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OSCILLATION CRITERIA FOR EVEN ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract. Some new oscillation results are obtained for the even order nonlinear neutral difference equation of the form

$$\Delta(a_n\Delta^{m-1}z_n)+q_nf(x_{n-\sigma})=0$$

where $z_n=x_n+p_nx_{\tau(n)}$. Our results generalize and improve some of the existing results. Two examples are provided to illustrate the main results.

1. Introduction

In this paper, we are concerned with the oscillation of all solutions of even order nonlinear neutral difference equation of the form

$$\Delta(a_n\Delta^{m-1}z_n)+q_nf(x_{n-\sigma})=0,\ n\geqslant n_0 \tag{1.1}$$

where $m\geqslant 2$ is an even integer and $z_n=x_n+p_nx_{\tau(n)}$ subject to the following conditions:

(c_1) $\{a_n\}$ is a positive increasing sequence of real numbers for all $n\geqslant n_0$;

(c_2) $\{q_n\}$ and $\{p_n\}$ with $0\leqslant p_n\leqslant p<\infty, q_n>0$ are sequences of real numbers for all $n\geqslant n_0$;

(c_3) $\{\tau(n)\}$ is sequence of integers such that $\lim\limits_{n\to\infty}\tau(n)=\infty$ and σ is a positive integer;

(c_1) f is a continuous real valued function such that $\dfrac{f(y)}{y}>L>0$ for $y\neq 0$ and L is a constant.

Let $\theta=\max\{\sigma,\min\limits_{n\geqslant n_0}\tau(n)\}$. By a solution of equation (1.1) we mean a sequence $\{x_n\}$ which is defined for all $n\geqslant n_0-\theta$ and satisfies equation (1.1) for sufficiently large value of n. As a customary, a nontrivial solution $\{x_n\}$ of equation (1.1) is said to be

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nonoscillatory if all the terms of $\{x_n\}$ are eventually of one sign; otherwise the solution $\{x_n\}$ is called oscillatory.

In recent years, there is an increasing interest in studying the oscillatory and asymptotic behavior of solutions of higher order neutral difference equations, since such type of equations naturally arise in the applications including problems in population dynamics or in cobweb models in economics. The problem of finding sufficient conditions which ensure that all solutions or all bounded solutions of difference equations of neutral type are oscillatory has been studied by many authors, see for example [1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14], and the references contained there in. Most of the authors consider the case when sequence $\{p_n\}$ in the neutral part satisfying

$$0 \leqslant p_n \leqslant p < 1 \text{ and } \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty.$$

But the results on the oscillation of equation (1.1) when the sequence $\{p_n\}$ satisfying

$$0 \leqslant p_n \leqslant p < \infty \text{ or } \sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$$

are relatively scarce, see [1, 2].In particular in [9], the authors considered a continuous analog of equation(1.1) and obtained two results with a restriction on the neutral term.Further in one theorem they obtained a criteria which implies that every solution is either oscillatory or tends to zero and will not say when all solutions are oscillatory. Motivated by these observations in this paper, we establish some sufficient conditions for the oscillation of all solutions of equation (1.1) when

$$0 \leqslant p_n \leqslant p < \infty \text{ and either } \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty \text{ or } \sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$$

satisfied. Therefore our results extend and improve the results in [9] and some of the the results in [1-5,7,8,10-14].

In Section 2, we present some preliminary lemmas which are needed for our subsequent discussion. Section 3 deals with oscillation results for the equation (1.1) and in Section 4, we provide some examples to illustrate the main results.

2. Some Preliminary Lemmas

In this section, we present lemmas which will be useful in proving our main results. Throughout this paper we use the following notation without further mention:

$$P_n = \min\{q_n, q_{\tau(n)}\}, \; Q_n = LP_n,$$

and

$$\delta_n = \sum_{s=n}^{\infty} \frac{1}{a_s}.$$

LEMMA 1. *Let $\{u_n\}$ be a sequence of real numbers and $u_n > 0$ with $\{\Delta^m u_n\}$ be of constant sign eventually and not identically zero eventually. Then there exist integers $l \in \{0,1,2,\dots,m\}$ with $(m+l)$ odd for $\Delta^m u_n \leqslant 0$, and $(m+l)$ even for $\Delta^m u_n \geqslant 0$ and $N > 0$ such that*

$$\Delta^j u_n > 0 \text{ for } j = 1,2,3\dots,l, \tag{2.1}$$

and

$$(-1)^{j+l}\Delta^j u_n > 0 \text{ for } j = l+1, l+2,\dots,m \tag{2.2}$$

for all $n \geqslant N$.

