Differential
Equations
& Applications
Volume 6, Number 3 (2014), 353–360

doi:10.7153/dea-06-19

OSCILLATION CRITERIA FOR HALF–LINEAR DIFFERENTIAL EQUATIONS WITH $p(t)$–LAPLACIAN

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(*Communicated by Qihu Zhang*)

Abstract. This paper presents suffient conditions for oscillation of solutions of half-linear differential equations with $p(t)$-Laplacian. As an application of generalized Riccati-type inequality, some new oscillation theorems are established.

Dedicated to Professor Norio Yoshida on the occasion of his 65-th birthday

1. Introduction

The differential equations and variational problems with $p(x)$-growth conditions arise from nonlinear elasticity theory, electrorheological fluids, etc. (see [3]–[5]). Especially, we have much interest in studying oscillation problem for the $p(\cdot)$-Laplacian equation.

In 2007, Zhang [6] was treated oscillation problem for nonlinear equations with $p(t)$-Laplacian

$$(|u'(t)|^{p(t)-2}u'(t))' + c(t)g(t,u) = 0,\ t > 0.$$

Motivated by this article [6], Yoshida[10],[11] established Picone identities and Strumian comparison theorems for half-linear elliptic inequalities with $p(x)$-Laplacian

$$vQ[v] \leqslant 0,$$

where

$$Q[v] \equiv \nabla \cdot (A(x)|\nabla v|^{p(x)-1}\nabla v) - A(x)\nabla p(x)(\log|v|)|\nabla v|^{p(x)-1}\nabla v \\ + B(x)|\nabla v|^{p(x)-1}\nabla v + C(x)|v|^{p(x)-1}v.$$

It notes in above inequality that $\log|v|$ has singularities at zeros of $v(x)$, but $v\log|v|$ is continuous at every zero x_0, that is, $\lim_{\varepsilon\to+0}\varepsilon\log\varepsilon = 0$ when $v\log|v| = 0$ at $x = x_0$.

Mathematics subject classification (2010): 34K11, 35B05, 35R10.
Keywords and phrases: $p(t)$-Laplacian, oscillation theory, half-linear.

Paper DEA-06-19

There are few papers dealing the oscillation problems of $p(\cdot)$-Laplacian type equation[7],[8]. It is easy to consider that $p(t)$-Laplacian equation

$$(|u'(t)|^{p(t)-2}u'(t))' + c(t)|u|^{p(t)-2}u = 0,\ t > 0$$

is not half-linear, if $p(t)$ is not a constant.

Recently, Adamowicz and Hästö introduced a new varient of the $p(\cdot)$-Laplacian in [1], [2]. They were baced on the strong form of the p-Laplace equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$, i.e.

$$\Delta_p u = |\nabla u|^{p-4}\left[(p-2)\sum_{i,j} u_{x_i,x_j}u_{x_i}u_{x_j} + |\nabla u|^2\Delta u\right] = 0.$$

If p is replaced by $p(x)$ and defines this operator as $\tilde{\Delta}_{p(\cdot)}$, then we note that

$$\tilde{\Delta}_{p(\cdot)}u \equiv \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) - |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p.$$

Furthermore we found in [1] that the following simple result:

PROPOSITION 1. *Let p be Lipschitz with $1 < p^- \leqslant p^+ < \infty$. Then*

$$\tilde{\Delta}_{p(\cdot)}(\lambda u) = \lambda^{p(\cdot)-1}\tilde{\Delta}_{p(\cdot)}u$$

in the sense of distributions for $u \in W^{1,p(\cdot)}(\Omega)$ and $\lambda \in [0,\infty)$. In particular, if u is a solution, then so is λu.

Therefore, our aim in this article is to obtain sufficient conditions for the oscillation of solutions of half-linear differential equations with $p(t)$-Laplacian

$$(|u'(t)|^{p(t)-2}u'(t))' - p'(t)(\log|u'(t)|)|u'(t)|^{p(t)-2}u'(t) \\ + c(t)|u(t)|^{p(t)-2}u(t) = 0,\ t > 0. \qquad \text{(E)}$$

DEFINITION 1. A solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros, otherwise it is nonoscillatory.

