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EXISTENCE OF POSITIVE SOLUTION FOR A SINGULAR SYSTEM INVOLVING GENERAL QUASILINEAR OPERATORS

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Abstract. In this paper we study a result of existence of positive solution the following class of singular system:

$$(P)\qquad \begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = h_1(x)v^{-\gamma_1} + k_1(x)v^{\alpha_1} \text{ in } \Omega,\\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = h_2(x)u^{-\gamma_2} + k_2(x)u^{\alpha_2} \text{ in } \Omega,\\ u,v>0 \text{ in } \Omega,\\ u=v=0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geqslant 3$, $2 \leqslant p_1, p_2 < N$. For $i = 1,2$, $\alpha_i, \gamma_i \in (0, p_i - 1)$ and h_i and k_i are continuous functions. The hypotheses on the functions $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ allow to show that (P) includes a large class of systems. We use topological arguments to show the main result.

1. Introduction

This paper concerns with the existence of solution of singular elliptic systems of the type

$$(P)\qquad \begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = h_1(x)v^{-\gamma_1} + k_1(x)v^{\alpha_1} \text{ in } \Omega,\\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = h_2(x)u^{-\gamma_2} + k_2(x)u^{\alpha_2} \text{ in } \Omega,\\ u,v>0 \text{ in } \Omega,\\ u=v=0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geqslant 3$, $2 \leqslant p_1, p_2 < N$. For $i = 1,2$, $\alpha_i, \gamma_i \in (0, p_i - 1)$, $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1- function and h_i and k_i are continuous functions. More precisely, we will suppose that the functions h_i, k_i and a_i satisfy the following assumptions:

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(A_1) There are real constants $\xi_0 > 0$, $\xi_1, \xi_2, \xi_3 \geqslant 0$ and $p_i < q_i < N$ for $(i = 1,2)$ such that

$$\xi_0 + H(\xi_3)\xi_1 t^{\frac{q_i - p_i}{p_i}} \leqslant a_i(t) \leqslant \xi_2 + \xi_3 t^{\frac{q_i - p_i}{p_i}}, \ \forall\, t \geqslant 0$$

where $H : [0,+\infty) \to \{0,1\}$ is the function given by

$$H(t) = \begin{cases} 1 \text{ if } t > 0 \\ 0 \text{ if } t = 0. \end{cases}$$

(A_2) The mapping $g_i : \mathbb{R}^+ \to \mathbb{R}^+$, given by $g_i(t) = a_i(t^{p_i})t^{p_i - 2}$, is increasing, for $i = 1,2$.

(A_3) The mappings $h_i, k_i : \overline{\Omega} \to (0,+\infty)$ are continuous functions, for $i = 1,2$.

Our main result is the following:

THEOREM 1. *Assume that conditions (A_1), (A_2), (A_3), $2 \leqslant p_1, p_2 < N$ and $\alpha_i, \gamma_i \in (0, p_i - 1)$ hold true. Then problem (P) has a solution.*

A considerable effort has been devoted during the last years in studying singular elliptic problems, as it can be seen, for instance, in [1], [2] [3], [5], [6], [10], [12], [16], [17], [19], [20] and references therein. This is due to their significance in applications (fluid mechanics pseudoplastics flow, chemical heterogeneous catalysts, non-Newtonian fluids, biological pattern formation) as well as to their mathematical relevance. Some of these applications can be seen in [1] and the references therein.

Theorem 1 is related to results of [1] and [20]. In [1] the authors studies the system

$$\begin{cases} -\Delta_p u = v^{-\gamma_1} + v^{\alpha_1} \text{ in } \Omega, \\ -\Delta_q v = u^{-\gamma_2} + u^{\alpha_2} \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

and shows the existence of solution by using theorem a due to Rabinowitz [13] (see Proposition 3.1) and a Hardy-Sobolev inequality (see Proposition 2.1).

In [20] the author studies the system

$$\begin{cases} -\Delta u = u^{-p} v^{-q} \text{ in } \Omega, \\ -\Delta v = u^{-r} v^{-s} \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

and shows some existence, nonexistence and uniqueness results for different values of p, q, r, s and using sub-supersolution methods.

We have completed the studies found in [1] and [20] because, in this work, we have more general operators than those considered in these articles.

Just to illustrate the degree of generality of the problem (P) let us consider some special cases, depending on the functions a_i, that are covered in this article, i.e., a_i satisfies assumptions $(A_1) - (A_2)$.

EXAMPLE 1. If $a_i \equiv 1$ with $i = 1,2$, then

$$-\mathrm{div}(a_i(|\nabla w_i|^{p_i})|\nabla w_i|^{p_i-2}\nabla w_i) = -\Delta_{p_i} w_i,$$

where $\xi_0 = \xi_1 = 1$ and $\xi_3 = 0$ and $\xi_2 > 0$.