LEMMA 2. *Let $\{u_n\}$ be a sequence of real numbers and $u_n > 0$ and $\Delta^m u_n \leqslant 0$ and not identically equal to zero. Then there exists a large integer $N > 0$ such that*

$$u_n \geqslant \frac{(n-N)^{(m-1)}}{(m-1)!}\Delta^{m-1}u_{2^{m-l-1}n} \text{ for } n \geqslant N,\ l \in \{0,1,2,\dots,m\}, \tag{2.3}$$

where $u^{(j)} = u(u-1)(u-2)\dots(u-j+1)$. Note if further $\{u_n\}$ is increasing, then

$$u_n \geqslant \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{(m-1)}\Delta^{m-1}u_n \text{ for all } n \geqslant 2^{m-1}n.$$

The proofs of last two lemmas can be found in [1].

LEMMA 3. *Assume that*

$$\sum_{s=n_0}^{\infty}\frac{1}{a_s} = \infty$$

and let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists $n_1 \geqslant n_0$ such that

$$z_n > 0, \Delta z_n > 0, \Delta^{m-1}z_n > 0 \text{ and } \Delta^m z_n \leqslant 0 \text{ for all } n \geqslant n_1$$

Proof. Since $\{x_n\}$ is a positive solution of equation (1.1) there exists $n_1 \geqslant n_0$ such that $x_n > 0$ and $x_{\tau(n)} > 0$ for all $n \geqslant n_1$. Then by the definition of z_n, we have $z_n > 0$ for all $n \geqslant n_1$. Also from the equation (1.1), we have

$$\Delta(a_n\Delta^{m-1}z_n) = -q_n f(x_{n-\sigma}) < 0 \text{ for all } n \geqslant n_0. \tag{2.4}$$

Therefore $a_n\Delta^{m-1}z_n$ is decreasing and of one sign for all $n \geqslant n_1$. Since $\{a_n\}$ is positive, we have either $\Delta^{m-1}z_n < 0$ or $\Delta^{m-1}z_n > 0$ eventually. We shall prove that $\Delta^{m-1}z_n > 0$. If not, then there exists a constant $c < 0$ such that

$$a_n\Delta^{m-1}z_n \leqslant c < 0 \text{ for all } n \geqslant n_1.$$

Dividing the last inequality by a_n and summing from n_1 to n we get

$$\Delta^{m-2}z_n - \Delta^{m-2}z_{n_1} \leqslant c\sum_{s=n_1}^{n}\frac{1}{a_s}.$$

Letting $n \to \infty$ in the last inequality, we see that $\Delta^{m-2} z_n \to -\infty$. That is $\Delta^{m-2} z_n < 0$ eventually. Now $\Delta^{m-2} z_n < 0$ eventually implies $\Delta^{m-3} z_n < 0$ eventually. Continuing this process we get $z_n < 0$ eventually which is a contradiction. Hence $\Delta^{m-1} z_n > 0$ eventually. Moreover $\{a_n\}$ is positive and increasing and $\Delta(a_n \Delta^{m-1} z_n) < 0$ for all $n \geqslant n_1$, we have $\Delta^m z_n \leqslant 0$ for all $n \geqslant n_1$. $\square$

The proof of the following lemma can be found in [6].

LEMMA 4. *The first order difference inequality*

$$\Delta y_n + p_n y_{n-\tau} \leqslant 0$$

has no eventually positive solution if

$$\liminf_{n\to\infty} \sum_{s=n-\tau}^{n-1} p_s > \left(\frac{\tau}{\tau+1}\right)^{\tau+1} \tag{2.5}$$

or

$$\limsup_{n\to\infty} \sum_{s=n-\tau}^{n} p_s > 1. \tag{2.6}$$

3. Oscillation Results

In this section, we present some sufficient conditions for the oscillation of all solutions of equation (1.1).

