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BIFURCATION TYPE PHENOMENA FOR POSITIVE SOLUTIONS OF NONLINEAR NEUMANN EIGENVALUE PROBLEMS

EVGENIA H. PAPAGEORGIOU

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Abstract. We consider a parametric nonlinear problem driven by the p-Laplace differential operator and with a reaction which is p-superinear near $+\infty$ but need not satisfy the usual in such cases Ambrosetti-Rabinowitz condition. Using critical point theory and truncation and comparison techniques, we prove a bifurcation-type theorem for such problems.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Neumann eigenvalue problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda f(z,u(z)) \ \text{ in } \ \Omega, \\ \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega, \ \ \lambda > 0, \ \ u > 0. \end{array} \right\} \tag{$1)_\lambda$}$$

Here Δ_p denotes the p-Laplace differential operator define by

$$\Delta_p u = \operatorname{div}\big(||Du||^{p-2}Du\big), \ (1 < p < \infty) \ \text{ for all } \ u \in W^{1,p}(\Omega).$$

Also $\beta \in L^\infty$, $\beta \geqslant 0$, $\beta \neq 0$ and $f(z,x)$ is a Caratheodory function which exhibits a (p-1)-superlinear growth in $x \in \mathbb{R}$ near $+\infty$. However, we do not assume that it satisfies the usual in such cases Ambrosetti-Rabinowitz condition (AR-condition for short). The aim of this work is to show the existence, nonexistence and multiplicity of positive solutions for various values of the parameter $\lambda > 0$. More precisely, we show that for problem $(1)_\lambda$ we have a bifurcation-type result, namely there exists a critical value $\lambda^* > 0$ of the parameter such that if $\lambda \in (0,\lambda^*)$, problem $(1)_\lambda$ has at least two nontrivial positive smooth solutions, if $\lambda = \lambda^*$ problem $(1)_\lambda$ has at least one positive smooth solution and if $\lambda > \lambda^*$, then problem $(1)_\lambda$ has no positive solution.

This problem, has been investigated primarily in the context of Dirichlet boundary value problems. For $p = 2$ (semilinear equations), we mention the works of Ambrosetti-Brezis-Cerami [2], Delgado-Suarez [7], Maya-Shivaji [18] and Rabinowith [25]. Extensions to the case of the Dirichlet p-Laplacian can be found in the works of Ambrosetti-

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Azorero- Alonso [3], Brock- Itturiaga -Ubilla [5], Dong [8], Azorero-Manfredi-Alonso [4], Gasinski-Papageorgiou [10], Guo [11], Guo-Zhang [12], Hu-Papageorgiou [13] and [14], Perera [24] and Takeuchi [27]. From the aforementioned works Ambrosetti-Azorero-Alonso [3], Azorero-Manfredi- Peral Alonso [4], Guo-Zhang [12] extend the semilinear work of Ambrosetti-Brezis-Cerami [2] and consider problems with the combined effect of concave and convex terms. So, their reaction term has the form $f_\lambda(z,x) = f_\lambda(x) = \lambda|x|^{q-2}x + |x|^{r-2}x$ with $q < p < r$. Hu-Papageorgiou [14] also considered problems with concave and p-superlinear nonlinearities, but had a more general reaction of the form $f_\lambda(z,x) = \lambda|x|^{q-2}x + f_0(z,x)$ with $f_0(z,x)$ a Caratheodory function satisfying the AR-condition on the positive semiaxis. Dong [8] and Takeuchi [27] deal with logistic equations of the superdiffusive type and so their reaction term has the form $f_\lambda(z,x) = f_\lambda(x) = \lambda x^{q-1}(1-x^r)$ with $q > p$ and $r > 0$ (in Takeuchi [27] $p > 2$). The works of Gasinski-Papageorgiou [10], Hu-Papageorgiou [13] and Perera [24] extend to the p-Lapalcian the semilinear work of Maya-Shivaji [18]. In Hu-Papageorgiou [13] the potential function is nonsmooth (hemivariational inequality) and the approach is degree theoretic, while in Perera [24] the approach is variational based on the critical point theory. Both works relax considerably the hypotheses of Maya-Shivaji [18] who had a sublinear reaction. Nevertheless their framework of analysis does not incorporate p-superlinear problems. The same can be said about the work of Brock- Itturiaga -Ubilla [5], which also excludes the possibility of p-superlinear reaction.

For the Neumann p-Laplacian, we only have the works of Motreanu-Motreanu -Papageorgiou [20] and Wu-Chen [28]. In Motreanu-Motreanu-Papageorgiou [20] the authors deal with problems near resonance both below and above and prove existence and multiplicity results. In Wu-Chen [28] $\operatorname*{essinf}_{\Omega} \beta > 0$ and $p > N$ (low dimensional problem). By the Sobolev embedding theorem, this last restriction implies that the Sobolev space $W^{1,p}(\Omega)$ is embedded compactly in $C(\bar{\Omega})$ and this is an essential tool in the reasoning of Wu-Chen [28]. Moreover, their approach is completely different and it is based on KKM-theorem (see, for example, Papageorgiou-Kyritsi [23], p.80).

2. Mathematical Background

In this section, for the convenience of the reader, we recall some of the main mathematical tools which we will use in the sequel.

We start with critical point theory. So, let X be a Banach space and X^* its topological dual. By $<\cdot,\cdot>$ we denote the duality brackets for the pair (X^*,X). Given $\varphi \in C^1(X)$, we say that φ satisfies the "C-condition", if the following is true:

> "Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\} \subseteq \mathbb{R}$ is bounded and $(1+\|u_n\|)\varphi'(u_n) \to 0 \in X^*$, admits a strongly convergent subsequence".

Using this compactness-type condition on φ, we can have the following minimax characterization of certain critical values of φ. The result is known in the literature as the "mountain pass theorem".

THEOREM 1. *If X is Banach space, $\varphi \in C^1(X)$ and satisfies the C-condition,* $u_0,\ u_1 \in X,\ r > 0,\ ||u_1 - u_0|| > r,$

$$max\{\varphi(u_0), \varphi(u_1)\} < inf[\varphi(u) : ||u - u_0|| = r] = \eta_r,$$
$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)) \text{ where } \Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = u_0,\ \gamma(1) = u_1\},$$

then $c \geqslant \eta_r$ and c is a critical value of φ.