EXAMPLE 2. If $a_i(t) = 1 + t^{\frac{q_i-p_i}{p_i}}$ with $i = 1,2$, we obtain

$$-\mathrm{div}(a_i(|\nabla w_i|^{p_i})|\nabla w_i|^{p_i-2}\nabla w_i) = -\Delta_{p_i} w_i - \Delta_{q_i} w_i$$

with $\xi_0 = \xi_1 = \xi_2 = \xi_3 = 1$.

EXAMPLE 3. Taking

$$a_i(t) = 1 + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$$

with $i = 1,2$, we get

$$-\mathrm{div}(a_i(|\nabla w_i|^{p_i})|\nabla w_i|^{p_i-2}\nabla w_i) = -\mathrm{div}\left(|\nabla w_i|^{p_i-2}\nabla w_i + \frac{|\nabla w_i|^{p_i-2}\nabla w_i}{(1+|\nabla w_i|^{p_i})^{\frac{p_i-2}{p_i}}}\right)$$

with $\xi_0 = 1, \xi_1 = 2, \xi_3 = 0$ and $\xi_2 > 0$.

EXAMPLE 4. We now consider

$$a_i(t) = 1 + t^{\frac{q_i-p_i}{p_i}} + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$$

with $i = 1,2$ to obtain

$$-\mathrm{div}(a_i(|\nabla w_i|^{p_i})|\nabla w_i|^{p_i-2}\nabla w_i) = -\Delta_{p_i} w_i - \Delta_{q_i} w_i - \mathrm{div}\left(\frac{|\nabla w_i|^{p_i-2}\nabla w_i}{(1+|\nabla w_i|^{p_i})^{\frac{p_i-2}{p_i}}}\right),$$

where $\xi_0 = 1, \xi_1 = 2$ and $\xi_3 = \xi_2 = 1$.

Other combinations can be made with the functions presented in the examples above, generating very interesting elliptic systems from the mathematical point of view and applications.

Due to the presence of the general operator some more estimates refined are need, such as in Lemmas 1, 2 and 3.

This class of operators was studied in [8], [9] and some reference given there.

The plan of this paper is as follows. In the section 2 we show some preliminary results on the general operator and a comparison principle. In the section 3 we show an existence result for an auxiliary problem. We prove the main result in section 4.

2. Preliminary results

We define $X = W_0^{1,\beta_1}(\Omega) \times W_0^{1,\beta_2}(\Omega)$ equipped with the norm

$$\|(u,v)\| = \|u\|_{1,\beta_1} + \|v\|_{1,\beta_2},$$

where

$$\|u\|_{1,\beta_i} = \|u\|_{1,p_i} + H(\xi_3)\|u\|_{1,q_i}.$$

In this text $|.|_\infty$ denotes the norm in $L^\infty(\Omega)$. Moreover, we say that the pair $(u,v) \in X$ is a solution of the problem (P) if

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla \phi dx = \int_\Omega \left[\frac{h_1(x)}{v^{\gamma_1}} + k_1(x)v^{\alpha_1}\right]\phi dx,$$

and

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla \varphi dx = \int_\Omega \left[\frac{h_2(x)}{u^{\gamma_2}} + k_2(x)u^{\alpha_2}\right]\varphi dx,$$

for all $(\phi,\varphi) \in X$.

In this article we work with the operator $T_i : W_0^{1,\beta_i}(\Omega) \to W_0^{1,\beta_i}(\Omega)'$ such that $i = 1,2$, $\langle T_i w_i, \phi_i\rangle$ is given by

$$\langle T_i w_i, \phi_i\rangle = \int_\Omega a_i(|\nabla w_i|^{p_i})|\nabla w_i|^{p_i-2}\nabla w_i \nabla \phi_i \, dx.$$

A straightforward calculation shows that T_i is continuous. In what follows we prove that T_i is monotone and coercive.

LEMMA 1. *The operator T_i satisfies the following conditions:*

$$\langle T_i u_i - T_i v_i, u_i - v_i\rangle > 0 \text{ for all } u_i, v_i \in W_0^{1,\beta_i}(\Omega) \text{ with } u_i \neq v_i$$

and

$$\lim_{\|u_i\|_{1,\beta_i}\to\infty} \frac{\langle T_i u_i, u_i\rangle}{\|u_i\|_{1,\beta_i}} = +\infty.$$

