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EXISTENCE OF SOLUTION FOR BIHARMONIC SYSTEMS WITH INDEFINITE WEIGHTS

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Abstract. In this article we deal with the existence questions to the nonlinear biharmonic systems. Using theory of monotone operators, we show the existence of a unique weak solution to the weighted biharmonic systems. We also show the existence of a positive solution to weighted biharmonic systems in the unit ball in $\mathbb{R}^n$, using Leray Schauder fixed point theorem. In this study we allow sign-changing weights.

1. Introduction

We consider the following biharmonic system

$$\begin{cases} \Delta^2 u = a(x)g(x,v) \text{ in } \Omega, \\ \Delta^2 v = b(x)h(x,u) \text{ in } \Omega, \\ u = 0 = v \text{ on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain. The single fourth order nonlinear equations arise in various physical phenomenon such as study of travelling waves in suspension bridges [25], micro electro mechanical systems (MEMS) [29], radar imaging [2], bending behaviour of a thin elastic rectangular plate [33], geometric and functional design [4, 5, 31] etc. Consider the model problem

$$\Delta^2 u = \lambda g(u) \text{ in}\,\Omega, u = 0 = \frac{\partial u}{\partial \nu} \text{ on } \partial\Omega, \tag{1.2}$$

where $g\colon \mathbb{R} \to \mathbb{R}$. The Existence of a solution to (1.2) was proved by F. Tomi [34] when $\lambda = 1$, under monotonicity assumptions on nonlinearity g . Arioli et al. [3] proved the existence of a regular as well as a singular solution to (1.2) when $g(u) = e^u$ and Ω is the unit ball in $\mathbb{R}^n$, using comparison principle and method of monotone iteration. Y. Liu and Z. Wang [27] showed the existence of a non trivial solution to the following problem

$$\Delta^2 u = g(x,u) \text{ in}\,\Omega, u = 0 = \frac{\partial u}{\partial \nu} \text{ on } \partial\Omega,$$

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with asymptotic linear nonlinearity in an open, smooth and bounded subset of $\mathbb{R}^n$ for $n \geqslant 5$, using a variant of mountain pass theorem. H.C. Grunau [19] showed the existence of a solution to following problem

$$\begin{cases} Lu(x) + g(u) = f(x) \text{ in } \Omega, \\ u = 0 = D^\alpha u \text{ for } |\alpha| \leqslant m-1 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where L is a uniformly elliptic operator of order $2m$. He proved the existence of a classical solution using a local maximum principle when g satisfies sign condition, $tg(t) \geqslant 0$. H.C. Grunau and G. Sweers [20] proved the existence of a classical solution to the problem

$$\begin{cases} Lu(x) + g(x,u) = f(x) \text{ in } \Omega, \\ u = 0 = D^\alpha u \text{ for } |\alpha| \leqslant m-1 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where L is a uniformly elliptic operator of order $2m$, using a local maximum principle under some growth conditions on g. R. Wolfgang and T. Weth [37] considered (1.4) with $f = 0$ and

$$L = \left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right)^m + \sum_{1 \leqslant |\alpha| \leqslant 2m-1} b_\alpha(x) D^\alpha + b_0(x),$$

where

$$b_\alpha \in L^\infty(\Omega),\ a_{ij} \in C^{2m-2,\alpha}(\bar{\Omega}).$$

They proved the existence of a solution when nonlinearity g is superlinear at origin and

$$\lim_{s \to \infty} \frac{g(x,s)}{s^q} = h(x),\ \lim_{s \to -\infty} \frac{g(x,s)}{|s|^q} = k(x),$$

where $h,k \in C(\bar{\Omega})$ are positive and $q > 1$ is subcritical, using degree theory. N. Lam and G. Lu [24] discussed the following problem for polyharmonic operator

$$\begin{cases} (-\Delta)^m u = f(x,u) \text{ in } \Omega, \\ u = \nabla u = \ldots = \nabla^{m-1} u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^{2m}$ and f is of exponential growth. They proved the existence of a nontrivial solution to (1.5), using mountain pass theorem. Motivated by the above references, in the present article we establish the existence of a unique weak solution to (1.1), without monotonicity assumptions on g. To prove our result we follow the ideas introduced in Section 7.6 [16].

Next, we consider the singular biharmonic system

$$\begin{cases} \Delta^2 u = a(x) \dfrac{g(x,\, v)}{|x|^4} \ \text{ in } \ \Omega, \\ \Delta^2 v = b(x) \dfrac{h(x,\, u)}{|x|^4} \ \text{ in } \ \Omega, \\ u = 0 = v \ \text{ on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \ \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

where

$$0 \in \Omega \subseteq \mathbb{R}^n,\, n \geqslant 5 \text{ and } a,b \in L^\infty(\Omega).$$

Consider the model problem

$$\begin{cases} \Delta^2 u = \mu V(x) f(x,u) + g(x,u) \ \text{ in } \Omega, \\ u = 0 = \frac{\partial u}{\partial v} \ \text{ on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\mu > 0$ is a parameter and V is a singular potential. Y. Wang and Y. Shen [36] showed the existence of a nontrivial solution for critical exponent

$$\mu = \mu^* = \frac{n^2(n-4)^2}{16}$$

and

$$V(x) = \frac{1}{|x|^4},\ f(x,u) = q(x)u,\ g(x,u) = u,$$

using a variational method. For $\mu > \mu^*$ the operator becomes unbounded and (1.7) does not have any solution. For $\mu < \mu^*$ this problem was studied by H. Xiong and Y.T. Shen [39] and references cited therein. Y.T. Shen and Y.X. Yao [32] proved the existence of a solution to (1.7) in a new Hilbert space when

$$V(x) = \frac{1}{|x|^4},\ f(x,u) = u,\ g(x,u) = u,$$

using variational methods. N.T. Chung [12] established the existence of multiple solutions to (1.7), when

$$V(x) = \frac{1}{|x|^4},\ f(x,u) = a(x)u,\ g(x,u) = \lambda b(x)h(u),$$

where a is sign changing, b is nonnegative, $\lambda > 0$ is a parameter and h is sublinear, using a variant of three critical point theorem of G. Bonanno [7]. Yao et al. [40] proved existence and nonexistence of a nontrivial solution to (1.7), using Mountain Pass theorem and Hardy-Sobolev inequality, when

$$V(x) = \frac{1}{|x|^s},\ f(x,u) = u|u|^{q-2},\ g(x,u) = u|u|^{2_*-2},$$

where

$$2 \leqslant q \leqslant 2_*(s) = \frac{2(n-s)}{n-4} \leqslant 2_* = \frac{2n}{n-4},$$

and 2_* is the critical Sobolev exponent for the embedding $H^2(\mathbb{R}^n) \hookrightarrow L^{2_*}(\mathbb{R}^n)$. H. Xie and J. Wang [38] considered the following p-biharmonic equation

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \frac{\mu|u|^{p-2}}{|x|^{2p}} = f(x,u) \text{ in } \Omega, \\ u = 0 = \frac{\partial u}{\partial v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.8}$$

They proved the existence of a sign changing solution for $\mu < \mu_{np}$ using variational method, where

$$\mu_{np} = \left(\frac{n(p-1)(n-2p)}{p^2}\right)^p$$

is the best constant in Rellich inequality [28]. Motivated by the above references and [16], we use theory of monotone operators to prove the existence of a unique weak solution to the Problem (1.6).