In our study of problem $(1)_\lambda$ we will make use of the following two space:

$$W_n^{1,p}(\Omega) = \Big\{u \in W^{1,p}(\Omega) : u = \lim_{k\to+\infty} u_k \text{ in } W^{1,p}(\Omega), u_k \in C^\infty(\bar{\Omega}), \frac{\partial u_k}{\partial n} = 0 \text{ on } \partial\Omega\Big\}$$

and

$$C_n^1(\bar{\Omega}) = \Big\{u \in C^1(\bar{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\Big\}.$$

Both these spaces are ordered Banach spaces, with order cones given by

$$W_+ = \{u \in W_n^{1,p}(\Omega) : u(z) \geqslant 0 \text{ a.e. on } \Omega\}$$

and

$$C_+ = \{u \in C_n^1(\bar{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \bar{\Omega}\}.$$

Moreover, $intC_+ \neq \emptyset$ and more precisely we have

$$intC_+ = \{u \in C_n^1(\bar{\Omega}) : u(z) > 0 \text{ for all } z \in \bar{\Omega}\}.$$

DEFINITION 1. A map $A : X \to X^*$ is said to be *of type* $(S)_+$ if for any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ for which $x_n \xrightarrow{w} x$ in X and $\limsup_{n\to\infty} < A(x_n), x_n - x > \leqslant 0$, one has $x_n \to x$ in X.

Let $A : W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ be the nonlinear map corresponding to $-\Delta_p$, namely

$$< A(x), y > = \int_\Omega ||Dx||^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } x, y \in W_n^{1,p}(\Omega). \tag{1}$$

Hereafter, by $< \cdot, \cdot >$ we denote the duality brackets for the pair $(W_n^{1,p}(\Omega)^*, W_n^{1,p}(\Omega))$. For the map A, we have the following result (see, for example, Aizicovici-Papageorgiou-Staicu [1], Proposition 2).

PROPOSITION 1. *The map $A : W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ defined by (1) is of type* $(S)_+$.

3. The bifurcation-type result

The hypotheses on the reaction function $f(z,x)$ are:

$\underline{H}$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function such that $f(z,0)=0$ a.e. in Ω and

(i) $|f(z,x)|\leqslant a(z)+c|x|^{r-1}$ for a.a $z\in\Omega$, all $x\in\mathbb{R}$, with $a\in L^{\infty}(\Omega)_{+}$, $c>0$ and $p<r<p^{*}$, where $p^{*}=\begin{cases}\dfrac{Np}{N-p}, & p<N\\ +\infty, & p\geqslant N\end{cases}$

(ii) if $F(z,x)=\int_0^x f(z,s)ds$, then

$$\lim_{x\to+\infty}\frac{F(z,x)}{x^{p}}=+\infty \text{ uniformly for a.a } z\in\Omega \tag{2}$$

and there exist $\gamma\in((r-p)\max\{1,\frac{N}{p}\},p^{*})$ and $\beta_0>0$ s.t

$$\beta_0\leqslant \liminf_{x\to+\infty}\frac{f(z,x)x-pF(z,x)}{x^{\gamma}} \text{ uniformly for a.a } z\in\Omega;$$

(iii) $f(z,x)>0$ for a.a $z\in\Omega$, all $x>0$, $\inf[f(z,x):x\geqslant s]\geqslant\mu_s>0$ for a.a $z\in\Omega$, $s>0$ and there exist $\eta\in L^{\infty}(\Omega)_{+}$, $\eta\neq 0$ and $\tau\in(1,p)$ s.t

$$f(z,x)\geqslant\eta(z)x^{\tau-1} \text{ for a.a } z\in\Omega,\ \text{all } x\geqslant 0;$$

(iv) for every $\theta>0$ and every bounded interval I of $(0,+\infty)$, there exists $\sigma_{\theta,I}>0$ s.t $x\to\lambda f(z,x)+\sigma_{\theta,I}x^{p-1}$ is nondecreasing on $[0,\theta]$ for a.a $z\in\Omega$ and all $\lambda\in I$.

REMARK 1. Since we are interested in positive solutions and both the conditions near infinity (see H(ii)) and near zero (see H(iii)), involve only the positive semiaxis, we may (and will) assume without any loss of generality that $f(z,x)=0$ for a.a $z\in\Omega$, all $x\leqslant 0$. Hypothesis H(ii) classifies the problem as "p-superlinear". However, note that we do not use AR-condition which is a common feature in most superlinear problems studied in the literature. We remind the reader that AR-condition (unilateral version, since in our case $f(z,x)=0$ for a.a $z\in\Omega$, all $x\leqslant 0$) says that there exist $\mu>p$ and $M>0$ s.t

$$0<\mu F(z,x)\leqslant f(z,x)x \text{ for a.a } z\in\Omega,\ \text{all } x\geqslant M. \tag{3}$$

Integrating (3) we obtain the weaker condition

$$c_0x^{\mu}\leqslant F(z,x) \text{ for a.a } z\in\Omega,\ \text{all } x\geqslant M \text{ and some } c_0>0. \tag{4}$$

Clearly (4) implies (2). In fact (2) is much weaker than (4) and permits a much slower growth for $F(z,\cdot)$ (see the example below). Analogous conditions were also used by Costa-Magalhaes [6] (Dirichlet pde's) and Fei [9] (Hamiltonian systems). Other generalizations of AR-condition can be found in the works of Jeanjean [16], Miyagaki-Souto [19] and Schechter-Zou [26].

EXAMPLE 1. The following function satisfies hypotheses H (for the sake of simplicity we drop the z-dependence):

$$f(x)=\begin{cases}0, & x<0,\\ \eta x^{\tau-1}-\eta_0 x^{q-1}, & 0\leqslant x\leqslant 1 \text{ with } 1<\tau<q, \eta>\eta_0>0,\\ x^{p-1}\ln x+\eta-\eta_0, & 1<x.\end{cases}$$

Note that $f(\cdot)$ does not satisfies the AR-condition (see (3)).

Let $\mathscr{J}=\{\lambda>0:$ problem $(1)_\lambda$ has a positive solution$\}$. Using variational methods combined with suitable truncation techniques, we show that $\mathscr{J}\neq\emptyset$. We will need the following simple lemma (see [23]):

LEMMA 1. *If $\beta\in L^\infty(\Omega)$, $\beta\geqslant 0$, $\beta\neq 0$, then there exists $\xi_0>0$ s.t.*

$$||Du||_p^p+\int_\Omega \beta|u|^p dz\geqslant \xi_0||u||^p \text{ for all } u\in W^{1,p}(\Omega).$$

PROPOSITION 2. *If hypotheses H hold, then $\mathscr{J}\neq\emptyset$ and if $\lambda\in\mathscr{J}$ and $\mu\in(0,\lambda)$, then $\mu\in\mathscr{J}$.*

