+\widetilde{k}(x,y)\geqslant 0 \quad \text{a.a. } x,\,y\in\mathbb{R}^N, \tag{1.6}$$

where $\widetilde{k}$ is defined as

$$\widetilde{k}(x,y) := \frac{1}{2}[x\cdot\nabla_x k(x,y) + y\cdot\nabla_y k(x,y)].$$

Then we have the following blowup result:

THEOREM 1.2. *Let $N \geqslant 3$ and $a > -(N-2)^2/4$. Assume that k satisfies* **(K1)**, **(K2)** *and* **(K4)**. *Then for every $u_0 \in H^1(\mathbb{R}^N)$ with $|x|u_0 \in L^2(\mathbb{R}^N)$ and $E(u_0) < 0$ there exist $T_1, T_2 > 0$ such that $u \in C(\overline{I}; H^1(\mathbb{R}^N)) \cap C^1(\overline{I}; H^{-1}(\mathbb{R}^N))$ is a (unique) local weak solution to* **(HE)**$_a$ *for every open interval I with $0 \in I$ and $\overline{I} \subset (-T_1, T_2)$ and u satisfies*

$$\lim_{t \to -T_1+0} \|\nabla u(t)\|_{L^2} = \infty = \lim_{t \to T_2-0} \|\nabla u(t)\|_{L^2},$$

that is, the weak solution blows up in finite time.

Here we remark the conditions **(K3)** and **(K4)**. Let $k(x,y) := |x-y|^{-1}$. Then **(K3)** is satisfied. Moreover, $\widetilde{k}(x,y) = (-1/2)k(x,y)$. Thus we see that $k(x,y) + \widetilde{k}(x,y) = (1/2)k(x,y) \geqslant 0$ and hence k satisfies also **(K4)**. In this case the energy $E(\varphi)$ is not negative for any $\varphi \in H^1(\mathbb{R}^N)$. Therefore the negativity of the energy functional with **(K4)** concerns the blowup in finite time.

This paper is divided into five sections. In Section 2 we give some preliminary results. Notations are prepared in Section 2.1. Section 2.2 is devoted to the linear operator $-\Delta + a|x|^{-2}$ and related results. Analysis of the integral operator with kernel k in the class $L_x^\beta(L_y^\alpha)$ and the approximation are discussed in Sections 2.3. Section 3 is devoted to the justification of the virial identity. Theorem 1.2 is proved in Section 4. Finally, in Section 5 some remarks to our result are in order.

2. Notations and preliminaries

2.1. Notations

First $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with norm

$$\|u\|_{L^p} := \Big(\int_{\mathbb{R}^N} |u(x)|^p \, dx\Big)^{1/p}, \quad u \in L^p(\mathbb{R}^N) \ (1 \leqslant p < \infty),$$
$$\|u\|_{L^\infty} := \operatorname{ess\,sup} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

Let $p \in [1,\infty]$. Then $p' \in [1,\infty]$ denotes the Hölder conjugate $p' := p/(p-1)$. $H^1(\mathbb{R}^N)$ is the usual L^2-type Sobolev space with the norm

$$\|u\|_{H^1} := (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}, \quad u \in H^1(\mathbb{R}^N).$$

On the other hand, $H^{-1}(\mathbb{R}^N)$ is the dual of $H^1(\mathbb{R}^N)$. Note that we have a usual triplet

$$H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N),$$

where the inclusion is continuous and dense. In particular, we have

$$H^1(\mathbb{R}^N) \subset L^q(\mathbb{R}^N),\ L^{q'}(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N), \quad 2 \leqslant q \leqslant \frac{2N}{N-2},\ N \geqslant 3.$$

We use another L^2-type space. $D(x)$ is defined as

$$D(x) := \{u \in L^2(\mathbb{R}^N); |x|u \in L^2(\mathbb{R}^N)\}.$$

In fact, $D(x)$ is Hilbert space with the norm $\|u\|_{D(x)} := \|(1+|x|^2)^{1/2}u\|_{L^2}$, $u \in D(x)$.

Let $I \subset \mathbb{R}$ be an open interval and Y a Banach space. Then $C(\bar{I};Y)$ is a family of the (strongly) continuous Y-valued function on $\bar{I}$. On the other hand, the vector-valued Lebesgue space $L^p(I;Y)$ is equipped with norm

$$\|u\|_{L^p(I;Y)} := \big\| \|u(\cdot)\|_Y \big\|_{L^p(I)} < \infty.$$

Moreover the vector-valued Sobolev space $W^{1,p}(I;Y)$ is equipped with norm

$$\|u\|_{W^{1,p}(I;Y)} := \|u\|_{L^p(I;Y)} + \|u'\|_{L^p(I;Y)} < \infty.$$

Here u' denotes the weak derivative of u respect to time variable $t \in I$. Then it is wellknown that $W^{1,p}(I;Y) \subset C(\bar{I};Y)$ for $1 < p \leqslant \infty$.

To simplify the notation, we write

$$P_a := -\Delta + a|x|^{-2}, \quad a > -(N-2)^2/4. \tag{2.1}$$

Also we denote $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$.

2.2. The properties of the operator P_a and some closedness lemmas

First we note that P_a is nonnegative and selfadjoint in $H^{-1}(\mathbb{R}^N)$ (see [9, Theorem 5.2]). Thus we can consider the Schrödinger group $\{e^{-itP_a}\}$ in $H^{-1}(\mathbb{R}^N)$. We use the Strichartz estimates for $\{e^{-itP_a}\}$ established by Burq, Planchon, Stalker and Tahvildar-Zadeh [2] (see also [10, Lemma 2.2]):

LEMMA 2.1. *Let $N \geqslant 3$ and (p,q) be a Schrödinger admissible pair, i.e.,*

$$\frac{2}{p} + \frac{N}{q} = \frac{N}{2}, \quad p,q \geqslant 2.$$

Assume that (p_j,q_j) $(j=1,\ 2)$ are Schrödinger admissible pairs. Then

$$\Big\| \int_0^t e^{-i(t-s)P_a}\Phi(s,x)\,ds \Big\|_{L^{p_2}(\mathbb{R};L^{q_2})} \leqslant C_{p_1,p_2} \|\Phi\|_{L^{p_1'}(\mathbb{R};L^{q_1'})}, \tag{2.2}$$
$$\Phi \in L^{p_1'}(\mathbb{R};L^{q_1'}(\mathbb{R}^N)).$$

It follows from the Hardy inequality

$$\Big\| \frac{\varphi}{|x|} \Big\|_{L^2} \leqslant \frac{2}{N-2} \|\nabla\varphi\|_{L^2}, \quad \forall\, \varphi \in H^1(\mathbb{R}^N),\ N \geqslant 3, \tag{2.3}$$

that for $a > -(N-2)^2/4$, $\delta \geqslant 0$ and $\varphi \in H^1(\mathbb{R}^N)$

$$\Big(1 - \frac{4a_-}{(N-2)^2}\Big)\|\nabla\varphi\|_{L^2}^2 \leqslant \|\nabla\varphi\|_{L^2}^2 + a\int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^2+\delta}\,dx \tag{2.4}$$

$$\leqslant \Big(1+\frac{4a_+}{(N-2)^2}\Big)\|\nabla\varphi\|_{L^2}^2.$$

Also we see that for every $u \in H^1(\mathbb{R}^N)$

$$\Big(-\Delta+\frac{a}{|x|^2+\delta}\Big)u \to \Big(-\Delta+\frac{a}{|x|^2}\Big)u \quad (\delta\to 0) \text{ strongly in } H^{-1}(\mathbb{R}^N). \tag{2.5}$$

Hence it is useful to consider the approximate operator $-\Delta+a(|x|^2+\delta)^{-1},\ \delta>0$ of P_a. We use the following closedness lemma.

LEMMA 2.2. *Let $N\geqslant 3$ and $\{\delta_n\}_n\subset(0,\infty)$ be a sequence such that $\delta_n>\delta_{n+1}$ and $\delta_n\to 0$ $(n\to\infty)$. Assume that $a>-(N-2)^2/4$, $\{u_n\}_n\subset H^1(\mathbb{R}^N)$ and $u\in H^1(\mathbb{R}^N)$ satisfy*

$$u_n \to u \quad (n\to\infty) \text{ weakly in } H^1(\mathbb{R}^N),$$

$$\|\nabla u_n\|_{L^2}^2+a\Big\|\frac{u_n}{\sqrt{|x|^2+\delta_n}}\Big\|_{L^2}^2 \to \|\nabla u\|_{L^2}^2+a\Big\|\frac{u}{|x|}\Big\|_{L^2}^2 \quad (n\to\infty).$$

Then $\nabla u_n\to\nabla u$ $(n\to\infty)$ strongly in $L^2(\mathbb{R}^N)^N$.

Proof. It follows from (2.4) that

$$\begin{aligned}
\Big(1-\frac{4a_-}{(N-2)^2}\Big)&\|\nabla(u-u_n)\|_{L^2}^2\\
&\leqslant \|\nabla(u_n-u)\|_{L^2}^2+a\Big\|\frac{u_n-u}{\sqrt{|x|^2+\delta_n}}\Big\|_{L^2}^2\\
&=\Big(\|\nabla u_n\|_{L^2}^2+a\Big\|\frac{u_n}{\sqrt{|x|^2+\delta_n}}\Big\|_{L^2}^2\Big)-\Big(\|\nabla u\|_{L^2}^2+a\Big\|\frac{u}{\sqrt{|x|^2+\delta_n}}\Big\|_{L^2}^2\Big)\\
&\qquad -2\operatorname{Re}\langle\nabla u,\nabla(u_n-u)\rangle_{L^2}-2a\operatorname{Re}\Big\langle\frac{u}{|x|^2+\delta_n},u_n-u\Big\rangle_{H^{-1},H^1}\\
&=I_{1n}-I_{2n}-2\operatorname{Re}I_{3n}-2a\operatorname{Re}I_{4n}.
\end{aligned}$$

We see from the assumption that $I_{1n}\to\|\nabla u\|_{L^2}^2+a\||x|^{-1}u\|_{L^2}^2$ $(n\to\infty)$ and $I_{3n}\to 0$ $(n\to\infty)$. The dominated convergence theorem implies that $(|x|^2+\delta_n)^{-1/2}u\to|x|^{-1}u$ $(n\to\infty)$ strongly in $L^2(\mathbb{R}^N)$. Thus we have

$$I_{2n}\to\|\nabla u\|_{L^2}^2+a\Big\|\frac{u}{|x|}\Big\|_{L^2}^2 \quad (n\to\infty).$$

In a way similar to I_{2n}, (2.5) and the weak convergence u_n to u in $H^1(\mathbb{R}^N)$ yield that $I_{4n}\to 0$ $(n\to\infty)$. Therefore we obtain

$$I_{1n}-I_{2n}-2\operatorname{Re}I_{3n}-2a\operatorname{Re}I_{4n}\to 0 \quad (n\to\infty)$$

and hence we conclude $\nabla u_n\to\nabla u$ $(n\to\infty)$ strongly in $L^2(\mathbb{R}^N)^N$. □

Next lemma is used in the verification of the convergence $xu_n(t)$ to $xu(t)$.