PROPOSITION 2. *The $p(t)$-Laplacian equation* (E) *is half-linear in the sense that u is solution of* (E)*, then ku is also solution of* (E) *for any constant k.*

Proof. Let u be any solution of (E), and k be any constant. We see that

$$\begin{aligned}&(|k|^{p(t)-2}k \cdot |u'(t)|^{p(t)-2}u'(t))' \\ &\quad - p'(t)\log(|k| \cdot |u'(t)|) \cdot |k|^{p(t)-2}k \cdot |u'(t)|^{p(t)-2}|u'(t)| \\ &\quad + c(t) \cdot |k|^{p(t)-2}k \cdot |u(t)|^{p(t)-2}u = 0.\end{aligned} \qquad (1.1)$$

By direct calculation, it is easy to see that

$$(|k|^{p(t)-2}k \cdot |u'(t)|^{p(t)-2}u'(t))'$$

$$
\begin{aligned}
&= |k|^{p(t)-2}k\cdot(|u'(t)|^{p(t)-2}u'(t))' \\
&\quad + p'(t)\log|k|\cdot|k|^{p(t)-2}k\cdot|u'(t)|^{p(t)-2}u'(t).
\end{aligned} \tag{1.2}
$$

Substituting (1.1) into (1.2) yields

$$
\begin{aligned}
|k|^{p(t)-2}k\Big\{ &(|u'(t)|^{p(t)-2}u'(t))' - p'(t)(\log|u'(t)|)|u'(t)|^{p(t)-2}u'(t) \\
&+ c(t)|u(t)|^{p(t)-2}u(t)\Big\} = 0
\end{aligned}
$$

for any constant k. Hence, we conclude that (E) is half-linear.

2. Main results

First let us present a direct extension of the work [9] to the generalized Riccati inequality.

PROPOSITION 3. *If there exists a function* $\phi(t)\in C^1([T_0,\infty);(0,\infty))$ *such that*

$$
\int_{T_1}^{\infty}\left(\frac{\alpha(t)-1}{\alpha(t)}\right)\left(\frac{\bar{p}(t)|\phi'(t)|^{\alpha(t)}}{\phi(t)}\right)^{\frac{1}{\alpha(t)-1}}dt<\infty,
$$

$$
\int_{T_1}^{\infty}\phi(t)\bar{q}(t)dt=\infty
$$

for some $T_1 \geqslant T_0$, *then the Riccati inequality*

$$
x'(t)+\frac{1}{\bar{p}(t)}|x(t)|^{\alpha(t)}\leqslant-\bar{q}(t), \tag{2.1}
$$

has no positive solution on $[T,\infty)$ *for all large* T, *where* $\alpha(t)\in C(\mathbb{R};(1,\infty))$, $\bar{p}(t)\in C([T_0,\infty);(0,\infty))$ *and* $\bar{q}(t)\in C([T_0,\infty);\mathbb{R})$.

Proof. Suppose that (2.1) has a positive solution $x(t)$, then there exists a $T_0>0$ such that $x(t)>0$ for $t\geqslant T_0$. Mutiplying (2.1) by $\phi(t)$ and integrating both sides over $[T_0,t]$, we have

$$
\int_{T_0}^{t}\left(\frac{\phi(s)}{\bar{p}(s)}\right)x(s)^{\alpha(s)}ds-\int_{T_0}^{t}x(s)\phi'(s)ds+\int_{T_0}^{t}\bar{q}(s)\phi(s)ds\leqslant x(T_0)\phi(T_0). \tag{2.2}
$$

By Young's inequality we obtain

$$
\begin{aligned}
x(s)|\phi'(s)| &\leqslant x(s)\left(\frac{\phi(s)}{\bar{p}(s)}\right)^{\frac{1}{\alpha(s)}}\cdot\left(\frac{\bar{p}(s)}{\phi(s)}\right)^{\frac{1}{\alpha(s)}}|\phi'(t)| \\
&\leqslant \frac{1}{\alpha(s)}\left(\frac{\phi(s)}{\bar{p}(s)}\right)x(s)^{\alpha(s)}
\end{aligned}
$$

$$+\frac{\alpha(s)-1}{\alpha(s)}\left(\left(\frac{\bar{p}(s)}{\phi(s)}\right)^{\frac{1}{\alpha(s)}}|\phi'(s)|\right)^{\frac{\alpha(s)}{\alpha(s)-1}}.$$

This and (2.2) imply that

$$\begin{aligned}&\int_{T_0}^{t}\left(1-\frac{1}{\alpha(s)}\right)\left(\frac{\phi(s)}{\bar{p}(s)}\right)x(s)^{\alpha(s)}ds\\&\quad-\int_{T_0}^{t}\left(\frac{\alpha(s)-1}{\alpha(s)}\right)\left(\left(\frac{\phi(s)}{\bar{p}(s)}\right)|\phi'(s)|^{\alpha(s)}\right)^{\frac{1}{\alpha(s)-1}}ds\\&\quad+\int_{T_0}^{t}\bar{q}(s)\phi(s)ds\leqslant x(T_0)\phi(T_0).\end{aligned}\tag{2.3}$$

Therefore, the left hand side of (2.3) is finite, which cotradicts the condition. The proof is complete.