THEOREM 1. *Assume that* $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$. *If*

$$\sum_{n=n_0}^{\infty} P_n = \infty, \tag{3.1}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $\{x_n\}$ is a positive solution of equation (1.1). Then there exists a $n_1 \geqslant n_0$ such that $x_n > 0$, $x_{\tau(n)} > 0$ and $x_{n-\sigma} > 0$ for all $n \geqslant n_1$. Then from Lemma 3, we have $z_n > 0, \Delta z_n > 0, \Delta^{m-1} z_n > 0$ and $\Delta^m z_n \leqslant 0$ for all $n \geqslant n_1$.

Now , using the condition (c_4) in equation (1.1), we see that

$$\Delta(a_n \Delta^{m-1} z_n) = -q_n f(x_{n-\sigma}) \leqslant -L q_n x_{n-\sigma} < 0 \text{ for all } n \geqslant n_1. \tag{3.2}$$

Therefore $a_n \Delta^{m-1} z_n$ is decreasing. Also from the last inequality, we have

$$\Delta(a_n \Delta^{m-1} z_n) + L q_n x_{n-\sigma} + p \Delta(a_{\tau(n)} \Delta^{m-1} z_{\tau(n)}) + L q_{\tau(n)} p x_{\tau(n-\sigma)} \leqslant 0 \ \text{ for all } \ n \geqslant n_1.$$

That is

$$\Delta(a_n\Delta^{m-1}z_n)+LP_nz_{n-\sigma}+p\Delta(a_{\tau(n)}\Delta^{m-1}z_{\tau(n)})\leqslant 0 \text{ for all } n\geqslant n_1. \tag{3.3}$$

Now summing the last inequality from n_1 to $n-1$, we obtain

$$a_n\Delta^{m-1}z_n-a_{n_1}\Delta^{m-1}z_{n_1}+L\sum_{s=n_1}^{n-1}P_sz_{s-\sigma}+pa_{\tau(n)}\Delta^{m-1}z_{\tau(n)} \\ -pa_{\tau(n_1)}\Delta^{m-1}z_{\tau(n_1)}\leqslant 0 \text{ for all } n\geqslant n_1.$$

That is

$$L\sum_{s=n_1}^{n-1}P_sz_{s-\sigma}\leqslant a_{n_1}\Delta^{m-1}z_{n_1}-a_n\Delta^{m-1}z_n+pa_{\tau(n_1)}\Delta^{m-1}z_{\tau(n_1)} \\ -pa_{\tau(n)}\Delta^{m-1}z_{\tau(n)}\leqslant 0 \text{ for all } n\geqslant n_1. \tag{3.4}$$

Since $\Delta z_n>0$ and $z_n>0$ eventually there exists a positive constant c such that $z_{n-\sigma}\geqslant c$ for all $n\geqslant n_1$. Using this and the monotonicity of $a_n\Delta^{m-1}z_n$ in the last inequality and letting $n\to\infty$, we get

$$\sum_{s=n_1}^{n-1}P_s<\infty, \tag{3.5}$$

which is a contradiction to (3.1). Now the proof is complete. $\square$

REMARK 1. In the last theorem we did not impose any condition on the sequence $\{\tau(n)\}$. That is , $\tau(n)$ may be delay or advanced argument. Hence our result is more general than some of the existing results in the literature.

THEOREM 2. *Assume that* $\sum_{s=n_0}^{\infty}\frac{1}{a_n}=\infty$ *and let* $\tau(n)=n+\tau$. *If either*

$$\liminf_{n\to\infty}\sum_{s=n-\sigma}^{n-1}\frac{(s-\sigma)^{(m-1)}Q_s}{a_{s-\sigma}}\geqslant\frac{(1+p)(m-1)!}{\lambda}\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \tag{3.6}$$

or

$$\limsup_{n\to\infty}\sum_{s=n-\sigma}^{n}\frac{(s-\sigma)^{(m-1)}Q_s}{a_{s-\sigma}}\geqslant\frac{(1+p)(m-1)!}{\lambda}, \tag{3.7}$$

where $\lambda\in(0,1)$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. If possible let us assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that there exists $n_1\geqslant n_0$ such that $x_n>0$, $x_{\tau(n)}>0$ and $x_{n-\sigma}>0$ for all $n\geqslant n_1$. Now proceeding as in the previous theorem, we obtain (3.3). That is,

$$\Delta(a_n\Delta^{m-1}z_n)+LP_nz_{n-\sigma}+p\Delta(a_{\tau(n)}\Delta^{m-1}z_{\tau(n-\sigma)})\leqslant 0 \text{ for all } n\geqslant n_1. \tag{3.8}$$