LEMMA 2.3. *Let $u, u_n \in C([-T,T];H^1(\mathbb{R}^N))$ satisfy*

$$\|x\varphi(t)\|_{L^2}^2 = \|x\varphi(0)\|_{L^2}^2 + 4\,\mathrm{Im}\int_0^t \int_{\mathbb{R}^N} \overline{\varphi(s,x)}\, x\cdot\nabla\varphi(s,x)\,dx\,ds. \tag{2.6}$$

Assume that $\|u_n\|_{L^\infty(-T,T;H^1)} \leqslant C$ for all $n\in\mathbb{N}$, $u_n(t)\to u(t)$ $(n\to\infty)$ strongly in $H^1(\mathbb{R}^N)$ for all $t\in[-T,T]$ and $xu_n(0)\to xu(0)$ $(n\to\infty)$ strongly in $L^2(\mathbb{R}^N)^N$. Then $xu_n(t)\to xu(t)$ $(n\to\infty)$ strongly in $L^2(\mathbb{R}^N)^N$ for all $t\in[-T,T]$.

Proof. Step 1. Note that $\|xu_n(t)\|_{L^2}$ is uniformly bounded in $n\in\mathbb{N}$ and $t\in[-T,T]$. In fact, (2.6) implies

$$\|x\varphi(t)\|_{L^2}^2 \leqslant \|x\varphi(0)\|_{L^2}^2 + 2\left|\int_0^t \|x\varphi(s)\|_{L^2}^2\,ds\right| + 2\left|\int_0^t \|\nabla\varphi(s)\|_{L^2}^2\,ds\right|.$$

Hence the Gronwall inequality implies that

$$\|x\varphi(t)\|_{L^2}^2 \leqslant \left|\int_0^t e^{2|t-s|}(\|x\varphi(0)\|_{L^2}^2 + \|\nabla\varphi(s)\|_{L^2}^2)\,ds\right|.$$

It follows from the uniform boundedness of u_n in $L^\infty(-T,T;H^1(\mathbb{R}^N))$ that the uniform boundedness of xu_n in $L^\infty(-T,T;L^2(\mathbb{R}^N)^N)$.

Step 2. Since x is bounded and linear operator in $L^2(B(0,R))$, $xu_n(t)\to xu(t)$ $(n\to\infty)$ strongly in $L^2(B(0,R))^N$ for every $R>0$ and $t\in[-T,T]$.

Step 3. Next we show that $xu_n(t)\to xu(t)$ $(n\to\infty)$ weakly in $L^2(\mathbb{R}^N)^N$. First, let $\varphi\in C_0^\infty(\mathbb{R}^N)^N$. Then there exists $R>0$ such that $\mathrm{supp}\,\varphi\subset B(0,R)$. *Step 2* implies that

$$\langle xu_n(t),\varphi\rangle_{L^2} = \int_{B(0,R)} xu_n(t)\cdot\overline{\varphi}\,dx \to \int_{B(0,R)} xu(t)\cdot\overline{\varphi}\,dx = \langle xu(t),\varphi\rangle_{L^2}\ (n\to\infty)$$

for all $t\in[-T,T]$. In general case, fix $v\in L^2(\mathbb{R}^N)^N$. Then there exists $\{\varphi_m\}_m\in C_0^\infty(\mathbb{R}^N)^N$ such that $\varphi_m\to v$ $(m\to\infty)$ strongly in $L^2(\mathbb{R}^N)^N$.

$$\begin{aligned}
&|\langle xu_n(t)-xu(t),v\rangle_{L^2}|\\
&\qquad\leqslant \|xu_n(t)\|_{L^2}\|v-\varphi_m\|_{L^2} + |\langle xu_n(t)-xu(t),\varphi_m\rangle_{L^2}| + \|xu(t)\|_{L^2}\|v-\varphi_m\|_{L^2}\\
&\qquad\to C\|v-\varphi_m\|_{L^2}\ (n\to\infty)\quad \forall\, t\in[-T,T].
\end{aligned}$$

Here C is independent of n and m. Thus letting $m\to\infty$ ensures $xu_n(t)\to xu(t)$ $(n\to\infty)$ weakly in $L^2(\mathbb{R}^N)^N$ for every $t\in[-T,T]$.

Step 4. Since $\nabla u_n(t)\to\nabla u(t)$ $(n\to\infty)$ strongly in $L^2(\mathbb{R}^N)^N$ and $xu_n(t)\to xu(t)$ $(n\to\infty)$ weakly in $L^2(\mathbb{R}^N)^N$ for $t\in[-T,T]$, we see that for every $t\in[-T,T]$

$$\int_{\mathbb{R}^N} x\overline{u_n(t)}\cdot\nabla u_n(t)\,dx \to \int_{\mathbb{R}^N} x\overline{u(t)}\cdot\nabla u(t)\,dx \quad (n\to\infty).$$

On the other hand, $\|\nabla u_n(t)\|_{L^2}$ and $\|xu_n(t)\|_{L^2}$ are uniformly bounded in $n\in\mathbb{N}$ and $t\in[-T,T]$. Thus the dominated convergence theorem implies that $\|xu_n(t)\|_{L^2}^2\to\|xu(t)\|_{L^2}^2$ $(n\to\infty)$. Therefore we conclude that $xu_n(t)\to xu(t)$ $(n\to\infty)$ strongly in $L^2(\mathbb{R}^N)^N$. $\square$

REMARK 2.1. Let $u, u_n \in C([-T,T];H^1(\mathbb{R}^N))$ and $f, f_n \in L^\infty(-T,T)$. Assume that

$$xu_n(t) \to xu(t),\ \nabla u_n(t) \to \nabla u(t) \quad (n\to\infty) \text{ strongly in } L^2(\mathbb{R}^N)^N,\ t\in[-T,T]$$
$$f_n(t) \to f(t) \quad (n\to\infty), \quad |f_n(t)| \leqslant M, \quad t\in[-T,T],$$

and u_n satisfies

$$\|xu_n(t)\|_{L^2}^2 = \|xu_n(0)\|_{L^2}^2 + 4t\,\mathrm{Im}\int_{\mathbb{R}^N} x\overline{u_n(0)}\cdot\nabla u_n(0)\,dx + \int_0^t (t-s)f_n(s)\,ds.$$

Then the dominated convergence theorem asserts that

$$\|xu(t)\|_{L^2}^2 = \|xu(0)\|_{L^2}^2 + 4t\,\mathrm{Im}\int_{\mathbb{R}^N} x\overline{u(0)}\cdot\nabla u(0)\,dx + \int_0^t (t-s)f(s)\,ds.$$

2.3. Properties of kernels

In this section we consider the kernel k in the integral operator (1.1). First, we define the kernel k_R and the index γ as

$$k_R(x,y) := \begin{cases} k(x,y) & |k(x,y)| \leqslant R, \\ R & k(x,y) > R, \\ -R & k(x,y) < -R, \end{cases} \tag{2.7}$$

$$\gamma := \left[1 - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right]^{-1} \in \left[1, \frac{N}{N-2}\right]. \tag{2.8}$$

Note that $k_R \in L_x^\infty(L_y^\infty)$ with $\|k_R\|_{L_x^\infty(L_y^\infty)} \leqslant R$ and $\|k - k_R\|_{L_x^\beta(L_y^\alpha)} \to 0$ $(R\to\infty)$. Next we consider the smooth approximation of k. Let ρ_ε be the Friedrichs mollifier. Then k_ε is defined

$$k_\varepsilon(x,y) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-\xi)\rho_\varepsilon(y-\eta)k(\xi,\eta)\,d\xi\,d\eta. \tag{2.9}$$

Define $(k_R)_\varepsilon$ and $(k-k_R)_\varepsilon$ in a way similar to (2.9). The Young inequality implies that

$$\begin{aligned} \|(k_R)_\varepsilon\|_{L_x^\infty(L_y^\infty)} &\leqslant \|k_R\|_{L_x^\infty(L_y^\infty)} \leqslant R, \\ \|(k-k_R)_\varepsilon\|_{L_x^\infty(L_y^\infty)} &\leqslant \varepsilon^{-N(\alpha^{-1}+\beta^{-1})}\|\rho_1\|_{L^{\alpha'}}\|\rho_1\|_{L^{\beta'}}\|k-k_R\|_{L_x^\beta(L_y^\alpha)}, \\ \|(k-k_R)_\varepsilon\|_{L_x^\beta(L_y^\alpha)} &\leqslant \|k-k_R\|_{L_x^\beta(L_y^\alpha)}. \end{aligned} \tag{2.10}$$

Thus we see that $k_\varepsilon \in L_x^\infty(L_y^\infty)$.

Next we define two functionals G_ε and G as for every $\varphi \in H^1(\mathbb{R}^N)$

$$G_\varepsilon(\varphi) := \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k_\varepsilon(x,y)|\varphi(x)|^2|\varphi(y)|^2\,dxdy, \tag{2.11}$$

$$G(\varphi) := \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)|\varphi(x)|^2|\varphi(y)|^2\,dxdy. \tag{2.12}$$

By virtue of [10, Lemma 3.1], **(K1)** and **(K2)** imply **(G3)**:

$$|G(u)-G(v)|,\,|G_\varepsilon(u)-G_\varepsilon(v)| \leqslant c^4M^4\,\|k-k_R\|_{L^\beta_x(L^\alpha_y)} + RM^3\,\|u-v\|_{L^2} \tag{2.13}$$

for every $u, v \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1} \leqslant M$, $\|v\|_{H^1} \leqslant M$.

Next we show that k_ε is an approximation of k.

LEMMA 2.4. *Let k satisfy* **(K1)** *and* **(K2)**. *Then for every $\varphi \in H^1(\mathbb{R}^N)$*

$$\varphi(x)\int_{\mathbb{R}^N}(k_R)_\varepsilon(x,y)|\varphi(y)|^2\,dy \to \varphi(x)\int_{\mathbb{R}^N}k_R(x,y)|\varphi(y)|^2\,dy \tag{2.14}$$
$$(\varepsilon\to 0)\ \textit{strongly in}\ L^2(\mathbb{R}^N),$$

$$\varphi(x)\int_{\mathbb{R}^N}(k-k_R)_\varepsilon(x,y)|\varphi(y)|^2\,dy \to \varphi(x)\int_{\mathbb{R}^N}(k-k_R)(x,y)|\varphi(y)|^2\,dy \tag{2.15}$$
$$(\varepsilon\to 0)\ \textit{strongly in}\ L^{2\gamma}(\mathbb{R}^N).$$

Proof. Step 1. First we show that for every $f \in L^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N}(k_R)_\varepsilon(x,y)f(y)\,dy \to \int_{\mathbb{R}^N}k_R(x,y)f(y)\,dy \quad (\varepsilon\to 0)\ \text{a.a.}\ x\in\mathbb{R}^N. \tag{2.16}$$

It follows from (2.10) that $|(k_R)_\varepsilon(x,y)f(y)| \leqslant R|f(y)|$ a.a. $x,y\in\mathbb{R}^N$. Thus we see that $(k_R)_\varepsilon(x,y) \to k_R(x,y)$ $(\varepsilon\to 0)$ a.a. $x,y\in\mathbb{R}^N$. Hence the dominated convergence theorem ensures (2.16).