Proof. The map $A+\beta K_p: W_n^{1,p}(\Omega)\to W_n^{1,p}(\Omega)^*$ is maximal monotone, strictly monotone and for all $u\in W_n^{1,p}(\Omega)$ we have

$$\begin{aligned}<(A+\beta K_p)(u),u> &= ||Du||_p^p+\int_\Omega \beta|u|^p dz\\ &\geqslant \xi_0||u||^p \quad \text{(see Lemma 1)}\end{aligned}$$

$$\Rightarrow u\to(A+\beta K_p)(u) \text{ is coercive.}$$

Hence $u\to(A+\beta K_p)(u)$ is surjective (see, for example, Papageorgiou-Kyritsi [23], p. 172), where $K_p: L^p(\Omega)\to L^{p'}(\Omega)$ $(1/p+1/p'=1)$ is the map defined by $K_p(u)=|u|^{p-2}u$. Therefore we can find unique (due to the strict monotonicity) $\bar{u}\in W_n^{1,p}(\Omega)\setminus\{0\}$ s.t

$$A(\bar{u})+\beta K_p(\bar{u})=1, \tag{5}$$

$$\Rightarrow -\Delta_p\bar{u}(z)+\beta(z)|\bar{u}(z)|^{p-2}\bar{u}(z)=1 \text{ a.e in } \Omega, \frac{\partial\bar{u}}{\partial n}=0 \text{ on } \partial\Omega \tag{6}$$

(see Motreanu-Papageorgiou [21]).

From (6) and the nonlinear regularity theory (see Hu-Papageorgiou [15] and Lieberman [17]), we have $\bar{u}\in C_n^1(\bar{\Omega})$. On (5) we act with $-\bar{u}^-\in W_n^{1,p}(\Omega)$ and obtain

$$||D\bar{u}^-||_p^p+\int_\Omega \beta(\bar{u}^-)^p dz\leqslant 0,$$

$$\Rightarrow \bar{u}^- = 0, \text{ i.e., } \bar{u} \in C_+ \setminus \{0\} \text{ (see Lemma 1).}$$

From (6) we have

$$\Delta_p \bar{u}(z) \leqslant ||\beta||_\infty \bar{u}(z)^{p-1} \text{ a.e. in } \Omega,$$
$$\Rightarrow \bar{u} \in intC_+ \text{ (see Vazquez [29]).}$$

Hence there exists $\xi_1 > 0$ s.t. $\xi_1 \leqslant \bar{u}(z)$ for all $z \in \bar{\Omega}$. Let $\bar{\lambda} = 1/||N_f(\bar{u})||_\infty$ where $N_f(\bar{u})(\cdot) = f(\cdot, \bar{u}(\cdot))$ (see hypothesis H(i)). Then we have

$$A(\bar{u}) + \beta \bar{u}^{p-1} = 1 \geqslant \bar{\lambda} N_f(\bar{u}). \tag{7}$$

We consider the following truncation of the nonlinearity $f(z,\cdot)$:

$$g(z,x) = \begin{cases} 0, & x < 0, \\ f(z,x), & 0 \leqslant x \leqslant \bar{u}(z), \\ f(z,\bar{u}(z)), & \bar{u}(z) < x. \end{cases} \tag{8}$$

Clearly, this is a Caratheodory function. We set $G(z,x) = \int_0^x g(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_{\bar{\lambda}} : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_{\bar{\lambda}}(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta |u|^p dz - \bar{\lambda}\int_\Omega G(z,u)dz \text{ for all } u \in W_n^{1,p}(\Omega).$$

Exploiting the compact embedding of $W_n^{1,p}(\Omega)$ into $L^r(\Omega)$ (recall $r < p^*$), we can easily check that $\hat{\varphi}_{\bar{\lambda}}$ is sequentially weakly lower semicontinuous. Moreover, because of (8) and Lemma 1, it is clear that $\hat{\varphi}_{\bar{\lambda}}$ is coercive. So, by virtue of the Weierstrass theorem we can find $u_0 \in W_n^{1,p}(\Omega)$ s.t.

$$\hat{\varphi}_{\bar{\lambda}}(u_0) = \inf[\hat{\varphi}_{\bar{\lambda}}(u) : u \in W_n^{1,p}(\Omega)] = \hat{m}_{\bar{\lambda}}. \tag{9}$$

Let $\xi \in (0, \xi_1)$ (recall $\xi_1 \leqslant \min_{\bar{\Omega}} \bar{u}$). Then

$$\begin{aligned} \hat{\varphi}_{\bar{\lambda}}(\xi) &= \frac{\xi^p}{p}\int_\Omega \beta dz - \bar{\lambda}\int_\Omega F(z,\xi)dz \text{ (see (8))} \\ &\leqslant \frac{\xi^p}{p}\int_\Omega \beta dz - \frac{\bar{\lambda}}{\tau}\xi^\tau \int_\Omega \eta dz \text{ (see H(iii))} \\ &= \xi^\tau [\frac{\xi^{p-\tau}}{p}\int_\Omega \beta dz - \frac{\bar{\lambda}}{\tau}\int_\Omega \eta dz]. \end{aligned} \tag{10}$$

Since $\tau < p$, by choosing $\xi \in (0, \xi_1)$ even small if necessary, from (10) we infer that

$$\hat{\varphi}_{\bar{\lambda}}(\xi) < 0,$$

$$\Rightarrow \hat{\varphi}_{\bar{\lambda}}(u_0) = \hat{m}_{\bar{\lambda}} < 0 = \hat{\varphi}_{\bar{\lambda}}(0) \text{ (see (9))},$$
$$\Rightarrow u_0 \neq 0.$$

From (9), we have

$$\begin{aligned} &\hat{\varphi}'_{\bar{\lambda}}(u_0) = 0, \\ \Rightarrow &A(u_0) + \beta|u_0|^{p-2}u_0 = \bar{\lambda}N_g(u_0) \\ &\text{(where } N_g(u)(\cdot) = g\big(\cdot, u(\cdot)\big) \text{ for all } u \in W_n^{1,p}(\Omega)). \end{aligned} \tag{11}$$

On (11) we act with $-u_0^- \in W_n^{1,p}(\Omega)$ and obtain

$$||Du_0^-||_p^p + \int_\Omega \beta |u_0^-|^p dz = 0, \text{ (see (8))},$$
$$\Rightarrow \xi_0||u_0^-||^p \leqslant 0, \text{ (see Lemma 1), i.e., } u_0 \geqslant 0, u_0 \neq 0.$$

Also, on (11) we act with $(u_0 - \bar{u})^+ \in W_n^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &< A(u_0), (u_0 - \bar{u})^+ > + \int_\Omega \beta u_0^{p-1}(u_0 - \bar{u})^+ dz \\ &= \bar{\lambda} \int_\Omega g(z, u_0)(u_0 - \bar{u})^+ dz \\ &= \bar{\lambda} \int_\Omega f(z, \bar{u})(u_0 - \bar{u})^+ dz \text{ (see (8))} \\ &\leqslant < A(\bar{u}), (u_0 - \bar{u})^+ > + \int_\Omega \beta \bar{u}^{p-1}(u_0 - \bar{u})^+ dz \text{ (see (7))}, \\ \Rightarrow &< A(\bar{u}) - A(u_0), (u_0 - \bar{u})^+ > + \int_\Omega \beta(\bar{u}^{p-1} - u_0^{p-1})(u_0 - \bar{u})^+ dz \geqslant 0 \\ \Rightarrow &u_0 \leqslant \bar{u}. \end{aligned}$$