Now, since $\Delta^{m-1}z_n > 0, \Delta^m z_n \leqslant 0$, using Lemma 2 there exists $n_2 \geqslant n_1$ such that

$$\Delta(a_n\Delta^{m-1}z_n)+Q_n\frac{1}{(m-1)!}\left(\frac{n-\sigma}{2^{m-1}}\right)^{(m-1)}\Delta^{m-1}z_{n-\sigma} + p\Delta(a_{\tau(n)}\Delta^{m-1}z_{\tau(n-\sigma)}) \leqslant 0 \text{ for all } n \geqslant n_2 \geqslant 2^{m-1}. \tag{3.9}$$

Put $u_n = a_n\Delta^{m-1}z_n$. Then $u_n > 0$ and $\Delta u_n \leqslant 0$ and the last inequality becomes

$$\Delta(u_n + pu_{\tau(n)}) + \frac{\lambda Q_n}{(m-1)!}\frac{(n-\sigma)^{(m-1)}}{a_{n-\sigma}}u_{n-\sigma} \leqslant 0 \text{ for all } n \geqslant n_2, \text{ for every } \lambda, \tag{3.10}$$

where

$$0 < \lambda = \left(\frac{1}{2^{m-1}}\right)^{(m-1)} < 1.$$

Now put $w_n = u_n + pu_{\tau(n)}$. Then $w_n > 0$. Since u_n is decreasing and $\tau(n) = n+\tau \geqslant n$, we have

$$w_n \leqslant (1+p)u_n. \tag{3.11}$$

Using (3.11) in (3.10), we see that w_n is a positive solution of

$$\Delta w_n + \frac{\lambda Q_n}{(m-1)!}\frac{(n-\sigma)^{(m-1)}}{(1+p)a_{n-\sigma}}w_{n-\sigma} \leqslant 0 \text{ for all } n \geqslant n_2. \tag{3.12}$$

Now we consider two cases when (3.6) or (3.7) holds.
Case (i). If the condition (3.6) holds, then Lemma 4 implies that the inequality (3.12) has no positive solution, which is a contradiction.
Case (ii). If condition (3.7) holds, again by Lemma 4 we conclude that the inequality (3.12) has no positive solution, which is a contradiction. This completes the proof. □

THEOREM 3. *Assume that* $\sum_{n=n_0}^{\infty}\frac{1}{a_n} = \infty$ *and* $n-\sigma \leqslant \tau(n) \leqslant n$. *If either*

$$\liminf_{n\to\infty}\sum_{s=n-\sigma}^{n-1}\frac{(s-\sigma)^{(m-1)}Q_s}{a_{s-\sigma}} > (1+p)(m-1)!\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \tag{3.13}$$

or when $\tau^{-1}(n-\sigma)$ *is nondecreasing,and*

$$\limsup_{n\to\infty}\sum_{s=n-\sigma}^{n}\frac{(s-\sigma)^{(m-1)}Q_s}{a_{s-\sigma}} > (1+p)(m-1)!, \tag{3.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality of we may assume that x_n is a positive solution of equation (1.1).

Then there exists an integer $n_1 \geqslant n_0$ such that $x_n > 0$, $x_{\tau(n)} > 0$ and $x_{n-\sigma} > 0$ for all $n \geqslant n_1$. Now proceeding as in the previous theorem, we obtain

$$\Delta(u_n + pu_{\tau(n)}) + \frac{\lambda Q_n}{(m-1)!} \frac{(n-\sigma)^{(m-1)}}{a_{n-\sigma}} u_{n-\sigma} \leqslant 0 \text{ for all } n \geqslant n_1. \tag{3.15}$$

Put $w_n = u_n + pu_{\tau(n)}$. Then $w_n > 0$. Since u_n is decreasing, we have

$$w_n = u_n + pu_{\tau(n)} \leqslant (1+p)u_{\tau(n)} \text{ for } \tau(n) \leqslant n. \tag{3.16}$$

Using (3.16) in (3.15), we get

$$\Delta w_n + \frac{\lambda Q_n}{(m-1)!} \frac{(n-\sigma)^{(m-1)}}{a_{n-\sigma}} w_{\tau^{-1}(n-\sigma)} \leqslant 0 \text{ for all } n \geqslant n_1. \tag{3.17}$$

