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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS FOR A CLASS OF $(p(x),q(x))$– LAPLACIAN SYSTEMS

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Abstract. In this paper, our main purpose is to establish the existence of positive solution of the following system

$$\begin{cases} -\Delta_{p(x)}u = u^{\alpha(x)} + \lambda^{p(x)}v^{m(x)}, & x \in \Omega \\ -\Delta_{q(x)}v = v^{\beta(x)} + \theta^{q(x)}u^{n(x)}, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, $p(x), q(x)$ are functions which satisfy some conditions, $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. We give the existence results of positive solutions and consider the asymptotic behavior of the solutions near the boundary. The approach is based on the sub- and super-solution method.

1. Introduction

In this paper, our main purpose is to establish the existence results of positive solutions of the following system

$$\begin{cases} -\Delta_{p(x)}u = u^{\alpha(x)} + \lambda^{p(x)}v^{m(x)}, & x \in \Omega \\ -\Delta_{q(x)}v = v^{\beta(x)} + \theta^{q(x)}u^{n(x)}, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, and $p, q \in C^1(\overline{\Omega})$ are positive functions, the operator $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian and the corresponding problem is called a variable exponent problem.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It aries from nonlinear elasticity theory, electro-rheological fluids, etc. (see [16,23]). Many results have been obtained on such problems, for example [1-2,5-8,10,14]. For the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions, we refer to [1-2,5-7]. For the existence results for the elliptic systems with variable exponents, we refer to [8,14,19-22].

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When $p(x) \equiv p, q(x) \equiv q(p,q$ are positive constants), system (1.1) becomes the well known (p,q)-Laplacian system, such problems have been considered widely, see [3,13,18] and the reference therein.

In [13], the authors studied the existence of positive weak solutions for the following problem

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega \\ -\Delta_p v = \lambda g(u), & x \in \Omega \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Under the condition of

$$\lim_{s\to\infty} \frac{f(M[g(s)]^{\frac{1}{p-1}})}{s^{p-1}} = 0, \ \forall M > 0, \tag{1.3}$$

the authors gave the existence of positive solutions for problem (1.2).

In [3], the author considered the existence and nonexistence of positive weak solutions to the following problem

$$\begin{cases} -\Delta_p u = \lambda u^{\alpha} v^{\gamma}, & x \in \Omega \\ -\Delta_q v = \lambda u^{\delta} v^{\beta}, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Recently, in [18], the authors considered the existence and nonexistence of entire positive solutions to the following problem

$$\begin{cases} -\Delta_p u = a(x)u^{\alpha} + \lambda c(x)v^m, & x \in \mathbb{R}^N \\ -\Delta_q v = b(x)v^{\beta} + \theta c(x)v^n, & x \in \mathbb{R}^N \\ u, v > 0, & x \in \mathbb{R}^N \text{ and } u \to 0, v \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{1.5}$$

Under suitable conditions, they obtained the following results:

(i) there exists $(0,0) < (\lambda_*, \theta_*) < (\infty, \infty)$ such that the system (1.5) has at least one positive solution, if $(0,0) \leqslant (\lambda, \theta) \leqslant (\lambda_*, \theta_*)$

(ii) there exists $(0,0) < (\lambda^*, \theta^*) < (\infty, \infty)$ such that the system (1.5) has no positive solution, if $(\lambda^*, \theta^*) < (\lambda, \theta)$.

Here we use $(\lambda, \theta) > (\lambda^*, \theta^*)$ to denote $\lambda > \lambda^*, \theta > \theta^*$ and the same meaning for other cases in this paper.

We note that in order to obtain the existence results, the first eigenfunction of $-\Delta_p$ is used to construct the sub-solution for problems (1.2),(1.4) and (1.5). But for the variable exponent problems, maybe the first eigenvalue and the first eigenfunction of the operator $-\Delta_{p(x)}$ do not exist. Even if the first eigenfunction of $-\Delta_{p(x)}$ exists, because of the nonhomogeneity of $-\Delta_{p(x)}$, we still cannot to construct the sub-solution of variable exponent problems with the first eigenfunction. In many cases, the radial symmetric conditions are affective to deal with variable exponent problems, there are many papers about the radial variable exponent problems, see [8-9,20,22] and reference therein. In [19,20], with a condition similar to (1.3), the author discussed the existence of positive solutions of the following problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(v), & x \in \Omega \\ -\Delta_{p(x)} v = \lambda g(u), & x \in \Omega \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Recently, in [21], the author considered the existence and asymptotic behavior of positive solutions of the following system

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)}(u^{\alpha(x)}v^{\gamma(x)} + h_1(x)), & x \in \Omega \\ -\Delta_{q(x)}v = \lambda^{q(x)}(u^{\delta(x)}v^{\beta(x)} + h_2(x)), & x \in \Omega \\ u = v = 0, \quad x \in \partial\Omega. \end{cases}$$

Motivated by the above results, we study problem (1.1) in this paper. Our aim is to give the existence and asymptotic behavior of positive weak solutions for problem (1.1). By a new method to construct sub-supersolution, we obtain the existence of positive weak solutions for problem (1.1) via sub-supersolution method.

