Differential
Equations
& Applications
Volume 6, Number 3 (2014), 429–440
doi:10.7153/dea-06-24

# RESULTS OF LOCAL AND GLOBAL MILD SOLUTION FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATION WITH STATE DEPENDENT DELAY

GANGA RAM GAUTAM AND JAYDEV DABAS

(*Communicated by Michal Fečkan*)

*Abstract.* In this paper, we establish the existence of local and global mild solution for an impulsive fractional integro-differential equation with state dependent delay subject to nonlocal initial condition. The existence results for local mild solution are proved by applying the Schauder, nonlinear Larey Schauder alternative and Banach fixed point theorems. Then, we prove global existence result. An example is presented to demonstrate the application of the established results.

## 1. Introduction

It is observed that the convolution integral is known as the Riemann–Liouville fractional integral. The differential equations with non integer order have been proved the valuable tools in the modeling phenomena of several areas of science and technology. Actually, fractional differential equations are more accurate than to integer differential equations in modeling of several real world problems such as nonlinear oscillation of earthquakes, seepage flow in porous media, fluid dynamics, viscoelasticity, electrochemistry, electromagnetic, control and food science, etc.

Let $X$ be a complex Banach space and $PC_t := PC([-d,t];X), d > 0, 0 \leqslant t \leqslant T < \infty$, be a Banach space of all such functions $\phi : [-d,t] \to X$, which are continuous every where except for a finite number of points $t_i$, $i = 1,2,\dots,m$, at which $\phi(t_i^+)$ and $\phi(t_i^-)$ exists and $\phi(t_i) = \phi(t_i^-)$, endowed with the norm

$$\|\phi\|_t = \sup_{-d\leqslant s\leqslant t} \|\phi(s)\|_X,\ \phi \in PC_t,$$

where $\|\cdot\|_X$ is the norm in $X$.

Consider the following nonlocal impulsive fractional functional integro-differential equation with state dependent delay in a Banach space $X$:

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)ds + f(t, y_{\rho(t,y_t)}, By(t)), t \in J, t \neq t_k, \tag{1.1}$$

*Mathematics subject classification* (2010): 26A33, 47H10, 34A37, 34A12.

*Keywords and phrases*: fractional order differential equation, fixed point theorems, impulsive condition, nonlocal condition.

The research of J. Dabas has been partially supported by Department of Science & Technology, project No.SR/FTP/MS-030/2011.

© ELEMENT, Zagreb
Paper DEA-06-24

$$y(t)+g(y)(t)=\phi(t),\ t\in[-d,0], \tag{1.2}$$

$$\triangle y(t_k)=I_k(y(t_k^-)),\quad k=1,2,\dots,m, \tag{1.3}$$

where $J=[0,T], \alpha\in(1,2), T<\infty$, $A:D(A)\subset X\to X$ is a closed linear operator of sectorial type on $X$, and $f:J\times PC_0\times X\to X$, $\rho:J\times PC_0\to(-\infty,T]$, $g:X\to X$, $I_k: X\to X,(k=1,2,\dots,m)$ and $\phi\in PC_0$ are given functions. Here $0=t_0<t_1<\ldots<t_m<t_{m+1}=T$, $\triangle y(t_k)=y(t_k^+)-y(t_k^-)$, $y(t_k^+)=\lim_{h\to0}y(t_k+h), y(t_k^-)=\lim_{h\to0}y(t_k-h)$, represents the right and left hand limits of $y(t)$ at $t=t_k$ respectively and also we assume $y(t_k^-)=y(t_k)$. The history function $y_t:[-d,0]\to X$ is an element of $PC_0$ and defined by $y_t(\theta)=y(t+\theta)$, $\theta\in[-d,0]$.

The term $By(t)$ is stand for $By(t)=\int_0^t K(t,s)y(s)ds$, where $K\in C(D,\mathbb{R}^+)$, the set of all positive functions which are continuous on $D=\{(t,s)\in\mathbb{R}^2:0\leqslant s\leqslant t<T\}$ and $B^*=\sup_{t\in[0,t]}\int_0^t K(t,s)ds<\infty$.

Several evolution processes are defined by the certain moments of the time with changes of the state abruptly such changes are in the form of impulses. The impulsive effects can be shown in many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, etc. For details, we cite the papers [10, 11, 12, 16, 17, 26, 27].

The hypothesis of functional differential equations with non integer order has emerged as an important branch of nonlinear analysis. The problem with state dependent delay has many applications in modeling of the problems as dynamics system, adaptive control, etc due to this fact such equations play an important role in mathematics. For more details, we refers the papers [1, 2, 4, 6, 7, 8, 9, 18, 19, 23].

Many of the physical systems can better be described by the nonlocal conditions such conditions are encountered in various applications such as chemical engineering, heat conduction, population dynamics and blood flow models. The nonlocal initial condition in diffusion phenomenon of a small amount of gas in a transparent tube can give a better results than using the Cauchy Problem $x(0)=x_0$, for more details, we refer the papers [5, 10, 24].