Next, we consider the following biharmonic system

$$\begin{cases} \Delta^2 u = \lambda a(x) f(v) \text{ in } B, \\ \Delta^2 v = \lambda b(x) g(u) \text{ in } B, \\ u = 0 = v \text{ on } \partial B, \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on } \partial B, \end{cases} \tag{1.9}$$

where, B denotes the unit ball in $\mathbb{R}^n$ with boundary ∂B, λ is a positive parameter, $a, b : \Omega \to \mathbb{R}$ are sign changing, $f, g : [0, \infty) \to \mathbb{R}$ are continuous with $f(0) > 0, g(0) > 0$.

P.-L. Lions [26] studied the existence of a positive solution to the Dirichlet problem

$$\begin{aligned} -\Delta u &= \lambda a(x) f(u), && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

with weight function and nonlinearity satisfy $a \geqslant 0$, $f \geqslant 0$, respectively. Problem with indefinite weights was studied by Brown et al. [8, 9], Cac at al. [10], D.D. Hai [21] and references cited therein. Consider the following problem

$$\begin{aligned} -\Delta u &= \lambda a(x) f(u), && \text{in } \Omega, \\ -\Delta v &= \lambda b(x) g(v), && \text{in } \Omega, \\ u &= 0 = v && \text{on } \partial\Omega. \end{aligned} \tag{1.11}$$

R. Dalmasso [14] established the existence of a positive solution to (1.11) when,

$$a(x) = 1,\ b(x) = 1,\ \lambda = 1,$$

using Schauder fixed point theorem. D. G. de Figueiredo and B. Ru [15] proved the existence of a nontrivial solution to (1.11), when

$$a(x) = 1,\ b(x) = 1,\ \lambda = 1,\ f(s) = s^p,\ 0 < p < \frac{2}{n-2}$$

and g is superlinear, using variational methods. Clément et al. [13] proved the existence of a solution to (1.11) in an Orlicz-Sobolev space setting, when

$$a(x) = 1 = b(x) \text{ and } \lambda = 1.$$

D.D. Hai and R. Shivaji [22] proved the existence of a positive solution to (1.11) with $a(x) = b(x) = 1$, using method of sub and super solutions and Schauder fixed point theorem. They also proved the uniqueness of the solution [23], when Ω is a ball in $\mathbb{R}^n$.

J. Tyagi [35] established the existence of a solution to (1.11) with indefinite weights by method of monotone iteration and Schauder fixed point theorem. R. Chen [11] proved the existence of a solution to (1.11) with indefinite weights by Leray-Schauder fixed point theorem. Motivated by the above references, purpose of the present article is to study the existence of positive solutions to (1.9) by using Leray-Schauder fixed point theorem, following the ideas introduced by D.D. Hai [21].

In the present study we assume following hypotheses on nonlinearities and weights:

(H1) $f,g:[0,\infty)\to\mathbb{R}$ are continuous with $f(0)>0, g(0)>0$.

(H2) There exists $\mu_1>0$ such that

$$\int_B G(x,y)a^+(y)dy \geqslant (1+\mu_1)\int_B G(x,y)a^-(y)dy, \forall x\in B.$$

(H3) There exists $\mu_2>0$ such that

$$\int_B G(x,y)b^+(y)dy \geqslant (1+\mu_2)\int_B G(x,y)b^-(y)dy, \forall x\in B,$$

where $G(x,y)$ is the Green's function of Δ^2 associated with the Dirichlet boundary conditions. Here a^+,b^+ are positive parts of a and b respectively while a^- and b^- are the negative parts.

(H4) $a,b\in L^\infty(\Omega)$, g and h satisfy Carathéodory property on $\Omega\times\mathbb{R}$.

(H5) $|g(x,s_1)-g(x,s_2)|\leqslant c_1|s_1-s_2|$ and $|h(x,s_1)-h(x,s_2)|\leqslant c_2|s_1-s_2|$, where $c_1,c_2<\frac{\lambda_1}{M}$ and λ_1 denotes the principle eigenvalue of biharmonic operator with Dirichlet boundary conditions and $M=\max\{\|a\|_\infty, \|b\|_\infty\}$.

(H6) $|g(x,s_1)-g(x,s_2)|\leqslant c_1|s_1-s_2|$ and $|h(x,s_1)-h(x,s_2)|\leqslant c_2|s_1-s_2|$, where $c_1,c_2<\frac{1}{16M}n^2(n-4)^2$, where $M=\max\{\|a\|_\infty, \|b\|_\infty\}$.

(H7) For $n\leqslant 3$:

$$|g(x,s)|\leqslant r(x)+C(|s|) \text{ and } |h(x,s)|\leqslant q(x)+C(|s|),$$

where $q,r\in L^1(\Omega)$ and $C(t)$ is a nonnegative continuous function of the variable $t\geqslant 0$.

(H8) For $n=4$:

$$|g(x,s)|\leqslant r(x)+c_1|s|^{q-1} \text{ and } |h(x,s)|\leqslant q(x)+c_2|s|^{q-1},$$

where $q,r\in L^{\frac{q}{q-1}}(\Omega)$, $c_1,c_2>0, q\geqslant 1$ is arbitrary.

(H9) For $n\geqslant 5$:

$$|g(x,s)|\leqslant r(x)+c_1|s|^{\frac{n+4}{n-4}} \text{ and } |h(x,s)|\leqslant q(x)+c_2|s|^{\frac{n+4}{n-4}},$$

where $q,r\in L^{\frac{2n}{n+4}}, c_1,c_2>0$.

We now state the main results that we prove in the next sections:

THEOREM 1. *Let (H4), (H5) hold, then* (1.1) *has a unique weak solution* $(u,v) \in H_0^2(\Omega) \times H_0^2(\Omega)$, *provided*

(i) $n \leqslant 3$ *and* $(H7)$ *hold.*

(ii) $n = 4$ *and* $(H8)$ *hold.*

(iii) $n \geqslant 5$ *and* $(H9)$ *hold.*

THEOREM 2. *Let (H4), (H6) and (H9) hold then* (1.6) *has a unique weak solution.*

THEOREM 3. *Let* a, b *be non zero continuous functions on* $\bar{B}$ *and (H1)-(H3) hold. Then there exists a positive number* λ^*, *depending on weights* a, b *and nonlinearities* f, g *such that* (1.9) *has a positive solution for* $0 < \lambda < \lambda^*$.

REMARK 1. Theorem 1 holds for all space dimension under different growth conditions on nonlinearities g and h, while Theorem 2 for singular system holds only for $n \geqslant 5$.