Proof. To prove the first part, it is enough to show the inequality below:

$$C|x-y|^{p_i} \leqslant \left\langle a_i(|x|^{p_i})|x|^{p_i-2}x - a_i(|y|^{p_i})|y|^{p_i-2}y, x-y\right\rangle,$$

for all $x, y \in \mathbb{R}^N$ and $i = 1,2$. Indeed, firstly note that

$$\left\langle a_i(|x|^{p_i})|x|^{p_i-2}x - a_i(|y|^{p_i})|y|^{p_i-2}y, x-y\right\rangle \\ = \sum_{j=1}^N (a_i(|x|^{p_i})|x|^{p_i-2}x_j - a_i(|y|^{p_i})|y|^{p_i-2}y_j)(x_j - y_j)$$

and for all $z, \xi \in \mathbb{R}^N$ we get

$$\sum_{k,j=1}^{N} \frac{\partial}{\partial z_k}(a_i(|z|^{p_i})|z|^{p_i-2}z_j)\xi_k\xi_j = (p_i-2)|z|^{p_i-4}\sum_{k,j=1}^{N} a_i(|z|^{p_i})z_kz_j\xi_k\xi_j$$
$$+\sum_{k,j=1}^{N} a_i(|z|^{p_i})|z|^{p_i-2}\delta_{k,j}\xi_k\xi_j + p_i\sum_{k,j=1}^{N} a_i'(|z|^{p_i})|z|^{2p_i-4}z_kz_j\xi_k\xi_j.$$

Hence

$$\sum_{k,j=1}^{N} \frac{\partial}{\partial z_k}(a_i(|z|^{p_i})|z|^{p_i-2}z_j)\xi_k\xi_j = (p_i-2)|z|^{p_i-4}a_i(|z|^{p_i})\sum_{k,j=1}^{N} z_kz_j\xi_k\xi_j$$
$$+a_i(|z|^{p_i})|z|^{p_i-2}|\xi|^2 + pa_i'(|z|^{p_i})|z|^{2p_i-4}\sum_{k,j=1}^{N} z_kz_j\xi_k\xi_j.$$

Since

$$\sum_{k,j=1}^{N} z_kz_j\xi_k\xi_j = (\sum_{j=1}^{N} z_j\xi_j)^2,$$

we have

$$\sum_{k,j=1}^{N} \frac{\partial}{\partial z_k}(a_i(|z|^{p_i})|z|^{p_i-2}z_j)\xi_k\xi_j$$
$$=(\sum_{j=1}^{N} z_j\xi_j)^2|z|^{p_i-4}\Big[(p_i-2)a_i(|z|^{p_i})+p_ia_i'(|z|^{p_i})|z|^{p_i}\Big]$$
$$+a_i(|z|^{p_i})|z|^{p_i-2}|\xi|^2.$$

By (A_2), we derive

$$\sum_{k,j=1}^{N} \frac{\partial}{\partial z_k}(a_i(|z|^{p_i})|z|^{p_i-2}z_j)\xi_k\xi_j \geqslant a_i(|z|^{p_i})|z|^{p_i-2}|\xi|^2. \tag{2.1}$$

Moreover, if $|y| \geqslant |x|$, we have $\frac{1}{2}|x-y| \leqslant |y|$ and for $t \in [0,\frac{1}{4}]$ we get

$$|y+t(x-y)| \geqslant |y| - t|x-y| \geqslant \frac{1}{4}|x-y|.$$

Making $z = x-y$ and $\xi = x-y$, from direct calculations we get

$$\sum_{j=1}^{N}(a_i(|x|^{p_i})|x|^{p_i-2}x_j - a_i(|y|^{p_i})|y|^{p_i-2}y_j)(x_j-y_j)$$
$$=\int_0^1\sum_{k,j=1}^{N} \frac{\partial}{\partial z_k}(a_i(|z|^{p_i})|z|^{p_i-2}z_j)\xi_k\xi_j\,dt.$$

Using (2.1) we get

$$\begin{aligned}\langle a_i(|x|^{p_i})|x|^{p_i-2}x - a_i(|y|^{p_i})|y|^{p_i-2}y, x-y\rangle \\ \geqslant a_i(|y+t(x-y)|^{p_i})|y+t(x-y))|^{p_i-2}|x-y|^2\end{aligned}$$

By (A_1) we conclude

$$\langle a_i(|x|^{p_i})|x|^{p_i-2}x - a_i(|y|^{p_i})|y|^{p_i-2}y, x-y\rangle \geqslant \frac{\xi_0}{4}|x-y|^{p_i-2}|x-y|^2 = \frac{\xi_0}{4}|x-y|^{p_i}.$$

The second part follows by using the growth of the operator T_i given by hypothesis (A_1). $\square$

Lemma 1 provides the monotonicity and coerciveness of the operator T_i. Thus, by Minty-Browder's Theorem [4, Teorema V. 15], given $f_i \in W_0^{1,\beta_i}(\Omega)'$ with $i = 1,2$, there exists a unique $u_i \in W_0^{1,\beta_i}(\Omega)$ enjoying

$$\begin{cases} -\mathrm{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) = f_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases}$$

in the weak sense.