The different version of differential equation (1.1) has been studied by several authors. Cuevas et al. [15] have considered the following abstract integro-differential equation with infinite delay

$$v'=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}Av(s)ds+f(t,v_t),\ t\geqslant 0,\ v_0=\phi_0\in\mathfrak{B}, \tag{1.4}$$

where $\mathfrak{B}$ is a phase space, $\alpha\in(1,2)$, $A:D(A)\subset X\to X$ is a linear densely defined operator of sectorial type on a Banach space X. The author's of [20] have established the existence and uniqueness of $S$-asymptotically $\omega$-periodic mild solution of (1.4). In [14], Cueva et al. have established the above said results for the problem (1.4) without delay.

Agarwal et al. [2] study the following class of fractional integro-differential equations with state dependant delay

$$u'(t)=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}Au(s)ds+f(t,u_{\rho(t,u_t)}),\ t\in[0,b],u(0)=\phi\in\mathfrak{B}, \tag{1.5}$$

and established the sufficient conditions for the existence of the mild solution. Benchohra et al. [7] have investigated the existence of solution on a compact interval for the following fractional integro-differential inclusion with state dependent delay in a Banach space when the delay is infinite

$$y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)ds \in F(t, y_{\rho(t,y_t)}), a.e.\ t \in [0,b], y(0) = \phi \in \mathfrak{B}. \quad (1.6)$$

Our present work motivated by the papers [2, 3, 7, 12, 13, 14, 15, 16, 26]. In this paper, we concerned with the local and global existence of mild solutions for impulsive fractional integro-differential equations with state dependent delay and nonlocal conditions. To the best of the author's knowledge the concept of local and global existence of mild solutions of the considered problem in this paper is an untreated topic in the literature.

We define the mild solution of the system (1.1)-(1.3) using the concept introduced in [16, 21]. The mild solution is associated with Mittag–Leffler function, solution and resolvent operators. The results are obtained by using the fixed point techniques and solution operator on a complex Banach space.

We organize the rest of this paper as follows: in Section 2, we present some necessary definitions, preliminary results that will be used to prove our main results. The proof of local and global existence of mild solution is given in Section 3, and Section 4 contains an illustrative example.

## 2. Preliminaries

This section is equipped with preliminaries and some definitions, which are required in this paper. Let $L(X)$ represents the Banach space of all bounded linear operator from $X$ into $X$, and the corresponding norm is denoted by $\|\cdot\|_{L(X)}$. The function spaces defined in the introduction section are same in rest of the paper and the notations for the function spaces have their usual meaning if it is not specified.

DEFINITION 1. A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^{\alpha} - z} d\mu,\ \alpha, \beta > 0, z \in \mathbb{C},$$

where $C$ is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leqslant |z|^{1/\alpha}$ counter clockwise.

The most interesting properties of the Mittag-Lefller functions are associated with their Laplace transform

$$\int_0^{\infty} e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^{\alpha}) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \omega},\ Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega > 0,$$

see [25] for more details.

To avoid the repetitions of some definitions used in this paper we refer the readers: such as sectorial operator one can see the paper [22], and for solution operator (see definition 2.1 in [2]). Now, we present the definition of mild solution for the system (1.1)-(1.3) based on the paper [16].

DEFINITION 2. A piecewise continuous function $y \in PC_T$ is called the mild solution of the system (1.1)-(1.3) if it satisfies the following equivalent integral equation

$$y(t) = \begin{cases} \phi(t) - g(y)(t), & t \in [-d,0], \\ S_\alpha(t)[\phi(0) - g(y)(0)] + \int_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}, By(s)) ds, & t \in (0,t_1], \\ S_\alpha(t)[\phi(0) - g(y)(0)] + S_\alpha(t-t_1) I_1(y(t_1^-)) & \\ + \int_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}, By(s)) ds, & t \in (t_1,t_2], \\ \cdots & \\ S_\alpha(t)[\phi(0) - g(y)(0)] + \sum_{i=1}^m S_\alpha(t-t_i) I_i(y(t_i^-)) & \\ + \int_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}, By(s)) ds, & t \in (t_m, T], \end{cases} \tag{2.1}$$

where

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda,$$

is called solutions operator and $\Gamma$ is a suitable path lying on $\sum_{\theta,\omega}$.

For more detail of $S_\alpha(t)$ one can see the paper [2] and steps of proof of the definition of mild solution, we refer the reader to see [2, 12, 16].

## 3. Main Results

In this section, we give the main results on the existence of mild solution of the system (1.1)-(1.3). Also we use the following results from the paper [28]. If $A \in \mathscr{A}^\alpha(\theta_0, \omega_0)$, then $\|S_\alpha(t)\| \leqslant Me^{\omega t}$. Let $\widetilde{M_S} := \sup_{0 \leqslant t \leqslant T} \|S_\alpha(t)\|_{L(X)}$, so we have $\|S_\alpha(t)\|_{L(X)} \leqslant \widetilde{M_S}$.