THEOREM 1. *Assume that the following hypothesis holds*:

(H) $c(t)\in C((0,\infty);(0,\infty))$, *and increasing function* $p(t)\in C^1(\mathbb{R};(1,\infty))$ *satisfying* $1+p'(t)<p(t)$ *and*

$$1<\inf_{t\in\mathbb{R}}p(t)\leqslant\sup_{t\in\mathbb{R}}p(t)<\infty.$$

If there exists $\phi(t)\in C^1((0,\infty);(0,\infty))$ *such that*

$$\int^{\infty}\left\{\frac{|\phi'(t)|^{\frac{p(t)+1}{p(t)}}}{\left(p'(t)\big(p(t)-1-p'(t)\big)^{p(t)-1}\right)^{\frac{1}{p(t)}}\phi(t)}\right\}^{p(t)}dt<\infty,$$

$$\int^{\infty}\phi(t)c(t)dt=\infty,$$

then every solution $u(t)$ *of* (E) *is oscillatory.*

Proof. Suppose to the contrary that (E) admits a positive solution $u(t)$. This shows that there exists a $t_0>0$ satisfying $u(t)>0$ for $t\geqslant t_0$. Here we see that

$$\begin{aligned}&\left(e^{-\int_{t_0}^{t}p'(s)\log|u'(s)|ds}|u'(t)|^{p(t)-2}u'(t)\right)'\\&\qquad=-c(t)e^{-\int_{t_0}^{t}p'(s)\log|u'(s)|ds}|u(t)|^{p(t)-2}u(t)<0,\ t\geqslant t_0.\end{aligned}\tag{2.4}$$

That is, for some $t_1>t_0$, we see that $u'(t)>0$ or $u'(t)\leqslant 0$, $t\geqslant t_1$. In the latter case, it follows from (2.4) that

$$e^{-\int_{t_0}^{t}p'(s)\log|u'(s)|ds}|u'(t)|^{p(t)-2}u'(t)\leqslant e^{-\int_{t_0}^{t_1}p'(s)\log|u'(s)|ds}|u'(t_1)|^{p(t_1)-2}u'(t_1),$$

and then

$$-(-u'(t))^{p(t)-1}\leqslant -e^{\int_{t_1}^{t}p'(s)\log|u'(s)|ds}|u'(t_1)|^{p(t_1)-1}.$$

Hence it is easy to see that

$$u'(t) \leqslant -e^{\frac{1}{p(t)-1}\int_{t_1}^{t} p'(s)\log|u'(s)|ds}|u'(t_1)|^{\frac{p(t_1)-1}{p(t)-1}}. \tag{2.5}$$

On the other hand, equation (E) shows

$$\left(|u'(t)|^{p(t)-2}u'(t)\right)' \leqslant p'(t)\log|u'(t)|\left(|u'(t)|^{p(t)-2}u'(t)\right).$$

So we derive

$$-\frac{\left(|u'(t)|^{p(t)-1}\right)'}{|u'(t)|^{p(t)-1}} \leqslant -p'(t)\log|u'(t)|,$$

which can be rewritten as

$$-\left\{p'(t)\log|u'(t)| + (p(t)-1)\frac{|u'(t)|'}{|u'(t)|}\right\} \leqslant -p'(t)\log|u'(t)|.$$

Consequently it is easy to see that

$$\frac{|u'(t)|'}{|u'(t)|} \geqslant 0,\ t \geqslant t_1. \tag{2.6}$$

Integrating (2.6) over $[t_1,t]$ yields

$$\log|u'(t)| \geqslant \log|u'(t_1)|. \tag{2.7}$$

Combining (2.4) with (2.6), we obtain

$$\begin{aligned} u'(t) &\leqslant -e^{\frac{p(t)-p(t_1)}{p(t)-1}\log|u'(t_1)|}|u'(t_1)|^{\frac{p(t_1)-1}{p(t)-1}} \\ &\leqslant -\min_{t\geqslant t_1}\left\{e^{\frac{p(t)-p(t_1)}{p(t)-1}\log|u'(t_1)|}|u'(t_1)|^{\frac{p(t_1)-1}{p(t)-1}}\right\} \equiv -a < 0. \end{aligned}$$

Integrating the above inequality yields

$$u(t) \leqslant -a(t-t_1) + u(t_1) \to -\infty$$

by letting as $t \to \infty$. This is a contradiction. This leads to $u'(t) > 0$, $t \geqslant t_2$ for some $t_2 \geqslant t_1$. Let the function $w(t)$ be defined by

$$w(t) = \left(\frac{u'(t)}{u(t)}\right)^{p(t)-1} > 0,\ t \geqslant t_2.$$

Differentiating $w(t)$ shows that

$$\begin{aligned} w'(t) = p'(t)\log|u'(t)|w(t) - c(t) \\ - w(t)\left\{p'(t)\log u(t) + (p(t)-1)\left(\frac{u'(t)}{u(t)}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= -c(t) + p'(t)w(t)\log\left(\frac{u'(t)}{u(t)}\right) \\
&\qquad - (p(t)-1)w(t)^{\frac{p(t)}{p(t)-1}},\ t \geqslant t_2.
\end{aligned}
$$

