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EXISTENCE OF A NON–TRIVIAL SOLUTION FOR NONLINEAR DIFFERENCE EQUATIONS

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Abstract. The existence of a non-trivial solution for a discrete non-linear Dirichlet problem involving p-Laplacian is investigated. The technical approach is based on a local minimum theorem for differentiable functionals due to Bonanno.

1. Introduction

There seems to be increasing interest in the existence of solutions to boundary value problems for finite difference equations with p-Laplacian operator, because of their applications in many fields. Results on this topic are usually achieved by using various fixed point theorems in cone; see [4, 18, 22, 23] and references therein for details. This kind of problems play a fundamental role in different fields of research, such as mechanical engineering, control systems, economics, computer science, physics, artificial or biological neural networks, cybernetics, ecology and many others. Important tools in the study of nonlinear difference equations are fixed point theorems and upper and lower solution techniques; see, for instance, [16, 19, 20] and references therein. It is well known that critical point theory is an important tool to deal with the problems for differential equations. More, recently, in [5, 6, 8, 9, 13, 14, 15, 17, 21] by starting from the seminal papers [1, 2], the existence and multiplicity of solutions for nonlinear discrete boundary value problems have been investigated by adopting variational methods.

The aim of this paper is to establish the existence of at least one non-trivial solution for the following discrete boundary-value problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) + q_k\phi_p(u(k)) = \lambda f(k,u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where T is a fixed positive integer, $[1,T]$ is the discrete interval $\{1,...,T\}$, $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $\lambda > 0$ is a parameter, $\Delta u(k) = u(k+1)-u(k)$ is the forward difference operator and $q_k \in \mathbb{R}_0^+$ for all $k\in[1,T]$, $\phi_p(s) = |s|^{p-2}s$ and $1 < p < +\infty$.

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More precisely, based on a local minimum theorem (Theorem 2) due to Bonanno [7], we ensure an exact interval of the parameter λ, in which the problem (1.1) admits at least one non-trivial solution.

As an example, here, we point out the following special case of our main results.

THEOREM 1. *Let* $h:[1,T]\to\mathbb{R}$ *be a positive function and* $g:\mathbb{R}\to\mathbb{R}$ *be a non-negative continuous function such that*

$$\lim_{t\to 0^+}\frac{g(t)}{t^{p-1}}=+\infty.$$

Then, for each

$$\lambda\in\left]0,\frac{2^p}{p(T+1)^{p-1}\sum_{k=1}^T h(k)}\sup_{c>0}\frac{c^p}{\int_0^c g(\xi)d\xi}\right[,$$

the problem

$$\begin{cases}-\Delta(\phi_p(\Delta u(k-1)))+q_k\phi_p(u(k))=\lambda h(k)g(u(k)), & k\in[1,T],\\ u(0)=u(T+1)=0,\end{cases}$$

admits at least one positive solution in the space $\{u:[0,T+1]\to\mathbb{R}:u(0)=u(T+1)=0\}$.

We refer to the paper [3, 10, 11, 12] in which Theorem 2 has been successfully employed to the existence of at least one non-trivial solution for two-point boundary value problems.

The rest of this paper is arranged as follows. In section 2, we recall some basic definitions and the main tool (Theorem 2) and in section 3, we provide our main results that contains several theorems and finally, we illustrate the results by giving examples.

2. Preliminaries

Our main tool is a local minimum theorem due to Bonanno (see [7, Theorem 5.1]), which is recalled below (see also [7, Proposition 2.1]). Such a result is more general than [24, Theorem 2.5] since the critical point, surely, is not zero.

First, for given Φ, $\Psi:X\to\mathbb{R}$, we defined the following functions

$$\beta(r_1,r_2)=\inf_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u)-\Psi(v)}{r_2-\Phi(v)} \tag{2.1}$$

and

$$\rho(r_1,r_2)=\sup_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\Psi(v)-\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{\Phi(v)-r_1} \tag{2.2}$$

for all $r_1,r_2\in\mathbb{R}$, with $r_1<r_2$.

THEOREM 2. *([7, Theorem 5.1]) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a sequentially weakly semicontinuous coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1,r_2) < \rho(r_1,r_2),$$

where β and ρ are given by (2.1) and (2.2). Then, for each $\lambda \in \Lambda =]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1,r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1,r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

In order to give the variational formulation of the problem (1.1), on T-dimensional Banach space

$W := \{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\}$, equipped with the norm

$$||u|| := \left\{ \sum_{k=1}^{T+1} |\Delta u(k-1)|^p + q_k |u(k)|^p \right\}^{1/p}.$$