THEOREM 1. (Mild Solution) *Let $f$ and $I_k$ be the bounded continuous functions then for every $\phi \in PC_0$ there exist a $\tau = \tau(\phi), 0 < \tau < T$ such that the system (1.1)-(1.3) has a local mild solution $y \in PC([-d,\tau], X)$.*

*Proof.* Let us take $y_0 \in PC_T$ such that $y_0 + g(y_0) = \phi(t)$ on $[-d,0]$ and $t', r > 0$ be such that

$$B_r(y_0) = \{y \in PC_T : \|y(t) - y_0(t)\|_{t'} \leqslant r\}$$

and $\|f(t,\psi,y)\|_X \leqslant N$, $\|I_k(y)\|_X \leqslant \tilde{\sigma}$, $k = 1,2,\cdots,m$, for $0 \leqslant t \leqslant t'$ and let $y \in B_r(y_0)$. Choose $t'' > 0$ such that $\|S_\alpha(t)y_0(0) - y_0(0)\|_X \leqslant \frac{r}{3}$ for $0 \leqslant t \leqslant t''$ and $\|y_0(t) - y_0(0)\|_X \leqslant \frac{r}{3}$ for $0 \leqslant t \leqslant t''$. Now let

$$\tau = \min\left\{ t', t'', T, \frac{r - 3m\widetilde{M_S}\tilde{\sigma}}{3\widetilde{M_S}N} \right\}.$$

Set $Y = PC_\tau = PC([-d,\tau],X)$ and

$$Y_0 = \{y : y \in Y \text{ on } [-d,0] \text{ and } y(t) \in B_r(y_0) \text{ for } 0 \leqslant t \leqslant \tau\}.$$

It is obvious that $Y_0$ is a bounded closed convex subset of $Y$. We define the mapping $P : Y_0 \to Y$ by

$$\begin{aligned} P(y) &= S_\alpha(t)y_0(0) + \sum_{0<t_i<t} S_\alpha(t-t_i)I_i(y(t_i^-)) \\ &\quad + \int_0^t S_\alpha(t-s)f(s, y_{\rho(s,y_s)}, By(s))ds. \end{aligned} \tag{3.1}$$

We need to show that $P : Y_0 \to Y_0$. For this, let $y(t) \in Y_0, t \in [0,\tau]$, we have

$$\begin{aligned} \|P(y(t)) - y_0(t)\|_X &\leqslant \|S_\alpha(t)y_0(0) - y_0(0)\|_X + \|y_0(t) - y_0(0)\|_X \\ &\quad + \sum_{0<t_i<t} \|S_\alpha(t-t_i)\|_X \|I_i(y(t_i^-))\|_X \\ &\quad + \int_0^t \|S_\alpha(t-s)\|_X \|f(s, y_{\rho(s,y_s)}, By(s))\|_X ds \\ &\leqslant \frac{r}{3} + \frac{r}{3} + m\widetilde{M_S}\tilde{\sigma} + \widetilde{M_S}N\tau \leqslant r. \end{aligned}$$

Thus $P : Y_0 \to Y_0$. Now, we show that $P$ is continuous, for this propose, we consider a sequence $y^n \to y \in Y_0$, then

$$\begin{aligned} \|Py^n - Py\|_X &\leqslant \sum_{0<t_i<t} \|S_\alpha(t-t_i)\|_{L(X)} \|I_i(y^n(t_i^-)) - I_i(y(t_i^-))\|_X \\ &\quad + \int_0^t \|S_\alpha(t-s)\|_X \|f(s, y^n_{\rho(s,y^n_s)}, By^n(s)) - f(s, y_{\rho(s,y_s)}, By(s))\|_X ds \\ &\leqslant m\widetilde{M_S}\|I_i(y^n(t_i^-)) - I_i(y(t_i^-))\|_X \\ &\quad + \widetilde{M_S}\int_0^t \|f(s, y^n_{\rho(s,y^n_s)}, By^n(s)) - f(s, y_{\rho(s,y_s)}, By(s))\|_X ds. \end{aligned}$$

Since the functions $f$ and $I_i$ are continuous so $\|Py^n - Py\|_X \to 0$ as $n \to \infty$, implies $P$ is continuous. To prove that $P$ maps bounded set into bounded set in $Y_0$. To do this, we have

$$\|Py\|_X \leqslant \widetilde{M_S}(\|y_0(0)\| + m\tilde{\sigma} + \tau N).$$

Next, we shall show that $P$ is a family of equi-continuous functions. Let $l_1, l_2 \in [0,\tau]$ such that $0 \leqslant l_1 < l_2 \leqslant \tau$. Then

$$\begin{aligned} \|P(y)(l_2) - P(y)(l_1)\|_X &\leqslant \|S_\alpha(l_2) - S_\alpha(l_1)\|_{L(X)} \|y_0(0)\|_X \\ &\quad + \sum_{0<t_i<t} \|S_\alpha(l_2-t_i) - S_\alpha(l_2-t_i)\|_{L(X)} \|I_i(y)\|_X \\ &\quad + \int_0^{l_1} \|S_\alpha(l_2-s) - S_\alpha(l_1-s)\|_{L(X)} \|f(s, y_{\rho(s,y_s)}, By(s))\|_X ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{l_1}^{l_2}\|S_\alpha(l_2-s)\|_{L(X)}\|f(s,y_{\rho(s,y_s)},By(s))\|_X ds\\
\leqslant\ &\|S_\alpha(l_2)-S_\alpha(l_1)\|_{L(X)}\|y_0(0)\|_X\\
&+\sum_{0<t_i<t}\|S_\alpha(l_2-t_i)-S_\alpha(l_2-t_i)\|_{L(X)}\|I_i(y)\|_X\\
&+N\int_0^{l_1}\|S_\alpha(l_2-s)-S_\alpha(l_1-s)\|_{L(X)}ds+\widetilde{M_S}N(l_2-l_1).
\end{aligned}
$$