REMARK 2. If we consider the polyharmonic system for $m \geqslant 2$

$$\begin{cases} (-\Delta)^m u = a(x) g(x, v) \ \text{in} \ \Omega, \\ (-\Delta)^m v = b(x) h(x, u) \ \text{in} \ \Omega, \\ D^\alpha u = 0 = D^\alpha v, \ |\alpha| \leqslant m-1, \ \text{on} \ \partial\Omega, \end{cases} \tag{1.12}$$

then under the hypotheses of Theorem 1, (1.12) has unique weak solution $(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega)$.

REMARK 3. If we consider the polyharmonic system, for $m \geqslant 2$

$$\begin{cases} (-\Delta)^m u = a(x) \dfrac{g(x, v)}{|x|^{2m}} \ \text{in} \ \Omega, \\ (-\Delta)^m v = b(x) \dfrac{h(x, u)}{|x|^{2m}} \ \text{in} \ \Omega, \\ D^\alpha u = 0 = D^\alpha v, \ |\alpha| \leqslant m-1, \ \text{on} \ \partial\Omega, \end{cases} \tag{1.13}$$

then under the hypotheses of Theorem 2, (1.13) has a unique weak solution $(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega)$.

REMARK 4. If we consider the polyharmonic system for $m \geqslant 2$

$$\begin{cases} (-\Delta)^m u = \lambda a(x) f(v) \ \text{in} \ B, \\ (-\Delta)^m v = \lambda b(x) g(u) \ \text{in} \ B, \\ D^\alpha u = 0 = D^\alpha v, \ |\alpha| \leqslant m-1, \ \text{on} \ \partial B, \end{cases} \tag{1.14}$$

then under the hypotheses of Theorem 3, there exists a positive number λ^*, depending on weights a, b and nonlinearities f, g such that (1.14) has a positive solution for $0 < \lambda < \lambda^*$.

We organize the article as follows: Section 2 deals with preliminaries. Section 3 deals with proofs of Theorem 1, 2, 3. In Section 4 we give some examples illustrating main results.

2. Preliminaries

In this section we recall some definitions and results. Throughout this article Ω denotes an open, smooth and bounded subset of $\mathbb{R}^n$. Let $m \geqslant 1$ be an integer. We denote by

$$H_0^m(\Omega) := \big\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leqslant m \text{ and } D^\alpha u = 0, \text{ on } \partial\Omega, \\ \text{for all } |\alpha| \leqslant m-1\big\}.$$

$H_0^m(\Omega)$ is a Hilbert space equipped with inner product

$$(u,v)_{H_0^m(\Omega)} = \begin{cases} \int_\Omega \Delta^k u \Delta^k v dx, & m = 2k; \\ \int_\Omega \nabla(\Delta^k u)\cdot\nabla(\Delta^k v) dx, & m = 2k+1. \end{cases}$$

and norm

$$\|u\|_{H_0^m(\Omega)} = \begin{cases} \|\Delta^k u\|_{L^2(\Omega)}, & m = 2k; \\ \|\nabla(\Delta^k u)\|_{L^2(\Omega)}, & m = 2k+1. \end{cases}$$

Throughout this article we denote $\|\cdot\|_{H_0^2(\Omega)}$ by $\|\cdot\|$ and $(\cdot,\cdot)_{H_0^2(\Omega)}$ by $(\cdot,\cdot)$.

DEFINITION 1. Principle eigenvalue $\lambda_{m,1}$ of polyharmonic operator with Dirichlet boundary conditions, is defined as

$$\lambda_{m,1} = \inf_{0 \neq v \in H_0^m(\Omega)} \frac{\|v\|^2_{H_0^m(\Omega)}}{\|v\|^2_{L^2(\Omega)}}. \tag{2.1}$$

For details, see [17].

From the above definition for $m = 2$, we get

$$\|v\|_{L^2(\Omega)} \leqslant \frac{1}{\sqrt{\lambda_{2,1}}}\|v\|_{H_0^2(\Omega)}, \ \forall v \in H_0^2(\Omega). \tag{2.2}$$

Throughout this article we denote $\lambda_{2,1}$ by λ_1.

DEFINITION 2. (Strongly Monotone Operator [16]) Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $T : H \to H$. T is said to be strongly monotone if there exists $c > 0$ such that

$$\langle Tu_1 - Tu_2, u_1 - u_2\rangle \geqslant c\|u_1 - u_2\|^2, \ \forall u_1, u_2 \in H.$$

REMARK 5. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, then $H\times H$ is also a Hilbert space with inner product defined by

$$\langle(u_1,v_1),(u_2,v_2)\rangle_{H\times H}=\langle u_1,u_2\rangle+\langle v_1,v_2\rangle.$$

DEFINITION 3. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $L:H\times H\to H\times H$. L is said to be strongly monotone if there exists $c>0$ such that $\forall u_1,u_2,v_1,v_2\in H$ we have

$$\langle L(u_1,v_1)-L(u_2,v_2),(u_1,v_1)-(u_2,v_2)\rangle_{H\times H}\geqslant c\left(\|u_1-u_2\|^2+\|v_1-v_2\|^2\right).$$

THEOREM 4. *Let H be a real Hilbert space and $S:H\longrightarrow H$ be continuous and strongly monotone operator. Then for any $h\in H$ the equation $Tu=h$ has a unique solution.*

Proof. For proof we refer to Corollary 5.3.9 [16].

LEMMA 1. (Hardy- Rellich Inequality) *For all $u\in H_0^2(\Omega)$ the Hardy Rellich inequality says,*

$$\int_\Omega|\Delta u|^2dx\geqslant\frac{n^2(n-4)^2}{16}\int_\Omega\frac{u^2}{|x|^4}dx,\ n\geqslant 5, \tag{2.3}$$

where $\frac{n^2(n-4)^2}{16}$ is the best constant in (2.3) and it is never achieved in any domain $\Omega\subseteq\mathbb{R}^n$.

For a proof, we refer to [30].
Next we state the Hardy-Rellich inequality for polyharmonic operators:

LEMMA 2. *Let $0<k<\frac{n}{2}$ be an integer and $u\in W_0^{k,2}(\Omega)$. Then if $k=2m$*

$$\int_\Omega(\Delta^m u)^2\,dx\geqslant\left(\prod_{l=1}^{2m}\frac{(n+4m-4l)^2}{4}\right)\int_\Omega\frac{u^2}{|x|^{4m}}dx. \tag{2.4}$$

If $k=2m+1$,

$$\int_\Omega|\nabla\Delta^m u|^2dx\geqslant\left(\prod_{l=1}^{2m+1}\frac{(n+4m+2-4l)^2}{4}\right)\int_\Omega\frac{u^2}{|x|^{4m+2}}dx. \tag{2.5}$$

For details, we refer to [1].

DEFINITION 4. (Carathéodory Property [16]) Let Ω be an open subset in $\mathbb{R}^n$. A function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is said to have Carathéodory property if
(i) for all $y\in\mathbb{R}$ the function $x\mapsto f(x,y)$ is Lebesgue measurable on Ω.
(ii) for $a.a.x\in\Omega$ the function $y\mapsto f(x,y)$ is continuous on $\mathbb{R}$.