Thus $\{w_n\}$ is a positive solution of the inequality (3.17).
Case (i). Suppose (3.13) holds, then Lemma 4 implies that the inequality (3.17) has no positive solution, which is a contradiction.
Case (ii). Suppose (3.14) holds. then again Lemma 4 implies that the inequality (3.17) has no positive solution, which is a contradiction. Now the proof is complete. □

THEOREM 4. *Assume that*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$$

and $n-\sigma \leqslant \tau(n) \leqslant n$. *If either* (3.13) *or when* $\tau^{-1}(n-\sigma)$ *is nondecreasing with* (3.14) *holds and for sufficiently large* $n_1 \geqslant n_0$

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[\frac{\lambda}{(n-2)!} \delta_s Q_s (s-\sigma)^{(m-2)} - \frac{(1+p)}{4a_{s+1}\delta_{s+1}} \right] = \infty \tag{3.18}$$

where $0 < \lambda < 1$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. If possible let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that x_n is a positive solution of equation (1.1). Then there exists a $n_1 \geqslant n_0$ such that $x_n > 0$, $x_{\tau(n)} > 0$ and $x_{n-\sigma} > 0$ for all $n \geqslant n_1$. From equation (1.1) we see that $\Delta(a_n \Delta^{m-1} z_n) \leqslant 0$ for all $n \geqslant n_1$. Since $\{a_n\}$ is positive, $\Delta^{m-1} z_n$ is of one sign for all $n \geqslant n_1$.
Case (i). Suppose $\Delta^{m-1} z_n > 0$ eventually. The proof for this case is similar to that of Case (i) of Theorem 3 and hence the details are omitted.
Case (ii). Suppose $\Delta^{m-1} z_n < 0$ eventually. Then by Lemma 1, we have $\Delta^{m-2} z_n > 0$ and $\Delta z_n > 0$. Now define w_n by

$$w_n = \frac{a_n \Delta^{m-1} z_n}{\Delta^{m-2} z_n} \text{ for all } n \geqslant n_2 \geqslant n_1. \tag{3.19}$$

Then $w_n < 0$ and

$$\Delta w_n = \frac{\Delta(a_n\Delta^{m-1}z_n)}{\Delta^{m-2}z_n} - \frac{a_{n+1}\Delta^{m-1}z_{n+1}}{\Delta^{m-2}z_{n+1}\Delta^{m-2}z_n}\Delta^{m-1}z_n \text{ for all } n \geqslant n_2.$$

Since $a_n\Delta^{m-1}z_n$ is decreasing and $\Delta^{m-2}z_n$ is increasing, we have

$$\Delta w_n \leqslant \frac{\Delta(a_n\Delta^{m-1}z_n)}{\Delta^{m-2}z_n} - \frac{w_{n+1}^2}{a_{n+1}}. \tag{3.20}$$

Using the decreasing nature of $a_n\Delta^{m-1}z_n$ we have

$$a_l\Delta^{m-1}z_l \leqslant a_n\Delta^{m-1}z_n \quad \text{for all } l \geqslant n \geqslant n_2.$$

Dividing the last inequality by a_l and summing the resulting inequality from n to $l-1$, we get

$$\Delta^{m-2}z_l - \Delta^{m-2}z_n \leqslant a_n\Delta^{m-1}z_n\sum_{s=n}^{l-1}\frac{1}{a_s}.$$

Letting $l \to \infty$, we obtain

$$0 \leqslant \Delta^{m-2}z_n + a_n\Delta^{m-1}z_n\delta_n$$

$$\text{or} \qquad -1 \leqslant \frac{a_n\Delta^{m-1}z_n\delta_n}{\Delta^{m-2}z_n} = w_n\delta_n \leqslant 0 \text{ for all } n \geqslant n_2. \tag{3.21}$$

Define v_n by

$$v_n = \frac{a_{\tau(n)}\Delta^{m-1}z_{\tau(n)}}{\Delta^{m-2}z_n} \text{ for all } n \geqslant n_2. \tag{3.22}$$

We obtain $v_n \leqslant 0$ and

$$-1 \leqslant v_n\delta_n \leqslant 0 \ \text{ for all } \ n \geqslant n_2. \tag{3.23}$$