Since $S_\alpha(t)$ is strongly continuous and the continuity of the function $t\mapsto\|S(t)\|_{L(X)}$ allows us to conclude that $\lim_{l_2\to l_1}\|S_\alpha(l_2-t_i)-S_\alpha(l_1-t_i)\|_{L(X)}=0,$ which implies that $\|P(y)(l_2)-P(y)(l_1)\|_X\to 0$ as $l_2\to l_1$. This proves that $P$ is a family of equicontinuous functions. So, we conclude by Arzela-Ascoli's theorem that $P$ is a completely continuous map. Finally, it follows from the Schauder's fixed point theorem that the map $P$ has a fixed point in $Y_0$ which is a local mild solution of (1.1)-(1.3) on $[0,\tau]$, satisfying $y(t)\in PC_T$ for $0\leqslant t\leqslant\tau$. This completes the proof of the theorem. □

To prove our second result, we use nonlinear alternative of Leray-Schauder theorem and we assume:

(H1) Function $f:J\times PC_0\times X\to X$ is continuous and there exist $m\in PC_T([0,\tau],[0,\infty))$ and a continuous non-decreasing function $W:[0,\infty)\to(0,\infty),$ such that

$$
\|f(t,\psi,y)\|_X\leqslant m(t)W(\|\psi\|_{PC_0}+\|y\|_X),\ \forall\ (t,\psi,y)\in J\times PC_0\times X.
$$

THEOREM 2. *Let the assumption $(H1)$ hold and*

$$
\widetilde{M_S}(1+B^*)\int_0^\tau m(s)ds<\int_C^\infty\frac{1}{W(s)}ds, \tag{3.2}
$$

*where $C=(1+B^*)\widetilde{M_S}\|y_0(0)\|+m\widetilde{M_S}(1+B^*)\tilde{\sigma}$, then for every $\phi\in PC_0$ there exist a $\tau=\tau(\phi),0<\tau<T$ such that the system (1.1)-(1.3) has at least one local mild solution $y\in PC([-d,\tau],X)$.*

*Proof.* Let $P:Y_0\to Y_0$ be the operator defined in Eq. (3.1). If $y=\lambda Py$, $\lambda\in(0,1)$ then we have

$$
\|y\|_X\leqslant\widetilde{M_S}\|y_0(0)\|+m\widetilde{M_S}\tilde{\sigma}+\widetilde{M_S}\int_0^t m(s)W(\|y\|+B^*\|y\|)ds,
$$

since $\rho(s,y_s)\leqslant s$ for every $s\in[0,\tau]$. If $\gamma(t)=(1+B^*)\|y\|$, we obtain

$$
\begin{aligned}
\gamma(t)\leqslant\ &(1+B^*)\widetilde{M_S}\|y_0(0)\|+m\widetilde{M_S}(1+B^*)\tilde{\sigma}\\
&+\widetilde{M_S}(1+B^*)\int_0^t m(s)W(\gamma(s))ds.
\end{aligned}\tag{3.3}
$$

Denoting by $\beta(t)$ the right-hand side of the inequality (3.3), we find

$$
\beta'(t)\leqslant\widetilde{M_S}(1+B^*)m(t)W(\beta(t))ds,
$$

and hence we obtain

$$\int_C^{\beta(t)} \frac{1}{W(s)} ds \leqslant \widetilde{M_S}(1+B^*) \int_0^\tau m(s)ds. \tag{3.4}$$

The inequality (3.2) and (3.4) permit us to conclude that the set of functions $\{\beta : \lambda \in (0,1)\}$ is bounded, which in turn shows that $\{y : \lambda \in (0,1)\}$ is bounded. The rest of the proof is similar to that of Theorem 1. This completes the proof of the theorem. □

Further, we assume the following assumptions to prove our next result.

(H2) The function $f : J \times PC_0 \times X \to X$ is continuous and there exists constants $N_1, N_2$ such that

$$\|f(t,\psi,y) - f(t,\chi,z)\|_X \leqslant N_1\|\psi - \chi\|_{PC_0} + N_2\|y-z\|_X, \ \forall\, \psi,\chi \in PC_0, \ \forall\, y,z \in X.$$

(H3) The functions $I_k : X \to X$ are continuous and there exists a constant $\sigma > 0$ such that $\|I_k(y) - I_k(z)\|_X \leqslant \sigma\|y-z\|_X$, $\forall\, y,z \in X; k = 1,2,\ldots,m$.