Next we define the weak solution for (1.1). Multiply first equation in (1.1) by $\phi \in C_c^\infty(\Omega)$ and second equation by $\psi \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ denotes the set of infinitely differentiable function having compact support in Ω. Then on integrating using Green's theorem, we get

$$\int_\Omega \Delta u \Delta \phi \, dx = \int_\Omega a(x) g(x,v) \phi \, dx,$$
$$\int_\Omega \Delta v \Delta \psi \, dx = \int_\Omega b(x) h(x,u) \phi \, dx,$$

for all $\phi, \psi \in C_c^\infty(\Omega)$. Above equations can be rewritten as

$$\int_\Omega \Delta u \Delta \phi \, dx - \int_\Omega a(x) g(x,v) \phi \, dx = 0, \tag{2.6}$$
$$\int_\Omega \Delta v \Delta \psi \, dx - \int_\Omega b(x) h(x,u) \phi \, dx = 0. \tag{2.7}$$

Since $C_c^\infty(\Omega)$ is dense in $H_0^2(\Omega)$ therefore (2.6), (2.7) hold for each $\phi,\ \psi \in H_0^2(\Omega)$ respectively.

DEFINITION 5. (Weak Solution) $(u,v) \in H_0^2(\Omega) \times H_0^2(\Omega)$ is called weak solution of (1.1) if

$$\left(\int_\Omega (\Delta u \Delta \phi \; - a(x) g(x,v) \phi \;) \, dx, \int_\Omega (\Delta v \Delta \psi - b(x) h(x,u) \psi) \, dx \right) = 0,$$

for all $\phi,\ \psi \in H_0^2(\Omega)$.

Let us consider the boundary value problem

$$\begin{cases} (-\Delta)^m u = f \ \text{ in} \, \Omega \\ D^\alpha u = 0 \, \text{on} \, \partial\Omega, \ \text{for} \, |\alpha| \leqslant m-1, \end{cases} \tag{2.8}$$

where $f \in H^{-m}(\Omega)$, the dual space of $H_0^m(\Omega)$.

It is well known that the exact form of Green's function for $(-\Delta)^m$ is not easily determined, however T. Boggio explicitly calculated the Green's function [6, 17], when Ω is the unit ball in $\mathbb{R}^n$ (denoted by B). T. Boggio [6] proved the following:

LEMMA 3. *The Green's function for the Dirichlet problem* (2.8) *with* $\Omega = B$ *is positive and given by*

$$G_{m,n}(x,y) = k_{m,n} |x-y|^{2m-n} \int_1^{\left| |x| y - \frac{x}{|x|} \right| / |x-y|} (v^2-1)^{m-1} v^{1-n} dv. \tag{2.9}$$

The positive constant $k_{m,n}$ *is defined by*

$$k_{m,n} = \frac{1}{4^{m-1} n e_n ((m-1)!)^2}, \ e_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)},$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Using the Green's function $G_{m,n}(x,y)$, solution for (2.8) can be expressed as

$$u(x)=\int_{\Omega}G_{m,n}(x,y)f(y)\,dy. \tag{2.10}$$

For details we refer to p.p. 48[17].

Throughout this article we denote $G_{2,n}(x,y)$, the Green's function for biharmonic operator with Dirichlet boundary conditions by $G(x,y)$.

LEMMA 4. (Leray- Schauder fixed point theorem) *Let X be a Banach space and $T: X \longrightarrow X$ a completely continuous (continuous and compact) operator. Suppose that there exists a constant $M>0$, such that each solution $(x,\sigma)\in X\times[0,1]$ of*

$$x=\sigma Tx, \sigma\in[0,1], x\in X$$

satisfies $\|x\|_X \leqslant M$. Then T has a fixed point.

Proof. For a proof, we refer to p.p. 280 [18].

3. Proofs of Main Results

3.1. Proof of Theorem 1

Proof. We want to find $(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)$ such that for all $\phi,\ \psi\in H_0^2(\Omega)$,

$$\left(\int_{\Omega}(\Delta u\Delta\phi-a(x)g(x,v)\phi\ )\,dx,\int_{\Omega}(\Delta v\Delta\psi-b(x)h(x,u)\psi)\,dx\right)=0 \tag{3.1}$$

(3.1) can be rewritten as

$$(Tu-S_1v, Tv-S_2u)=0, \tag{3.2}$$

where

$$\begin{aligned}(Tu,\phi) &= \int_{\Omega}\Delta u\Delta\phi\,dx,\\ (S_1v,\psi) &= \int_{\Omega}a(x)g(x,v)\psi\,dx,\\ (S_2u,\psi) &= \int_{\Omega}b(x)h(x,u)\psi\,dx,\end{aligned}$$

for all $\phi,\psi\in H_0^2(\Omega)$. Since T is just identity operator on $H_0^2(\Omega)$, therefore, T is continuous. Under the hypotheses $(H7)-(H9)$, S_1 and S_2 are also continuous. Thus the operator

$$L(u,v)=(Tu-S_1v, Tv-S_2u) \tag{3.3}$$

is continuous on $H_0^2(\Omega)\times H_0^2(\Omega)$. Next we claim that L is strongly monotone. We set $H=H_0^2(\Omega)\times H_0^2(\Omega)$, then

$$\langle L(u_1,v_1)-L(u_2,v_2),(u_1,v_1)-(u_2,v_2)\rangle_H$$

$$
\begin{aligned}
&= \langle L(u_1,v_1)-L(u_2,v_2),(u_1-u_2,v_1-v_2)\rangle_H \\
&= \langle (Tu_1-S_1v_1,Tv_1-S_2u_1)-(Tu_2-S_1v_2,Tv_2-S_2u_2),(u_1-u_2,v_1-v_2)\rangle_H \\
&= (Tu_1-Tu_2,u_1-u_2)-(S_1v_1-S_1v_2,u_1-u_2)+(Tv_1-Tv_2,v_1-v_2) \\
&\quad -(S_2u_1-S_2u_2,v_1-v_2) \\
&= \|u_1-u_2\|^2+\|v_1-v_2\|^2-\int_\Omega a(x)(g(x,v_1)-g(x,v_2))(u_1-u_2)\,dx \\
&\quad -\int_\Omega b(x)(h(x,u_1)-h(x,u_2))(v_1-v_2)\,dx \\
&\geqslant \|u_1-u_2\|^2+\|v_1-v_2\|^2-\int_\Omega |a(x)|\,|g(x,v_1)-g(x,v_2)|\,|u_1-u_2|\,dx \\
&\quad -\int_\Omega |b(x)|\,|h(x,u_1)-h(x,u_2)|\,|v_1-v_2|\,dx \\
&\geqslant \|u_1-u_2\|^2+\|v_1-v_2\|^2-2Mc\int_\Omega |v_1-v_2|\,|u_1-u_2|\,dx, \\
&\quad \text{(by (H5), M=}\max\{\|a\|_\infty,\|b\|_\infty\}), \\
&\geqslant \|u_1-u_2\|^2+\|v_1-v_2\|^2-\frac{Mc}{\lambda_1}\left(\|u_1-u_2\|^2+\|v_1-v_2\|^2\right) \\
&= \left(1-\frac{Mc}{\lambda_1}\right)\left(\|u_1-u_2\|^2+\|v_1-v_2\|^2\right).
\end{aligned}
$$

Since by $(H5)$, $\left(1-\frac{Mc}{\lambda_1}\right)>0$, therefore L is strongly monotone. Thus by Theorem 4, Equation (3.2) has a unique solution, that is, (1.1) has a unique weak solution.