Step 2. Next we show that for every $f\in L^\gamma(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N}(k-k_R)_\varepsilon(x,y)f(y)\,dy \to \int_{\mathbb{R}^N}(k-k_R)(x,y)f(y)\,dy \quad (\varepsilon\to 0) \tag{2.17}$$

strongly in $L^{\gamma'}(\mathbb{R}^N)$. To end this, we divide $(k-k_R)_\varepsilon-(k-k_R)$ into

$$\rho_\varepsilon(x)*[\rho_\varepsilon(y)*\ell_R(x,y)-\ell_R(x,y)]+[\rho_\varepsilon(x)*\ell_R(x,y)-\ell_R(x,y)],$$

where $\ell_R := k-k_R$. Since $\rho_\varepsilon(y)*\ell_R(x,\cdot)\to\ell_R(x,\cdot)$ $(\varepsilon\to 0)$ strongly in $L^\alpha(\mathbb{R}^N)$ a.a. $x\in\mathbb{R}^N$, we have

$$\rho_\varepsilon(x)*\int_{\mathbb{R}^N}[\rho_\varepsilon(y)*\ell_R(x,y)-\ell_R(x,y)]f(y)\,dy \to 0 \quad (\varepsilon\to 0)\ \text{strongly in}\ L^{\gamma'}(\mathbb{R}^N).$$

On the other hand, since $\int_{\mathbb{R}^N}\ell_R(x,y)f(y)\,dy \in L^{\gamma'}(\mathbb{R}^N)$, we obtain

$$\rho_\varepsilon(x)*\int_{\mathbb{R}^N}\ell_R(x,y)f(y)\,dy \to \int_{\mathbb{R}^N}\ell_R(x,y)f(y)\,dy \quad (\varepsilon\to 0)\ \text{strongly in}\ L^{\gamma'}(\mathbb{R}^N).$$

Therefore we conclude (2.17).

Step 3. Fix $\varphi\in H^1(\mathbb{R}^N)$. Then it follows from (2.16) and the dominated convergence theorem that (2.14). On the other hand, (2.17) implies (2.15). □

REMARK 2.2. Lemma 2.4 implies that if k satisfies **(K1)** and **(K2)**, then

$$G_\varepsilon(\varphi) \to G(\varphi) \quad (\varepsilon \to 0), \quad \varphi \in H^1(\mathbb{R}^N). \tag{2.18}$$

Finally, we consider $x \cdot \nabla_x k(x,y)$ and $x \cdot \nabla_x k_\varepsilon(x,y)$. First note that **(K1)** implies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x \cdot \nabla_x k(x,y) f(x) f(y)\, dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} y \cdot \nabla_y k(x,y) f(x) f(y)\, dxdy.$$

Therefore we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x \cdot \nabla_x k(x,y) f(x) f(y)\, dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \widetilde{k}(x,y) f(x) f(y)\, dxdy. \tag{2.19}$$

Now we define

$$\widetilde{k_\varepsilon}(x,y) := \frac{1}{2}[x \cdot \nabla_x k_\varepsilon(x,y) + y \cdot \nabla_y k_\varepsilon(x,y)]. \tag{2.20}$$

Note that if k satisfies **(K1)**, then $\widetilde{k_\varepsilon}$ also satisfies **(K1)**. Here $\widetilde{k_\varepsilon}$ is not the approximation of $\widetilde{k}$ defined as in (2.9), but $\widetilde{k_\varepsilon}$ can be rewritten by using k and $\widetilde{k}$. In fact, we obtain

$$\widetilde{k_\varepsilon}(x,y) = (\widetilde{k})_\varepsilon(x,y) + \frac{1}{2}\widetilde{\rho}_\varepsilon(x) * \rho_\varepsilon(y) * k(x,y) + \frac{1}{2}\rho_\varepsilon(x) * \widetilde{\rho}_\varepsilon(y) * k(x,y),$$

where $(\widetilde{k})_\varepsilon$ is the smooth approximation of $\widetilde{k}$ defined as in (2.9) and $\widetilde{\rho}_\varepsilon(x) := N\rho_\varepsilon(x) + x \cdot \nabla_x \rho_\varepsilon(x)$. As like the Friedrichs mollifier, we see that

$$\widetilde{\rho}_\varepsilon * f \to 0 \quad (\varepsilon \to 0) \text{ strongly in } L^p(\mathbb{R}^N), \quad f \in L^p(\mathbb{R}^N), (1 \leqslant p < \infty),$$
$$\widetilde{\rho}_\varepsilon * f \to 0 \quad (\varepsilon \to 0) \text{ a.e. on } \mathbb{R}^N, \quad f \in L^\infty(\mathbb{R}^N).$$

Therefore Lemma 2.4 and (2.13) yield the following lemma.

LEMMA 2.5. *Let k satisfy* **(K1)**, **(K2)** *and* **(K4)**. *Define*

$$\widetilde{G}_\varepsilon(\varphi) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \widetilde{k_\varepsilon}(x,y)|\varphi(x)|^2|\varphi(y)|^2\, dxdy, \tag{2.21}$$

$$\widetilde{G}(\varphi) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \widetilde{k}(x,y)|\varphi(x)|^2|\varphi(y)|^2\, dxdy, \tag{2.22}$$

for $\varphi \in H^1(\mathbb{R}^N)$. Then for any $\varphi \in H^1(\mathbb{R}^N)$

$$\widetilde{G}_\varepsilon(\varphi) \to \widetilde{G}(\varphi) \quad (\varepsilon \to 0) \tag{2.23}$$

and for every $\varepsilon > 0$ and $u, v \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1} \leqslant M$ and $\|v\|_{H^1} \leqslant M$

$$\begin{aligned} |\widetilde{G}_\varepsilon(u) - \widetilde{G}_\varepsilon(v)| \leqslant 4M^4 \,(\|\widetilde{k} - \widetilde{k}_{\widetilde{R}}\|_{L_x^{\widetilde{\beta}}(L_y^{\widetilde{\alpha}})} + \|\widetilde{\rho}_1\|_{L^1}\|k - k_R\|_{L_x^\beta(L_y^\alpha)}) \\ + 4M^3 \,(\|\widetilde{\rho}_1\|_{L^1} R + \widetilde{R})\, \|u - v\|_{L^2}. \end{aligned} \tag{2.24}$$

REMARK 2.3. More precisely, we have

$$|\widetilde{G}_\varepsilon(u) - \widetilde{G}_\varepsilon(v)| \leqslant C(k) M^3 \|u - v\|_{H^1} \tag{2.25}$$

for every $\varepsilon > 0$ and $u, v \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1} \leqslant M$ and $\|v\|_{H^1} \leqslant M$. Here we do not need $k + \widetilde{k} \geqslant 0$ as in **(K4)** and $\widetilde{k} - \widetilde{k}_{\widetilde{R}} \to 0$ ($\widetilde{R} \to \infty$) for (2.25).

3. Proof of the virial identity

In this section we show the key identity which is called *the virial identity* of **(HE)** $_a$:

$$\|xu(t)\|_{L^2}^2 = \|xu_0\|_{L^2}^2 + 4t\,\mathrm{Im}\int_{\mathbb{R}^N} x\overline{u_0}\cdot\nabla u_0\,dx + \int_0^t (t-s)V(u(s))\,ds \tag{3.1}$$

for any local weak solutions to **(HE)** $_a$ with initial value $u(0) = u_0 \in H^1(\mathbb{R}^N)\cap D(x)$. Here

$$V(\varphi) := 8\,\|\nabla\varphi\|_{L^2}^2 + 8a\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 - 4\widetilde{G}(\varphi) \quad \varphi\in H^1(\mathbb{R}^N)\cap D(x) \tag{3.2}$$

($\widetilde{G}$ is defined in (2.22)). We divide the proof of (3.1) into four stages:

Stage 1. First we construct approximate solutions to **(HE)** $_a$:

$$\begin{cases} i\dfrac{\partial v_{\varepsilon,\delta}}{\partial t} = -\Delta v_{\varepsilon,\delta} + \dfrac{a\,v_{\varepsilon,\delta}}{|x|^2+\delta} + v_{\varepsilon,\delta}K_\varepsilon(|v_{\varepsilon,\delta}|^2) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ v_{\varepsilon,\delta}(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{HE$_a^{\varepsilon,\delta}$}$$

where $a\in\mathbb{R}$, $\varepsilon>0$, $\delta>0$ and

$$K_\varepsilon f(x) := K_\varepsilon(f)(x) = \int_{\mathbb{R}^N} k_\varepsilon(x,y)f(y)\,dy,$$

where k_ε is defined in (2.9).

Stage 2. We derive the virial identity for **(HE)** $_a^{\varepsilon,\delta}$.

Stage 3. Next we consider

$$\begin{cases} i\dfrac{\partial u_\varepsilon}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u_\varepsilon + u_\varepsilon K_\varepsilon(|u_\varepsilon|^2) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u_\varepsilon(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{HE$_a^{\varepsilon}$}$$

The solution is the limit of $v_{\varepsilon,\delta}$ ($\delta\to 0$). By letting $\delta\to 0$ of **(HE)** $_a^{\varepsilon,\delta}$ we confirm the virial identity for **(HE)** $_a^{\varepsilon}$.

Stage 4. We verify the virial identity for **(HE)** $_a$ by letting $\varepsilon\to 0$ of **(HE)** $_a^{\varepsilon}$.

Now we begin to prove (3.1).

Stage 1 of proof (3.1). First we consider the approximate problem **(HE)** $_a^{\varepsilon,\delta}$ of **(HE)** $_a$ to obtain the virial identity. Note that $a(|x|^2+\delta)^{-1}\in L^\infty(\mathbb{R}^N)$. Since $k_\varepsilon \in L_x^\infty(L_y^\infty)$, K_ε is locally Lipschitz continuous in $L^2(\mathbb{R}^N)$. Therefore [3, Theorem 3.3.1] yields that for every $u_0\in H^1(\mathbb{R}^N)$ there exists a global unique weak solution $v_{\varepsilon,\delta}\in C(\mathbb{R};H^1(\mathbb{R}^N))\cap C^1(\mathbb{R};H^{-1}(\mathbb{R}^N))$ to **(HE)** $_a^{\varepsilon,\delta}$. Moreover, $v_{\varepsilon,\delta}$ satisfies the conservation laws:

$$\|v_{\varepsilon,\delta}(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E_{\varepsilon,\delta}(v_{\varepsilon,\delta}(t)) = E_{\varepsilon,\delta}(u_0) \quad \forall\, t\in\mathbb{R}, \tag{3.3}$$

where

$$E_{\varepsilon,\delta}(\varphi)=\frac{1}{2}\|\nabla\varphi\|_{L^2}^2+\frac{a}{2}\left\|\frac{\varphi}{\sqrt{|x|^2+\delta}}\right\|_{L^2}^2+G_\varepsilon(\varphi),\quad \varphi\in H^1(\mathbb{R}^N).$$

Furthermore, if $u_0\in H^2(\mathbb{R}^N)$, then $v_{\varepsilon,\delta}\in C(\mathbb{R};H^2(\mathbb{R}^N))\cap C^1(\mathbb{R};L^2(\mathbb{R}^N))$.