Also from (3.22), we get

$$\begin{aligned}\Delta v_n &= \frac{\Delta(a_{\tau(n)}\Delta^{m-1}z_{\tau(n)})}{\Delta^{m-2}z_n} - \frac{a_{\tau(n+1)}\Delta^{m-1}z_{\tau(n+1)}}{\Delta^{m-2}z_{n+1}\Delta^{m-2}z_n}\Delta^{m-1}z_n \\ &\leqslant \frac{\Delta(a_{\tau(n)}\Delta^{m-1}z_{\tau(n)})}{\Delta^{m-2}z_n} - \frac{v_{n+1}^2}{a_{\tau(n+1)}}.\end{aligned} \tag{3.24}$$

Combining (3.20) and (3.24), we obtain

$$\Delta w_n + p\Delta v_n \leqslant \frac{\Delta(a_n\Delta^{m-1}z_n)}{\Delta^{m-2}z_n} - \frac{w_{n+1}^2}{a_{n+1}} + p\frac{\Delta(a_{\tau(n)}\Delta^{m-1}z_{\tau(n)})}{\Delta^{m-2}z_n} - p\frac{v_{n+1}^2}{a_{\tau(n+1)}}.$$

Using the inequality (3.3) in the last inequality, we have

$$\Delta w_n + p\Delta v_n \leqslant \frac{-LP_nz_{n-\sigma}}{\Delta^{m-2}z_n} - \frac{w_{n+1}^2}{a_{n+1}} - p\frac{v_{n+1}^2}{a_{\tau(n+1)}}. \tag{3.25}$$

Now from Lemma 2,

$$z_{n-\sigma} \geqslant \frac{\lambda}{(m-2)!}(n-\sigma)^{(m-2)}\Delta^{m-2}z_{n-\sigma}. \tag{3.26}$$

Since $\Delta^{m-1}z_n < 0$ and $n-\sigma \leqslant n$, we have

$$\Delta^{m-2}z_n < \Delta^{m-2}z_{n-\sigma}. \tag{3.27}$$

Combining the inequalities (3.25), (3.26) and (3.27), we have

$$\Delta w_n + p\Delta v_n \leqslant -\frac{\lambda Q_n}{(m-2)!}(n-\sigma)^{(m-2)} - \frac{w_{n+1}^2}{a_{n+1}} - p\frac{v_{n+1}^2}{a_{n+1}}. \tag{3.28}$$

Now multiplying (3.28) by δ_n and taking summation on the resulting inequality from n_2 to $n-1$, we obtain

$$\begin{aligned}\delta_n w_n - \delta_{n_2}w_{n_2} + \sum_{s=n_2}^{n-1}\frac{w_{s+1}}{a_s} + p\delta_n v_n - p\delta_{n_2}v_{n_2} + p\sum_{s=n_2}^{n-1}\frac{v_{s+1}}{a_s}&\\ + \sum_{s=n_2}^{n-1}\frac{w_{s+1}^2}{a_{s+1}}\delta_s + p\sum_{s=n_2}^{n-1}\frac{v_{s+1}^2}{a_{s+1}}\delta_s&\\ + \frac{\lambda}{(m-2)!}\sum_{s=n_2}^{n-1}Q_s(s-\sigma)^{m-2}\delta_s \leqslant 0.&\end{aligned} \tag{3.29}$$

Using the increasing nature of $\{a_n\}$, decreasing nature of $\{\delta_n\}$ and then completion of square, we have

$$\begin{aligned}\delta_n w_n - \delta_{n_2}w_{n_2} + p\delta_n v_n - p\delta_{n_2}v_{n_2} + \frac{\lambda}{(m-2)!}\sum_{s=n_2}^{n-1}Q_s(s-\sigma)^{(m-2)}\delta_s&\\ -\frac{1}{4}\sum_{s=n_2}^{n-1}\frac{1}{a_{s+1}\delta_{s+1}} - \frac{p}{4}\sum_{s=n_2}^{n-1}\frac{1}{a_{s+1}\delta_{s+1}} \leqslant 0&\end{aligned}$$

or

$$\begin{aligned}\delta_n w_n + p\delta_n v_n + \sum_{s=n_2}^{n-1}\left[\frac{\lambda}{(m-2)!}Q_s(s-\sigma)^{(m-2)}\delta_s - \frac{(1+p)}{4a_{s+1}\delta_{s+1}}\right]&\\ \leqslant \delta_{n_2}w_{n_2} + \delta_{n_2}v_{n_2}.&\end{aligned}$$