3.2. Proof of Remark 2

Proof. Define the operator $T\colon H_0^m(\Omega)\to H_0^m(\Omega)$ by

$$
(Tu,\phi)=\begin{cases}\int_\Omega \Delta^k u\Delta^k v\,dx, & m=2k;\\ \int_\Omega \nabla(\Delta^k u)\cdot\nabla(\Delta^k v)\,dx, & m=2k+1.\end{cases}
$$

Using this definition of T, rest of the proof is on the same lines as of the proof of Theorem 1. For sake of brevity, we omit the details.

3.3. Proof of Theorem 2

Proof. We want to find $(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)$ such that for all $\phi,\ \psi\in H_0^2(\Omega)$,

$$
\left(\int_\Omega\left(\Delta u\Delta\phi-a(x)\frac{g(x,v)}{|x|^4}\phi\right)dx,\int_\Omega\left(\Delta v\Delta\psi-b(x)\frac{h(x,u)}{|x|^4}\psi\right)dx\right)=0 \tag{3.4}
$$

This can be rewritten as

$$
(Tu-S_1v,Tv-S_2u)=0, \tag{3.5}
$$

where

$$
\begin{aligned}
(Tu,\phi) &= \int_\Omega \Delta u \Delta \phi \, dx, \\
(S_1 v,\psi) &= \int_\Omega a(x)\frac{g(x,v)}{|x|^4}\psi \, dx, \\
(S_2 u,\psi) &= \int_\Omega b(x)\frac{h(x,u)}{|x|^4}\psi \, dx,
\end{aligned}
$$

for all $\phi,\psi \in H_0^2(\Omega)$. Since T is just identity operator on $H_0^2(\Omega)$, therefore, T is continuous. Under the hypothesis $(H9)$, S_1 and S_2 are also continuous. Thus the operator

$$L(u,v) = (Tu - S_1 v, Tv - S_2 u) \tag{3.6}$$

is continuous operator on $H_0^2(\Omega)\times H_0^2(\Omega)$. Next we claim that L is strongly monotone. We set $H = H_0^2(\Omega)\times H_0^2(\Omega)$, then

$$
\begin{aligned}
&\langle L(u_1,v_1) - L(u_2,v_2), (u_1,v_1)-(u_2,v_2)\rangle_H \\
&= \langle L(u_1,v_1) - L(u_2,v_2), (u_1-u_2,v_1-v_2)\rangle_H \\
&= \langle (Tu_1 - S_1v_1, Tv_1 - S_2u_1) - (Tu_2 - S_1v_2, Tv_2 - S_2u_2), (u_1-u_2,v_1-v_2)\rangle_H \\
&= (Tu_1 - Tu_2, u_1-u_2) - (S_1v_1 - S_1v_2, u_1-u_2) + (Tv_1 - Tv_2, v_1-v_2) \\
&\qquad - (S_2u_1 - S_2u_2, v_1-v_2) \\
&= \|u_1-u_2\|^2 + \|v_1-v_2\|^2 - \int_\Omega a(x)(g(x,v_1)-g(x,v_2))\frac{(u_1-u_2)}{|x|^4}\,dx \\
&\qquad - \int_\Omega b(x)(h(x,u_1)-h(x,u_2))\frac{(v_1-v_2)}{|x|^4}\,dx \\
&\geqslant \|u_1-u_2\|^2 + \|v_1-v_2\|^2 - \int_\Omega |a(x)|\,|g(x,v_1)-g(x,v_2)|\,\frac{|u_1-u_2|}{|x|^4}\,dx \\
&\qquad - \int_\Omega |b(x)|\,|h(x,u_1)-h(x,u_2)|\,\frac{|v_1-v_2|}{|x|^4}\,dx \\
&\geqslant \|u_1-u_2\|^2 + \|v_1-v_2\|^2 - 2Mc\int_\Omega \frac{|v_1-v_2|}{|x|^2}\,\frac{|u_1-u_2|}{|x|^2}\,dx, \\
&\qquad (\text{by (H5), M=}\max\{\|a\|_\infty,\|b\|_\infty\}), \\
&\geqslant \|u_1-u_2\|^2 + \|v_1-v_2\|^2 - 2Mc\left(\int_\Omega \frac{|v_1-v_2|^2}{|x|^4}dx\right)^{\frac12}\left(\int_\Omega \frac{|u_1-u_2|^2}{|x|^4}\right)^{\frac12} dx \\
&\geqslant \|u_1-u_2\|^2 + \|v_1-v_2\|^2 - \frac{32Mc}{n^2(n-4)^2}\|v_1-v_2\|\|u_1-u_2\|, \ (\text{from}\,(2.3)) \\
&\geqslant \|u_1-u_2\|^2 + \|v_1-v_2\|^2 - \frac{16Mc}{n^2(n-4)^2}\left(\|u_1-u_2\|^2 + \|v_1-v_2\|^2\right) \\
&= \left(1 - \frac{16Mc}{n^2(n-4)^2}\right)\left(\|u_1-u_2\|^2 + \|v_1-v_2\|^2\right).
\end{aligned}
$$

Since by $(H6)$, $\left(1-\frac{16Mc}{n^2(n-4)^2}\right)>0$, therefore L is strongly monotone. Thus by Theorem 4, Equation (3.5) has a unique solution, that is, (1.6) has a unique weak solution.

3.4. Proof of Remark 3

Proof. On using Lemma 2, the proof is on the same lines as the proof of Theorem 2. For sake brevity, we omit the details.

3.5. Proof of Theorem 3

Let

$$C(\bar{B})\times C(\bar{B}) := \{(u,v) : u,v \text{ are continuous on } \bar{B}\}$$

with norm $\|(u,v)\|_\infty = \max_{x\in\bar{B}}(|u(x)|,|v(x)|)$. Then $(C(\bar{B})\times C(\bar{B}),\|(\cdot,\cdot)\|_\infty)$ is a Banach space.

We assume that

$$f(v)=f(0),\ v\leqslant 0;\ g(u)=g(0),\ u\leqslant 0.$$

To prove the main result we need the following lemma.