Therefore, we have that $u_0 \in [0, \bar{u}] = \{u \in W_n^{1,p}(\Omega) : 0 \leqslant u(z) \leqslant \bar{u}(z) \text{ a.e. in } \Omega\}$. Hence because of (8), we see that (11) becomes

$$\begin{aligned} &A(u_0) + \beta u_0^{p-1} = \bar{\lambda} N_f(u_0), \\ \Rightarrow &-\Delta_p u_0(z) + \beta(z) u_0(z)^{p-1} = \bar{\lambda} f(z, u_0(z)) \text{ a.e. in } \Omega, \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega, \\ \Rightarrow &u_0 \in intC_+ \text{ (nonlinear regularity and Vazquez [29]), and } \mathscr{J} \neq \emptyset. \end{aligned}$$

Now, let $\lambda \in \mathscr{J}$ and $\mu \in (0, \lambda)$. We can find a nontrivial solution $u \in W_n^{1,p}(\Omega)$ of $(1)_\lambda$ satisfying $u \geqslant 0$. Nonlinear regularity and the Vazquez maximum principle imply that $u \in intC_+$. We have

$$A(u) + \beta u^{p-1} = \lambda N_f(u) \geqslant \mu N_f(u) \text{ (see H(iii) and recall } \mu < \lambda).$$

Then truncating $f(z, \cdot)$ at $u(z)$ and reasoning as above via the direct method we obtain the solution $\hat{u} \in [0, u] \cap intC_+$ of $(1)_\mu$. Therefore $\mu \in \mathscr{J}$. $\square$

Next $\lambda^* = \sup \mathscr{J}$. Next we show that $\lambda^* < +\infty$. In what follows $\hat{u}_1$ denotes the positive L^p-normalized (that is $||\hat{u}_1||_p = 1$) principal eigenfunction of $-\Delta u + \beta|u|^{p-2}u$. We know that $\hat{u}_1 \in intC_+$.

PROPOSITION 3. *If hypotheses H hold, then $\lambda^* < +\infty$.*

Proof. Hypotheses H imply that we can find $\bar{\lambda} > 0$ s.t.

$$\bar{\lambda} f(z,x) \geqslant ||\beta||_\infty x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{12}$$

Let $\lambda > \bar{\lambda}$ and suppose that problem $(1)_\lambda$ admits a positive solution u. We know that $u \in intC_+$. We can find $t > 0$ s.t. $t\hat{u}_1 \leqslant u$. Let $t > 0$ be the biggest such coefficient. Set $\theta = ||u||_\infty$, $I = [\bar{\lambda}, \bar{\lambda} + v]$ with $v \geqslant \lambda - \bar{\lambda} > 0$ (so $\lambda \in I$) and let $\sigma_\theta > 0$ be as postulated by hypothesis H(iv). Let $\delta \in (0, \min_{\bar{\Omega}} u)$ (recall $u \in intC_+$) and set $u_\delta = u - \delta \in intC_+$. We have

$$\begin{aligned}
&-\Delta_p u_\delta(z) + (\beta(z) + \sigma_\theta) u_\delta(z)^{p-1} \\
=&-\Delta_p u(z) + (\beta(z) + \sigma_\theta) u(z)^{p-1} - w(\delta) \text{ (with } w(\delta) \to 0^+ \text{ as } \delta \to 0^+) \\
=&\lambda f(z,u(z)) + \sigma_\theta u(z)^{p-1} - w(\delta) \\
=&\bar{\lambda} f(z,u(z)) + \sigma_\theta u(z)^{p-1} + (\lambda - \bar{\lambda}) f(z,u(z)) - w(\delta) \\
\geqslant&\bar{\lambda} f(z,u(z)) + \sigma_\theta u(z)^{p-1} + (\lambda - \bar{\lambda})\mu_s - w(\delta) \\
&\qquad\qquad \text{(with } s = \min_{\bar{\Omega}} u \text{ and } \mu_s > 0 \text{ as postulated by H(iii))} \\
\geqslant&\bar{\lambda} f(z,t\hat{u}_1(z)) + \sigma_\theta (t\hat{u}_1(z))^{p-1} + (\lambda - \bar{\lambda})\mu_s - w(\delta).
\end{aligned}$$

Recall that $w(\delta) \to 0^+$ as $\delta \to 0^+$. So, for $\delta \in (0,1)$ small we will have $w(\delta) \leqslant (\lambda - \bar{\lambda})\mu_s$. Hence

$$\begin{aligned}
&-\Delta_p u_\delta(z) + (\beta(z) + \sigma_\theta) u_\delta(z)^{p-1} \\
&\geqslant \bar{\lambda} f(z,t\hat{u}_1(z)) + \sigma_\theta (t\hat{u}_1(z))^{p-1} \\
&\geqslant (\beta(z) + \sigma_\theta)(t\hat{u}_1(z))^{p-1} \text{ (see (12))} \\
&= -\Delta_p (t\hat{u}_1(z)) + (\beta(z) + \sigma_\theta)(t\hat{u}_1(z))^{p-1}, \\
&\Rightarrow u_\delta \geqslant t\hat{u}_1 \text{ for } \delta \in (0,1) \text{ small}, \\
&\Rightarrow u - t\hat{u}_1 \in intC_+, \text{ contradicting the maximality of } t > 0.
\end{aligned}$$

Therefore it follows that for $\lambda > \bar{\lambda}$ problem $(1)_\lambda$ has no positive solution and so $\lambda^* \leqslant \bar{\lambda} < +\infty$. □

We show that in fact $\lambda^* \in \mathscr{J}$.