By taking limit supremum as $n \to \infty$ in the last inequality we obtain a contradiction to (3.18). This completes the proof. □

THEOREM 5. *Assume that*

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n} < \infty$$

and let $\tau(n) \geqslant n$. *If either* (3.6) *holds or when* $\tau^{-1}(n-\sigma)$ *is nondecreasing with* (3.7) *holds and for sufficiently large* $n_1 \geqslant n_0$

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1} \left[\frac{\lambda}{(m-2)!} Q_s (s-\sigma)^{(m-2)} \delta_{\tau(s)} - \frac{(1+p)}{4a_{s+1}\delta_{\tau(s+1)}} \right] = \infty \tag{3.30}$$

where $0 < \lambda < 1$ *is a constant, then every solution of equation* (1.1) *is oscillatory.*

Proof. On the contrary let us assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $\{x_n\}$ is a positive solution of equation (1.1). Then there exists $n_1 \geqslant n_0$ such that $x_n > 0$, $x_{\tau(n)} > 0$ and $x_{n-\sigma} > 0$ for all $n \geqslant n_1$. From equation (1.1), we see that $\{a_n \Delta^{m-1} z_n\}$ is decreasing for all $n \geqslant n_1$. Then there are two cases for $\Delta^{m-1} z_n$, namely, either $\Delta^{m-1} z_n > 0$ eventually or $\Delta^{m-1} z_n < 0$ eventually.

Case (i). $\Delta^{m-1x} z_n > 0$ for all $n \geqslant n_1$. Then the proof is similar to that of case (i) in Theorem 2 and the details are omitted.

Case (ii). $\Delta^{m-1} z_n < 0$ for all $n \geqslant n_1$. Then by Lemma 1, we have $\Delta^{m-2} z_n > 0$ and $\Delta z_n > 0$. Define γ_n by

$$\gamma_n = \frac{a_{\tau(n)} \Delta^{m-1} z_{\tau(n)}}{\Delta^{m-2} z_n} \text{ for all } n \geqslant n_2 \geqslant n_1. \tag{3.31}$$

Then $\gamma_n < 0$ for all $n \geqslant n_2$. Since $a_n \Delta^{m-1} z_n$ is decreasing we have

$$a_{\tau(s)} \Delta^{m-1} z_{\tau(s)} \leqslant a_{\tau(n)} \Delta^{m-1} z_{\tau(n)} \text{ for all } s \geqslant n \geqslant n_2.$$

Dividing the last inequality by $a_{\tau(s)}$ and then summing the resulting inequality from n to $l-1$, we get

$$\Delta^{m-2} z_{\tau(l)} - \Delta^{m-2} z_n \leqslant a_{\tau(n)} \Delta^{m-1} z_{\tau(n)} \sum_{s=n}^{l-1} \frac{1}{a_{\tau(s)}}. \tag{3.32}$$

Letting, $l \to \infty$ we see that

$$0 \leqslant \Delta^{m-2} z_{\tau(n)} \leqslant a_{\tau(n)} \Delta^{m-1} z_{\tau(n)} \delta_{\tau(n)}. \tag{3.33}$$

Since $\Delta^{m-1} z_n < 0$, $\Delta^{m-2} z_n$ is decreasing and therefore for $\tau(n) \geqslant n$, we have

$$\Delta^{m-2} z_{\tau(n)} \leqslant \Delta^{m-2} z_n \text{ for all } n \geqslant n_2. \tag{3.34}$$

Combining (3.33) and (3.34), we obtain

$$-1 \leqslant v_n \delta_{\tau(n)} \leqslant 0 \text{ for all } n \geqslant n_2. \tag{3.35}$$