Stage 2 of proof (3.1). Now we show the virial identity for $\textbf{(HE)}_a^{\varepsilon,\delta}$. First we calculate the time derivative of $\|xv_{\varepsilon,\delta}(t)\|_{L^2}^2$. As in [3, Lemma 6.5.2] we obtain

$$\frac{d}{dt}\|xv_{\varepsilon,\delta}(t)\|_{L^2}^2=4\,\mathrm{Im}\int_{\mathbb{R}^N}x\overline{v_{\varepsilon,\delta}(t)}\cdot\nabla v_{\varepsilon,\delta}(t)\,dx. \tag{3.4}$$

Next we consider the second derivative of $\|xv_{\varepsilon,\delta}(t)\|_{L^2}^2$ respect to t.

LEMMA 3.1. *Let $v_{\varepsilon,\delta}$ be a global weak solution to $\textbf{(HE)}_a^{\varepsilon,\delta}$ with $v_{\varepsilon,\delta}(0)=u_0$. Assume that $u_0\in H^1(\mathbb{R}^N)\cap D(x)$. Then $v_{\varepsilon,\delta}$ satisfies*

$$\frac{d^2}{dt^2}\|xv_{\varepsilon,\delta}(t)\|_{L^2}^2=V_{\varepsilon,\delta}(v_{\varepsilon,\delta}(t))\quad\forall\,t\in\mathbb{R}, \tag{3.5}$$

where $V_{\varepsilon,\delta}$ is defined as

$$V_{\varepsilon,\delta}(\varphi):=8\,\|\nabla\varphi\|_{L^2}^2+8a\left\|\frac{x\varphi}{|x|^2+\delta}\right\|_{L^2}^2-4\widetilde{G}_\varepsilon(\varphi),\quad\varphi\in H^1(\mathbb{R}^N). \tag{3.6}$$

Proof. Step 1. Assume further that $u_0\in H^2(\mathbb{R}^N)\cap D(x)$. Then the weak solution $v_{\varepsilon,\delta}$ belongs to $C(\mathbb{R};H^2(\mathbb{R}^N))\cap C^1(\mathbb{R};L^2(\mathbb{R}^N))$. Fix $\mu>0$. Now we calculate

$$\frac{d^2}{dt^2}\left\|\frac{xv_{\varepsilon,\delta}(t)}{\sqrt{1+\mu|x|^2}}\right\|_{L^2}^2=\frac{d}{dt}\mathrm{Im}\int_{\mathbb{R}^N}\frac{4x\overline{v_{\varepsilon,\delta}(t)}}{(1+\mu|x|^2)^2}\cdot\nabla v_{\varepsilon,\delta}(t)\,dx$$

and let $\mu\to0$. Applying the integral by parts we obtain

$$\begin{aligned}\frac{d^2}{dt^2}\left\|\frac{xv_{\varepsilon,\delta}(t)}{\sqrt{1+\mu|x|^2}}\right\|_{L^2}^2&=-\mathrm{Im}\int_{\mathbb{R}^N}\Big(M_\mu(x)\overline{v_{\varepsilon,\delta}(t)}+\frac{8x\cdot\overline{\nabla v_{\varepsilon,\delta}(t)}}{(1+\mu|x|^2)^2}\Big)v'_{\varepsilon,\delta}(t)\,dx\\&=I_0(t;\mu)+I_1(t;\mu)+I_2(t;\mu)+I_3(t;\mu),\end{aligned} \tag{3.7}$$

where

$$M_\mu(x):=\mathrm{div}\Big(\frac{4x}{(1+\mu|x|^2)^2}\Big)=\frac{4N-16}{(1+\mu|x|^2)^2}+\frac{16}{(1+\mu|x|^2)^3},$$

$$I_0(t;\mu):=\mathrm{Re}\int_{\mathbb{R}^N}M_\mu(x)\overline{v_{\varepsilon,\delta}(t)}\Big[-\Delta v_{\varepsilon,\delta}(t)+\frac{av_{\varepsilon,\delta}(t)}{|x|^2+\delta}+v_{\varepsilon,\delta}(t)K_\varepsilon(|v_{\varepsilon,\delta}(t)|^2)\Big]\,dx,$$

$$I_1(t;\mu):=\mathrm{Re}\int_{\mathbb{R}^N}\frac{8x\cdot\overline{\nabla v_{\varepsilon,\delta}(t)}}{(1+\mu|x|^2)^2}\,[-\Delta v_{\varepsilon,\delta}(t)]\,dx,$$

$$I_2(t;\mu) := \mathrm{Re}\int_{\mathbb{R}^N} \frac{8x\cdot\overline{\nabla v_{\varepsilon,\delta}(t)}}{(1+\mu|x|^2)^2}\,\frac{a\,v_{\varepsilon,\delta}(t)}{|x|^2+\delta}\,dx,$$

$$I_3(t;\mu) := \mathrm{Re}\int_{\mathbb{R}^N} \frac{8x\cdot\overline{\nabla v_{\varepsilon,\delta}(t)}}{(1+\mu|x|^2)^2}\,v_{\varepsilon,\delta}(t)\,K_\varepsilon(|v_{\varepsilon,\delta}(t)|^2)\,dx.$$

Since $|M_\mu(x)| \leqslant 4N$ and $M_\mu(x)\to 4N$ $(\mu\to 0)$ for all $x\in\mathbb{R}^N$, we have

$$I_0(t;\mu) \to 4N\,\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2 + 4aN\left\|\frac{v_{\varepsilon,\delta}(t)}{\sqrt{|x|^2+\delta}}\right\|_{L^2}^2 + 4N\int_{\mathbb{R}^N} |v_{\varepsilon,\delta}(t)|^2 K_\varepsilon(|v_{\varepsilon,\delta}(t)|^2)\,dx \quad (\mu\to 0)\ \forall\, t\in\mathbb{R}. \tag{3.8}$$

Next we consider $I_1(t;\mu)$. Integrating by parts we have $I_1(t;\mu) = I_{11}(t;\mu)+I_{12}(t;\mu)$, where

$$\begin{aligned} I_{11}(t;\mu) &= \sum_{j,l=1}^N \mathrm{Re}\int_{\mathbb{R}^N} \frac{8\delta_{jl}(1+\mu|x|^2) - 32\mu x_l x_j}{(1+\mu|x|^2)^3}\,\overline{\frac{\partial v_{\varepsilon,\delta}(t)}{\partial x_l}}\,\frac{\partial v_{\varepsilon,\delta}(t)}{\partial x_j}\,dx \\ &= \int_{\mathbb{R}^N}\Big[\frac{8}{(1+\mu|x|^2)^2}|\nabla v_{\varepsilon,\delta}(t)|^2 - \frac{32\mu|x\cdot\nabla v_{\varepsilon,\delta}(t)|^2}{(1+\mu|x|^2)^3}\Big]\,dx, \end{aligned}$$

$$\begin{aligned} I_{12}(t;\mu) &= \sum_{j,l=1}^N \mathrm{Re}\int_{\mathbb{R}^N} \frac{8x_l}{(1+\mu|x|^2)^2}\,\overline{\frac{\partial^2 v_{\varepsilon,\delta}(t)}{\partial x_l\partial x_j}}\,\frac{\partial v_{\varepsilon,\delta}(t)}{\partial x_j}\,dx \\ &= \sum_{j,l=1}^N \int_{\mathbb{R}^N} \frac{4x_l}{(1+\mu|x|^2)^2}\,\frac{\partial}{\partial x_l}\left|\frac{\partial v_{\varepsilon,\delta}(t)}{\partial x_j}\right|^2 dx = -\int_{\mathbb{R}^N} M_\mu(x)\,|\nabla v_{\varepsilon,\delta}(t)|^2\,dx. \end{aligned}$$

Thus we obtain

$$I_1(t;\mu) \to (8-4N)\,\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2 \quad (\mu\to 0)\ \forall\, t\in\mathbb{R}. \tag{3.9}$$

Integrating by parts we see the convergence of I_2:

$$\begin{aligned} I_2(t;\mu) &= -\int_{\mathbb{R}^N} |v_{\varepsilon,\delta}(t)|^2\,\mathrm{div}\Big(\frac{4x}{(1+\mu|x|^2)^2}\,\frac{a}{|x|^2+\delta}\Big)\,dx \\ &\to -\int_{\mathbb{R}^N} |v_{\varepsilon,\delta}(t)|^2\Big(\frac{4aN}{|x|^2+\delta} - \frac{8a|x|^2}{(|x|^2+\delta)^2}\Big)\,dx \quad (\mu\to 0), \quad t\in\mathbb{R}. \end{aligned} \tag{3.10}$$

Finally we consider I_3. In a way similar to I_2, we calculate

$$I_3(t;\mu) = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big[M_\mu(x)k_\varepsilon(x,y) + \frac{4x\cdot\nabla_x k_\varepsilon(x,y)}{(1+\mu|x|^2)^2}\Big]|v_{\varepsilon,\delta}(t,x)|^2|v_{\varepsilon,\delta}(t,y)|^2\,dxdy.$$

Thus applying (2.19) we obtain for $t\in\mathbb{R}$

$$I_3(t;\mu) \to -\int_{\mathbb{R}^N} [4N\,k_\varepsilon(x,y) + 4\widetilde{k_\varepsilon}(x,y)]|v_{\varepsilon,\delta}(t,x)|^2|v_{\varepsilon,\delta}(t,y)|^2\,dxdy \tag{3.11}$$

as $\mu \to 0$. Therefore we see that for $t \in \mathbb{R}$

$$\begin{aligned} &I_0(t;\mu)+I_1(t;\mu)+I_2(t;\mu)+I_3(t;\mu) \\ &\quad \to 8\,\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2 + 8a\left\|\frac{x\,v_{\varepsilon,\delta}(t)}{|x|^2+\delta}\right\|_{L^2}^2 - 4\widetilde{G}_\varepsilon(v_{\varepsilon,\delta}(t)) \quad (\mu \to 0) \\ &\quad = V_{\varepsilon,\delta}(v_{\varepsilon,\delta}(t)). \end{aligned} \tag{3.12}$$

Step 2. Next we show the uniform boundedness:

$$|I_j(t;\mu)| \leqslant C_j(\varepsilon, \|u_0\|_{H^1}) \quad \forall\, \delta>0, \forall\, \mu>0, \forall\, t\in\mathbb{R}. \tag{3.13}$$

To end this first we confirm

$$\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2} \leqslant C(\varepsilon, \|u_0\|_{H^1}) \quad \forall\, t\in\mathbb{R}, \forall\, \delta>0. \tag{3.14}$$

By using the conservation laws (3.3) we calculate

$$\begin{aligned} &\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2 + a\int_{\mathbb{R}^N} \frac{|v_{\varepsilon,\delta}(t)|^2}{|x|^2+\delta}\,dx \\ &\qquad = \|\nabla u_0\|_{L^2}^2 + a\int_{\mathbb{R}^N} \frac{|u_0|^2}{|x|^2+\delta}\,dx + 2\,[G_\varepsilon(u_0) - G_\varepsilon(v_{\varepsilon,\delta}(t))]. \end{aligned} \tag{3.15}$$

It follows from (2.13) with $k_\varepsilon \in L^\infty_x(L^\infty_y)$ and (3.3) that

$$|G_\varepsilon(v_{\varepsilon,\delta}(t))| \leqslant 4\,\|k_\varepsilon\|_{L^\infty_x(L^\infty_y)}\,\|u_0\|_{L^2}^4.$$

Applying (2.4) to (3.15) we obtain

$$\left(1-\frac{4a_-}{(N-2)^2}\right)\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2 \leqslant \left(1+\frac{4a_+}{(N-2)^2}\right)\|\nabla u_0\|_{L^2}^2 + 16\,\|k_\varepsilon\|_{L^\infty_x(L^\infty_y)}\|u_0\|_{L^2}^4.$$

This is nothing but (3.14).