The paper is organized as follows. In section 2, we recall some facts that will be needed in the paper. In section 3, we give the proofs of main result. We will show the asymptotic behavior of the positive solutions of problem (1.1) in the last section.

2. Notations and preliminaries

In order to deal with $p(x)$-Laplacian problem, we need some theories on spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later(see[6,15-17]). For any $f \in C(\overline{\Omega})$, we write

$$f^+ = \max_{x\in\overline{\Omega}} f(x),\ f^- = \min_{x\in\overline{\Omega}} f(x).$$

Denote

$$L^{p(x)}(\Omega) = \Big\{u|\ u \text{ is a measurable real-valued funcion, } \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\Big\{\lambda > 0|\int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leqslant 1\Big\}$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. We call it variable exponent Lebesgue space.

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega)|\ |\nabla u| \in L^{p(x)}(\Omega)\},$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},\ \ \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and we call it variable exponent Sobolev space. From [6], we know that spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces.

We define

$$(L(u),v)=\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx,\ \ \forall u,v\in W_0^{1,p(x)}(\Omega),$$

then $L:W_0^{1,p(x)}(\Omega)\to (W_0^{1,p(x)}(\Omega))^*$ is a continuous, bounded and strictly monotone operator, and it is a homeomorphism(see[10,Theorem 3.1]).

DEFINITION 2.1. (1) $(u,v)\in W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$ is called a (weak) solution of problem (1.1) if it satisfies

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx=\int_\Omega (u^{\alpha(x)}+\lambda^{p(x)}v^{m(x)})\varphi dx\\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi dx=\int_\Omega (v^{\beta(x)}+\theta^{q(x)}u^{n(x)})\psi dx\end{cases}$$

for any $(\varphi,\psi)\in W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$.

(2) $(u,v)\in W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$ is called a sub-solution (super-solution) of problem (1.1) if $(u,v)\leqslant(\geqslant)(0,0)$ on $\partial\Omega$ and

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx\leqslant(\geqslant)\int_\Omega (u^{\alpha(x)}+\lambda^{p(x)}v^{m(x)})\varphi dx\\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi dx\leqslant(\geqslant)\int_\Omega (v^{\beta(x)}+\theta^{q(x)}u^{n(x)})\psi dx\end{cases}$$

for any $(\varphi,\psi)\in W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$ with $(\varphi,\psi)\geqslant(0,0)$.

DEFINITION 2.2. Let $u_1,u_2\in W^{1,p(x)}(\Omega)$. We say that $-\Delta_{p(x)}u_1\leqslant-\Delta_{p(x)}u_2$ if for all $\varphi\in W_0^{1,p(x)}(\Omega)$ with $\varphi\geqslant 0$, we have

$$\int_\Omega |\nabla u_1|^{p(x)-2}\nabla u_1\nabla\varphi dx\leqslant\int_\Omega |\nabla u_2|^{p(x)-2}\nabla u_2\nabla\varphi dx.$$

Now we give a comparison principle as follows.

LEMMA 2.1. (see [4,Lemma 2.2]) *Let* $u_1,u_2\in W^{1,p(x)}(\Omega)$. *If*

$$-\Delta_{p(x)}u_1\leqslant-\Delta_{p(x)}u_2 \text{ and } u_1\leqslant u_2 \text{ on } \partial\Omega \text{ (i.e. } (u_1-u_2)^+\in W_0^{1,p(x)}(\Omega)),$$

then $u_1\leqslant u_2$ *in* Ω.

Throughout this paper, we assume the following conditions:

(D1) $\Omega\subset\mathbb{R}^N$ is an open bounded domain with C^2 boundary $\partial\Omega$;

(D2) $p,q\in C^1(\overline{\Omega})$ and $1<p^-\leqslant p^+, 1<q^-\leqslant q^+$;

(D3) $\alpha,\beta,m,n\in C(\overline{\Omega})$ satisfying $\alpha(x),\beta(x)\geqslant 0$ and $m(x),n(x)>0$ on $\overline{\Omega}$;

(D4) $0<\alpha^+<p^--1$, $0<\beta^+<q^--1$ and $(p^--1)(q^--1)>m^+n^+$.