In view of the inequality $e^x \geqslant ex$ for all $x > 0$, we obtain

$$
\begin{aligned}
w'(t) &\leqslant -c(t) + p'(t)w(t)\left\{\left(\frac{u'(t)}{u(t)}\right) - 1\right\} \\
&\qquad - (p(t)-1)w(t)^{\frac{p(t)}{p(t)-1}} \\
&\leqslant -c(t) - \left\{p'(t)w(t) + (p(t)-1-p'(t))w(t)^{\frac{p(t)}{p(t)-1}}\right\}.
\end{aligned}
$$

Applying Young's inequality and following notation:

$$
p^- \equiv \inf_{t \geqslant T} p(t), \quad p^+ \equiv \sup_{t \geqslant T} p(t),
$$

it is not difficult to see that

$$
\begin{aligned}
w'(t) &\leqslant -c(t) - p(t)^{\frac{1}{p(t)}}\left(\frac{p(t)}{p(t)-1}\right)^{\frac{p(t)-1}{p(t)}} \times \\
&\qquad \times \left(p'(t)\right)^{\frac{1}{p(t)}}\left(p(t)-1-p'(t)\right)^{\frac{p(t)-1}{p(t)}} w(t)^{\frac{p(t)+1}{p(t)}} \\
&\leqslant -c(t) - (p^-)^{\frac{1}{p^+}}\left(\frac{p^-}{p^+-1}\right)^{\frac{p^--1}{p^+}} \times \\
&\qquad \times \left\{p'(t)\Big(p(t)-1-p'(t)\Big)^{p(t)-1}\right\}^{\frac{1}{p(t)}} w(t)^{\frac{p(t)+1}{p(t)}}.
\end{aligned}
$$

Therefore, the conclusion follows from Proposition 2. The proof is complete.

REMARK 1. If $u(t)$ is an eventually negative solution of (E), then we set $v(t) \equiv -u(t)$. Clearly, $v(t)$ is an eventually positive solution of (E).

EXAMPLE 1. Consider the equation

$$
\begin{aligned}
(|u'(t)|^{\frac{t}{t+1}}u'(t))' - \frac{1}{(t+1)^2}(\log|u'(t)|)|u'(t)|^{\frac{t}{t+1}}u'(t)& \\
+ |u(t)|^{\frac{t}{t+1}}u(t) = 0,& \qquad (2.8)
\end{aligned}
$$

where $p(t) = 3 - \frac{1}{t+1}$ and $c(t) = 1$ for $t > 0$. Let $\phi(t) = (t+1)^{-1}$, then it is easy to verify that the conditions of Theorem 1 hold, since

$$
\int^{\infty}\left\{\frac{(t+1)^{1-\frac{1}{t+1}}}{t^{2-\frac{1}{t+1}}(2t+3)^{2-\frac{1}{t+1}}}\right\}dt \leqslant \int^{\infty}\frac{1}{t^{2-\frac{1}{t+1}}(t+1)}dt \leqslant \int^{\infty}\frac{dt}{t^2} < \infty
$$

and

$$\int^{\infty}(t+1)^{-1}dt=\infty.$$

Hence, every solution of (2.8) is oscillatory.

EXAMPLE 2. Consider the equation

$$(|u'(t)|^{-\frac{e^{-t}}{1+e^{-t}}}u'(t))'-\frac{e^{-t}}{(1+e^{-t})^2}(\log|u'(t)|)|u'(t)|^{-\frac{e^{-t}}{1+e^{-t}}}u'(t) \\ +|u(t)|^{-\frac{e^{-t}}{1+e^{-t}}}u(t)=0, \tag{2.9}$$

where $p(t)=1+\frac{1}{1+e^{-t}}$ and $c(t)=1$ for $t>0$. If we set $\phi(t)=(1+e^{-t})^{-1}$, then we obtain the following conditions:

$$\int^{\infty}\left\{\frac{(e^{-t}+e^{-2t})^{1+\frac{1}{1+e^{-t}}}}{(1+e^{-t})^2}\right\}dt\leqslant\int^{\infty}(e^{-t}+e^{-2t})dt<\infty$$

and

$$\int^{\infty}(1+e^{-t})^{-1}dt=\infty.$$

Thus by Theorem 1, every solution of (2.9) is oscillatory.

Acknowledgements. The author is sincerely thank to Professor Norio Yoshida for continuing interest and encouragement, without him this study could not have been carried out. Also, the author thank referees for careful reading the manuscript and for giving useful comments.

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(Received August 29, 2013)
(Revised May 28, 2014)

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