Now we evaluate I_j ($j=0,1,2,3$). We divide I_0 into $I_{01}+I_{02}+I_{03}$, where

$$\begin{aligned} I_{01}(t;\mu) &:= \mathrm{Re}\int_{\mathbb{R}^N} M_\mu(x)\overline{v_{\varepsilon,\delta}(t)}[-\Delta v_{\varepsilon,\delta}(t)]\,dx, \\ I_{02}(t;\mu) &:= \int_{\mathbb{R}^N} \frac{a M_\mu(x)}{|x|^2+\delta}\,|v_{\varepsilon,\delta}(t)|^2\,dx, \\ I_{03}(t;\mu) &:= \int_{\mathbb{R}^N} M_\mu(x)|v_{\varepsilon,\delta}(t)|^2 K_\varepsilon(|v_{\varepsilon,\delta}(t)|^2)\,dx. \end{aligned}$$

For I_{01}, integrating by parts we see

$$\begin{aligned} &\int_{\mathbb{R}^N} M_\mu(x)\overline{v_{\varepsilon,\delta}(t)}[-\Delta v_{\varepsilon,\delta}(t)]\,dx \\ &\qquad = \int_{\mathbb{R}^N} [\nabla M_\mu(x)\cdot\overline{v_{\varepsilon,\delta}(t)}\nabla v_{\varepsilon,\delta}(t) + M_\mu(x)|\nabla v_{\varepsilon,\delta}(t)|^2]\,dx. \end{aligned}$$

We see that $|\nabla M_\mu(x)| \leqslant 16(N+2)$ by a simple calculation. Thus we obtain

$$|I_{01}(t;\mu)| \leqslant 16(N+2)\,\|v_{\varepsilon,\delta}(t)\|_{L^2}\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2} + 4N\,\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2.$$

For I_{02} and I_{03}, we see

$$|I_{02}(t;\mu)| \leqslant \int_{\mathbb{R}^N} |M_\mu(x)| \frac{|a|\,|v_{\varepsilon,\delta}(t)|^2}{|x|^2+\delta}\,dx \leqslant \frac{16N\,|a|}{(N-2)^2}\,\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2,$$

$$\begin{aligned}|I_{03}(t;\mu)| &\leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |M_\mu(x)|\,|k_\varepsilon(x,y)|\,|v_{\varepsilon,\delta}(t,x)|^2|v_{\varepsilon,\delta}(t,y)|^2\,dxdy \\ &\leqslant 4N\,\|k_\varepsilon\|_{L^\infty_x(L^\infty_y)}\,\|v_{\varepsilon,\delta}(t)\|_{L^2}^4.\end{aligned}$$

Therefore we obtain the uniform boundedness of I_0:

$$\begin{aligned}|I_0(t;\mu)| \leqslant N\Big[4+\frac{16|a|}{(N-2)^2}\Big]\,C(\varepsilon,\,\|u_0\|_{H^1})^2 + 16(N+2)\,\|u_0\|_{L^2}\,C(\varepsilon,\,\|u_0\|_{H^1}) \\ + 4N\,\|k_\varepsilon\|_{L^\infty_x(L^\infty_y)}\,\|u_0\|_{L^2}^4 \quad \forall\, t\in\mathbb{R}. \qquad (3.16)\end{aligned}$$

For I_1, we can calculate

$$\begin{aligned}|I_1(t;\mu)| &\leqslant \int_{\mathbb{R}^N} \frac{8\,|\nabla v_{\varepsilon,\delta}(t)|^2}{(1+\mu|x|^2)^2}\,dx \qquad (3.17)\\ &\quad + \int_{\mathbb{R}^N} \frac{32\mu\,|x\cdot\nabla v_{\varepsilon,\delta}(t)|^2}{(1+\mu|x|^2)^3}\,dx + \int_{\mathbb{R}^N} |M_\mu(x)|\,|\nabla v_{\varepsilon,\delta}(t)|^2\,dx \\ &\leqslant 4(N+10)\,C(\varepsilon,\,\|u_0\|_{H^1})^2 \quad \forall\, t\in\mathbb{R}.\end{aligned}$$

For I_2, using (2.3) and (3.14) we have

$$\begin{aligned}|I_2(t;\mu)| &\leqslant \int_{\mathbb{R}^N} \frac{|a|M_\mu(x)\,|v_{\varepsilon,\delta}(t)|^2}{|x|^2+\delta}\,dx + \int_{\mathbb{R}^N} \frac{8|a|\,|x|^2\,|v_{\varepsilon,\delta}(t)|^2}{(1+\mu|x|^2)^2(|x|^2+\delta)^2}\,dx \qquad (3.18)\\ &\leqslant \frac{16(N+2)|a|}{(N-2)^2}\,C(\varepsilon,\,\|u_0\|_{H^1})^2 \quad \forall\, t\in\mathbb{R}.\end{aligned}$$

For I_3, note that $|x\cdot\nabla_x k_\varepsilon(x,y)|$ can be evaluated as $\|x\cdot\nabla_x k_\varepsilon\|_{L^\infty_x(L^\infty_y)} \leqslant C_k(\varepsilon)$. Hence we see that

$$|I_3(t;\mu)| \leqslant 4[N\,\|k_\varepsilon\|_{L^\infty_x(L^\infty_y)} + C_k(\varepsilon)]\,\|u_0\|_{L^2}^4 \quad \forall\, t\in\mathbb{R}. \qquad (3.19)$$

Since (3.16)–(3.19) are proved, we obtain (3.13),

Step 3. Combining (3.13) and (3.12) into (3.7), Remark 2.1 ensures the virial identity for $\mathbf{(HE)}_a^{\varepsilon,\delta}$ (3.5) when $u_0 \in H^2(\mathbb{R}^N)\cap D(x)$.

Step 4. We remark that *Step 2* yields

$$|V_{\varepsilon,\delta}(v_{\varepsilon,\delta}(t))| \leqslant \widetilde{C}(\varepsilon,\,\|u_0\|_{H^1}) \quad \forall\, t\in\mathbb{R}, \forall\, \delta>0. \qquad (3.20)$$

Step 5. Let $u_0 \in H^1(\mathbb{R}^N) \cap D(x)$. Then there exists $\{u_{0m}\}_m \subset H^2(\mathbb{R}^N) \cap D(x)$ such that $u_{0m} \to u_0$ $(m \to \infty)$ in $H^1(\mathbb{R}^N) \cap D(x)$. Denote $v^m_{\varepsilon,\delta}$ as a global weak solution to **(HE)**$_a$ with initial value $v^m_{\varepsilon,\delta}(0) = u_{0m}$. Since **(HE)**$_a^{\varepsilon,\delta}$ verifies the continuous dependence of initial value (see [3, Theorem 3.3.1]), we see that $v^m_{\varepsilon,\delta}(t) \to v_{\varepsilon,\delta}(t)$ $(m \to \infty)$ in $C([-T,T];H^1(\mathbb{R}^N))$. Thus $V_{\varepsilon,\delta}(v^m_{\varepsilon,\delta}(t)) \to V_{\varepsilon,\delta}(v_{\varepsilon,\delta}(t))$ $(m \to \infty)$ for $t \in [-T,T]$. Also (3.20) ensures that $|V_{\varepsilon,\delta}(v^m_{\varepsilon,\delta}(t))| \leqslant C$ for $t \in [-T,T]$ and $m \in \mathbb{N}$. Finally, Lemma 2.3 implies that $xv^m_{\varepsilon,\delta}(t) \to xv_{\varepsilon,\delta}(t)$ $(m \to \infty)$ strongly in $L^2(\mathbb{R}^N)^N$ for every $t \in \mathbb{R}$. Thus Remark 2.1 asserts that (3.5) holds even if $u_0 \in H^1(\mathbb{R}^N) \cap D(x)$. □

Stage 3 of proof (3.1). First note that $\|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}$ is uniformly bounded in $t \in \mathbb{R}$ and $\delta > 0$ [see (3.14)]. Next we prove that $\|v'_{\varepsilon,\delta}(t)\|_{H^{-1}}$ is uniformly bounded in $t \in \mathbb{R}$ and $\delta > 0$. By using (2.4) and **(G2)** in [10, Lemma 3.1] we have

$$\|v'_{\varepsilon,\delta}(t)\|_{H^{-1}} \leqslant \Big(1 + \frac{4|a|}{(N-2)^2}\Big)\|v_{\varepsilon,\delta}(t)\|_{H^1} + \|k_\varepsilon\|_{L^\infty_x(L^\infty_y)}\|v_{\varepsilon,\delta}(t)\|^3_{L^2}.$$

Applying (3.14) and (3.3), we obtain

$$\|v'_{\varepsilon,\delta}(t)\|_{H^{-1}} \leqslant C'(\varepsilon, \|u_0\|_{H^1}) \quad \forall\, t \in \mathbb{R}, \forall\, \delta > 0. \tag{3.21}$$

Since (3.14) and (3.21) are verified, [3, Proposition 1.1.2] yields that for every $T > 0$ there exist $\{\delta_j\}_j \subset (0,\infty)$ and $v_\varepsilon \in C_{\mathrm{w}}([-T,T];H^1(\mathbb{R}^N))$ such that $\delta_j \to 0$ $(j \to \infty)$ and

$$v_{\varepsilon,\delta_j}(t) \to v_\varepsilon(t)\ (j \to \infty) \quad \text{weakly in } H^1(\mathbb{R}^N)\ \forall\, t \in [-T,T], \tag{3.22}$$

$$v'_{\varepsilon,\delta_j} \to v'_\varepsilon\ (j \to \infty) \quad \text{weakly}^* \text{ in } L^\infty(-T,T;H^{-1}(\mathbb{R}^N)). \tag{3.23}$$

In particular, we see from (2.5) and (3.22) that for every $t \in [-T,T]$

$$\Big(-\Delta + \frac{a}{|x|^2 + \delta_j}\Big)v_{\varepsilon,\delta_j}(t) \to \Big(-\Delta + \frac{a}{|x|^2}\Big)v_\varepsilon(t)\ (j \to \infty) \quad \text{weakly in } H^{-1}(\mathbb{R}^N). \tag{3.24}$$

Combining (3.24) and (3.23) with **(HE)**$_a^{\varepsilon,\delta}$, we see that there exists f such that

$$\begin{aligned} v_{\varepsilon,\delta_j} K_\varepsilon(|v_{\varepsilon,\delta_j}|^2) &= iv'_{\varepsilon,\delta_j} - \Big(-\Delta + \frac{a}{|x|^2 + \delta_j}\Big)v_{\varepsilon,\delta_j} \\ &\to iv'_\varepsilon - P_a v_\varepsilon =: f\ (j \to \infty) \text{ weakly}^* \text{ in } L^\infty(-T,T;H^{-1}(\mathbb{R}^N)). \end{aligned}$$