PROPOSITION 4. *If hypotheses H hold, then $\lambda^* \in \mathscr{J}$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathscr{J}$ be an increasing sequence s.t. $\lambda_n \to (\lambda^*)^-$ as $n\to\infty$. For every $n\geqslant 1$, we can find $u_n \in intC_+$ s.t.

$$A(u_n)+\beta u_n^{p-1} = \lambda_n N_f(u_n) \text{ for all } n\geqslant 1. \tag{13}$$

Since $f\geqslant 0$ (see H(iii)) and $\{\lambda_n\}_{n\geqslant 1}$ is increasing, we see that if $m>n$, then,

$$A(u_m)+\beta u_m^{p-1} \geqslant \lambda_n N_f(u_m).$$

Then truncating $f(z,\cdot)$ at $u_m \in intC_+$ and reasoning as in the proof of Proposition 2, we obtain a solution $u_n\in intC_+$ of $(1)_{\lambda_n}$ s.t. $\varphi_{\lambda_n}(u_n)<0$. Therefore, we may assume without any loss of generality that

$$\varphi_{\lambda_n}(u_n)<0 \text{ for all } n\geqslant 1. \tag{14}$$

On (13) we act with $u_n\in intC_+$ and obtain

$$-||Du_n||_p^p - \int_\Omega \beta u_n^p dz + \lambda_n \int_\Omega f(z,u_n)u_n dz = 0 \text{ for all } n\geqslant 1. \tag{15}$$

Also, from (14) we have

$$||Du_n||_p^p + \int_\Omega \beta u_n^p dz - \lambda_n \int_\Omega pF(z,u_n)dz < 0 \text{ for all } n\geqslant 1. \tag{16}$$

We add (15) and (16)

$$\lambda_n \int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz < 0 \text{ for all } n\geqslant 1. \tag{17}$$

By virtue of hypothesis H(ii), we can find $\beta_1\in(0,\beta_0)$ and $M_1>0$ s.t.

$$\beta_1 x^\gamma \leqslant f(z,x)x - pF(z,x) \text{ for a.a. } z\in\Omega, \text{ all } x\geqslant M_1. \tag{18}$$

Moreover, hypothesis H(i) implies that

$$|f(z,x)x-pF(z,x)| \leqslant M_2 \text{ for a.a. } z\in\Omega, \text{ all } x<M_1 \text{ and some } M_2>0. \tag{19}$$

From (18) and (19) it follows that

$$\beta_1(x^+)^\gamma - M_3 \leqslant f(z,x)x-pF(z,x) \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R} \text{ and some } M_3>0. \tag{20}$$

Returning to (17) and using (20), we obtain

$$\beta_1\int_\Omega u_n^\gamma dz \leqslant M_3 \text{ for all } n\geqslant 1 \text{ (recall } u_n\in intC_+),$$
$$\Rightarrow \{u_n\}_{n\geqslant 1}\subseteq L^\gamma(\Omega) \text{ is bounded.} \tag{21}$$

It is clear that in hypothesis H(ii), we may assume $\gamma\leqslant r<p^*$. Suppose $N\neq p$. Then we can find $t\in[0,1)$ s.t. $1/r=(1-t)/\gamma+t/p^*$. Invoking the interpolation inequality, we have

$$||u_n||_r \leqslant ||u_n||_\gamma^{1-t}||u_n||_{p^*}^t,$$

$$\Rightarrow ||u_n||_r \leqslant M_4 ||u_n||^t \text{ for some } M_4 > 0, \text{ all } n \geqslant 1 \text{ (see (21))}$$
$$\Rightarrow ||u_n||_r^r \leqslant M_4 ||u_n||^{rt} \tag{22}$$

Hypothesis H(i) implies that

$$|f(z,x)x| \leqslant \hat{a}(z) + \hat{c}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } \hat{a} \in L^\infty(\Omega)_+, \hat{c} > 0. \tag{23}$$

On (13) we act with $u_n \in intC_+$ and obtain

$$\begin{aligned} ||Du_n||_p^p + \int_\Omega \beta u_n^p dz &= \lambda_n \int_\Omega f(z,u_n)u_n dz \\ &\leqslant \lambda_n c_1(1 + ||u_n||_r^r) \text{ for some } c_1 > 0, \text{ all } n \geqslant 1 \text{ (see (23))} \\ &\leqslant \lambda_n c_2(1 + ||u_n||^{tr}) \text{ for some } c_2 > 0, \text{ all } n \geqslant 1 \text{ (see (22))}, \\ \Rightarrow \xi_0 ||u_n||^p &\leqslant \lambda^* c_2(1 + ||u_n||^{tr}) \text{ for all } n \geqslant 1 \text{ (see Lemma 1)}. \end{aligned} \tag{24}$$

From the restriction on γ (see H(ii)), we have that $tr < p$ and so from (24), it follows that

$$\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega) \text{ is bounded.} \tag{25}$$

If $N = p$, then by definition $p^* = +\infty$ and by the Sobolev embedding theorem, $W_n^{1,p}(\Omega)$ is embedded compactly into $L^s(\Omega)$ for all $s \in [1,\infty)$. Let $\gamma \leqslant r < s$ and choose $t \in [0,1)$ s.t. $1/r = (1-t)/\gamma + /s$, hence $tr = s(r-\gamma)/(s-\gamma)$. Note that $s(r-\gamma)/(s-\gamma) \to r-\gamma$ as $s \to +\infty = p^*$ and by hypothesis H(ii), $r - \gamma < p$ (recall $N = p$). Hence for $s > p$ large enough, we will have $tr < p$ and so the previous argument with p^* replaced by this large $s > p$ works and gives again (25). Because of (25), we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_n^{1,p}(\Omega) \text{ and } u_n \to u^* \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{26}$$

On (13) we act with $u_n - u^* \in W_n^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (26). Then

$$\lim_{n\to\infty} < A(u_n), u_n - u^* >= 0,$$
$$\Rightarrow u_n \to u^* \text{ in } W_n^{1,p} \text{ as } n \to \infty \text{ (since } A \text{ is of type } (S)_+). \tag{27}$$

So, if in (13) we pass to the limit as $n \to \infty$ and use (27), we obtain

$$A(u^*) + \beta(u^*)^{p-1} = \lambda^* N_f(u^*),$$
$$\Rightarrow u^* \in C_+ \text{ solves } (1)_{\lambda^*}.$$

It remains to show that $u^* \neq 0$, hence $u^* \in intC_+$ (by the nonlinear maximum principle of Vazquez). To this end, we consider the following auxiliary nonlinear Neumann problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \hat{\mu}\eta(z)u(z)^{\tau-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u > 0. \end{array} \right\} \tag{28}$$

Here $\hat{\mu} \in (0,\lambda_1)$. Let $\psi_0 : W_n^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (28) defined by

$$\psi_0 = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \frac{\hat{\mu}}{\tau}\int_\Omega \eta(u^+)^\tau dz$$

$$\geqslant \frac{\xi_0}{p}||u||^p - \frac{c_3}{\tau}||u||^\tau \text{ for some } c_3 > 0, \text{ all } u \in W_n^{1,p}(\Omega). \tag{29}$$

Since $\tau < p$ (see H(iii)), from (29) we infer that ψ_0 is coercive. It is also sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\tilde{u} \in W_n^{1,p}(\Omega)$ s.t.

$$\psi_0(\tilde{u}) = \inf[\psi_0(u) : u \in W_n^{1,p}(\Omega)] = \tilde{m}. \tag{30}$$