LEMMA 5. *Let $0<\delta<1$. Then there exists a positive number $\bar{\lambda}$ such that for $0<\lambda<\bar{\lambda}$,*

$$\begin{aligned}
\Delta^2 u &= \lambda a^+(x)f(v),\ \text{in} B,\\
\Delta^2 v &= \lambda b^+(x)g(u),\ \ \text{in} B,\\
u &= 0 = v \quad \text{on } \partial B,\\
\frac{\partial u}{\partial \nu} &= 0 = \frac{\partial v}{\partial \nu} \quad \text{on } \partial B,
\end{aligned} \tag{3.7}$$

(3.7) has a positive solution $(\tilde{u_\lambda},\tilde{v_\lambda})$ with $\|(\tilde{u_\lambda},\tilde{v_\lambda})\|\to 0$ as $\lambda\to 0$ and

$$\tilde{u}_\lambda(x)\geqslant \lambda\delta f(0)p_1(x),\ x\in B;\ \tilde{v}_\lambda(x)\geqslant \lambda\delta g(0)p_2(x),\ x\in B,$$

where

$$p_1(x)=\int_B G(x,y)a^+(y)\,dy \quad p_2(x)=\int_B G(x,y)b^+(y)\,dy$$

and $G(x,y)$ is Green's function for biharmonic operator with Dirichlet boundary conditions defined by (2.9).

Proof. The proof is adapted from [21]. Let $A: C(\bar{B})\times C(\bar{B}) \longrightarrow C(\bar{B})\times C(\bar{B})$ be defined by

$$A(u,v)(x) = (\lambda\int_B G(x,y)a^+(y)f(v)dy, \lambda\int_B G(x,y)b^+(y)g(u)dy)$$

then $A: C(\bar{B})\times C(\bar{B}) \longrightarrow C(\bar{B})\times C(\bar{B})$ is completely continuous and fixed points of A are solutions to (3.7). We shall apply Lemma 4 to prove that A has a fixed point for λ small.

Let $\varepsilon > 0$ be such that

$$f(x) \geqslant \delta f(0), g(x) \geqslant \delta g(0), 0 \leqslant x \leqslant \varepsilon, \tag{3.8}$$

this follows from the (H1). Now define

$$\tilde{f}(t) = \max_{s\in[0,t]} f(s),\ \tilde{g}(t) = \max_{s\in[0,t]} g(s). \tag{3.9}$$

Then $\tilde{f}$ and $\tilde{g}$ are continuous and non-decreasing. Let

$$\tilde{h}(t) = \max\{\tilde{f}(t), \tilde{g}(t)\} \tag{3.10}$$

Then $\tilde{h}$ is continuous.

Suppose that $\lambda < \dfrac{\varepsilon}{\|p\|_\infty \tilde{h}(\varepsilon)}$, then

$$\frac{\tilde{h}(\varepsilon)}{\varepsilon} < \frac{1}{2\lambda\|p\|_\infty}, \tag{3.11}$$

where $\|p\|_\infty = \max\{\|p_1\|_\infty, \|p_2\|_\infty\}$.

(H1),(3.9) and (3.10) imply that $\tilde{h}(0) > 0$, and therefore

$$\lim_{t\to 0+} \frac{\tilde{h}(t)}{t} = +\infty. \tag{3.12}$$

Inequality (3.11) and (3.12) imply that there exists $A_\lambda \in (0,\varepsilon)$ such that

$$\frac{\tilde{h}(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda\|p\|_\infty}. \tag{3.13}$$

Now let $(u,v) \in C(\bar{B})\times C(\bar{B})$ and $\theta \in (0,1)$ be such that $(u,v) = \theta A(u,v)$. Then we have

$$\begin{aligned}
\|(u,v)\| &= \max\{\|u\|_\infty, \|v\|_\infty\} \\
&\leqslant \max\{\lambda\|p_1\|_\infty \tilde{f}(\|v\|_\infty), \lambda\|p_2\|_\infty \tilde{g}(\|u\|_\infty)\} \\
&\leqslant \max\{\lambda\|p_1\|_\infty \tilde{f}(\|(u,v)\|), \lambda\|p_2\|_\infty \tilde{g}(\|(u,v)\|)\} \\
&\leqslant \max\{\lambda\|p\|_\infty \tilde{f}(\|(u,v)\|), \lambda\|p\|_\infty \tilde{g}(\|(u,v)\|)\} \\
&\leqslant \lambda\|p\|_\infty \tilde{h}(\|(u,v)\|),
\end{aligned}$$

which implies that $\|(u,v)\| \neq A_\lambda$. Note that $A_\lambda \to 0$ as $\lambda \to 0$. By Lemma 4, A has a fixed point $(\tilde{u}_\lambda, \tilde{v}_\lambda)$ with $\|(\tilde{u}_\lambda, \tilde{v}_\lambda)\| \leqslant A_\lambda < \varepsilon$. Consequently, from (3.8) it follows that

$$\tilde{u}_\lambda(x) \geqslant \lambda\delta f(0)p_1(x), x\in B;\ \tilde{v}_\lambda(x) \geqslant \lambda\delta g(0)p_2(x),\ x\in B. \tag{3.14}$$

This completes the proof.

Proof of Theorem 3. Let

$$q_1(x) = \int_B G(x,y)a^-(y)dy, \qquad q_2(x) = \int_B G(x,y)b^-(y)dy.$$

It follows from (H2), (H3) and Lemma 5, that there exists four positive constants $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in (0,1)$ such that

$$q_1(x)|f(s)| \leqslant \gamma_1 p_1(x) f(0), \quad \text{for } s \in [0,\alpha_1],\, x \in B,$$

$$q_2(x)|g(s)| \leqslant \gamma_2 p_2(x) g(0), \quad \text{for } s \in [0,\alpha_2],\, x \in B.$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$. Then

$$q_1(x)|f(s)| \leqslant \gamma_1 p_1(x) f(0), \quad \text{for } s \in [0,\alpha],\, x \in B, \tag{3.15}$$

$$q_2(x)|g(s)| \leqslant \gamma_2 p_2(x) g(0), \quad \text{for } s \in [0,\alpha],\, x \in B. \tag{3.16}$$

Fix $\delta \in (\gamma, 1)$, where $\gamma = \max\{\gamma_1, \gamma_2\}$. Let $h(0) = \max\{f(0), g(0)\}$ and let λ_1^*, λ_2^* be so small such that

$$\|\tilde{u}_\lambda\|_\infty + \lambda\delta h(0)\|p\|_\infty \leqslant \alpha, \quad \text{for}\, \lambda \in (0, \lambda_1^*),$$

$$\|\tilde{v}_\lambda\|_\infty + \lambda\delta h(0)\|p\|_\infty \leqslant \alpha, \quad \text{for}\, \lambda \in (0, \lambda_2^*),$$

where $\tilde{u}_\lambda$ and $\tilde{v}_\lambda$ are given by Lemma 5 and

$$|f(t) - f(s)| \leqslant f(0)\frac{\delta - \gamma_1}{2}, \quad \text{for } t,s \in [-\alpha,\alpha], |t-s| \leqslant \lambda_1^*\delta h(0)\|p\|_\infty,$$

$$|g(t) - g(s)| \leqslant g(0)\frac{\delta - \gamma_2}{2}, \quad \text{for } t,s \in [-\alpha,\alpha], |t-s| \leqslant \lambda_2^*\delta h(0)\|p\|_\infty.$$