(G5) in [10, Lemma 3.1] asserts that

$$\mathrm{Im} \int_0^t \langle f(s), v_\varepsilon(s)\rangle_{H^{-1},H^1}\, ds = 0 \quad \forall\, t \in [-T,T].$$

Now v_ε satisfies $iv'_\varepsilon = P_a v_\varepsilon + f$ in $L^\infty(-T,T;H^{-1}(\mathbb{R}^N))$. Thus we obtain the conservation law of charge for v_ε. Combining this with (3.3) we have

$$\|v_\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2} = \|v_{\varepsilon,\delta_j}(t)\|_{L^2} \quad \forall\, t \in [-T,T]. \tag{3.25}$$

Hence we see from (3.22) and (3.25) that $v_{\varepsilon,\delta_j}(t) \to v_\varepsilon(t)$ $(j \to \infty)$ strongly in $L^2(\mathbb{R}^N)$ for $t \in [-T,T]$. Therefore **(G5)** in [10, Lemma 3.1] ensures that $f = v_\varepsilon K_\varepsilon(|v_\varepsilon|^2)$ and v_ε satisfies

$$\begin{cases} iv_\varepsilon' = P_a v_\varepsilon + v_\varepsilon K_\varepsilon(|v_\varepsilon|^2) & \text{in } L^\infty(-T,T;H^{-1}(\mathbb{R}^N)). \\ v_\varepsilon(0) = u_0 \in H^1(\mathbb{R}^N). \end{cases}$$

On the other hand, Theorem 1.1 implies that there exists a unique weak solution to **(HE)**$_a^\varepsilon$. Thus the uniqueness for **(HE)**$_a^\varepsilon$ implies $v_\varepsilon = u_\varepsilon$. Moreover, u_ε satisfies the conservation laws:

$$\|u_\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E_\varepsilon(u_\varepsilon(t)) = E_\varepsilon(u_0) \quad \forall\, t \in \mathbb{R}, \tag{3.26}$$

where the energy of **(HE)**$_a^\varepsilon$ is defined as

$$E_\varepsilon(\varphi) := \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{a}{2}\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 + G_\varepsilon(\varphi), \quad \varphi \in H^1(\mathbb{R}^N). \tag{3.27}$$

We have proved that $v_{\varepsilon,\delta_j}(t) \to u_\varepsilon(t)$ $(j \to \infty)$ strongly in $L^2(\mathbb{R}^N)$ for $t \in [-T,T]$. More precisely, we see that

$$v_{\varepsilon,\delta_j}(t) \to u_\varepsilon(t) \quad (j \to \infty) \text{ strongly in } H^1(\mathbb{R}^N), \quad t \in [-T,T]. \tag{3.28}$$

In fact, it follows from (3.3), (3.26) and (2.13),

$$\begin{aligned} \|\nabla v_{\varepsilon,\delta}(t)\|_{L^2}^2 + a\left\|\frac{v_{\varepsilon,\delta}(t)}{\sqrt{|x|^2+\delta}}\right\|_{L^2}^2 &= 2E_{\varepsilon,\delta}(v_{\varepsilon,\delta}(t)) - 2G_\varepsilon(v_{\varepsilon,\delta}(t)) \\ &\to 2E(u_0) - 2G_\varepsilon(u_\varepsilon(t)) \\ &= \|\nabla u_\varepsilon(t)\|_{L^2}^2 + a\left\|\frac{u_\varepsilon(t)}{|x|}\right\|_{L^2}^2 \ (j \to \infty). \end{aligned}$$

Lemma 2.2 yields (3.28). On the other hand, [3, Lemma 6.5.2] implies

$$\|xu_\varepsilon(t)\|_{L^2}^2 - \|xu_0\|_{L^2}^2 = 4\,\mathrm{Im}\int_0^t \int_{\mathbb{R}^N} x\overline{u_\varepsilon(s)} \cdot \nabla u_\varepsilon(s)\,dx\,ds.$$

Thus Lemma 2.3 with (3.14) and (3.28) ensures

$$xv_{\varepsilon,\delta}(t) \to xu_\varepsilon(t) \quad (j \to \infty) \text{ strongly in } L^2(\mathbb{R}^N), \quad t \in [-T,T].$$

Now we can derive the virial identity for **(HE)**$_a^\varepsilon$ by letting $\delta \to 0$ of (3.5):

$$\frac{d^2}{dt^2}\|xu_\varepsilon(t)\|_{L^2}^2 = V_\varepsilon(u_\varepsilon(t)), \quad t \in \mathbb{R}, \tag{3.29}$$

where V_ε is defined for $\varphi \in H^1(\mathbb{R}^N)$

$$V_\varepsilon(\varphi) := 8\,\|\nabla\varphi\|_{L^2}^2 + 8a\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 - 4\widetilde{G}_\varepsilon(\varphi). \tag{3.30}$$

Since $\widetilde{k_\varepsilon} \in L_x^\infty(L_y^\infty)$, (2.13) implies that

$$|\widetilde{G}_\varepsilon(v_{\varepsilon,\delta_j}(t)) - \widetilde{G}_\varepsilon(u_\varepsilon(t))| \leqslant 4\,\|\widetilde{k_\varepsilon}\|_{L_x^\infty(L_y^\infty)}\|u_0\|_{L^2}^3\|v_{\varepsilon,\delta_j}(t)-u_\varepsilon(t)\|_{L^2} \\ \to 0 \quad (j\to\infty)\ \forall\, t\in[-T,T]. \tag{3.31}$$

On the other hand, (3.28) implies that for all $t\in[-T,T]$

$$\|\nabla v_{\varepsilon,\delta_j}(t)\|_{L^2}^2 + a\left\|\frac{v_{\varepsilon,\delta_j}(t)}{\sqrt{|x|^2+\delta_j}}\right\|_{L^2}^2 \to \|\nabla u_\varepsilon(t)\|_{L^2}^2 + a\left\|\frac{u_\varepsilon(t)}{|x|}\right\|_{L^2}^2 \quad (j\to\infty). \tag{3.32}$$

Hence (3.31) and (3.32) yield that $V_{\varepsilon,\delta_j}(v_{\varepsilon,\delta_j}(t)) \to V_\varepsilon(u_\varepsilon(t))$ $(j\to\infty)$ for $t\in[-T,T]$. Moreover, applying (3.14) and (2.4) we have

$$|V_{\varepsilon,\delta_j}(v_{\varepsilon,\delta_j}(t))| \leqslant 8\Big(1+\frac{4a_+}{(N-2)^2}\Big)\,\|\nabla v_{\varepsilon,\delta_j}(t)\|_{L^2}^2 + 4\,\|\widetilde{k_\varepsilon}\|_{L_x^\infty(L_y^\infty)}\|v_{\varepsilon,\delta_j}(t)\|_{L^2}^4 \\ \leqslant 8\Big(1+\frac{4a_+}{(N-2)^2}\Big)\,C(\varepsilon,\,\|u_0\|_{H^1}) + 4\,\|\widetilde{k_\varepsilon}\|_{L_x^\infty(L_y^\infty)}\|u_0\|_{L^2}^4.$$

Therefore Remark 2.1 ensures (3.29).

Stage 4 of proof (3.1). First we show $u_\varepsilon \to u$ $(\varepsilon\to 0)$.

LEMMA 3.2. *Let u be a local weak solution to* **(HE)**$_a$ *in* $(-T_1,T_2)$. *Then* $u_\varepsilon \to u$ $(\varepsilon\to 0)$ *strongly in* $L^\infty(-T_1,T_2;H^1(\mathbb{R}^N))$.

Proof. Step 1. First we show the uniform boundedness of u_ε:

$$\|u_\varepsilon(t)\|_{H^1} \leqslant M_0 \quad \forall\,\varepsilon>0, \forall\, t\in[-T,T]. \tag{3.33}$$

Now we denote

$$\|\varphi\|_{\widetilde{H}} := \Big(\|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + a\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2\Big)^{1/2} = \|(1+P_a)^{1/2}\varphi\|_{L^2}, \quad \varphi\in H^1(\mathbb{R}^N)$$

and set $M := 2\|u_0\|_{\widetilde{H}}$. Note that $\|\cdot\|_{\widetilde{H}}$ is equivalent to $\|\cdot\|_{H^1}$ [see (2.4) with $\delta = 0$]. Define

$$\tau_\varepsilon := \sup_{T>0}\{\|u_\varepsilon(t)\|_{\widetilde{H}} \leqslant M,\ t\in[-T,T]\}.$$

If $\tau_\varepsilon = \infty$, then we have proved the uniform boundedness. Thus we assume $\tau_\varepsilon < \infty$. Since $u_\varepsilon \in C(\mathbb{R};H^1(\mathbb{R}^N))$, τ_ε satisfies

$$\|u_\varepsilon(\tau_\varepsilon)\|_{\widetilde{H}} = M \text{ or } \|u_\varepsilon(-\tau_\varepsilon)\|_{\widetilde{H}} = M. \tag{3.34}$$

It follows from (3.26) and (2.13) that for $t\in[-\tau_\varepsilon,\tau_\varepsilon]$

$$\|u_\varepsilon(t)\|_{\widetilde{H}}^2 - \|u_0\|_{\widetilde{H}}^2 = 2\,[G_\varepsilon(u_0) - G_\varepsilon(u_\varepsilon(t))] \\ \leqslant 2c^4M^4\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)} + 2RM^3\,\|u_0-u_\varepsilon(t)\|_{L^2}. \tag{3.35}$$

On the other hand, we see from **(G2)** in [10, Lemma 3.1] that

$$\begin{aligned}\|u_\varepsilon'(t)\|_{\widetilde{H}^*} &\leqslant \|P_a u_\varepsilon(t)\|_{\widetilde{H}^*} + \|u_\varepsilon(t)\,K_\varepsilon(|u_\varepsilon(t)|^2)\|_{\widetilde{H}^*}\\ &\leqslant \|u_\varepsilon(t)\|_{\widetilde{H}} + R_0\,\|u_\varepsilon(t)\|_{L^2}^3 + C\,\|k-k_{R_0}\|_{L_x^\beta(L_y^\alpha)}\|u_\varepsilon(t)\|_{\widetilde{H}}^3\\ &\leqslant M + R_0\,\|u_0\|_{L^2}^3 + C\,\|k-k_{R_0}\|_{L_x^\beta(L_y^\alpha)}M^3 =: C(M) \quad \forall\, t\in[-\tau_\varepsilon,\tau_\varepsilon],\end{aligned}$$

where $\|\cdot\|_{\widetilde{H}^*} := \|(1+P_a)^{-1/2}\cdot\|_{L^2}$; note that $\|\cdot\|_{\widetilde{H}^*}$ is equivalent to $\|\cdot\|_{H^{-1}}$. Applying [3, Lemma 3.3.6] we obtain

$$\|u_\varepsilon(t)-u_\varepsilon(s)\|_{L^2} \leqslant \sqrt{2}C(M)\,|t-s|^{1/2}, \quad t,s\in[-\tau_\varepsilon,\tau_\varepsilon]. \tag{3.36}$$

Combining (3.36) with setting $s=0$ into (3.35), we see that

$$\|u_\varepsilon(t)\|_{\widetilde{H}}^2 - \|u_0\|_{\widetilde{H}}^2 \leqslant 2c^4M^4\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)} + 2\sqrt{2}RM^3C(M)|t|^{1/2}.$$