3. Existence of positive solutions

Now, we consider problem (1.1) in a bounded domain Ω with C^2 boundary, we use $d(x)$ denote the distance of $x \in \Omega$ to the boundary of Ω. From Lemma 14.16 in [12], there exists a constant $0 < \delta$ small enough such that $d(x) \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)| \equiv 1$, where $\Omega_\varepsilon = \{x \in \Omega | d(x) < \varepsilon\}$.

Then we can denote

$$v_1(x) = \begin{cases} \xi d(x), \quad d(x) < \delta; \\ \xi\delta + \int_\delta^{d(x)} \xi (\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} dt, \ \ \delta \leqslant d(x) < 2\delta; \\ \xi\delta + \int_\delta^{2\delta} \xi (\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} dt, \ \ 2\delta \leqslant d(x), \end{cases}$$

where $\xi > 0$ is a constant. Obviously, $0 \leqslant v_1 \in C^1(\overline{\Omega})$ (see [21]).

Now we consider the following problem

$$\begin{cases} -\Delta_{p(x)} w(x) = \mu, \ \ x \in \Omega \\ w = 0, \ \ x \in \partial\Omega \end{cases} \tag{3.1}$$

and have the following results.

LEMMA 3.1. (see [11]) *If the positive parameter μ is large enough and w is the unique solution of (3.1), then for any $\nu \in (0,1)$, there exist positive constants C_1, C_2 such that*

$$C_1 \mu^{\frac{1}{p^+ - 1 + \nu}} \leqslant \max_{x \in \overline{\Omega}} w(x) \leqslant C_2 \mu^{\frac{1}{p^- - 1}}.$$

Proof. By computation, we have

$$-\Delta_{p(x)} v_1(x) = \begin{cases} -\xi^{p(x)-1}[(\nabla p \nabla d)\ln\xi + \Delta d], \quad d(x) < \delta; \\ \left\{ \frac{2(p(x)-1)}{\delta(p^- - 1)} - (\frac{2\delta - d}{\delta})[(\ln \xi (\frac{2\delta - d}{\delta})^{\frac{2}{p^- - 1}}) \nabla p \nabla d + \Delta d] \right\} \\ \times \xi^{p(x)-1} (\frac{2\delta - d}{\delta})^{\frac{2(p(x)-1)}{p^- - 1} - 1}, \ \ \delta < d(x) < 2\delta; \\ 0, 2\delta < d(x). \end{cases}$$

From Lemma 2.2 of [21], we know that for any $\nu \in (0,1)$, there exists a positive constant $C = C(\delta, \nu, \Omega, p)$ which is independent on ξ such that

$$|-\Delta_{p(x)} v_1(x)| \leqslant C \xi^{p(x)-1+\nu} \text{ a.e. on } \Omega.$$

If we let $C\xi^{p^+ - 1 + \nu} = \frac{1}{2}\mu$, then $v_1(x)$ is a sub-solution of (3.1). By the definition of $v_1(x)$ and Lemma 2.1, there exists a positive constant C_1 such that

$$\xi\delta = C_1 \mu^{\frac{1}{p^+ - 1 + \nu}} \leqslant \max_{x \in \overline{\Omega}} v_1(x) \leqslant \max_{x \in \overline{\Omega}} w(x).$$

The right inequality can be obtained from Lemma 2.1 of [11]. □

Now we have the following result.

THEOREM 3.1. *If (D1)-(D4) hold, then there exists* $(\lambda_*,\theta_*)>(0,0)$ *such that problem (1.1) possesses a positive solution for any* $(\lambda,\theta)\geqslant(\lambda_*,\theta_*)$.

Proof. According to the sub-super solution method for variable exponent problems(see [11]), we only need to construct a positive sub-solution (ϕ_1,ϕ_2) and a super-solution (z_1,z_2) of (1.1) such that $(\phi_1,\phi_2)\leqslant(z_1,z_2)$, then there exists a positive solution (u,v) of (1.1) satisfying $(\phi_1,\phi_2)\leqslant(u,v)\leqslant(z_1,z_2)$. That's complete the proof.