THEOREM 3. *Let the assumptions* $(H2),(H3)$ *hold and*

$$m\widetilde{M_S}\sigma + \widetilde{M_S}\tau(N_1 + N_2B^*) < 1,$$

*then for every* $\phi \in PC_0$ *there exist a* $\tau = \tau(\phi), 0 < \tau < T$ *such that the system (1.1)-(1.3) has a unique local mild solution* $y \in PC([-d,\tau],X)$.

*Proof.* Consider the mapping $P : Y_0 \to Y_0$ defined as in Eq. (3.1). Let $y, y^* \in Y_0$, then we have

$$\begin{aligned}\|P(y) - P(y^*)\|_X &\leqslant \sum_{0<t_i<t} \|S_\alpha(t-t_i)\|_{L(X)}\|I_i(y(t_i^-)) - I_i(y^*(t_i^-))\|_X \\ &\quad + \int_0^t \|S_\alpha(t-s)\|_{L(X)}\|f(s,y_{\rho(s,y_s)},By(s)) - f(s,y^*_{\rho(s,y^*_s)},By^*(s))\|_X ds \\ &\leqslant [m\widetilde{M_S}\sigma + \widetilde{M_S}\tau(N_1+N_2B^*)]\|y-y^*\|_X.\end{aligned}$$

Since $m\widetilde{M_S}\sigma + \widetilde{M_S}\tau(N_1+N_2B^*) < 1$, it follows that there exists a unique $y \in Y_0$ such that $y$ is a unique local mild solution of the system (1.1)-(1.3) on $[-d,\tau]$. This completes the proof of the theorem. □

THEOREM 4. *Let* $f : J \times PC_0 \times X \to X$ *be the bounded continuous function then for every* $\phi \in PC_0$, *the system (1.1)-(1.3) has a global mild solution* $y$ *on a maximal interval of existence* $[0,t_{\max})$. *If* $t_{\max} < \infty$, *then* $\lim_{t\uparrow t_{\max}} \|y(t)\| = \infty$.

*Proof.* We have a mild solution $y$ of the system (1.1)-(1.3) defined on $[0,\tau]$ can be extended to a larger interval $[0,\tau+\tau_0], \tau_0>0$, by defining $y(t+\tau)=w(t)$, then the system (1.1)-(1.3) transform into the following form

$$\begin{cases} w'(t)=\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}Aw(s)ds+\widetilde{f}(t,w_{\rho(t,w_t)},\widetilde{B}w(t)),\ t\in[0,T-\tau],\ t\neq t_k,\\ w(t)-\widetilde{g}(w(t))=\widetilde{\phi(t)},\ t\in[-d-\tau,0],\\ \triangle w(\widetilde{t_k})=I_k(w(\widetilde{t_k^-})),\quad k=1,2,...m, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} \widetilde{f}(t,w_{\rho(t,w_t)},\widetilde{B}w(t)) &= f(t+\tau,w_{\rho(t+\tau,w_t)},Bw(t+\tau)),\\ \triangle w(\widetilde{t_k}) &= I_k(w(\widetilde{t_k^-})),\quad k=1,2,...m,\\ \widetilde{g}(w(t)) &= g(w(t+\tau)),\\ \widetilde{\phi(t)} &= \phi(t+\tau), \end{aligned}$$

and $\widetilde{t_k}=t_k-\tau$. Since the functions $f,g$ are bounded and continuous. So by Theorem 1, let us choose $w_0(t)$ such that $w_0(t)=\widetilde{\phi(t)}-\widetilde{g}(w(t))$, for $t\in[-d-\tau,0]$, and there exists a function $w\in PC([-d-\tau,T-\tau])$, such that $w(t)$ is a mild solution of (3.5) on $[-d-\tau,\tau_1]$ for some $0<\tau_1\leqslant T-\tau$ and define as

$$w(t)=\begin{cases} w_0(t), & t\in[-d-\tau,0],\\ S_\alpha(t)w_0(0)+\sum_{0<\widetilde{t_i}<t}S_\alpha(t-\widetilde{t_i})I_i(w(\widetilde{t_i^-})) & \\ +\int_0^t S_\alpha(t-s)\widetilde{f}(s,w_{\rho(s,w_s)},\widetilde{B}w(s))ds, & t\in[0,\tau_1]. \end{cases}$$

Then

$$\widetilde{y}(t)=\begin{cases} y(t), & t\in[-d,\tau],\\ w(t-\tau), & t\in[\tau,\tau+\tau_1], \end{cases} \tag{3.6}$$

is a mild solution of the system (1.1)-(1.3) on $[-d,\tau+\tau_1]$. Since $y(t+\tau)=w(t)$, for $t\in[\tau,\tau+\tau_1]$, we have

$$w(t-\tau)=y(t)=\begin{cases} S_\alpha(t-\tau)y_0(\tau)+\sum_{\tau<t_i<t}S_\alpha(t-t_i)I_i(y(t_i^-))\\ +\int_\tau^t S_\alpha(t-s)f(s,y_{\rho(s,y_s)},By(s))ds. \end{cases}$$

