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ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF nTH–ORDER HALF–LINEAR DYNAMIC EQUATIONS

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Abstract. In this paper, we study the nth-order half-linear dynamic equations

$$(x^{[n-1]})^{\Delta}(t)+p(t)\,\phi_{\alpha[1,n-1]}(x(g(t)))=0$$

on an above-unbounded time scale $\mathbb{T}$, where $n \geqslant 2$,

$$x^{[i]}(t) := r_i(t)\phi_{\alpha_i}\left[\left(x^{[i-1]}\right)^{\Delta}(t)\right],\ i=1,\dots,n-1,\ \text{ with } x^{[0]}=x,$$

$\phi_\beta(u) := |u|^{\beta}\operatorname{sgn} u$, and $\alpha[i,j] := \alpha_i \cdots \alpha_j$. Criteria are obtained for the asymptotics and oscillation of solutions for both even and odd order cases. This work extends several known results in the literature on second-order, third-order, and higher-order linear and half-linear dynamic equations.

1. Introduction

In this paper we consider the asymptotic behavior of solutions of the nth-order half-linear dynamic equation

$$(x^{[n-1]})^{\Delta}(t)+p(t)\,\phi_{\alpha[1,n-1]}(x(g(t)))=0 \tag{1.1}$$

on an above-unbounded time scale $\mathbb{T}$, where

(i) $n \geqslant 2$ is an integer, and $x^{[i]}(t) := r_i(t)\phi_{\alpha_i}\left[\left(x^{[i-1]}\right)^{\Delta}(t)\right]$, $i=1,2,\dots,n-1$, $t\in\mathbb{T}$, with $x^{[0]}=x$;

(ii) $\phi_\beta(u) := |u|^{\beta}\operatorname{sgn} u$ for $\beta>0$; and

(iii) $\alpha[i,j] := \alpha_i\cdots\alpha_j$ for $1 \leqslant i \leqslant j \leqslant n-1$.

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Without loss of generality we assume $0 \in \mathbb{T}$. For $A \subset \mathbb{T}$ and $B \subset \mathbb{R}$, we denote by $C_{rd}(A,B)$ the space of right-dense continuous functions from A to B; and by $C^1_{rd}(A,B)$ the set of functions in $C_{rd}(A,B)$ with right-dense continuous Δ-derivatives. Throughout this paper we make the following assumptions:

(iv) $\alpha_i > 0, i = 1,2,\ldots,n-1$, are constants and $r_i \in C_{rd}([0,\infty)_{\mathbb{T}},(0,\infty))$ for $i = 1,2,...,n-1$, such that

$$\int_0^\infty r_i^{-1/\alpha_i}(s)\Delta s = \infty, \; i = 1,2,...,n-1; \tag{1.2}$$

(v) $p \in C_{rd}([0,\infty)_{\mathbb{T}},[0,\infty))$ such that $p \not\equiv 0$;

(vi) $g \in C_{rd}(\mathbb{T},\mathbb{T})$ is nondecreasing such that $g(t) \leqslant t$ and $\lim_{t\to\infty} g(t) = \infty$.

By a solution of Eq. (1.1) we mean a nontrivial real-valued function

$$x \in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R}) \quad \text{for some } T_x \geqslant 0$$

such that $x^{[i]} \in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, $i = 1,2,...,n-1$ and $x(t)$ satisfies Eq. (1.1) on $[T_x,\infty)_{\mathbb{T}}$.

Note that if $x(t)$ is a solution of Eq. (1.1), then $cx(t)$ is also a solution of Eq. (1.1) for any $c \in \mathbb{R}$. Hence Eq. (1.1) is a half-linear equation.

In the last few years, there has been an increasing interest in the oscillation and nonoscillation of solutions of various dynamic equations. A large number of papers were devoted to second order linear and nonlinear dynamic equations on time scales. For example, Agarwal, Bohner, and Saker [1] discussed the linear delay dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x(g(t)) = 0;$$