Let $\sigma=\frac{\ln 2}{k},\tau=\frac{\ln 2}{l}$, then there exist $k_1=l_1>1$ such that for any $k>k_1,l>l_1$, we have $0<\sigma,\tau<\delta$. Now we assume

$$\phi_1(x)=\begin{cases} e^{kd(x)}-1, & d(x)<\sigma\\ e^{k\sigma}-1+\int_{\sigma}^{d(x)}ke^{k\sigma}(\frac{2\sigma-t}{\sigma})^{\frac{2}{p^--1}}dt, & \sigma\leqslant d(x)<2\sigma\\ e^{k\sigma}-1+\int_{\sigma}^{2\sigma}ke^{k\sigma}(\frac{2\sigma-t}{\sigma})^{\frac{2}{p^--1}}dt, & 2\sigma\leqslant d(x)\end{cases}$$

and

$$\phi_2(x)=\begin{cases} e^{ld(x)}-1, & d(x)<\tau\\ e^{l\tau}-1+\int_{\tau}^{d(x)}le^{l\tau}(\frac{2\tau-t}{\tau})^{\frac{2}{q^--1}}dt, & \tau\leqslant d(x)<2\tau\\ e^{l\tau}-1+\int_{\tau}^{2\tau}le^{l\tau}(\frac{2\tau-t}{\tau})^{\frac{2}{q^--1}}dt, & 2\tau\leqslant d(x).\end{cases}$$

It is easy to see $\phi_1,\phi_2\in C^1(\overline{\Omega})$. By computation, we have

$$-\Delta_{p(x)}\phi_1=\begin{cases} -k(ke^{kd(x)})^{p(x)-1}\Big[p(x)-1+(d(x)+\frac{\ln k}{k})\nabla p(x)\nabla d(x)+\frac{\Delta d(x)}{k}\Big], \\ \qquad\qquad d(x)<\sigma;\\ \Big\{\frac{2(p(x)-1)}{\sigma(p^--1)}-(\frac{2\sigma-d}{\sigma})[(\ln ke^{k\sigma}(\frac{2\sigma-d}{\sigma})^{\frac{2}{p^--1}})\nabla p(x)\nabla d(x)+\Delta d(x)]\Big\}\\ \times(ke^{k\sigma})^{p(x)-1}(\frac{2\sigma-d}{\sigma})^{\frac{2(p(x)-1)}{p^--1}-1}, \quad \sigma<d(x)<2\sigma;\\ 0, \quad 2\sigma<d(x).\end{cases}$$

When $d(x)<\sigma$, we can obtain a constant $k_2>0$ such that for any $k>k_2$, we have

$$\begin{aligned}\Big|\Big(d(x)+\frac{\ln k}{k}\Big)\nabla p(x)\nabla d(x)+\frac{\Delta d(x)}{k}\Big| &\leqslant \Big(\frac{\ln 2}{k}+\frac{\ln k}{k}\Big)\Big(\sup_{x\in\overline{\Omega_\sigma}}|\nabla p(x)|+1\Big)\\ &<p^--1.\end{aligned}$$

Then when $k>\max\{k_1,k_2\}$, we have

$$-\Delta_{p(x)}\phi_1\leqslant 0\leqslant\phi_1^{\alpha(x)}+\lambda^{p(x)}\phi_2^{m(x)},\ d(x)<\sigma. \tag{3.2}$$

When $\sigma<d(x)<2\sigma$, since $d(x)\in C^2(\overline{\Omega_{3\delta}})$, there exists $C_3>0$ such that

$$-\Delta_{p(x)}\phi_1\leqslant C_3(ke^{k\sigma})^{p(x)-1}\ln k.$$

Then there exists $k_3>0$ such that when $k>k_3$, we have

$$C_3(ke^{k\sigma})^{p(x)-1}\ln k\leqslant k^{p(x)}.$$

Let $k_*=\max\{k_1,k_2,k_3\}$. Similarly, we obtain l_2,l_3 and denote $l_*=\max\{l_1,l_2,l_3\}$. Now we let $\sigma=\frac{\ln 2}{k_*},\tau=\frac{\ln 2}{l_*}$ and denote

$$\lambda_*=\frac{k_*}{(e^{\sigma}-1)^{\frac{m^+}{p^-}}},\ \theta_*=\frac{l_*}{(e^{\tau}-1)^{\frac{n^+}{q^-}}}.$$

Then for any $\lambda>\lambda_*$, we have

$$-\Delta_{p(x)}\phi_1\leqslant k_*^{p(x)}\leqslant \phi_1^{\alpha(x)}+\lambda^{p(x)}\phi_2^{m(x)},\ \sigma<d(x)<2\sigma. \tag{3.3}$$

It is easy to see that

$$-\Delta_{p(x)}\phi_1=0\leqslant \phi_1^{\alpha(x)}+\lambda^{p(x)}\phi_2^{m(x)},\ 2\sigma<d(x). \tag{3.4}$$

Combining (3.2),(3.3) and (3.4), we can obtain that

$$-\Delta_{p(x)}\phi_1\leqslant \phi_1^{\alpha(x)}+\lambda^{p(x)}\phi_2^{m(x)},\ \text{a.e. on }\Omega. \tag{3.5}$$

Similarly, for any $\theta>\theta_*$, we have

$$-\Delta_{q(x)}\phi_2\leqslant \phi_2^{\beta(x)}+\theta^{q(x)}\phi_1^{n(x)},\ \text{a.e. on }\Omega. \tag{3.6}$$

From (3.5) and (3.6), we can see that (ϕ_1,ϕ_2) is a sub-solution of (1.1).