Let $\lambda^* = \min\{\lambda_1^*, \lambda_2^*\}$. Then for $\lambda \in (0, \lambda^*)$, we have

$$\|\tilde{u}_\lambda\|_\infty + \lambda\delta h(0)\|p\|_\infty \leqslant \alpha, \quad \|\tilde{v}_\lambda\|_\infty + \lambda\delta h(0)\|p\|_\infty \leqslant \alpha \tag{3.17}$$

and for $t,s \in [-\alpha,\alpha], |t-s| \leqslant \lambda^*\delta h(0)\|p\|_\infty$, we have

$$|f(t) - f(s)| \leqslant f(0)\frac{\delta - \gamma_1}{2} \text{ and } |g(t) - g(s)| \leqslant g(0)\frac{\delta - \gamma_2}{2}. \tag{3.18}$$

Now, let $\lambda < \lambda^*$. We look for a solution (u_λ, v_λ) to (1.9) of the form $(\tilde{u}_\lambda + m_\lambda, \tilde{v}_\lambda + w_\lambda)$. Thus (m_λ, w_λ) solves the system

$$\begin{aligned}
\Delta^2 m_\lambda &= \lambda a^+(x)(f(\tilde{v}_\lambda + w_\lambda) - f(\tilde{v}_\lambda)) - \lambda a^-(x) f(\tilde{v}_\lambda + w_\lambda) \quad \text{in } B,\\
\Delta^2 w_\lambda &= \lambda b^+(x)(g(\tilde{u}_\lambda + m_\lambda) - g(\tilde{u}_\lambda)) - \lambda b^-(x) g(\tilde{u}_\lambda + m_\lambda) \quad \text{in } B,\\
m_\lambda &= 0 = w_\lambda \qquad \text{on } \partial B,
\end{aligned}$$

$$\frac{\partial m_\lambda}{\partial \nu} = 0 = \frac{\partial w_\lambda}{\partial \nu} \qquad \text{on } \partial B.$$

For each $(\psi, \phi) \in C(\bar{B}) \times C(\bar{B})$, let (m, w) be the solution of the system

$$\begin{aligned}
\Delta^2 m &= \lambda a^+(x)(f(\tilde{v}_\lambda + \phi) - f(\tilde{v}_\lambda)) - \lambda a^-(x) f(\tilde{v}_\lambda + \phi) \quad \text{in } B,\\
\Delta^2 w &= \lambda b^+(x)(g(\tilde{u}_\lambda + \psi) - g(\tilde{u}_\lambda)) - \lambda b^-(x) g(\tilde{u}_\lambda + \psi) \quad \text{in } B,\\
m &= 0 = w \ \ \text{on}\, \partial B,\\
\frac{\partial m}{\partial \nu} &= 0 = \frac{\partial w}{\partial \nu} \ \ \text{on}\, \partial B.
\end{aligned}$$

Then $A\colon C(\bar{B}) \times C(\bar{B}) \to C(\bar{B}) \times C(\bar{B})$ is completely continuous. Let $(m, w) \in C(\bar{B}) \times C(\bar{B})$ and $\theta \in (0,1)$ be such that

$$(m, w) = \theta A(m, w). \tag{3.19}$$

Then

$$\begin{aligned}
\Delta^2 m &= \lambda \theta a^+(x)(f(\tilde{v}_\lambda + w) - f(\tilde{v}_\lambda)) - \lambda \theta a^-(x) f(\tilde{v}_\lambda + w) \quad \text{in } B,\\
\Delta^2 w &= \lambda \theta b^+(x)(g(\tilde{u}_\lambda + m) - g(\tilde{u}_\lambda)) - \lambda \theta b^-(x) g(\tilde{u}_\lambda + m) \quad \text{in } B,\\
m &= 0 = w \ \ \text{on}\, \partial B,\\
\frac{\partial m}{\partial \nu} &= 0 = \frac{\partial w}{\partial \nu} \ \ \text{on}\, \partial B.
\end{aligned}$$

Now, we claim that $\|(m,w)\| \neq \lambda \delta h(0) \|p\|_\infty$.
Suppose to the contrary that $\|(m,w)\| = \lambda \delta h(0) \|p\|_\infty$, then there are three possible cases:
Case 1. $\|m\|_\infty = \|w\|_\infty = \lambda \delta h(0) \|p\|_\infty$. Then from (3.17), we have

$$\|\tilde{v}_\lambda + w\|_\infty \leqslant \|\tilde{v}_\lambda\|_\infty + \lambda \delta h(0) \|p\|_\infty \leqslant \alpha.$$

So $\|\tilde{v}_\lambda\|_\infty \leqslant \alpha$. Thus by (3.18), we obtain

$$|f(\tilde{v}_\lambda + w) - f(\tilde{v}_\lambda)| \leqslant f(0) \frac{\delta - \gamma_1}{2} \tag{3.20}$$

and on the other hand, (3.19) implies

$$\begin{aligned}
|m(x)| &\leqslant \lambda p_1(x) f(0) \frac{\delta - \gamma_1}{2} + \lambda \gamma_1 p_1(x) f(0)\\
&= \lambda p_1(x) f(0) \frac{\delta + \gamma_1}{2}\\
&< \lambda p_1(x) f(0) \delta\\
&\leqslant \lambda \delta h(0) \|p\|_\infty, \qquad \text{for} x \in B,
\end{aligned}$$

which implies that

$$\|m\|_\infty \leqslant \lambda \delta h(0) \|p\|_\infty,$$

which is a contradiction.
Case 2. $\|w\|_\infty < \|m\|_\infty = \lambda\delta h(0)\|p\|_\infty$. Then

$$\|\tilde{v}_\lambda + w\|_\infty < \|\tilde{v}_\lambda\|_\infty + \lambda\delta h(0)\|p\|_\infty \leqslant \alpha.$$

Thus

$$|f(\tilde{v}_\lambda + w) - f(\tilde{v}_\lambda)| \leqslant f(0)\frac{\delta - \gamma_1}{2}.$$

Now using the similar arguments as Case 1, we get

$$\|m\|_\infty < \lambda\delta h(0)\|p\|_\infty,$$

which is a contradiction.
Case 3. $\|m\|_\infty < \|w\|_\infty = \lambda\delta h(0)\|p\|_\infty$. Using arguments similar to Case 2, we obtain

$$\|w\|_\infty < \lambda\delta h(0)\|p\|_\infty,$$

which is a contradiction.
Thus the claim is proved.
By Lemma 4, A has a fixed point $(\tilde{m}_\lambda, \tilde{w}_\lambda)$ with

$$\|(\tilde{m}_\lambda, \tilde{w}_\lambda)\| \leqslant \lambda\delta h(0)\|p\|_\infty.$$

Using Lemma 5, we obtain

$$\begin{aligned} u_\lambda(x) &\geqslant \tilde{u}_\lambda(x) - |m(x)| \\ &\geqslant \lambda\delta p_1(x)f(0) - \lambda\frac{\delta+\gamma_1}{2}f(0)p_1(x) \\ &= \lambda\frac{\delta-\gamma_1}{2}f(0)p_1(x) \\ &> 0, \quad x \in B. \end{aligned}$$

Similarly, we can prove that $\tilde{v}_\lambda(x) > 0$, $x \in B$. This completes the proof.