Note that if $\xi \in (0,1)$, then

$$\psi_0(\xi) \leqslant \frac{\xi^p}{p}\int_\Omega \beta d\mu - \frac{\xi^\mu \mu}{\tau}\int_\Omega \eta dz.$$

Since $\tau < p$, by choosing $\xi \in (0,1)$ small enough we have $\psi_0(\xi) < 0 \Rightarrow \psi_0(\tilde{u}) = \tilde{m} < 0 = \psi_0(0) \Rightarrow \tilde{u} \neq 0$. Then from (30) we have

$$\begin{aligned}&\psi_0'(\tilde{u}) = 0,\\ \Rightarrow &A(\tilde{u}) + \beta\tilde{u}^{p-1} = \hat{\mu}\eta(\tilde{u}^+)^{p-1}.\end{aligned} \tag{31}$$

On (31) we act with $-\tilde{u}^- \in W_n^{1,p}(\Omega)$ and obtain

$$\begin{aligned}&||D\tilde{u}^-||_p^p + \int_\Omega \beta(\tilde{u}^-)^p dz = 0\\ \Rightarrow &\xi_0||\tilde{u}^-||^p \leqslant 0, \text{ i.e. } \tilde{u} \geqslant 0, \tilde{u} \neq 0 \text{ (see Lemma 1).}\end{aligned}$$

From (31), we have

$$\begin{aligned}&-\Delta_p\tilde{u}(z) + \beta(z)\tilde{u}(z)^{p-1} = \hat{\mu}\eta(z)\tilde{u}(z)^{\tau-1} \geqslant 0 \text{ a.e. in } \Omega, \frac{\partial\tilde{u}}{\partial\eta} = 0 \text{ on } \partial\Omega,\\ \Rightarrow &\tilde{u} \in intC_+ \text{ (nonlinear regularity theory and Vazquez).}\end{aligned}$$

Since $u_n \in intC_+$, we can find the biggest constant $t_n > 0$ s.t. $t_n\tilde{u} \leqslant u_n$. Suppose $t_n \in (0,1)$. Let $\theta_n = ||u_n||_\infty$ and let $\sigma_n > 0$ be the positive real postulated by hypothesis H(iv). For $\delta \in (0, \min_{\bar{\Omega}} u_n)$ (recall $u_n \in intC_+$), as before (see the proof of Proposition 3), we set $u_n^\delta = u_n - \delta \in intC_+$. We have

$$\begin{aligned}&-\Delta_p u_n^\delta(z) + (\beta(z) + \sigma_n)u_n^\delta(z)^{p-1}\\ =&-\Delta_p u_n(z) + (\beta(z) + \sigma_n)u_n(z)^{p-1} - w_n(\delta) \text{ (with } w_n(\delta) \to 0^+ \text{ as } \delta \to 0^+)\\ =&\lambda_n f(z, u_n(z)) + \sigma_n u_n(z)^{p-1} - w_n(\delta)\\ \geqslant&\hat{\mu} f(z, u_n(z)) + \sigma_n u_n(z)^{p-1} + (\lambda_n - \hat{\mu})f(z, u_n(z)) - w_n(\delta)\\ &\qquad\text{(since } \lambda_n \geqslant \lambda_1 > \hat{\mu} \text{ for all } n \geqslant 1 \text{ and } f \geqslant 0)\\ \geqslant&\hat{\mu} f(z, t_n\tilde{u}(z)) + \sigma_n(t_n\tilde{u}(z))^{p-1} + (\lambda_n - \hat{\mu})\mu_n - w_n(\delta)\\ &\qquad\text{(where } \mu_n = \mu_{s_n} > 0, s_n = \min_{\bar{\Omega}} u_n, \text{ see H(iii), (iv))}\end{aligned}$$

$$\geqslant \hat{\mu}\eta(z)(t_n\tilde{u}(z))^{\tau-1} + \sigma_n(t_n\tilde{u}(z))^{p-1} + (\lambda_n - \hat{\mu})\mu_n - w_n(\delta) \text{ (see H(iii)).}$$

Because $w_n(\delta) \to 0^+$ as $\delta \to 0^+$, for $\delta \in (0,1)$ small we have $(\lambda_n - \hat{\mu})\mu_n \geqslant w_n(\delta)$. Therefore

$$\begin{aligned}
&-\Delta_p u_n^\delta(z) + (\beta(z)+\sigma_n)u_n^\delta(z)^{p-1}\\
&\geqslant \hat{\mu}\eta(z)(t_n\tilde{u}(z))^{\tau-1} + \sigma_n(t_n\tilde{u}(z))^{p-1}\\
&\geqslant \hat{\mu}t_n^{p-1}\eta(z)(t_n\tilde{u}(z))^{\tau-1} + \sigma_n(t_n\tilde{u}(z))^{p-1} \text{ (since } t_n \in (0,1),\ \tau < p)\\
&= -\Delta_p(t_n\tilde{u}(z)) + (\beta(z)+\sigma_n)(t_n\tilde{u}(z))^{p-1} \text{ a.e. in } \Omega \text{ (see (28))},\\
\Rightarrow & u_n^\delta \geqslant t_n\tilde{u},\\
\Rightarrow & u_n - t_n\tilde{u} \in intC_+ \text{ contradicting the maximality of } t_n > 0.
\end{aligned}$$

So, $t_n \geqslant 1$ for all $n \geqslant 1$ and we have $\tilde{u} \leqslant u_n$ for all $n \geqslant 1$, hence $\tilde{u} \leqslant u$, i.e., $u \in intC_+$. □

Next we show that, for $\lambda \in (0,\lambda^*)$, we have two positive solutions.

PROPOSITION 5. *If hypotheses H hold and $\lambda \in (0,\lambda^*)$, then problem $(1)_\lambda$ has at least two positive smooth solutions*

$$u_0, \hat{u} \in intC_+,\ u_0 \leqslant \hat{u},\ \hat{u} \neq u_0.$$

Proof. Let $u^* \in intC_+$ be a solution for problem $(1)_{\lambda^*}$ (it exists by virtue of Proposition 4). Then

$$A(u^*) + \beta(u^*)^{p-1} = \lambda^* N_f(u^*) \geqslant \lambda N_f(u^*) \text{ (since } f \geqslant 0 \text{ and } \lambda < \lambda^*). \tag{32}$$

As before, truncating the reaction $f(z,\cdot)$ at $u^*(z)$ and using the direct method and (32), we obtain a solution $u_0 \in [0,u^*] \cap intC_+$ for problem $(1)_\lambda$. In fact we may assume that u_0 is the biggest solution of $(1)_\lambda$ in the order interval $[0,u^*]$. The existence of this extremal solution can be established as in Aizicovici-Papageorgiou-Staicu [1], Proposition 8) using the Kuratowski-Zorn lemma.