In the sequel, we will use the following inequality

$$\max_{k\in[1,T]} |u(k)| \leqslant \frac{(T+1)^{(p-1)/p}}{2} ||u||, \tag{2.3}$$

for every $u \in W$. It immediately follows, for instance, from Lemma 2.2 of [21]. Moreover, put

$$\Phi(u) := \frac{||u||^p}{p},\ \Psi(u) := \sum_{k=1}^{T} F(k,u(k)) \text{ and } I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \tag{2.4}$$

for every $u \in W$, where $F(k,t) := \int_0^t f(k,\xi)d\xi$ for every $(k,t) \in [1,T] \times \mathbb{R}$. An easy computation ensures that I_λ turns out to be of class C^1 on W with

$$\begin{aligned} I'_\lambda(u)(v) &= \sum_{k=1}^{T+1} \left[\phi_p(\Delta u(k-1))\Delta v(k-1) + q_k|u(k)|^{p-2}u(k)v(k) - \lambda f(k,u(k))v(k)\right] \\ &= -\sum_{k=1}^{T} \left[\Delta(\phi_p(\Delta u(k-1))v(k) - q_k|u(k)|^{p-2}u(k)v(k) + \lambda f(k,u(k))v(k)\right] \end{aligned}$$

for all $u, v \in W$. It is clear that the critical points of I_λ are exactly the solutions of the problem (1.1).

3. Main Results

First, for a given non-negative constant c and a given positive constant d with

$$\frac{(2c)^p}{(T+1)^{p-1}} \neq d^p(2 + \sum_{k=1}^{T} q_k),$$

put

$$a_d(c) := \frac{\sum_{k=1}^{T} \max_{|\xi|\leqslant c} F(k,\xi) - \sum_{k=1}^{T} F(k,d)}{\frac{(2c)^p}{(T+1)^{p-1}} - d^p(2+\sum_{k=1}^{T} q_k)}.$$

We state our main result as follows.

THEOREM 3. *Assume that there exist a non-negative constant c_1 and two positive constants c_2 and d with $c_1 < \frac{d}{2}(T+1)^{\frac{p-1}{p}}(2+\sum_{k=1}^{T} q_k)^{\frac{1}{p}} < c_2$ such that*

(A1) $\quad a_d(c_2) < a_d(c_1).$

Then, for any $\lambda \in]\frac{1}{pa_d(c_1)}, \frac{1}{pa_d(c_2)}[$ the problem (1.1) has at least one non-trivial solution $u_0 \in W$ such that $2c_1(T+1)^{\frac{1-p}{p}} < ||u_0|| < 2c_2(T+1)^{\frac{1-p}{p}}$.

Proof. Our aim is to apply Theorem 2 to study the problem (1.1). To this end, take $X = W$, and put Φ, Ψ and I_λ as in (2.4). Put

$$\bar{v}(t) = \begin{cases} d, & k \in [1,T], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$r_1 = \frac{(2c_1)^p}{p(T+1)^{p-1}} \qquad \text{and} \qquad r_2 = \frac{(2c_2)^p}{p(T+1)^{p-1}}.$$

Clearly $\bar{v} \in W$, and $\Phi(\bar{v}) = \frac{d^p}{p}(2+\sum_{k=1}^{T} q_k)$ and

$$\Psi(\bar{v}) = \sum_{k=1}^{T} F(k,\bar{v}(k)) = \sum_{k=1}^{T} F(k,d).$$

Moreover, for all $u \in W$ such that $\Phi(u) < r_i$, $i = 1,2$, taking (2.3) into account, one has $\max_{k\in[1,T]} |u(k)| \leqslant c_i$, $i = 1,2$. Therefore,

$$\sup_{u\in\Phi^{-1}(-\infty,r_i)} \Psi(u) = \sup_{||u||<(pr_i)^{\frac{1}{p}}} \sum_{k=1}^{T} F(k,u(k)) \leqslant \sum_{k=1}^{T} \max_{|\xi|\leqslant c_i} F(k,u(k)), \quad i = 1,2.$$

Hence,

$$\begin{aligned} 0 \leqslant \beta(r_1,r_2) &\leqslant \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u) - \Psi(\bar{v})}{r_2 - \Phi(\bar{v})} \\ &\leqslant p\frac{\sum_{k=1}^{T} \max_{|\xi|\leqslant c_2} F(k,u(k)) - \sum_{k=1}^{T} F(k,d)}{\frac{(2c)^p}{(T+1)^{p-1}} - d^p(2+\sum_{k=1}^{T} q_k)} \\ &\leqslant pa_d(c_2). \end{aligned}$$

On the other hand, one has

$$\rho(r_1,r_2) \geqslant p\frac{\sum_{k=1}^{T} F(k,d) - \sum_{k=1}^{T} \max_{|\xi|\leqslant c_1} F(k,u(k))}{\frac{(2c)^p}{(T+1)^{p-1}} - d^p(2+\sum_{k=1}^{T} q_k)} = pa_d(c_1).$$

Hence, from Assumption (A1), we get $\beta(r_1,r_2) < \rho(r_1,r_2)$.