Letting $t=\pm\tau_\varepsilon$ and applying (3.34) we have

$$\tau_\varepsilon^{1/2} \geqslant \frac{3M^2 - 8c^4M^4\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)}}{8\sqrt{2}RM^3C(M)} > 0;$$

note that $\|k-k_R\|_{L_x^\beta(L_y^\alpha)} \to 0$ $(R\to\infty)$ implies the positivity. Thus we obtain (3.33) by putting

$$T_M := \Big[\frac{3-8c^4M^2\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)}}{8\sqrt{2}RMC(M)}\Big]^2 > 0.$$

Step 2. Next we show that $u_\varepsilon \to u$ $(\varepsilon\to 0)$ strongly in $L^\infty(-T_1,T_2;L^2(\mathbb{R}^N))$ and in $L^{r(\gamma)}(-T_1,T_2;L^{2\gamma}(\mathbb{R}^N))$, where $r(\gamma) := 4\gamma/[N(\gamma-1)]$. Note that u and u_ε satisfy the following integral equations:

$$\begin{aligned}u(t) &= e^{-itP_a}u_0 - i\int_0^t e^{-i(t-s)P_a}[u(s)K(|u(s)|^2)]\,ds,\\ u_\varepsilon(t) &= e^{-itP_a}u_0 - i\int_0^t e^{-i(t-s)P_a}[u_\varepsilon(s)K_\varepsilon(|u_\varepsilon(s)|^2)]\,ds.\end{aligned}$$

We divide $u(t)-u_\varepsilon(t)$ into $J_1(t;\varepsilon)+J_2(t;\varepsilon)+J_3(t;\varepsilon)$, where

$$\begin{aligned}J_1(t;\varepsilon) &:= -i\int_0^t e^{-i(t-s)P_a}[u(s)K(|u(s)|^2) - K_\varepsilon(|u(s)|^2)]\,ds,\\ J_2(t;\varepsilon) &:= -i\int_0^t e^{-i(t-s)P_a}[u(s)-u_\varepsilon(s)]K_\varepsilon(|u(s)|^2)\,ds,\\ J_3(t;\varepsilon) &:= -i\int_0^t e^{-i(t-s)P_a}u_\varepsilon(s)K_\varepsilon(|u(s)|^2 - |u_\varepsilon(s)|^2)]\,ds.\end{aligned}$$

For simply we denote

$$K[\tilde{k}](f) := \int_{\mathbb{R}^N} \tilde{k}(x,y)f(y)\,dy, \qquad \|f\|_{L_t^\tau(L^\rho)} := \|f\|_{L^\tau(-T,T;L^\rho)}.$$

For J_1 applying the Strichartz estimates (2.2) we have

$$\|J_1\|_{L_t^\tau(L^\rho)} \leqslant; C_{\infty,\tau}\,\|u\,K[k_R-(k_R)_\varepsilon](|u|^2)\|_{L_t^1(L^2)} + C_{r(\gamma),\tau}\,\|u\,K[(k-k_R)-(k_\varepsilon-(k_R)_\varepsilon)](|u|^2)\|_{L_t^{r(\gamma)'}(L^{(2\gamma)'})}.$$

Applying (2.14), (2.15) and the dominated convergence theorem, we see that

$$\|J_1\|_{L_t^\tau(L^\rho)} \to 0 \quad (\varepsilon\to 0). \tag{3.37}$$

For I_2 applying the Strichartz estimates (2.2) we have

$$\begin{aligned}\|J_2\|_{L_t^\tau(L^\rho)} &\leqslant C_{\infty,\tau}\,\|(u-u_\varepsilon)\,K[(k_R)_\varepsilon](|u|^2)\|_{L_t^1(L^2)} \\ &\quad + C_{r(\gamma),\tau}\,\|(u-u_\varepsilon)\,K[k_\varepsilon-(k_R)_\varepsilon](|u|^2)\|_{L_t^{r(\gamma)'}(L^{(2\gamma)'})} \\ &\leqslant 2C_{\infty,\tau}RT\,\|u\|^2_{L_t^\infty(L^2)}\|u-u_\varepsilon\|_{L_t^\infty(L^2)} \\ &\quad + C_{r(\gamma),\tau}(2T)^{1-2/r(\gamma)}\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)}\|u\|^2_{L_t^\infty(L^{2\gamma})}\|u-u_\varepsilon\|_{L_t^{r(\gamma)}(L^{2\gamma})}.\end{aligned}$$

In a way similar to J_2, we can evaluate J_3 as follows:

$$\begin{aligned}\|J_3\|_{L_t^\tau(L^\rho)} &\leqslant C_{\infty,\tau}\,\|u_\varepsilon\,K[(k_R)_\varepsilon](|u|^2-|u_\varepsilon|^2)\|_{L_t^1(L^2)} \\ &\quad + C_{r(\gamma),\tau}\,\|u_\varepsilon\,K[k_\varepsilon-(k_R)_\varepsilon](|u|^2-|u_\varepsilon|^2)\|_{L_t^{r(\gamma)'}(L^{(2\gamma)'})} \\ &\leqslant 2C_{\infty,\tau}RT\,\|u_\varepsilon\|_{L_t^\infty(L^2)}(\|u\|_{L_t^\infty(L^2)}+\|u_\varepsilon\|_{L_t^\infty(L^2)})\|u-u_\varepsilon\|_{L_t^\infty(L^2)} \\ &\quad + C_{r(\gamma),\tau}(2T)^{1-2/r(\gamma)}\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)}\|u_\varepsilon\|_{L_t^\infty(L^{2\gamma})} \\ &\qquad \times\Big(\|u\|_{L_t^\infty(L^{2\gamma})}+\|u_\varepsilon\|_{L_t^\infty(L^{2\gamma})}\Big)\|u-u_\varepsilon\|_{L_t^{r(\gamma)}(L^{2\gamma})}.\end{aligned}$$

Set $(\tau,\rho)=(\infty,2)$ and $(r(\gamma),2\gamma)$. Now we put

$$M := \max\{\|u_0\|_{L^2}, \|u\|_{L^{r(\gamma)}(-T,T;L^{2\gamma})}, \sup_{\varepsilon\in(0,1)}\|u_\varepsilon\|_{L^{r(\gamma)}(-T,T;L^{2\gamma})}\} < \infty.$$

Case 1 ($\alpha^{-1}+\beta^{-1}<4/N$). Take $T_0\in(0,T)$ such that $6(C_{\infty,\infty}+C_{\infty,r(\gamma)})RM^2T_0\leqslant 1/2$ and $3(C_{r(\gamma),\infty}+C_{r(\gamma),r(\gamma)})\|k-k_R\|_{L_x^\beta(L_y^\alpha)}M^2(2T_0)^{1-2/r(\gamma)}\leqslant 1/2$. Then we obtain

$$\|u-u_\varepsilon\|_{L^{r(\gamma)}(-T_0,T_0;L^{2\gamma})}+\|u-u_\varepsilon\|_{L^\infty(-T_0,T_0;L^2)} \leqslant 2\,\|J_1\|_{L^\infty(-T_0,T_0;L^2)}+2\,\|J_1\|_{L^{r(\gamma)}(-T_0,T_0;L^{2\gamma})}. \tag{3.38}$$

It follows from (3.37) that

$$u_\varepsilon\to u \quad (\varepsilon\to 0) \text{ strongly in } L^\infty(-T_0,T_0;L^2(\mathbb{R}^N))\cap L^{r(\gamma)}(-T_0,T_0;L^{2\gamma}(\mathbb{R}^N)). \tag{3.39}$$

Case 2 ($\alpha^{-1}+\beta^{-1}=4/N$). Fix $R>0$ so that

$$3(C_{2,\infty}+C_{2,2})\|k-k_R\|_{L^\beta_x(L^\alpha_y)}M^2 \leqslant 1/2.$$

Next take $T_0 \in (0,T)$ such that $6(C_{\infty,\infty}+C_{\infty,r(\gamma)})RM^2T_0 \leqslant 1/2$. Then we have (3.39).

Extending the interval step by step, we conclude that $u_\varepsilon \to u$ $(\varepsilon \to 0)$ strongly in $L^\infty(-T_1,T_2;L^2(\mathbb{R}^N))$ and in $L^{r(\gamma)}(-T_1,T_2;L^{2\gamma}(\mathbb{R}^N))$.

Step 3. Assume that $u_\varepsilon \not\to u$ $(\varepsilon \to 0)$ in $C(\overline{I};H^1(\mathbb{R}^N))$. Then there exist $\varepsilon_0>0$ and bounded sequences $\{\varepsilon_m\}_m \subset (0,1)$ and $\{t_m\}_m \subset \overline{I}$ such that

$$\|u_{\varepsilon_m}(t_m)-u(t_m)\|_{H^1} \geqslant \varepsilon_0, \quad m \in \mathbb{N}.$$

We may also assume that $\varepsilon_m \to 0$ and $t_m \to t_0 \in \overline{I}$ $(m\to\infty)$. Since $u \in C(\overline{I};H^1(\mathbb{R}^N))$, we have $\|u(t_m)-u(t_0)\|_{H^1(\mathbb{R}^N)} < \varepsilon_0/2$ for sufficiently large m. Therefore we obtain

$$\|u_{\varepsilon_m}(t_m)-u(t_0)\|_{H^1} > \frac{\varepsilon_0}{2}.$$

On the other hand, it follows from *Step 2* that $\|u_{\varepsilon_m}(t_m)-u(t_m)\|_{L^2} \to 0$ $(m\to\infty)$. Since $u \in C(\overline{I};L^2(\mathbb{R}^N))$, we have $\|u(t_m)-u(t_0)\|_{L^2} \to 0$ $(m\to\infty)$. Thus we obtain $\|u_{\varepsilon_m}(t_m)-u(t_0)\|_{L^2} \to 0$ $(m\to\infty)$. This means that $u_{\varepsilon_m}(t_m) \to u(t_0)$ $(m\to\infty)$ strongly in $L^2(\mathbb{R}^N)$ but $u_{\varepsilon_m}(t_m) \not\to u(t_0)$ $(m\to\infty)$ strongly in $H^1(\mathbb{R}^N)$.