Using u_0, we will produce a second positive smooth solution for problem $(1)_\lambda$. To the end, we consider the following truncation of the reaction $f_0(z,x)$:

$$f_0(z,x) = \begin{cases} f(z,u_0(z)), & x \leqslant u_0(z),\\ f(z,x), & u_0(z) < x. \end{cases} \tag{33}$$

This is a Caratheodory function. We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0^\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0^\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \lambda\int_\Omega F_0(z,u)dz \text{ for all } u \in W_n^{1,p}(\Omega).$$

Claim 1: φ_0^λ satisfies the C- condition

Let $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ be a sequence s.t.

$$|\varphi_0^{\lambda}(u_n)| \leqslant M_5 \text{ for some } M_5 > 0, \text{ all } n \geqslant 1 \tag{34}$$

$$\text{and}(1+||u_n||)(\varphi_0^{\lambda})'(u_n) \to 0 \text{ in } W_n^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{35}$$

From (35) we have

$$|<(\varphi_0^{\lambda})'(u_n),h>| \leqslant \frac{\varepsilon_n||h||}{1+||u_n||} \text{ for all } h \in W_n^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+,$$

$$\Rightarrow |<A(u_n),h> + \int_\Omega \beta |u_n|^{p-2}u_n h dz - \lambda \int_\Omega f_0(z,u_n) h dz| \leqslant \frac{\varepsilon_n ||h||}{1+||u_n||} \text{ for all } n \geqslant 1. \tag{36}$$

In (36), we choose $h = -u_n^- \in W_n^{1,p}(\Omega)$ and obtain

$$\xi_0 ||u_n^-||^p - \lambda \int_\Omega f(z,u_0)(-u_n^-)dz \leqslant \varepsilon_n \text{ (see Lemma 1)}$$

$$\Rightarrow u_n^- \to 0 \text{ in } W_n^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see H(iii)).}$$

Next on (36), we use $h = u_n^+ \in W_n^{1,p}(\Omega)$ and obtain

$$-||Du_n^+||_p^p - \int_\Omega \beta (u_n^+)^p dz + \lambda \int_\Omega f_0(z,u_n)u_n^+ dz \leqslant \varepsilon_n \text{ for all } n \geqslant 1. \tag{37}$$

On the other hand from (34) and since $u_n^- \to 0$ for $n \to \infty$, we have

$$||Du_n^+||_p^p + \int_\Omega \beta (u_n^+)^p dz - \lambda \int_\Omega pF_0(z,u_n^+)dz \leqslant pM_5 \text{ for all } n \geqslant 1. \tag{38}$$

Adding (37) and (38), we obtain

$$\lambda \int_\Omega [pF_0(z,u_n^+) - f_0(z,u_n^+)u_n^+]dz \leqslant M_6 \text{ for some } M_6 > 0, \text{ all } n \geqslant 1,$$

$$\Rightarrow \int_\Omega [pF(z,u_n^+) - f(z,u_n^+)u_n^+]dz \leqslant M_7 \text{ for some } M_7 > 0, \text{ all } n \geqslant 1 \quad \text{(see (33))}. \tag{39}$$

Using (39) and reasoning as in the proof of Proposition 4, via hypothesis H(ii) and the interpolation inequality, we show that $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_n^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{40}$$

Therefore, if in (36) we set $h = u_n - u \in W_n^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (40) and the $(S)_+$-property of A, we obtain $u_n \to u \in W_n^{1,p}(\Omega)$ and so we

conclude that φ_0^λ satisfies the C-condition. This proves Claim 1.
Claim 2: $u_0 \in intC_+$ is a local minimizer of φ_0^λ
We consider the following truncation of the nonlinearity $f_0(z,\cdot)$:

$$g_0(z,x) = \begin{cases} f(z,x), & x \leqslant u^*(z), \\ f_0(z,u^*(z)), & u^*(z) < x. \end{cases} \tag{41}$$

This is a Caratheodory function. We set $G_0(z,x) = \int_0^x g_0(z,s)ds$ and introduce the C^1-functional $\psi_0^\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_0^\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \lambda\int_\Omega G_0(z,u)dz \text{ for all } u \in W_n^{1,p}(\Omega).$$

Clearly, ψ_0^λ is sequentially weakly lower semicintinuous. Moreover, because of (41) we see that ψ_0^λ is coercive. Hence, by the Weierstrass theorem, we can find $\hat{u}_0 \in W_n^{1,p}(\Omega)$ s.t.

$$\psi_0^\lambda(\hat{u}_0) = inf[\psi_0^\lambda(u) : u \in W_n^{1,p}(\Omega)]. \tag{42}$$

From (42), we have

$$\begin{aligned} &(\psi_0^\lambda)'(\hat{u}_0) = 0, \\ \Rightarrow &A(\hat{u}_0) + \beta|\hat{u}_0|^{p-2}\hat{u}_0 = \lambda N_{g_0}(\hat{u}_0). \end{aligned} \tag{43}$$

On (43) we act with $(u_0 - \hat{u}_0)^+ \in W_n^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &< A(\hat{u}_0), (u_0-\hat{u}_0)^+ > + \int_\Omega \beta|\hat{u}_0|^{p-2}\hat{u}_0(u_0-\hat{u}_0)^+ dz \\ =&\lambda\int_\Omega g_0(z,\hat{u}_0)(u_0-\hat{u}_0)^+ dz \\ =&\lambda\int_\Omega f(z,u_0)(u_0-\hat{u}_0)^+ dz \text{ (see (41))} \\ =&< A(u_0), (u_0-\hat{u}_0)^+ > + \int_\Omega \beta u_0^{p-1}(u_0-\hat{u}_0)^+ dz \\ &\text{(since } u_0 \in intC_+ \text{ is a solution of } (1)_\lambda\text{),} \end{aligned}$$

$$\begin{aligned} \Rightarrow &< A(\hat{u}_0) - A(u_0), (u_0-\hat{u}_0)^+ > + \int_\Omega \beta(|\hat{u}_0|^{p-2}\hat{u}_0 - u_0^{p-1})(u_0-\hat{u}_0)^+ dz = 0, \\ \Rightarrow &|\{u_0 > \hat{u}_0\}|_N = 0, \text{ i.e., } u_0 \leqslant \hat{u}_0 \end{aligned} \tag{44}$$