Therefore, owing to Theorem 2, for each $\lambda \in]\frac{1}{pa_d(c_1)}, \frac{1}{pa_d(c_2)}[$, the functional I_λ admits one critical point $u_0 \in W$ such that $r_1 < \Phi(u_0) < r_2$, that is

$$2c_1(T+1)^{\frac{1-p}{p}} < ||u_0|| < 2c_2(T+1)^{\frac{1-p}{p}}.$$

Hence, the proof is complete.

We now present an example to illustrate the result of Theorem 3.

EXAMPLE 1. Choose $c_1 = 1$, $c_2 = 10^4$, $d = 2$, $T = 9$, $p = 4$ and $\sum_{k=1}^{9} q_k = 14$. Clearly $a_2(c_1) = 0.0157$ and $a_2(c_2) = 0.00359$. We observe that all hypotheses of Theorem 3 are fulfilled. Hence, Theorem 3 follows that for every $\lambda \in]16,69[$ the problem

$$\begin{cases} -\Delta(\phi_4(\Delta u(k-1))) + q_k\phi_4(u(k)) = \frac{3}{4}\lambda(u(k))^2(\ln\frac{k+1}{k}), & k \in [1,9], \\ u(0) = u(10) = 0, \end{cases}$$

has at least one non-trivial solution u_0 such that $2\times 10^{-\frac{3}{4}} < ||u_0|| < 2\times 10^{\frac{13}{4}}$.

Here we point out an immediate consequence of Theorem 3 as follows.

THEOREM 4. *Assume that there exist two positive constants c and d with $\frac{d}{2}(T+1)^{\frac{p-1}{p}}(2+\sum_{k=1}^{T} q_k)^{\frac{1}{p}} < c$ such that*

(A2) $F(k,t) \geqslant 0$ for all $(k,t) \in [1,T]\times[0,d]$,
(A3) $\frac{\sum_{k=1}^{T}\max_{|\xi|\leqslant c} F(k,\xi)}{c^p} < \frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)}\frac{\sum_{k=1}^{T} F(k,d)}{d^p}$.

Then, for each

$$\lambda \in \left]\frac{d^p(2+\sum_{k=1}^{T} q_k)}{p\sum_{k=1}^{T} F(k,d)}, \frac{(2c)^p}{p(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leqslant c} F(k,\xi)}\right[$$

the problem (1.1) has at least one non-trivial solution $u_0 \in W$ such that $||u_0||_\infty \leqslant c$.

Proof. Applying Theorem 3 we have the conclusion, by picking $c_1 = 0$ and $c_2 = c$. Indeed, owing to our assumptions, one has

$$a_d(c) < \frac{\left(1 - \frac{(T+1)^{p-1}d^p(2+\sum_{k=1}^{T} q_k)}{(2c)^p}\right)\sum_{k=1}^{T}\max_{|\xi|\leqslant c} F(k,\xi)}{\frac{(2c)^p}{(T+1)^{p-1}} - d^p(2+\sum_{k=1}^{T} q_k)}$$

$$
\begin{aligned}
&= \frac{(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leqslant c}F(k,\xi)}{(2c)^p} \\
&< \frac{1}{(2+\sum_{k=1}^{T}q_k)}\frac{\sum_{k=1}^{T}F(k,d)}{d^p} \\
&= a_d(0).
\end{aligned}
$$

In particular, one has

$$
a_d(c) < \frac{(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leqslant c}F(k,\xi)}{(2c)^p}.
$$

Hence, Theorem 3, taking (2.3) into account, ensures the conclusion.

REMARK 1. If f is non-negative, then, thanks to [8, Theorem 2.2], the ensured solution u_0 in the conclusions of Theorems 3 and 4 is positive.

As a special case of the problem (1.1), we consider the following problem

$$
\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) + q_k\phi_p(u(k)) = \lambda\alpha(k)g(u(k)), & k\in[1,T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{3.1}
$$

where $\alpha : [1,T] \to \mathbb{R}$ and $g \in C(\mathbb{R},\mathbb{R})$ are nonnegative. Put $G(t) = \int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$. For a given non-negative constant c and a given positive constant d with

$$
\frac{(2c)^p}{(T+1)^{p-1}} \neq d^p(2+\sum_{k=1}^{T}q_k),
$$

put

$$
b_d(c) := \frac{G(c)-G(d)}{\frac{(2c)^p}{(T+1)^{p-1}} - d^p(2+\sum_{k=1}^{T}q_k)}
$$

Then, taking into account that in this case,

$$
\max_{|\xi|\leqslant c}\sum_{k=1}^{T}F(k,\xi) = G(c)\sum_{k=1}^{T}\alpha(k),
$$

Theorems 3 and 4 take the following forms, respectively.