Similarly defining w_n by

$$w_n = \frac{a_n \Delta^{m-1} z_n}{\Delta^{m-2} z_n} \text{ for all } n \geqslant n_2$$

we get

$$-1 \leqslant w_n \delta_{\tau(n)} \leqslant 0. \tag{3.36}$$

Now proceeding as in the proof of Theorem 4 we get (3.28). Multiplying (3.28) by $\delta_{\tau(n)}$ and then summing it from n_2 to $n-1$, we get

$$\begin{aligned}\delta_{\tau(n)} w_n - \delta_{\tau(n_2)} w_{n_2} + \sum_{s=n_2}^{n-1} \frac{w_{s+1}}{a_{\tau(s)}} + p\delta_{\tau(n)} v_n - p\delta_{\tau(n_2)} v_{n_2} \\ + p\sum_{s=n_2}^{n-1} \frac{v_{s+1}}{a_{\tau(s)}} + p\sum_{s=n_2}^{n-1} \frac{v_{s+1}^2}{a_{\tau(s)}} + \sum_{s=n_2}^{n-1} \frac{w_{s+1}^2}{a_{\tau(s)}} \\ + \frac{\lambda}{(m-2)!} \sum_{s=n_2}^{n-1} Q_s (s-\sigma)^{(m-2)} \delta_{\tau(s)} \leqslant 0\end{aligned} \tag{3.37}$$

Now using the increasing nature of $\{a_n\}$, decreasing nature of $\{\delta_n\}$ and then completing the square, we have

$$\begin{aligned}\delta_{\tau(n)} w_n - \delta_{\tau(n_2)} w_{n_2} + p\delta_{\tau(n)} v_n - p\delta_{\tau(n_2)} v_{n_2} \\ + \frac{\lambda}{(m-2)!} \sum_{s=n_2}^{n-1} Q_s (s-\sigma)^{(m-2)} \delta_{\tau(s)} \\ - \frac{1}{4} \sum_{s=n_2}^{n-1} \frac{1}{a_{\tau(s+1)} \delta_{\tau(s+1)}} - \frac{p}{4} \sum_{s=n_2}^{n-1} \frac{1}{a_{\tau(s+1)} \delta_{\tau(s+1)}} \leqslant 0\end{aligned}$$

or

$$\begin{aligned}\delta_{\tau(n)} w_n + p\delta_{\tau(n)} v_n \\ + \sum_{s=n_2}^{n-1} \left[\frac{\lambda}{(m-2)!} Q_s x(s-\sigma)^{(m-2)} \delta_{\tau(s)} - \frac{(1+p)}{4a_{\tau(s+1)} \delta_{\tau(s+1)}} \right] \\ \leqslant \delta_{\tau(n_2)} w_{n_2} + p\delta_{\tau(n_2)} v_{n_2}.\end{aligned}$$

By taking limit supremum as $n \to \infty$ in the last inequality we get a contradiction to (3.30). This completes the proof. □

4. Examples

In this section we present two examples to illustrate the main results

EXAMPLE 1. Consider the difference equation

$$\Delta(n\Delta^{m-1}(x_n + 4x_{n+2})) + 5(2^{m-1})(2n+1)x_{n-2} = 0, \; n \geqslant 2 \tag{4.1}$$

where $m \geqslant 4$ is an even integer. Here $p_n = 4 > 0$, $a_n = n$, $q_n = 5(2^{m-1})(2n+1)$, $\tau(n) = n+2$ and $\sigma = 2$. It is easy to see that all conditions of Theorem 2 are satisfied and hence every solution of equation (4.1) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (4.1).

EXAMPLE 2. Consider the difference equation

$$\Delta((n+1)n\Delta^{m-1}(x_n+2x_{n-1}))+2^m(n+1)^2x_{n-2}(1+x_{n-2}^2)=0,\ n\geqslant 1 \tag{4.2}$$

where $m\geqslant 4$ is an even integer. Here $a_n=n(n+1)$, $p_n=2$, $q_n=2^m(n+1)^2$, $\sigma=2$ and $\tau(n)=n-1$. It is easy to see that all conditions of Theorem 4 are satisfied and hence every solution of equation (4.2) is oscillatory.In fact $\{x_n\}=\{(-1)^n\}$ is one such oscillatory solution of equation (4.2).

We conclude this paper with the following remark.

REMARK 2. The results obtained in this paper extend and improve some of the existing results. First we extend the range of the neutral function p_n from the interval $0\leqslant p_n\leqslant p<1$ to $0\leqslant p_n\leqslant p<\infty$. Second ,in the case of $\sum\limits_{n=n_0}^{\infty}\frac{1}{a_n}<\infty$ most of the results obtained so far give the solutions of the equation are either oscillatory or tend to zero.But we improved this and obtained criteria under which all solutions are oscillatory.

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