Now using the Lemma 1 and arguing as [14, Lemma A2] and [14, Hopf Lemma] we get next two results.

LEMMA 2. *If Ω is a bounded domain and if $u_i, v_i \in W_0^{1,\beta_i}(\Omega)$ satisfy*

$$\begin{cases} -\mathrm{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) \leqslant -\mathrm{div}(a_i(|\nabla v_i|^{p_i})|\nabla v_i|^{p_i-2}\nabla v_i) \textit{ in } \Omega, \\ u_i \leqslant v_i \textit{ on } \partial\Omega \end{cases}$$

then $u_i \leqslant v_i$ a.e in Ω.

LEMMA 3. *Let Ω be a bounded domain with smooth boundary and $i = 1,2$. If $u_i \in C^1(\overline{\Omega}) \bigcap W_0^{1,\beta_i}(\Omega)$ and*

$$\begin{cases} -\mathrm{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) \geqslant 0 \\ u_i > 0 \textit{ in } \Omega, \\ u_i = 0 \textit{ on } \partial\Omega, \end{cases}$$

then $\dfrac{\partial u_i}{\partial \eta} < 0$ on $\partial\Omega$, where η is the outward normal to $\partial\Omega$.

Next, we recall the Hardy-Sobolev inequality which will play a key role in the proof of our main result:

PROPOSITION 2.1. *(Hardy-Sobolev inequality) If $u \in C^1(\overline{\Omega}) \bigcap W_0^{1,p}(\Omega)$ with $1 < p \leqslant N$, $u = 0$ on $\partial\Omega$ and $\frac{\partial u}{\partial \eta} < 0$, then $\frac{u}{Cd_x^\tau} \in L^r(\Omega)$, for $\frac{1}{r} = \frac{1}{p} - \frac{1-\tau}{N}, 0 \leqslant \tau \leqslant 1$, and*

$$\|\frac{u}{Cd_x^\tau}\|_{L^r} \leqslant \|\nabla u\|_{L^p},$$

where $d_x = dist\{x, \partial\Omega\}$, C is a positive constant, which does not depend on x. See [11].

3. The Approximate Problem

In what follows, we are going to show an existence result for an approximate problem.

Our first existence theorem rests heavily on the following result due to Rabinowitz [13].

PROPOSITION 3.1. *Let E be a Banach space and $T : \mathbb{R}^+ \times E \to E$ a continuous and compact mapping so that $T(0,u) = 0$, for all $u \in E$. Then the equation*

$$u = T(\lambda, u)$$

possesses an unbounded continuum $\mathscr{C} \subset \mathbb{R}^+ \times E$ of solutions with $(0,0) \in \mathscr{C}$.

For each $\varepsilon > 0$, let us consider the problem

$$(P_\varepsilon) \qquad \begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = \frac{h_1(x)}{(\varepsilon+|v|)^{\gamma_1}} + k_1(x)|v|^{\alpha_1} \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = \frac{h_2(x)}{(\varepsilon+|u|)^{\gamma_2}} + k_2(x)|u|^{\alpha_2} \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

For the approximate problem we have the following result

THEOREM 2. *For each $\varepsilon > 0$ and $\alpha_i, \gamma_i \in (0, p_i - 1)$ with $i = 1,2$, problem (P_ε) possesses a solution.*

Proof. Let us construct an operator. For each fixed $\varepsilon > 0$, let us construct the operator $T(\lambda, u, v)$ satisfying the assumptions imposed in Proposition 3.1. For this, let $\lambda \geqslant 0$ be and consider the problem

$$(\bar{P}) \qquad \begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = \lambda \left[\frac{h_1(x)}{(\varepsilon+|v|)^{\gamma_1}} + k_1(x)|v|^{\alpha_1}\right] \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = \lambda \left[\frac{h_2(x)}{(\varepsilon+|u|)^{\gamma_2}} + k_2(x)|u|^{\alpha_2}\right] \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases}$$

By Lemma 1, the operator $T_i : W_0^{1,\beta_i} \to (W_0^{1,\beta_i})'$ is monotone, continuous and coercive for $i = 1,2$. Thus, by the Minty-Browder's Theorem [4, Teorema V. 15] we have uniqueness of solution to the problems

$$(*) \qquad \begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = \lambda \left[\frac{h_1(x)}{(\varepsilon+|f|)^{\gamma_1}} + k_1(x)|f|^{\alpha_1}\right] \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

and

$$(**) \qquad \begin{cases} -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = \lambda \left[\frac{h_2(x)}{(\varepsilon+|g|)^{\gamma_2}} + k_2(x)|g|^{\alpha_2}\right] \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega \end{cases}$$

where $f \in W_0^{1,\beta_1}(\Omega)$ and $g \in W_0^{1,\beta_2}(\Omega)$.