3.6. Proof of Remark 4

Proof. The proof is on the same lines as of the proof of Theorem 3. For sake of brevity, we omit the details.

3.7. n × n SYSTEMS

Now we consider the following $n \times n$ system

$$\begin{aligned} \Delta^2 u_1 &= \lambda_1 a_1(x) f_1(u_2), && \text{in } B, \\ \Delta^2 u_2 &= \lambda_1 a_2(x) f_2(u_3), && \text{in } B, \\ &\vdots \end{aligned} \tag{3.21}$$

$$\Delta^2 u_{n-1} = \lambda_{n-1} a_{n-1}(x) f_{n-1}(u_n), \qquad \text{in } B,$$

$$\Delta^2 u_n = \lambda_n a_n(x) f_n(u_1), \qquad \text{in } B,$$

$$u_1 = u_2 = \ldots = u_n = 0, \qquad \text{on } \partial B,$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \ldots \frac{\partial u_n}{\partial \nu} = 0, \quad \text{on } \partial B,$$

where $a_i(x) \in L^\infty(B)$ $(i = 1,2,\ldots n)$ may be sign changing in B and $\lambda > 0$ is a parameter.

We assume the following hypotheses:
(H10) $f_i : [0,\infty) \longrightarrow \mathbb{R}$ which is continuous and $f_i(0) > 0\,(i = 1,2,\ldots,n)$.
(H11) $a_i(i = 1,2,\ldots,n)$ is continuous on B and there exist $k_i > 1\,(i = 1,2,\ldots,n)$, such that

$$\int_B G(x,y) a_i^+(y) dy \geqslant k_i \int_B G(x,y) a_i^-(y) dy \quad \forall x \in B,$$

where $G(x,y)$ is defined earlier.

Formulate the integral equation

$$(u_1,u_2,\ldots,u_n) = A(u_1,u_2,\ldots,u_n)$$

where $A : (C(\bar{B}))^n \longrightarrow (C(\bar{B}))^n$ is defined by

$$A(u_1,u_2,\ldots,u_n)(x) = \Big(\lambda \int_B G(x,y) a_1(y) f_1(u_2(y)) dy, \ldots, \\ \lambda \int_B G(x,y) a_n(y) f_n(u_1(y))\, dy\Big). \tag{3.22}$$

THEOREM 5. *Let (H10) and (H11) hold. Then there exists a positive number* λ^*, *depending on weights* $a_i, (i = 1,2,\ldots,n)$ *and nonlinearities* f_i, $(i = 1,2,\ldots,n)$ *such that* (3.21) *has a positive solution for* $0 < \lambda < \lambda^*$.

Proof. The proof is on the same lines as the proof of Theorem 3. For sake of brevity, we omit the details.

REMARK 6. Theorem 5 can be extended to $n \times n$ polyharmonic systems. The proof requires arguments similar to the proof of Theorem 3. We leave it as an exercise to an interested reader.

4. Examples

EXAMPLE 1. Consider $\Omega = B$, the unit ball in $\mathbb{R}^2$ and $a,b:\Omega \to \mathbb{R}$, defined by

$$a(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ -1, & |x| \geqslant \frac{1}{2} \end{cases} \text{ and } b(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| \geqslant \frac{1}{2} \end{cases}. \tag{4.1}$$

Define $g,h:\Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$g(x,s) = \begin{cases} cs^2 \sin x, & |s| < 1 \\ 0, & |s| \geqslant 1 \end{cases} \text{ and } h(x,s) = \begin{cases} cs^2 \cos x, & |s| < 1 \\ 0, & |s| \geqslant 1 \end{cases},$$

where $c < \lambda_1$, λ_1 is the principle eigenvalue of biharmonic operator defined by (2.1). Then

$$\begin{cases} \Delta^2 u = a(x)g(x,v) \text{ in } \Omega, \\ \Delta^2 v = b(x)h(x,u) \text{ in } \Omega, \\ u = 0 = v \text{ on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial\Omega, \end{cases} \tag{4.2}$$

has a unique weak solution.

Proof. Clearly, from (4.1), $\|a\|_\infty = 1 = \|b\|_\infty$. It is easy to verify that g,h satisfy (H1),(H2) and (H5), therefore, an application of Theorem 1 implies that the System (4.2) has a unique weak solution.

EXAMPLE 2. Let Ω be the unit ball in $\mathbb{R}^n$, $n \geqslant 5$. Consider the weights a,b defined by (4.1). Define $g,h:\Omega\times\mathbb{R}\to\mathbb{R}$ as follows

$$g(x,s) = \begin{cases} c\left(sinx + s^{\frac{2(n+4)}{n-4}}\right), & |s| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h(x,s) = \begin{cases} c\left(\cos x + s^{\frac{2(n+4)}{n-4}}\right), & |s| < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $c < \frac{1}{16}n^2(n-4)^2$. Then

$$\begin{cases} \Delta^2 u = a(x)\dfrac{g(x,v)}{|x|^4} \text{ in } \Omega, \\ \Delta^2 v = b(x)\dfrac{h(x,u)}{|x|^4} \text{ in } \Omega, \\ u = 0 = v \text{ on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial\Omega, \end{cases} \tag{4.3}$$

has a unique weak solution.

Proof. It is easy to verify that $(H4)$ holds. Clearly $M = \max\{\|a\|_\infty, \|b\|_\infty\} = 1$. Since $c < \frac{1}{16}n^2(n-4)^2$, therefore, $(H6)$ holds. From the definition of g and h, it is easy to see that (H9) holds. Therefore an application of Theorem 2 implies that the System (4.3) has a unique weak solution.

EXAMPLE 3. Let Ω denotes the unit ball in $\mathbb{R}^n$. Consider the functions a, b defined as

$$a(x) = |x|^2 \text{ and } b(x) = |x|^2, \ \forall x \in \Omega.$$

Define functions f and g as follows

$$f(x) = x^2 + 1, \ g(x) = x^2 + 1, \ \forall x \in [0, \infty).$$

Then there exists a positive number λ^* such that for $0 < \lambda < \lambda^*$ system

$$\begin{cases} \Delta^2 u = \lambda |x|^2 (v^2 + 1) \text{ in } \Omega, \\ \Delta^2 v = \lambda |x|^2 (u^2 + 1) \text{ in } \Omega, \\ u = 0 = v \ \text{ on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial\Omega, \end{cases} \tag{4.4}$$

has a positive solution.

Proof. From definition of f and g, it is clear that $(H1)$ holds. It is easy to verify that $(H2)$, $(H3)$ hold for any $\mu_1 > 0$ and $\mu_2 > 0$ respectively, therefore an application of Theorem 3 implies that there exists a positive number λ^* such that for $0 < \lambda < \lambda^*$ (4.4) has a positive solution.

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