Now we consider the following problem

$$\begin{cases}-\Delta_{p(x)}z_1=\lambda^{p^+}\mu_1, & x\in\Omega\\ -\Delta_{q(x)}z_2=\theta^{q^+}\mu_2, & x\in\Omega\\ z_1=z_2=0, & x\in\partial\Omega,\end{cases} \tag{3.7}$$

where μ_1,μ_2 are positive constants to be chosen. From Lemma 3.1, we have

$$\max_{x\in\overline{\Omega}} z_1(x)\leqslant C_2(\lambda^{p^+}\mu_1)^{\frac{1}{p^--1}}$$

and

$$\max_{x\in\overline{\Omega}} z_2(x)\leqslant C_2(\theta^{q^+}\mu_2)^{\frac{1}{q^--1}}.$$

For any $(\lambda,\theta)\geqslant(\lambda_*,\theta_*)$, if there exist positive constants μ_1,μ_2 satisfying

$$\lambda^{p^+}\mu_1=C_2(\lambda^{p^+}\mu_1)^{\frac{\alpha^+}{p^--1}}+\lambda C_2(\theta^{q^+}\mu_2)^{\frac{m^+}{q^--1}} \tag{3.8}$$

and

$$\theta^{q^+}\mu_2\geqslant C_2(\theta^{q^+}\mu_2)^{\frac{\beta^+}{q^--1}}+\theta C_2(\lambda^{p^+}\mu_1)^{\frac{n^+}{p^--1}}, \tag{3.9}$$

then (z_1,z_2) will be a super-solution of (1.1). Since $0<\alpha^+<p^--1$ and $0<\beta^+<q^--1$, from (3.8), we can see that μ_2 is large when μ_1 is large. Also from (3.8), we obtain

$$\lambda^{p^+}=\frac{C_2(\lambda^{p^+}\mu_1)^{\frac{\alpha^+}{p^--1}}}{\mu_1}+\frac{\lambda C_2(\theta^{q^+}\mu_2)^{\frac{m^+}{q^--1}}}{\mu_1}.$$

From $0<\alpha^+<p^- -1$ we have

$$\lim_{\mu_1\to\infty}\frac{C_2(\lambda^{p^+}\mu_1)^{\frac{\alpha^+}{p^--1}}}{\mu_1}=0.$$

Thus, when μ_1 is large, we have

$$\lambda^{p^+}\mu_1\leqslant 2\lambda C_2(\theta^{q^+}\mu_2)^{\frac{m^+}{q^--1}}.$$

Then by (D4) and when μ_1 is large, we obtain

$$\begin{aligned}\theta^{q^+}\mu_2&\geqslant C_2(\theta^{q^+}\mu_2)^{\frac{\beta^+}{q^--1}}+C_2^{1+\frac{n^+}{p^--1}}\theta(2\lambda)^{\frac{n^+}{p^--1}}(\theta^{q^+}\mu_2)^{\frac{m^+}{q^--1}\frac{n^+}{p^--1}}\\&=C_2(\theta^{q^+}\mu_2)^{\frac{\beta^+}{q^--1}}+\theta C_2(2\lambda C_2(\theta^{q^+}\mu_2)^{\frac{m^+}{q^--1}})^{\frac{n^+}{p^--1}}\\&\geqslant C_2(\theta^{q^+}\mu_2)^{\frac{\beta^+}{q^--1}}+\theta C_2(\lambda^{p^+}\mu_1)^{\frac{n^+}{p^--1}}.\end{aligned}$$

Thus (3.8) and (3.9) can be satisfied for some μ_1,μ_2 large enough. We obtained a super-solution of (1.1).

Now we will show that $(\phi_1,\phi_2)\leqslant(z_1,z_2)$ in Ω.