Then, we can define the operator

$$\begin{aligned} T : \mathbb{R}^+ \times X &\to X \\ (\lambda, f, g) &\mapsto T(\lambda, f, g) = (u, v), \end{aligned}$$

where u, v are the unique solutions of problems $(*)$ and $(**)$ respectively.

Let us show that T is a compact operator. For, let $((\lambda_n, f_n, g_n)) \subset \mathbb{R}^+ \times X$ be a bounded sequence and set $T(\lambda_n, f_n, g_n) = (u_n, v_n)$. It follows from the definition of the operator T that (u_n, v_n) satisfies

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) = \lambda_n \left[\frac{h_1(x)}{(\varepsilon+|f_n|)^{\gamma_1}} + k_1(x)|f_n|^{\alpha_1}\right] \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v_n|^{p_2})|\nabla v_n|^{p_2-2}\nabla v_n) = \lambda_n \left[\frac{h_2(x)}{(\varepsilon+|g_n|)^{\gamma_2}} + k_2(x)|g_n|^{\alpha_2}\right] \text{ in } \Omega, \\ u_n = v_n = 0 \text{ on } \partial\Omega. \end{cases}$$

Thus,

$$\begin{aligned} &\int_\Omega a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n \nabla\phi dx \\ &\qquad = \lambda_n \int_\Omega \left[\frac{h_1(x)}{(\varepsilon+|f_n|)^{\gamma_1}} + k_1(x)|f_n|^{\alpha_1}\right]\phi dx, \ \forall\ \phi \in W_0^{1,\beta_1}(\Omega) \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega a_2(|\nabla v_n|^{p_2})|\nabla v_n|^{p_2-2}\nabla v_n \nabla\varphi dx \\ &\qquad = \lambda_n \int_\Omega \left[\frac{h_2(x)}{(\varepsilon+|g_n|)^{\gamma_2}} + k_2(x)|g_n|^{\alpha_2}\right]\varphi dx, \ \forall\ \varphi \in W_0^{1,\beta_2}(\Omega). \end{aligned}$$

Considering, in particular, $\phi = u_n$ and $\varphi = v_n$ in the above equations we get

$$\int_\Omega a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1}\ dx = \lambda_n \int_\Omega \left[\frac{h_1(x)u_n}{(\varepsilon+|f_n|)^{\gamma_1}} + k_1(x)|f_n|^{\alpha_1}u_n\right]\ dx$$

and

$$\int_\Omega a_2(|\nabla v_n|^{p_2})|\nabla v_n|^{p_2}\ dx = \lambda_n \int_\Omega \left[\frac{h_2(x)v_n}{(\varepsilon+|g_n|)^{\gamma_2}} + k_2(x)|g_n|^{\alpha_2}v_n\right]\ dx.$$

Since $\alpha_i, \gamma_i \in (0, p_i - 1)$ with $i = 1,2$, we use Hölder's inequality with $\frac{p_1}{\alpha_1}$ and $\frac{p_1}{p_1-\alpha_1}$, assumption (A_1) and Sobolev's embedding to obtain

$$\begin{aligned} &\xi_0 \int_\Omega |\nabla u_n|^{p_1}\ dx + H(\xi_3)\xi_2 \int_\Omega |\nabla u_n|^{q_1}\ dx \\ &\qquad \leqslant \lambda_n \Big(C_\varepsilon |h_1|_\infty \|u_n\|_{1,\beta_1} + |k_1|_\infty \|f_n\|_{1,\beta_1}^{\alpha_1} \|u_n\|_{1,\beta_1} \Big). \end{aligned} \tag{3.1}$$

Since (λ_n) and $(\|f_n\|_{1,\beta_1}^{\alpha_1})$ are bounded in $\mathbb{R}^+$, we have (u_n) is bounded in $W_0^{1,\beta_1}(\Omega)$. In the same way we may show that (v_n) is bounded in $W_0^{1,\beta_2}(\Omega)$.