Continuing in this way, we get maximal interval $[-d,t_{\max})$ in which the solution of the system (1.1)-(1.3) can be extended. We shall now show that if $t_{\max}<\infty$, then $\|y(t)\|_X\to\infty$ as $t\to t_{\max}$. To do that, we shall prove that $t\to t_{\max}$ implies

$$\overline{\lim}_{t\to t_{\max}}\|y(t)\|_X=\infty.$$

Indeed, if $t\to t_{\max}$ and $\overline{\lim}_{t\to t_{\max}}\|y(t)\|_X<\infty$, we can assume that $S_\alpha(t)\leqslant\widetilde{M_S}$ and $\|y(t)\|\leqslant K_1$ for $0\leqslant t<t_{\max}$ where $\widetilde{M_S}$ and $K_1$ are constants. Now, if $0<r<t<t'<t_{\max}$, then

$$\|y(t')-y(t)\|_X\leqslant\|S_\alpha(t')-S_\alpha(t)\|_{L(X)}\|y_0(0)\|_{PC_0}$$

$$
\begin{aligned}
&+\sum_{0<t_i<t}\|S_\alpha(t'-t_i)-S_\alpha(t-t_i)\|_{L(X)}\|I_i(y(t_i^-))\|_X\\
&+\int_0^t\|S_\alpha(t'-s)-S_\alpha(t-s)\|_{L(X)}\|f(s,y_{\rho(s,y_s)},By(s))\|_X ds\\
&+\int_t^{t'}\|S_\alpha(t'-s)\|_{L(X)}\|f(s,y_{\rho(s,y_s)},By(s))\|_X ds\\
\leqslant\ &\|S_\alpha(t')-S_\alpha(t)\|_{L(X)}\|y_0(0)\|_{PC_0}\\
&+\tilde{\sigma}\sum_{0<t_i<t}\|S_\alpha(t'-t_i)-S_\alpha(t-t_i)\|_{L(X)}\\
&+N\int_0^t\|S_\alpha(t'-s)-S_\alpha(t-s)\|_{L(X)}ds+\widetilde{M_S}N(t'-t).
\end{aligned}\tag{3.7}
$$

Since $t>r>0$ is arbitrary and $S_\alpha(t)$ are continuous in the uniform operator topology for $t>r>0$, the right hand side of (3.7) tends to zero as $t',t$ tend to $t_{\max}$. Therefore $\lim_{t\to t_{\max}} y(t)=y(t_{\max})$ exists and by the first part of the proof the solution $y$ can be extended beyond $t_{\max}$, contradicting the maximality of $t_{\max}$. Therefore the assumption that $t_{\max}<\infty$ implies that $\overline{\lim}_{t\to t_{\max}}\|y(t)\|_X=\infty$. To conclude the proof of the theorem, now we will show that $\lim_{t\to t_{\max}}\|y(t)\|_X=\infty$. If this is false then there is a sequence $\tau_n\to t_{\max}$ and a constant $K_1$ such that $\|y(\tau_n)\|\leqslant K_1$ for all $n$. Let

$$
\beta=\sup\{\|f(t,y_{\rho(t,y_t)},By(t))\|:0\leqslant t\leqslant t_{\max},\|y\|\leqslant\widetilde{M_S}(K_1+1)\},
$$

and choose $0<\tilde{\sigma}<1$. Since $t\to\|y(t)\|$ is continuous and $\overline{\lim}_{t\to t_{\max}}\|y(t)\|=\infty$, we can find a sequence $\{\lambda_n\}$ with the following properties:

$$
\lambda_n\to 0 \text{ as } n\to\infty,\ \|y\|\leqslant\widetilde{M_S}(K_1+1),\text{ for } \tau_n\leqslant t\leqslant\tau_n+\lambda_n
$$

and $\|y(\tau_n+\lambda_n)\|=\widetilde{M_S}(K_1+1)$. But then, we have

$$
\begin{aligned}
\widetilde{M_S}(K_1+1)=\|y(\tau_n+\lambda_n)\|_X\leqslant\ &\|S_\alpha(\lambda_n)y(\tau_n)\|_X\\
&+\sum_{\tau<t_i<t}\|S_\alpha(\tau_n+\lambda_n-t_i)I_i(y(t_i^-))\|_X\\
&+\int_{\tau_n}^{\tau_n+\lambda_n}\|S_\alpha(\tau_n+\lambda_n-s)f(s,y_{\rho(s,y_s)},By(s))\|ds\\
\leqslant\ &\widetilde{M_S}K_1+m\widetilde{M_S}\tilde{\sigma}+\beta\widetilde{M_S}\lambda_n.
\end{aligned}
$$

Which is absurd as $\lambda_n\to 0$. Therefore, we have $\lim_{t\to t_{\max}}\|y(t)\|_X=\infty$. This completes the proof of the theorem. $\square$

## 4. An example

In this section, we apply our results to study the mild solution for following fractional integro-differential system.