THEOREM 5. *Assume that there exist a non-negative constant c_1 and two positive constants c_2 and d with $c_1 < \frac{d}{2}(T+1)^{\frac{p-1}{p}}(2+\sum_{k=1}^{T}q_k)^{\frac{1}{p}} < c_2$ such that*

(B1) $b_d(c_2) < b_d(c_1)$.

Then, for any

$$
\lambda \in \left]\frac{1}{p(\sum_{k=1}^{T}\alpha(k))b_d(c_1)}, \frac{1}{p(\sum_{k=1}^{T}\alpha(k))b_d(c_2)}\right[
$$

the problem (3.1) *has at least one positive solution* $u_0 \in W$ *such that* $2c_1(T+1)^{\frac{1-p}{p}} < ||u_0|| < 2c_2(T+1)^{\frac{1-p}{p}}$.

THEOREM 6. *Assume that there exist two positive constants* c *and* d *with* $\frac{d}{2}(T+1)^{\frac{p-1}{p}}(2+\sum_{k=1}^{T} q_k)^{\frac{1}{p}} < c$ *such that*

(B2) $\frac{G(c)}{c^p} < \frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)} \frac{G(d)}{d^p}$.

Then, for each

$$\lambda \in \left] \frac{d^p(2+\sum_{k=1}^{T} q_k)}{(p\sum_{k=1}^{T}\alpha(k))G(d)}, \frac{(2c)^p}{(p(T+1)^{p-1}\sum_{k=1}^{T}\alpha(k))G(c)} \right[$$

the problem (3.1) *has at least one positive solution* $u_0 \in W$ *such that* $||u_0||_\infty \leqslant c$.

We now prove the theorem in the introduction.

POOF OF THEOREM 1: For fixed λ as in the conclusion, there exists positive constant c such that

$$\lambda < \frac{2^p}{p(T+1)^{p-1}\sum_{k=1}^{T} h(k)} \frac{c^p}{\int_0^c g(\xi)d\xi}.$$

Moreover,

$$\text{the condition} \quad \lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty \quad \text{implies} \quad \lim_{t\to 0^+} \frac{\int_0^t g(\xi)d\xi}{t^p} = +\infty.$$

Therefore, a positive constant d satisfying $d < 2(T+1)^{-\frac{p-1}{p}}(2+\sum_{k=1}^{T} q_k)^{-\frac{1}{p}} c$ can be chosen such that

$$\frac{(2+\sum_{k=1}^{T} q_k)}{p\lambda\sum_{k=1}^{T} h(k)} < \frac{\int_0^d g(\xi)d\xi}{d^p}.$$

Hence, the conclusion follows from Theorem 3 with $c_1 = 0$, $c_2 = c$ and $f(k,t) = h(k)g(t)$ for every $(k,t) \in [1,T]\times\mathbb{R}$. □

REMARK 2. For fixed γ put

$$\lambda_\gamma := \frac{2^p}{p(T+1)^{p-1}\sum_{k=1}^{T} h(k)} \sup_{c\in]0,\gamma[} \frac{c^p}{\int_0^c g(\xi)d\xi}.$$

The result of Theorem 1 for every $\lambda \in]0,\lambda_\gamma[$ holds with $||u_0||_\infty < \gamma$ where u_0 is the ensured positive solution in W (see [10, Remark 4.3]).

Finally, we present the following example to illustrate the result of Theorem 1.

EXAMPLE 2. Consider the problem

$$\begin{cases} -\Delta(\phi_3(\Delta u(k-1))) + \phi_3(u(k)) = \lambda e^k(1+e^{-u^+}(u^+)^2(3-u^+)) & k \in [1,5], \\ u(0) = u(6) = 0, \end{cases} \tag{3.2}$$

where $u^+ := \max\{u,0\}$. Let $h(k) = e^k$ and $g(t) = 1 + e^{-t^+}(t^+)^2(3-t^+)$ for all $k \in [0,5]$ and $t \in \mathbb{R}$, where $t^+ := \max\{t,0\}$. It is clear that $\lim_{t\to 0^+} \frac{g(t)}{t^2} = +\infty$. Pick $\gamma = 1$. Hence, taking Remark 2 into account, by using Theorem 1, for every $\lambda \in \left]0, \frac{2}{27(\sum_{k=1}^5 e^k)} \frac{e}{e+1}\right[$, the problem (3.2) has at least one positive solution $u_0 \in \{u : [0,6] \to \mathbb{R} : u(0) = u(6) = 0\}$ such that $||u_0||_\infty < 1$.

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