Erbe, Peterson, and Saker [15], Saker [45], Agarwal, Regan, and Saker [2], and Hassan [33] investigated the pair of half-linear dynamic equations

$$(r(t)(x^\Delta(t))^{\alpha_1})^\Delta + p(t)x^{\alpha_1}(t) = 0$$

and

$$(r(t)(x^\Delta(t))^{\alpha_1})^\Delta + p(t)x^{\alpha_1}(\sigma(t)) = 0;$$

Erbe, Hassan, Peterson, and Saker [13] and [14] studied the half-linear delay dynamic equation

$$(r(t)(x^\Delta(t))^{\alpha_1})^\Delta + p(t)x^{\alpha_1}(g(t)) = 0$$

with $g(t) \leqslant t$ and

$$r^\Delta(t) \geqslant 0 \qquad \text{and} \qquad \int_{t_0}^\infty g^{\alpha_1}(t)p(t)\Delta t = \infty; \tag{1.3}$$

and Hassan [34] extended their results to the half-linear advanced dynamic equation (1) with $g(t) \geqslant t$.

Erbe, Peterson, and Saker [17, 18] and Yu and Wang [48] also derived oscillation criteria for the third order dynamic equations

$$\left(r_2(t)\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+p(t)x(t)=0,$$

$$\left(r_2(t)\left[(r_1(t)x^{\Delta}(t))^{\Delta}\right]^{\alpha_2}\right)^{\Delta}+p(t)x^{\gamma}(t)=0,$$

and

$$\left(r_2(t)\left[(r_1(t)\left(x^{\Delta}(t)\right)^{\alpha_1})^{\Delta}\right]^{\alpha_2}\right)^{\Delta}+p(t)x(t)=0;$$

and their work were further extended by Hassan [32] and Erbe, Hassan, and Peterson [19] to the equation with delay

$$\left(r_2(t)\left[(r_1(t)x^{\Delta}(t))^{\Delta}\right]^{\gamma}\right)^{\Delta}+p(t)x^{\gamma}(g(t))=0.$$

Also, Han, Li, Sun, and Zhang [31] discussed the third order delay dynamic equation

$$\left(r_2(t)\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+p(t)x(g(t))=0,$$

where $g(t)\leqslant t$ and

$$r_1^{\Delta}(t)\leqslant 0 \qquad \text{and} \qquad \int_{t_0}^{\infty} g(t)p(t)\Delta t=\infty. \tag{1.4}$$

Higher order dynamic equations have been studied by many authors. For instance, Grace, Agarwal, and Zafer [27] established oscillation and comparison criteria for the even order nonlinear dynamic equation

$$x^{\Delta^{2n}}(t)+p(t)(x^{\sigma}(t))^{\gamma}=0,$$

and Grace [29] developed oscillation criteria for the even order dynamic equation

$$\left[r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta}+p(t)(x^{\sigma}(t))^{\gamma}=0.$$

For more results on higher order dynamic equations, we refer the reader to the papers [10, 23, 44, 27, 46, 41, 29, 22, 28].

The purpose of this paper is to establish the asymptotic and oscillatory behavior of solutions of the nth order half-linear dynamic equation (1.1) without assuming the conditions (1.3) and (1.4). The results in this paper extend many results in the literature on the oscillation for second order, third order, and higher order linear and half-linear dynamic equations.

2. Asymptotic behavior

In this section, we discuss the asymptotic behavior of the solutions of Eq. (1.1).

LEMMA 1. *Assume Eq. (1.1) has an eventually positive solution $x(t)$. Then there exists an integer $m \in \{0,\ldots,n-1\}$ with $m+n$ odd such that*

$$x^{[k]}(t) > 0 \quad for \quad k = 0,1,\ldots,m \tag{2.1}$$

and

$$(-1)^{m+k} x^{[k]}(t) > 0 \quad for \quad k = m+1, m+2, ..., n \tag{2.2}$$

eventually.

Proof. Since $x(t)$ is an eventually positive solution of Eq. (1.1), there is a $t_0 \geqslant 0$ such that $x(g(t)) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. From (1.1), we have that for $t \in [t_0,\infty)_{\mathbb{T}}$,

$$\left(x^{[n-1]}\right)^{\Delta}(t) = -p(t)\,\phi_{\alpha[1,n-1]}(x(g(t))) < 0. \tag{2.3}$$

This implies that $x^{[i]}(t)$, $i = 1,2,...,n-1$, are eventually monotone and hence are of one sign. There are two possibilities:

(a) $x^{[k]}(t)$ and $x^{[k-1]}(t)$ have opposite signs eventually for $k = 1,2,\ldots,n$;

(b) there exists a largest $m \in \{1,2,\ldots,n-1\}$ such that $x^{[m]}(t)x^{[m-1]}(t) > 0$ eventually.