Also, on (43) we act with $(\hat{u}_0 - u^*)^+ \in W_n^{1,p}(\Omega)$ and have

$$\begin{aligned} &< A(\hat{u}_0), (\hat{u}_0-u^*)^+ > + \int_\Omega \beta\hat{u}_0^{p-1}(\hat{u}_0-u^*)^+ dz \\ =&\lambda\int_\Omega g_0(z,\hat{u}_0)(\hat{u}_0-u^*)^+ dz \end{aligned}$$

$$=\lambda \int_\Omega f_0(z,u^*)(\hat{u}_0-u^*)^+dz \text{ (see (41))}$$
$$=\lambda \int_\Omega f(z,u^*)(\hat{u}_0-u^*)^+dz \text{ (see (32) and recall that } u_0 \leqslant u^*)$$
$$\leqslant < A(u^*),(\hat{u}_0-u^*)^+ > + \int_\Omega \beta(u^*)^{p-1}(\hat{u}_0-u^*)^+dz \text{ (see (32))},$$

$$\Rightarrow 0 \leqslant < A(u^*)-A(\hat{u}_0),(\hat{u}_0-u^*)^+ > + \int_\Omega \beta((u^*)^{p-1}-\hat{u}_0^{p-1})(\hat{u}_0-u^*)^+dz,$$
$$\Rightarrow |\{\hat{u}>u^*\}|_N = 0, \text{ i.e., } \hat{u}_0 \leqslant u^*. \tag{45}$$

From (44) and (45), we see that $\hat{u}_0 \in [u_0,u^*]$ and so (43) becomes

$$A(\hat{u}_0)+\beta\hat{u}_0^{p-1} = \lambda N_f(\hat{u}_0)dz \text{ (see (41) and (33))},$$
$$\Rightarrow \hat{u}_0 \in intC_+ \text{ solves problem } (1)_\lambda \text{ (nonlinear regularity).}$$

Recall that u_0 is the biggest solution of $(1)_\lambda$ in the order interval $[0,u^*]$. Hence $\hat{u}_0 = u_0$. Let $\delta > 0$ and set $u_0^\delta = u_0+\delta \in intC_+$. Then for $\theta = ||u^*||_\infty$, $I=(0,\lambda^*]$, $\lambda \in I$ and $\sigma_\theta > 0$ as postulate by hypothesis H(iv), we have

$$-\Delta_p u_0^\delta(z)+\beta(z)u_0^\delta(z)^{p-1}+\sigma_\theta u_0^\delta(z)^{p-1}$$
$$=-\Delta_p u_0(z)+(\beta(z)+\sigma_\theta)u_0(z)^{p-1}-\rho_0(\delta) \text{ (with } \rho_0(\delta)\to 0^+ \text{ as } \delta \to 0^+)$$
$$=\lambda f(z,u_0(z))+\sigma_\theta u_0(z)^{p-1}+\rho_0(\delta)$$
$$=\lambda^* f(z,u_0(z))+\sigma_\theta u_0(z)^{p-1}-(\lambda^*-\lambda)f(z,u_0(z))+\rho_0(\delta)$$
$$\leqslant \lambda^* f(z,u^*(z))+\sigma_\theta u^*(z)^{p-1}-(\lambda^*-\lambda)\mu_s+\rho_0(\delta)$$
$$\text{(where } s=\min_{\bar{\Omega}} u_0, \text{ see H(iii)).}$$

Recall that $\rho_0(\delta)\to 0^+$ as $\delta \to 0^+$. So for $\delta \in (0,1)$ small, we have $\rho_0(\delta) \leqslant (\lambda^*-\lambda)\mu_s$. Hence

$$A(u_0^\delta)+(\beta+\sigma_\theta)(u_0^\delta)^{p-1}$$
$$\geqslant \lambda^* N_f(u^*)+\sigma_\theta(u^*)^{p-1}$$
$$=A(u^*)(\beta+\sigma_\theta)(u^*)^{p-1} \text{ (see (32))},$$
$$\Rightarrow u_0^\delta \leqslant u^*,$$
$$\Rightarrow u^*-u_0 \in intC_+.$$

Since $u_0 \in intC_+$, we see that $u_0 = \hat{u}_0 \in int_{C_n^1(\bar{\Omega})}[0,u^*]$. Therefore u_0 is a local $C_n^1(\bar{\Omega})$-minimizer of ψ_0^λ. But note that $\psi_0^\lambda\ |_{[0,u^*]} = \varphi_0^\lambda\ |_{[0,u^*]}$. Hence u_0 is a local $C_n^1(\bar{\Omega})$-minimizer of φ_0^λ, and this implies that u_0 is a local $W_n^{1,p}(\Omega)$-minimizer of φ_0^λ (see Motreanu-Papageorgiou [22]). This proves Claim 2.

We may assume that $u_0 \in intC_+$ is an isolated critical point of φ_0^λ. Indeed, otherwise we have a whole sequence of distinct solution of $(1)_\lambda$ belonging in $intC_+$ and so we are done. Then as in Aizicovici-Papageorgiou-Staicu [1], Proposition 5, we can find $\rho \in (0,1)$ small s.t.

$$\varphi_0^\lambda(u_0) < inf[\varphi_0^\lambda(u) : ||u-u_0|| = \rho] = \eta_\rho. \tag{46}$$

Hypothesis H(ii) implies that

$$\varphi_0^\lambda(\xi) \to -\infty \text{ as } \xi \to +\infty, \xi \in \mathbb{R}. \tag{47}$$

Then (46), (47) and Claim 1 permit the application of mountain pass theorem and so, we obtain $\hat{u} \in W_n^{1,p}(\Omega)$ s.t.

$$(\varphi_0^\lambda)'(\hat{u}) = 0 \tag{48}$$

$$\text{and } \varphi_0^\lambda(u_0) < \eta_\rho \leqslant \varphi_0^\lambda(\hat{u}). \tag{49}$$

From (49), it follows that $\hat{u} \neq u_0$. From (48), we have

$$A(\hat{u}) + \beta|\hat{u}|^{p-2}\hat{u} = \lambda N_f(\hat{u}). \tag{50}$$

As in the proof of Claim 2, acting on (50) with $(u_0 - \hat{u})^+ \in W_n^{1,p}(\Omega)$, we show that $\hat{u} \geqslant u_0$. Hence because of (33) we conclude that $\hat{u} \in intC_+$ (nonlinear regularity) solves $(1)_\lambda$. □

Summarizing the situation, we can formulate the following bifurcation-type result for problem $(1)_\lambda$.

THEOREM 2. *If hypotheses H hold,*
then there exists $\lambda^* > 0$ s.t. (a) *for all $\lambda \in (0,\lambda^*)$, problem $(1)_\lambda$ has at least two positive solutions $u_0, \hat{u} \in intC_+$, $u_0 \leqslant \hat{u}$, $u_0 \neq \hat{u}$;*

(b) *for $\lambda = \lambda^*$, problem $(1)_\lambda$ has at least one positive solution $u^* \in intC_+$;*

(c) *for all $\lambda > \lambda^*$, problem $(1)_\lambda$ has no positive solution.*

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Evgenia H. Papageorgiou
Hellenic Naval Academy
Hatzikyriakou Avenue
Peraeus 18539
Greece
e-mail: epap@snd.edu.gr