In the definition of $v_1(x)$, let $\xi=\frac{2}{\delta}(\max_{x\in\overline{\Omega}}\phi_1(x)+\max_{x\in\overline{\Omega}}|\nabla\phi_1(x)|)$. From the proof of Lemma 3.1, we know that when μ_1 is large enough, then

$$v_1(x)\leqslant z_1(x),\ x\in\overline{\Omega}.$$

So if we still have

$$\phi_1(x)\leqslant v_1(x),\ x\in\overline{\Omega},$$

the proof will be completed.

Obviously, we have

$$\phi_1(x)\leqslant 2\max_{x\in\overline{\Omega}}\phi_1(x)\leqslant v_1(x),\ d(x)\geqslant\delta.$$

Since $\phi_1-v_1\in C^1(\overline{\Omega_\delta})$, there exists a point $x_0\in\overline{\Omega_\delta}$ such that

$$\phi_1(x_0)-v_1(x_0)=\max_{x\in\overline{\Omega_\delta}}[\phi_1(x)-v_1(x)].$$

If $\phi_1(x_0)-v_1(x_0)>0$, then $0<d(x_0)<\delta$, so we have

$$\nabla\phi_1(x_0)-\nabla v_1(x_0)=0. \tag{3.10}$$

By the definition of $v_1(x)$ and ξ, we have

$$|\nabla v_1(x)|\equiv\xi>|\nabla\phi_1(x)|,\ 0<d(x)<\delta.$$

Contradicts to (3.10), then

$$\max_{x\in\overline{\Omega_\delta}}[\phi_1(x)-v_1(x)]\leqslant 0,$$

i.e.

$$\phi_1(x) \leqslant v_1(x),\ 0 \leqslant d(x) < \delta.$$

Then we obtain

$$\phi_1(x) \leqslant v_1(x),\ x \in \Omega.$$

By the proof of Lemma 3.1, there exists a positive constant C_4 such that

$$-\Delta_{p(x)} v_1(x) \leqslant C\xi^{p(x)-1+\nu} \leqslant C_4 \lambda_*^{p^+} \text{ a.e. on } \Omega.$$

We choose $\mu = C_4\lambda_*^{p^+}$ in problem (3.1) and according to Lemma 2.1, we have

$$v_1(x) \leqslant w(x),\ \ \forall x \in \Omega.$$

Again by Lemma 2.1, for any $\lambda > \lambda_*$ and μ_1 is large enough, we have

$$\phi_1(x) \leqslant v_1(x) \leqslant w(x) \leqslant z_1(x),\ x \in \Omega.$$

Similarly, for any $\theta > \theta_*$ and μ_2 is large enough, we have

$$\phi_2(x) \leqslant z_2(x),\ x \in \Omega.$$

That completes the proof of Theorem 3.1. □

REMARK 3.1. Under the conditions (D1)-(D4), if we still have $a,b,c,d \in C(\overline{\Omega})$, $a(x),b(x)$ are nonnegative functions and $c(x),d(x)$ are positive functions, then the following problem

$$\begin{cases} -\Delta_{p(x)}u = a(x)u^{\alpha(x)} + \lambda^{p(x)}c(x)v^{m(x)}, & x \in \Omega \\ -\Delta_{q(x)}v = b(x)v^{\beta(x)} + \theta^{q(x)}d(x)u^{n(x)}, & x \in \Omega \\ u = v = 0, \quad x \in \partial\Omega \end{cases}$$

has a positive solution when $(\lambda,\theta) > (\lambda_*,\theta_*)$ for some $(0,0) < (\lambda_*,\theta_*)$.

If $\Omega = B(0,r)$ and $p(x) = p(|x|), q(x) = q(|x|)$ are radial functions, where $B(0,r)$ denotes the open $N-$dimensional ball with center 0 and radius $r > 0$. It is a special case of problem (1.1). Denote by $\rho(x) = |x|$, then we have the following result.

COROLLARY 3.1. *If $\Omega = B(0,r)$ is a ball, $p(x) = p(|x|), q(x) = q(|x|)$ are radial functions and (D2)-(D4) hold, then there exists $(\lambda_*,\theta_*) > (0,0)$ such that for any $(\lambda,\theta) > (\lambda_*,\theta_*)$, problem (1.1) has at least one positive solution.*

Proof. The proof of Corollary 3.1 is along that of Theorem 3.1, but since we need not Lemma 3.1, the proof will be a little different. We give a sketch here.