Thus, up to a subsequence, we have

$$u_n \rightharpoonup u \ \text{ for some } \ u \in W_0^{1,\beta_1}(\Omega),$$

$$u_n \to u \ \text{ in } \ L^s(\Omega), \ \text{ for all } \ 1 \leqslant s < \beta_1^*,$$

$$v_n \rightharpoonup v \ \text{ for some } \ v \in W_0^{1,\beta_2}(\Omega),$$

$$v_n \to v \ \text{ in } \ L^s(\Omega), \ \text{ for all } \ 1 \leqslant s < \beta_2^*,$$

and

$$\lambda_n \to \lambda \geqslant 0.$$

Invoking the inequality

$$C|x-y|^{p_1} \leqslant \left\langle a_1(|x|^{p_1})|x|^{p_1-2}x - a_1(|y|^{p_1})|y|^{p_1-2}y, x-y \right\rangle,$$

for all $x,y \in \mathbb{R}^N$, we obtain

$$\begin{aligned}
C\|u_n-u\|_{1,p_1}^{p_1} \leqslant & \int_\Omega a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1}\,dx \\
& - \int_\Omega a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n \nabla u\,dx + o_n(1) \\
= & \lambda_n \int_\Omega \frac{h_1(x)u_n}{(|f_n|+\varepsilon)^{\gamma_1}}\,dx + \lambda_n \int_\Omega k_1(x)|f_n|^{\alpha_1} u_n\,dx \\
& - \lambda_n \int_\Omega \frac{h_1(x)u}{(|f_n|+\varepsilon)^{\gamma_1}}\,dx - \lambda_n \int_\Omega k_1(x)|f_n|^{\alpha_1} u\,dx = o_n(1),
\end{aligned}$$

where we conclude

$$\|u_n-u\|_{1,p_1} = o_n(1).$$

Following the same arguments treated above we conclude that

$$\|u_n-u\|_{1,q_1} = o_n(1).$$

Then, perhaps for a subsequence,

$$\|u_n-u\|_{1,\beta_1} = o_n(1).$$

Analogously, $v_n \to v$ in $W_0^{1,\beta_2}(\Omega)$.

This shows that T is a compact operator. Its continuity follows in a similar way. Because $T(0,u,v) = (0,0)$, we may use Proposition 3.1 to get a continuum $\mathscr{C} \subset \mathbb{R}^+ \times X$ of solutions of $T(\lambda,u,v) = (u,v)$, i.e., $T(\lambda,u,v)$ satisfies the equation

$$(\bar{P_\lambda}) \qquad \begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = \lambda\left[\frac{h_1(x)}{(\varepsilon+|v|)^{\gamma_1}} + k_1(x)|v|^{\alpha_1}\right] \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = \lambda\left[\frac{h_2(x)}{(\varepsilon+|u|)^{\gamma_2}} + k_2(x)|u|^{\alpha_2}\right] \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases}$$

So, if $(\lambda,0,0) \in \mathscr{C}$ then $\lambda = 0$, that is, $\mathscr{C} - \{(0,0,0)\}$ is constituted of nontrivial solutions.

Let us prove that for each $\lambda > 0$ there is $(\lambda,u,v) \in \mathscr{C}$. Suppose, on the contrary, that there is $\lambda^* > 0$ such that $(\lambda,u,v) \in \mathscr{C}$ implies $\lambda \leqslant \lambda^*$. Thus (λ,u,v) satisfies $(\bar{P_\lambda})$ and so

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1}\,dx = \lambda\int_\Omega\left[\frac{h_1(x)u}{(\varepsilon+|v|)^{\gamma_1}} + k_1(x)|v|^{\alpha_1}u\right]dx$$

and

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2}\,dx = \lambda\int_\Omega\left[\frac{h_2(x)v}{(\varepsilon+|u|)^{\gamma_2}} + k_2(x)|u|^{\alpha_2}v\right]dx.$$

From assumption (A_1) we have

$$\xi_0\int_\Omega|\nabla u_n|^{p_1}\,dx + H(\xi_3)\xi_2\int_\Omega|\nabla u_n|^{q_1}\,dx \leqslant \lambda\int_\Omega\left[\frac{h_1(x)u}{(\varepsilon+|v|)^{\gamma_1}} + k_1(x)|v|^{\alpha_1}u\right]dx. \qquad (3.2)$$

Moreover, using the Hölder inequality and Sobolev's embedding

$$\int_\Omega\frac{h_1(x)u}{(\varepsilon+|v|)^{\gamma_1}} \leqslant C_\varepsilon\|h\|_{L^{\beta_1}}\|u\|_{1,\beta_1}$$

and

$$\int_\Omega k_1(x)|v|^{\alpha_1}u \leqslant \|k_1\|_{L^{\frac{\beta_1}{\beta_1-\alpha_1-1}}}\|v\|_{1,\beta_2}^{\alpha_1}\|u\|_{1,\beta_1}.$$