$$\begin{cases} u'(t,x) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} u_{xx}(s,x)ds + \frac{e^{-t}u(t-\sigma_1(t)\sigma_2(\|u\|))}{(9+e^t)(1+u(t-\sigma_1(t)\sigma_2(\|u\|))} \\ + \int_0^t \cos(t-s)\frac{e^s u(t-\sigma_1(t)\sigma_2(\|u\|))}{25}ds, \ t \in [0,1], \, t \neq \frac{1^-}{2}, \\ u(t,0) = u(t,\pi) = 0, \ t \geqslant 0, \\ u(t,x) + \int_0^t c(x,\gamma)\cos(1+\|u(t,\gamma)\|)d\gamma = \phi(t,x), \, t \in [-d,0], \, 0 \leqslant x \leqslant \pi, \\ \Delta u|_{t=\frac{1}{2})} = \frac{u(t,\frac{1^-}{2})}{49+u(t,\frac{1^-}{2})}, \end{cases} \tag{4.1}$$

where $t \in [0,1]$. To represent this system in the abstract form (1.1)-(1.3). We choose $X = L^2([0,\pi])$ and consider the operator $A : D(A) \subset X \to X$ defined by $Aw = w''$ with the domain $D(A) := \{w \in X : \ w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = 0 = w(\pi)\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2(w,w_n)w_n, \ w \in D(A),$$

where $w_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of $A$. It is well known that the subordination principle of solution operator implies that $A$ is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geqslant 0}$ and hence A is a sectorial operator. Since $S_\alpha(t)$ is strongly continuous on $[0,\infty)$, by uniformly bounded theorem, there exists a constant $M > 0$ such that $\|S_\alpha(t)\|_{L(X)} \leqslant M$ for $t \in [0,1]$. Finally, we consider the following functions by setting $u(t)(x) = u(t,x)$, and $\rho = t - \sigma_1(t)\sigma_2(\|u(0)\|))$.

$$\begin{aligned} f(t,\phi,By) &= \frac{e^{-t}\phi}{(9+e^t)(1+\phi)} + \int_0^t \cos(t-s)\frac{e^s\phi}{25}ds, \\ I_k(u) &= \frac{u(t,\frac{1^-}{2})}{49+u(t,\frac{1^-}{2})}. \end{aligned}$$

Then, we have

$$\begin{aligned} &\|f(t,\phi,By) - f(t,\psi,Bz)\| \\ &\quad\leqslant \|\frac{e^{-t}\phi}{(9+e^t)(1+\phi)} - \frac{e^{-t}\psi}{(9+e^t)(1+\psi)}\| \\ &\qquad + \|\int_0^t \cos(t-s)\frac{e^s\phi}{25}ds - \int_0^t \cos(t-s)\frac{e^s\psi}{25}ds\| \\ &\quad\leqslant \frac{e^{-t}}{9+e^t}\|\frac{\phi}{1+\phi} - \frac{\psi}{1+\psi}\| + \int_0^t \cos(t-s)e^s\frac{1}{25}\|\phi-\psi\|ds \\ &\quad\leqslant \frac{e^{-t}\|\phi-\psi\|}{(9+e^t)(1+\phi)(1+\psi)} + \frac{1}{25}\|\phi-\psi\| \\ &\quad\leqslant \frac{e^{-t}}{9+e^t}\|\phi-\psi\| + \frac{1}{25}\|\phi-\psi\| \leqslant \frac{1}{10}\|\phi-\psi\| + \frac{1}{25}\|\phi-\psi\|, \end{aligned}$$

$$\|I_k(u) - I_k(v)\| \leqslant \frac{49\|u-v\|}{(49+u)(49+v)} \leqslant \frac{1}{49}\|u-v\|.$$

Thus the assumptions $(H1)$ and $(H2)$ hold. Furthermore, we have
$B^* = \sup_{t\in[0,1]} \int_0^t \cos(t-s)ds = 1,\ \ m=1, N_1 = \frac{1}{10},\ N_2 = \frac{1}{25},\ \sigma = \frac{1}{49}$ and $\widetilde{M_S} = 1.$
Then

$$m\widetilde{M_S}\sigma + \widetilde{M_S}\tau(N_1 + N_2 B^*) \approx 0.160 < 1.$$

Hence by the Theorem 3, we conclude that the problem (4.1) has a unique mild solution on $[0,1]$.

*Acknowledgements.* We are grateful to the referee's for their valuable suggestion to improve this paper.

REFERENCES


[1] S. ABBAS, M. BENCHOHRA, *Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay,* An. Int. J. for Theory and Appl., (2010) ISSN 1311–0454.

[2] R. P. AGARWAL, B. DE ANDRADE, *On fractional integro-differential equations with state-dependent delay,* Comp. and Math. with Appl., **62** (2011), 1143–1149.

[3] R. P. AGARWAL, B. DE ANDRADE, C. CUEVAS, *On type of periodicity and ergodicity to a class of fractional order differential equations,* Adv. Difference Equ., Article ID 179750, (2010), 1–25.

[4] M. M. ARJUNAN, V. KAVITHA, *Existence Results for impulsive neutral functional differential equations with state-dependent delay,* Elect. J. of Qual. Theory of Diff. Equ., **2009** (2009), 1143–1149.

[5] D. BAHUGUNA, *Existence uniqueness and regularity of solutions to semi-linear nonlocal functional differential problems,* Non. Anal., **57** (2004), 1021–1028.