To derive a contradiction it remains to show that

$$u_{\varepsilon_m}(t_m) \to u(t_0)\ (m\to\infty) \quad \text{strongly in } H^1(\mathbb{R}^N). \tag{3.40}$$

Now using the functions G, G_ε [see (2.12) and (2.11)] and

$$Q(\varphi) := \frac{1}{2}\|\nabla\varphi\|^2_{L^2(\mathbb{R}^N)} + \frac{a}{2}\left\|\frac{\varphi}{|x|}\right\|^2_{L^2(\mathbb{R}^N)}, \quad \varphi \in H^1(\mathbb{R}^N),$$

we can write as

$$E_\varepsilon(\varphi) = Q(\varphi)+G_\varepsilon(\varphi), \quad E(\varphi)=Q(\varphi)+G(\varphi), \quad \varphi \in H^1(\mathbb{R}^N). \tag{3.41}$$

Now we show

$$Q(u_{\varepsilon_m}(t_m)) \to Q(u(t_0)) \quad (m\to\infty). \tag{3.42}$$

To end this, first we see from the conservation laws (3.26) and (1.4) that

$$E_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) = E_{\varepsilon_m}(u_0) \to E(u_0) = E(u(t_0)) \quad (m\to\infty). \tag{3.43}$$

Next we prove

$$G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) \to G(u(t_0)) \quad (m\to\infty). \tag{3.44}$$

Applying (2.13) we calculate

$$|G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) - G(u(t_0))|$$

$$\begin{aligned}
&\leqslant |G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) - G_{\varepsilon_m}(u(t_0))| + |G_{\varepsilon_m}(u(t_0)) - G(u(t_0))| \\
&\leqslant c^4M^4 \|k-k_R\|_{L^\beta_x(L^\alpha_y)} + RM^3 \|u_{\varepsilon_m}(t_m) - u(t_0)\|_{L^2} + |G_{\varepsilon_m}(u(t_0)) - G(u(t_0))| \\
&\to c^4M^4 \|k-k_R\|_{L^\beta_x(L^\alpha_y)} \quad (m\to\infty).
\end{aligned}$$

Since $R>0$ is arbitrary, **(K2)** implies (3.44). Combining (3.43) and (3.44) into (3.41) we obtain (3.42).

On the other hand, by the boundedness of $\|u_{\varepsilon_m}(t_m)\|_{H^1}$ there exist $v\in H^1(\mathbb{R}^N)$ and a weak convergent subsequence $\{u_{\varepsilon_{m(j)}}(t_{m(j)})\}_j$ such that $u_{\varepsilon_{m(j)}}(t_{m(j)}) \to v$ $(j\to\infty)$ weakly in $H^1(\mathbb{R}^N)$. Since $u_{\varepsilon_{m(j)}}(t_{m(j)}) \to u(t_0)$ $(j\to\infty)$ strongly in $L^2(\mathbb{R}^N)$, we obtain $u_{\varepsilon_{m(j)}}(t_{m(j)}) \to u(t_0)$ $(j\to\infty)$ weakly in $H^1(\mathbb{R}^N)$. Therefore from the weak convergence in $H^1(\mathbb{R}^N)$ of $\{u_{\varepsilon_{m(j)}}(t_{m(j)})\}_j$ to $u(t_0)$ and the convergence of the corresponding norms we conclude (3.40), a contradiction. $\square$

Now we are the final position to prove (3.1). First note that Lemma 2.3 yields $xu_\varepsilon(t)\to xu(t)$ strongly in $L^2(\mathbb{R}^N)^N$. Next we show

$$V_\varepsilon(u_\varepsilon(t)) \to V(u(t)) \quad (\varepsilon\to 0),\ t\in[-T,T]. \tag{3.45}$$

Since $u_\varepsilon\to u$ $(\varepsilon\to 0)$ uniformly in $C([-T,T];H^1(\mathbb{R}^N))$, we see that

$$\|\nabla u_\varepsilon(t)\|_{L^2}^2 + a\left\|\frac{u_\varepsilon(t)}{|x|}\right\|_{L^2}^2 \to \|\nabla u(t)\|_{L^2}^2 + a\left\|\frac{u(t)}{|x|}\right\|_{L^2}^2 \quad (\varepsilon\to 0),\ t\in[-T,T].$$

On the other hand, (2.25) and (2.23) yield that

$$\begin{aligned}
\widetilde{G}_\varepsilon(u_\varepsilon(t)) &= [\widetilde{G}_\varepsilon(u_\varepsilon(t)) - \widetilde{G}_\varepsilon(u(t))] + [\widetilde{G}_\varepsilon(u(t)) - \widetilde{G}(u(t))] + \widetilde{G}(u(t)) \\
&\to \widetilde{G}(u(t)) \quad (\varepsilon\to 0),\ t\in[-T,T].
\end{aligned}$$

Thus we obtain (3.45).

Next we show the uniform boundedness of $V_\varepsilon(u_\varepsilon)$. It follows from (2.4) and (3.33) that

$$8\|\nabla u_\varepsilon(t)\|_{L^2}^2 + 8a\left\|\frac{u_\varepsilon(t)}{|x|}\right\|_{L^2}^2 \leqslant 8\Big(1+\frac{4a_+}{(N-2)^2}\Big)M_0^2 \quad \forall\, t\in[-T,T].$$

On the other hand, (2.25) implies that

$$4|\widetilde{G}_\varepsilon(u_\varepsilon(t))| \leqslant 4C(k)M_0^4 \quad \forall\, t\in[-T,T].$$

Thus we have $|V_\varepsilon(u_\varepsilon(t))|\leqslant C$ for $t\in[-T,T]$ and $\varepsilon>0$. Therefore Remark 2.1 asserts (3.1).

4. Proof of Theorem 1.2

Proof. Assume that u is a global weak solution to **(HE)**$_a$. Applying (3.1) and (1.4) we see that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 16E(u_0) - 4\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[k(x,y)+\widetilde{k}(x,y)]|u(t,x)|^2|u(t,y)|^2\,dxdy.$$

Define $\varphi(t) := \|xu(t)\|_{L^2}^2$. Thus **(K4)** implies that $\varphi''(t) \leqslant 16E(u_0)$. Integrating twice we have

$$\varphi(t) \leqslant \varphi(0) + \varphi'(0)t + 8E(u_0)t^2 =: \psi(t).$$

Since $E(u_0) < 0$, there exists $t_j > 0$ $(j = 1, 2)$ such that $\psi(-t_1) = 0 = \psi(t_2)$. Thus $\varphi(-t_1) \leqslant 0$ and $\varphi(t_2) \leqslant 0$. Since φ is continuous in t and $\varphi(0) > 0$, there exist $T_1, T_2 > 0$ such that $\varphi(-T_1) = 0 = \varphi(T_2)$. Applying the Hölder inequality and integrating by parts for $\int x\bar{v} \cdot \nabla v\, dx$, we have

$$\|v\|_{L^2}^2 \leqslant \frac{2}{N} \|xv\|_{L^2} \|\nabla v\|_{L^2} \quad \forall\, v \in H^1(\mathbb{R}^N) \cap D(x).$$

Let $v := u(t)$. Then (1.4) implies that

$$\|\nabla u(t)\|_{L^2} \geqslant \frac{N\,\|u_0\|_{L^2}^2}{2\,\|xu(t)\|_{L^2}}.$$

Letting $t \to -T_1 + 0$ or $t \to T_2 - 0$, we see that

$$\lim_{t \to -T_1+0} \|\nabla u(t)\|_{L^2} = \infty = \lim_{t \to T_2 - 0} \|\nabla u(t)\|_{L^2}.$$

This is a contradiction (see also [3, Remark 3.1.6 (ii)]). □

5. Concluding remarks

In a way similar to Sections 3 and 4, we can show the blowup in finite time for the nonlinear Schrödinger equations with inverse-square potentials:

$$\begin{cases} i\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + f(u) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \qquad \textbf{(NLS)}_a$$

where $f : \mathbb{C} \to \mathbb{C}$ is power type nonlinearities.

(N1) $f(0) = 0$ and there exist $p \in [1, (N+2)/(N-2))$ and $K \geqslant 0$ such that

$$|f(u) - f(v)| \leqslant K(1 + |u|^{p-1} + |v|^{p-1})|u - v| \quad \forall u, v \in \mathbb{C};$$

(N2) $f(x) \in \mathbb{R}$ $(x > 0)$ and $f(e^{i\theta}z) = e^{i\theta} f(z)$ $(z \in \mathbb{C},\ \theta \in \mathbb{R})$;

(N3) There exist $q \in [1, 1 + 4/N)$ and $L_1, L_2 \geqslant 0$ such that

$$F(x) := \int_0^x f(s)\,ds \geqslant -L_1 x^2 - L_2 x^{q+1} \quad \forall\, x > 0;$$

(N4) $2(N+2)F(x) - Nxf(x) \geqslant 0$ for $x > 0$.

In Okazawa-Suzuki-Yokota [9], unique and global existence of weak solutions to **(NLS)**$_a$ is verified under the assumption **(N1)–(N3)**. Note that unique and local existence of weak solutions to **(NLS)**$_a$ is assumed under **(N1)** and **(N2)**.

THEOREM 5.1. ([9, Theorem 5.1]) *Let* $N \geqslant 3$ *and* $a > -(N-2)^2/4$. *Assume that* f *satisfies* **(N1)–(N3)**. *Then for every* $u_0 \in H^1(\mathbb{R}^N)$ *there exists a unique global weak solution* u *to* **(NLS)**$_a$. *Moreover, u belongs to* $C(\mathbb{R};H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R};H^{-1}(\mathbb{R}^N))$ *and satisfies*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \widetilde{E}(u(t)) = \widetilde{E}(u_0) \quad \forall\, t \in \mathbb{R},$$

where $\widetilde{E}$ *is the energy defined as*

$$\widetilde{E}(\varphi) := \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{a}{2}\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 + \int_{\mathbb{R}^N} F(|\varphi(x)|)\,dx \quad \forall\, \varphi \in H^1(\mathbb{R}^N).$$

The blowup results in finite time for **(NLS)**$_a$ is as follows:

THEOREM 5.2. *Let* $N \geqslant 3$ *and* $a > -(N-2)^2/4$. *Assume that* k *satisfies* **(N1)**, **(N2)** *and* **(N4)**. *Then for every* $u_0 \in H^1(\mathbb{R}^N)$ *with* $|x|u_0 \in L^2(\mathbb{R}^N)$ *and* $\widetilde{E}(u_0) < 0$ *there exist* $T_1, T_2 > 0$ *such that* $u \in C(\overline{I};H^1(\mathbb{R}^N)) \cap C^1(\overline{I};H^{-1}(\mathbb{R}^N))$ *is a (unique) local weak solution to* **(NLS)**$_a$ *for every open interval* I *with* $0 \in I$ *and* $\overline{I} \subset (-T_1, T_2)$ *and* u *satisfies*

$$\lim_{t\to -T_1+0} \|\nabla u(t)\|_{L^2} = \infty = \lim_{t \to T_2 - 0} \|\nabla u(t)\|_{L^2},$$

that is, the weak solution blows up in finite time.

To derive the virial identity for **(NLS)**$_a$ we consider

$$\begin{cases} i\dfrac{\partial v_{\varepsilon,\delta}}{\partial t} = -\Delta v_{\varepsilon,\delta} + \dfrac{a v_{\varepsilon,\delta}}{|x|^2+\delta} + \rho_\varepsilon * [f(\rho_\varepsilon * v_{\varepsilon,\delta})] & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ v_{\varepsilon,\delta}(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{5.1}$$

In a way similar to Section 3, we obtain

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = \widetilde{V}(u(t)),$$

where

$$\widetilde{V}(\varphi) := 8\,\|\nabla\varphi\|_{L^2}^2 + 8a\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 - \int_{\mathbb{R}^N} [8N F(|\varphi|) - 4N f(|\varphi|)|\varphi|]\,dx.$$

REMARK 5.1. Let $k(x,y) := W(x-y)$. Then **(K4)** is rewritten as follows: $x \cdot \nabla W \in L^\infty(\mathbb{R}^N) + L^{1\vee(N/4)}(\mathbb{R}^N)$ and $W(x) + (1/2)x\cdot\nabla_x W(x) \geqslant 0$; see also [3, Theorem 6.5.4].

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