So, from (3.2) we have

$$\xi_0\|u\|_{1,p_1}^{p_1} + H(\xi_3)\xi_2\|u\|_{1,q_1}^{q_1} \leqslant C_\varepsilon'\left[\|u\|_{1,\beta_1} + \|v\|_{1,\beta_2}^{\alpha_1}\|u\|_{1,\beta_1}\right]$$

and

$$\xi_0\|v\|_{1,p_2}^{p_2} + H(\xi_3)\xi_2\|v\|_{1,q_2}^{q_2} \leqslant C_\varepsilon'\left[\|v\|_{1,\beta_2} + \|v\|_{1,\beta_2}\|u\|_{1,\beta_1}^{\alpha_2}\right],$$

which yields

$$\xi_0\|u\|_{1,p_1}^{p_1} + H(\xi_3)\xi_2\|u\|_{1,q_1}^{q_1} + \xi_0\|v\|_{1,p_2}^{p_2} + H(\xi_3)\xi_2\|v\|_{1,q_2}^{q_2} \leqslant C'_\varepsilon\left[\|u\|_{1,\beta_1} + \|v\|_{1,\beta_2}^{\alpha_1}\|u\|_{1,\beta_1} + \|v\|_{1,\beta_2} + \|v\|_{1,\beta_2}\|u\|_{1,\beta_1}^{\alpha_2}\right].$$

If $\|(u,v)\| \to +\infty$, it is enough to consider the cases:

(i) $\|u\|_{1,p_1} \to +\infty$ and $\|u\|_{1,q_1} \to +\infty$;

(ii) $\|u\|_{1,p_1}$ is bounded and $\|u\|_{1,q_1} \to +\infty$.

From the analysis of these two cases we conclude that $\|u\|_{1,\beta_1}$ is bounded and analogously $\|v\|_{1,\beta_2}$ is bounded, and thus $\mathscr{C}$ is bounded, which is a contradiction. Making $\lambda = 1$ we have a solution $(u_\varepsilon, v_\varepsilon)$ to the problem (P_ε). By the maximum principle, $u_\varepsilon, v_\varepsilon$ are positive in Ω. □

4. Proof of the Main Result

Proof the Theorem 1 For each $\varepsilon = \frac{1}{n}$, let $u_{\frac{1}{n}} = u_n$ and $v_{\frac{1}{n}} = v_n$ be the solution of problem (P_n) obtained in the previous theorem, that is,

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) = \frac{h_1(x)}{(\frac{1}{n}+|v_n|)^{\gamma_1}} + k_1(x)|v_n|^{\alpha_1} \text{ in } \Omega, \\ -\operatorname{div}(a_2(|\nabla v_n|^{p_2})|\nabla v_n|^{p_2-2}\nabla v_n) = \frac{h_2(x)}{(\frac{1}{n}+|u_n|)^{\gamma_2}} + k_2(x)|u_n|^{\alpha_2} \text{ in } \Omega, \\ u_n = v_n = 0 \text{ on } \partial\Omega. \end{cases}$$

So, from the equation

$$-\operatorname{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) = \frac{h_1(x)}{(\frac{1}{n}+|v_n|)^{\gamma_1}} + k_1(x)|v_n|^{\alpha_1} \text{ in } \Omega$$

we have

$$\begin{aligned} -\operatorname{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) &\geqslant \frac{h_1(x)}{(1+|v_n|)^{\gamma_1}} + k_1(x)|v_n|^{\alpha_1} \text{ in } \Omega \\ &\geqslant \frac{h_0}{(1+|v_n|)^{\gamma_1}} + k_0|v_n|^{\alpha_1}, \end{aligned}$$

where

$$h_0 = \min_{x\in\overline{\Omega}} h_1(x) \quad \text{and} \quad k_0 = \min_{x\in\overline{\Omega}} k_1(x).$$

Since the function $t \mapsto \frac{h_0}{(1+t)^{\gamma_1}} + k_0 t^{\alpha_1}$ is continuous and bounded from below for $t \geqslant 0$ it attains a positive minimum m_1. So,

$$-\operatorname{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) \geqslant m_1.$$

Let z_1 the only positive solution of

$$\begin{cases} -\operatorname{div}(a_1(|\nabla z_1|^{p_1})|\nabla z_1|^{p_1-2}\nabla z_1) = m_1 \text{ in } \Omega, \\ z_1 = 0 \text{ on } \partial\Omega. \end{cases}$$

Hence,

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u_n|^{p_1})|\nabla u_n|^{p_1-2}\nabla u_n) \geqslant -\mathrm{div}(a_1(|\nabla z_1|^{p_1})|\nabla z_1|^{p_1-2}\nabla z_1) \text{ in } \Omega, \\ u_n = z_1 \text{ on } \partial\Omega. \end{cases}$$

By Lemma 2, $u_n \geqslant z_1 > 0$ in Ω, for all $n \in \mathbb{N}$. Similarly we prove that $v_n \geqslant z_2 > 0$ in Ω, for all $n \in \mathbb{N}$, where z_2 satisfies

$$\begin{cases} -\mathrm{div}(a_2(|\nabla z_2|^{p_2})|\nabla z_2|^{p_2-2}\nabla z_2) = m_2 \text{ in } \Omega, \\ z_2 = 0 \text{ on } \partial\Omega \end{cases}$$

and m_2 is the positive minimum of the function $t \mapsto \frac{h_0}{(1+t)^{\gamma_2}} + k_0 t^{\alpha_2}$.