[6] M. BENCHOHRA, F. BERHOUN, *Impulsive fractional differential equations with state dependent delay,* Commun. of Appl. Anal., **14** (2010), 213–224.

[7] M. BENCHOHRA, S. LITIMEIN, G. N'GUEREKATA, *On fractional integro-differential inclusions with state-dependent delay in Banach spaces,* Appl. Anal., (2011) 1–16.

[8] J. P. CARVALHO DOS SANTOS, M. M. ARJUNAN, *Existence results for fractional neutral integro-differential equations with state-dependent delay,* Comp. and Math. with Appl., **62** (2011), 1275–1283.

[9] J. P. CARVALHO DOS SANTOS, C. CUEVAS, B. DE ANDRADE, *Existence results for a fractional equation with state dependent delay,* Adv. in Diff. Equ., Article ID 642013, (2011).

[10] A. CHAUHAN, J. DABAS, *Existence of mild solutions for impulsive fractional order semilinear evolution equations with nonlocal conditions,* Elect. J. Diff. Equ., **2011** (2011), 1–10.

[11] A. CHAUHAN, J. DABAS, M. KUMAR, *Integral boundary value problem for impulsive fractional functional differential equations with infinite delay,* Elect. J. Diff. Equ., **2012** (2012), 1–13.

[12] A. CHAUHAN, J. DABAS, *Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition,* Commun. Nonlinear Sci. Num. Simul., **19** (2014), 821–829.

[13] E. CUESTA, *Asymptotically behavior of the solutions of fractional integro-differential equations and some discretizations,* Discrete Cont. Dyn. Syst., (2007), 277–285.

[14] C. CUEVAS, J. C. DE SOUZA, *S-asymptotically $\omega$-periodic solutions of semilinear fractional integro-differential equations,* Appl. Math. Lett., **22** (2009), 865–870.

[15] C. CUEVAS, J. C. DE SOUZA, *Existence of S-asymptotically $\omega$-periodic solutions for fractional order functional integro-differential equations with infinite delay,* Nonlinear Anal. **72** (2010), 1683–1689.

[16] J. DABAS, A. CHAUHAN, *Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equations with infinity delay,* Math. and Computer Modelling, **57** (2013), 754–763.

[17] J. DABAS, A. CHAUHAN, M. KUMAR, *Existence of mild solution for impulsive fractional equation with infinity delay,* Int. J. Diff. Equ., Art ID 793023, (2011), 20 pages.

[18] M. A. DARWISH, S. K. NTOUYAS, *Functional differential equations of fractional order with state-dependent delay,* Dyna. Syst. and Appl., **18** (2009), 539–550.

[19] M. A. DARWISH, S. K. NTOUYAS, *Semilinear functional differential equations of fractional order with state-dependent delay,* Elect. J. of Diff. Equ., **2009** (2009), 1–10.

[20] B. DE ANDRADE, C. CUEVAS, *S-asymptotically $\omega$-periodic and asymptotically-periodic solutions to semilinear Cauchy problems with non dense domain,* Nonlinear Anal. **72** (2010), 3190–3208.

[21] M. FECKAN, Y. ZHOU, J. WANG, *On the concept and existence of solution for impulsive fractional differential equations,* Comm. Nonl. Sci. Num. Sum., **17** (2012), 3050–3060.

[22] M. HAASE, *The functional calculus for sectorial operators (Operator theory: Advances and Applications),* vol.169. Basel: Birkhauser-Verlag, 2006.

[23] V. KAVITHA, P. WANG, R. MURUGESU, *Existence results for neutral functional fractional differential equations with state dependent-delay,* Malaya J. of Mat., **1** (2012), 50–61.

[24] F. LI, *Nonlocal cauchy problem for delay fractional integro-differential equations of neutral type,* Advances in Difference Equ., **47** (2012).

[25] I. PODLUBNY, *Fractional Differential Equations*, New York, Acadmic Press, 1993.

[26] M. H. M. RASHID, A. AL-OMARI, *Local and global existence of mild solutions for impulsive fractional semilinear integro-differential equation,* Commun. Nonlinear Sci. Numer. Simulat., **16** (2011), 3493–3503.

[27] N. K. TOMAR, J. DABAS, *Controllability of impulsive fractional order semilinear evoluation equations with nonlocal conditions,* J. Non. Evol. Equ. Appl., **5** (2012), 57–67.

[28] X.B. SHU, Y. LAI, Y. CHEN, *The existence of mild solutions for impulsive fractional partial differential equations*, Nonli. Anal., **74** (2011), 2003–2011.


(Received December 5, 2013)

(Revised August 10, 2014)

*Ganga Ram Gautam*
*Department of Applied Science and Engineering*
*IIT Roorkee, Saharanpur Campus*
*Saharanpur-247001*
*India*
*e-mail:* gangaiitr11@gmail.com

*Jaydev Dabas*
*Department of Applied Science and Engineering*
*IIT Roorkee, Saharanpur Campus*
*Saharanpur-247001*
*India*
*e-mail:* jay.dabas@gmail.com