Let $\sigma = \frac{\ln 2}{k}, \tau = \frac{\ln 2}{l}$, then there exists $k_1 = l_1 > 1$ such that for any $k > k_1, l > l_1$, we have $\sigma,\tau \in (0,r)$, for any $\varepsilon > 0$ small enough, we denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, & d(x) < \sigma \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}(\frac{r-t}{r-\sigma})^{\frac{2}{p^- - 1}}dt, & \sigma \leqslant d(x) < r-\varepsilon \\ e^{k\sigma} - 1 + \int_\sigma^{r-\varepsilon} ke^{k\sigma}(\frac{r-t}{r-\sigma})^{\frac{2}{p^- - 1}}dt, & r-\varepsilon \leqslant d(x) \leqslant r \end{cases}$$

and

$$\phi_2(x)=\begin{cases} e^{ld(x)}-1, & d(x)<\tau \\ e^{l\tau}-1+\int_\tau^{d(x)} le^{l\tau}(\frac{r-t}{r-\tau})^{\frac{2}{q^--1}}dt, & \tau\leqslant d(x)<r-\varepsilon \\ e^{l\tau}-1+\int_\tau^{r-\varepsilon} le^{l\tau}(\frac{r-t}{r-\tau})^{\frac{2}{q^--1}}dt, & r-\varepsilon\leqslant d(x)\leqslant r. \end{cases}$$

It is easy to see that $\phi_1,\phi_2\in C^1(\overline{\Omega})$. By the same discussion in the proof of Theorem 3.1, we obtain $(\lambda_*,\theta_*)>(0,0)$, and for any $(\lambda,\theta)\geqslant(\lambda_*,\theta_*)$, (ϕ_1,ϕ_2) is a sub-solution of (1.1).

By directly computation, we can see

$$z_1=\int_\rho^r\Big(\frac{\lambda^{p^+}\mu_1}{N}t\Big)^{\frac{1}{p(t)-1}}dt,\ z_2=\int_\rho^r\Big(\frac{\theta^{q^+}\mu_2}{N}t\Big)^{\frac{1}{q(t)-1}}dt$$

is a positive solution of problem (3.7). Obviously, there exists a $\zeta\in[0,r]$ such that

$$\max_{x\in\overline{\Omega}} z_1=\int_0^r\Big(\frac{\lambda^{p^+}\mu_1}{N}t\Big)^{\frac{1}{p(t)-1}}dt=(\lambda^{p^+}\mu_1)^{\frac{1}{p(\zeta)-1}}\int_0^r\Big(\frac{t}{N}\Big)^{\frac{1}{p(t)-1}}\leqslant C_5(\lambda^{p^+}\mu_1)^{\frac{1}{p^--1}},$$

where C_5 is a positive constant and μ_1 is large. Similarly, we have

$$\max_{x\in\overline{\Omega}} z_2\leqslant C_6(\theta^{q^+}\mu_2)^{\frac{1}{q^--1}}.$$

Then for any $(\lambda,\theta)\geqslant(\lambda_*,\theta_*)$, we can see (3.8) and (3.9) can be satisfied for some μ_1,μ_2 large enough. Thus we obtain that (z_1,z_2) is a super-solution of (1.1).

Now, we show that $(\phi_1,\phi_2)\leqslant(z_1,z_2)$ in Ω.

Since $\Omega=B(0,r)$, we know $d(x)=r-\rho(x)$ for any $x\in\Omega$. For any $\varepsilon<\sigma$, when μ_1 is large enough, we have

$$\phi_1(x)\leqslant z_1(x),\ \ d(x)\leqslant\varepsilon.$$

When $\varepsilon<d(x)\leqslant r$, we can see that $\phi_1(x)$ is bounded and

$$z_1(x)=\int_\rho^r\Big(\frac{\lambda^{p^+}\mu_1}{N}t\Big)^{\frac{1}{p(t)-1}}dt\geqslant\int_{r-\varepsilon}^r\Big(\frac{\lambda^{p^+}\mu_1}{N}t\Big)^{\frac{1}{p(t)-1}}dt\to\infty,\ \text{as}\ \mu_1\to\infty.$$

Then

$$\phi_1(x)\leqslant z_1(x),\ \ x\in\Omega,$$

when μ_1 is large enough. Similarly, when μ_2 is large enough, we have

$$\phi_2(x)\leqslant z_2(x),\ \ x\in\Omega.$$

Thus $(\phi_1,\phi_2)\leqslant(z_1,z_2)$, we complete the proof of Corollary 3.1. $\square$

4. Asymptotic behavior of the positive solutions

In this section, we will discuss the asymptotic behavior of the positive solutions near the boundary. We shall establish the following theorems.