Since

$$\begin{cases} -\mathrm{div}(a_i(|\nabla z_i|^{p_1})|\nabla z_i|^{p_1-2}\nabla z_i) = m_i \text{ in } \Omega, \\ z_i > 0 \text{ in } \Omega, \\ z_i = 0 \text{ on } \partial\Omega. \end{cases}$$

Moreover, by (A_1) we can use the results in the paper by Serrin [15] about the L^∞ estimates and the results by Di Benedetto [7] and Tolksdorff [18], for the C^1 regularity in order to conclude $z_i \in C^1(\overline{\Omega}) \bigcap W_0^{1,p}(\Omega)$. By Lemma 3, we have that

$$\frac{\partial z_i}{\partial \eta} < 0 \text{ on } \partial\Omega.$$

Thus, for each $x \in \Omega$ we get,

$$u_n(x) \geqslant z_1(x) > Cd_x > 0 \text{ and } v_n(x) \geqslant z_2(x) > Cd_x > 0,$$

where $d_x = dist\{x, \partial\Omega\}$ and C is a positive constant that does not depend on x.

Hence,

$$\int_\Omega \frac{h_1(x)u_n}{(\frac{1}{n}+v_n)^{\gamma_1}} \leqslant \int_\Omega \frac{h_1(x)u_n}{v_n^{\gamma_1}} \leqslant \int_\Omega \frac{h_1(x)u_n}{Cd_x^{\gamma_1}} \leqslant \overline{h} \int_\Omega \frac{u_n}{Cd_x{}^{\gamma_1}},$$

where $\overline{h} = \max_{x\in\overline{\Omega}} h_1(x)$. We now use the Hardy-Sobolev inequality to get $\frac{u_n}{d_x^{\gamma_1}} \in L^r$ and

$$\int_\Omega \frac{h_1(x)u_n}{(\frac{1}{n}+v_n)^{\gamma_1}} dx \leqslant C_2 \|u_n\|_{1,\beta_1}.$$

Since $\alpha_1 \in (0, p_1 - 1)$ and using Hölder's inequality with $\frac{p_1}{\alpha_1}$ and $\frac{p_1}{p_1-\alpha_1}$ and Sobolev embedding we have

$$\int_\Omega v_n^{\alpha_1} u_n dx \leqslant C' \|u_n\|_{1,\beta_1} \|v_n\|_{1,\beta_2}^{\alpha_1}.$$

Consequently, by using (3.2) with $\lambda = 1$ and the previous two estimates we get that $\|u_n\|_{1,\beta_1}, \|v_n\|_{1,\beta_2}$ are bounded. Thus, up to a subsequence, we have

$$u_n \rightharpoonup u \text{ for some } u \in W_0^{1,\beta_1}(\Omega),$$

$$u_n \to u \text{ in } L^s(\Omega), \text{ for all } 1 \leqslant s < {\beta_1}^*,$$

$$v_n \rightharpoonup v \text{ for some } v \in W_0^{1,\beta_2}(\Omega),$$

$$v_n \to v \text{ in } L^s(\Omega), \text{ for all } 1 \leqslant s < {\beta_2}^*,$$

$$u_n(x) \to u(x) \text{ a.e. in } \Omega,$$

$$v_n(x) \to v(x) \text{ a.e. in } \Omega.$$

From the above a.e. convergence it can be proved that

$$u_n \to u \geqslant z_1 > 0 \text{ in } W_{loc}^{1,\beta_1}(\Omega),$$

$$v_n \to v \geqslant z_2 > 0 \text{ in } W_{loc}^{1,\beta_2}(\Omega)$$

and the following equalities

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}|\nabla u|\nabla\phi = \int_\Omega \frac{h_1(x)\phi}{v^{\gamma_1}} + \int_\Omega k_1(x)v^{\alpha_1}\phi, \ \forall\phi \in C_0^\infty(\Omega),$$

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}|\nabla v|\nabla\varphi = \int_\Omega \frac{h_2(x)\varphi}{u^{\gamma_2}} + \int_\Omega k_2(x)u^{\alpha_2}\varphi, \ \forall\varphi \in C_0^\infty(\Omega).$$

Using a density argument we have that (u,v) is a solution for (P) which concludes the proof of the theorem. □

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