THEOREM 4.1. *Under the conditions of (D1)-(D4) and (u,v) is a solution of (1.1) which has been obtained in Theorem 3.1, then for any $\nu \in (0,1)$, there exist positive constants C_7, C_8, C_9, C_{10} such that*

$$C_7 \lambda d(x) \leqslant u(x) \leqslant C_9 (\lambda^{p^+} \mu_1)^{\frac{1}{p^- - 1}} (d(x))^\nu, \text{ as } d(x) \to 0,$$

$$C_8 \theta d(x) \leqslant v(x) \leqslant C_{10} (\theta^{q^+} \mu_2)^{\frac{1}{q^- - 1}} (d(x))^\nu, \text{ as } d(x) \to 0,$$

where μ_1, μ_2 are large constants and satisfying (3.8) and (3.9).

Proof. Obviously, when $d(x) \to 0$, we have

$$u(x) \geqslant \phi_1(x) = e^{kd(x)} - 1 \geqslant C_7 \lambda d(x) \tag{4.1}$$

and

$$v(x) \geqslant \phi_2(x) = e^{ld(x)} - 1 \geqslant C_8 \theta d(x). \tag{4.2}$$

Define

$$v_3(x) = \kappa (d(x))^\nu, \ x \in \overline{\Omega_\varsigma},$$

where $0 < \varsigma < \delta$ is small enough and $\nu \in (0,1)$ is a constant.

By computation, we have

$$-\Delta_{p(x)} v_3(x) = -(\kappa\nu)^{p(x)-1} (\nu - 1)(p(x) - 1)(d(x))^{(\nu-1)(p(x)-1)-1} (1 + \Pi(x)), \ x \in \Omega_\varsigma, \tag{4.3}$$

where

$$\Pi(x) = \frac{d(x) \nabla p \nabla d \ln \kappa\nu}{(\nu - 1)(p(x) - 1)} + \frac{d(x) \nabla p \nabla d \ln d}{p(x) - 1} + \frac{d(x) \Delta d}{(\nu - 1)(p(x) - 1)}$$

and it is easy to see that $\Pi(x) \to 0$ as $d(x) \to 0$. Let $\kappa = \frac{C_4 (\lambda^{p^+} \mu_1)^{\frac{1}{p^- - 1}}}{\varsigma}$, when ς is small enough, from (4.3), we have

$$-\Delta_{p(x)} v_3(x) \geqslant (\kappa\nu)^{p(x)-1} \geqslant \lambda^{p^+} \mu_1.$$

Obviously $v_3(x) \geqslant z_1(x)$ when $d(x) = 0$ or $d(x) = \varsigma$ for ς is small enough.

On the other hand, when

$$\max\{(1-\nu)p^+, (1-\nu)q^+\} < 1,$$

we have $v_3 \in W^{1,p(x)}(\Omega_\varsigma) \cap W^{1,q(x)}(\Omega_\varsigma)$. According to Lemma 2.1, we have $v_3(x) \geqslant z_1(x)$. Thus

$$u(x) \leqslant C_9 \left(\lambda^{p^+} \mu_1\right)^{\frac{1}{p^- - 1}} (d(x))^\nu, \text{ as } d(x) \to 0.$$

Similarly, we have

$$u(x) \leqslant C_{10}(\lambda^{p^+}\mu_1)^{\frac{1}{p^- -1}}(d(x))^{\nu}, \text{ as } d(x) \to 0.$$

Combining with (4.1) and (4.2), we complete the proof. □

If $\Omega = B(0,r)$ and $p(x) = p(|x|), q(x) = q(|x|)$ are radial functions, we have the following stronger result.

THEOREM 4.2. *If $\Omega = B(0,r)$ is a ball, $p(x) = p(|x|), q(x) = q(|x|)$ are radial functions and (D2)-(D4) hold, (u,v) is a solution of (1.1) which has been obtained in Corollary 3.1, then*

$$u(x) = O(d(x)), \textit{ as } d(x) \to 0,$$

$$v(x) = O(d(x)), \textit{ as } d(x) \to 0.$$

Proof. When μ_1 is large enough, it is easy to see

$$\begin{aligned} u(x) \leqslant z_1(x) &= \int_{\rho}^{r} \Big(\frac{\lambda^{p^+}\mu_1}{N}t\Big)^{\frac{1}{p(t)-1}} dt \\ &= \int_{r-d(x)}^{r} \Big(\frac{\lambda^{p^+}\mu_1}{N}t\Big)^{\frac{1}{p(t)-1}} dt \leqslant \Big(\frac{\lambda^{p^+}\mu_1}{N}r\Big)^{\frac{1}{p^- -1}} d(x). \end{aligned}$$

Together with (4.1), we obtain

$$u(x) = O(d(x)), \text{ as } d(x) \to 0.$$

Similarly, we have

$$v(x) = O(d(x)), \text{ as } d(x) \to 0.$$

This completes the proof. □

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