If (a) holds, then (2.1) and (2.2) hold with $m = 0$.

Assume (b) holds with $x^{[m]}(t) < 0$ and $x^{[m-1]}(t) < 0$ for $t \geqslant t_1$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then

$$\begin{aligned} x^{[m-2]}(t) &= x^{[m-2]}(t_1) + \int_{t_1}^{t} \phi^{-1}_{\alpha_{m-1}}\left[x^{[m-1]}(s)\right] r_{m-1}^{-1/\alpha_{m-1}}(s)\,\Delta s \\ &< x^{[m-2]}(t_1) + \phi^{-1}_{\alpha_{m-1}}\left[x^{[m-1]}(t_1)\right] \int_{t_1}^{t} r_{m-1}^{-1/\alpha_{m-1}}(s)\,\Delta s. \end{aligned}$$

By (1.2) with $i = m-1$, $\lim_{t\to\infty} x^{[m-2]}(t) = -\infty$. Hence $x^{[m-2]}(t) < 0$ eventually. By the same reasoning we see that $x^{[k]}(t) < 0$ eventually for $k = m-2, m-3\ldots,0$. This contradicts the assumption that $x(t)$ is eventually positive.

Assume (b) holds with $x^{[m]}(t) > 0$ and $x^{[m-1]}(t) > 0$ eventually. By (2.3) we find that $m+n$ must be an odd number. With a similar argument to the above, we see that $x^{[k]}(t) > 0$ eventually for $k = m-2, m-3\ldots,0$. Therefore, (2.1) and (2.2) hold with this m. □

For further discussion, we introduce the following notation: For any $t,s \in \mathbb{T}$, define

$$R_i(t,s) := \int_s^t r_i^{-1/\alpha_i}(s)\Delta s, \; i = 1,2,...,n-1; \tag{2.4}$$

and for a fixed $m \in \{0,\ldots,n-1\}$, define the functions $\bar{R}_{m,i}(s,t)$, $i = 1,2\ldots,m$, and $p_i(t)$, $i = 1,\ldots,n$, by the following recurrence formulas:

$$\bar{R}_{m,i}(t,s) := \begin{cases} \left[\dfrac{1}{r_i(t)} \int_s^t \bar{R}_{m,i+1}(\tau,s)\Delta\tau\right]^{1/\alpha_i}, & i = 1,\ldots,m-1, \\ \left[\dfrac{1}{r_i(t)}\right]^{1/\alpha_i}, & i = m; \end{cases} \tag{2.5}$$

and

$$p_i(t) := \begin{cases} \left[\dfrac{1}{r_i(t)} \int_t^\infty p_{i+1}(\tau)\Delta\tau\right]^{1/\alpha_i}, & i = 1,\ldots,n-1, \\ p(t), & i = n; \end{cases} \tag{2.6}$$

provided the improper integrals involved are convergent.

Note that for $i = 1,\ldots,m$, $\bar{R}_{m,i}(t,s) \geqslant 0$ if $s \leqslant t$, and $(-1)^{n-i-1}\bar{R}_{m,i}(t,s) \geqslant 0$ if $s \geqslant t$.

THEOREM 1. *Assume Eq. (1.1) has an eventually positive solution $x(t)$ and $m \in \{0,\ldots,n-1\}$ is given in Lemma 1 such that (2.1) and (2.2) hold for $t > t_1 \in [0,\infty)_{\mathbb{T}}$. Then the following hold for $t \in (t_1,\infty)_{\mathbb{T}}$:*

(a) if $m \geqslant 1$, then

$$\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]^\Delta < 0; \tag{2.7}$$

(b) if $m \geqslant 2$, then for $i = 0,1,\ldots,m-2$

$$x^{[i]}(t) > \phi^{-1}_{\alpha[i+1,m-1]}\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]\int_{t_1}^t \bar{R}_{m,i+1}(s,t_1)\Delta s. \tag{2.8}$$

Proof. (a) From (2.1) and (2.2), we get for $t \in [t_1,\infty)_{\mathbb{T}}$

$$\begin{aligned} x^{[m-1]}(t) &= x^{[m-1]}(t_1) + \int_{t_1}^t \phi^{-1}_{\alpha_m}\left[x^{[m]}(s)\right] r_m^{-1/\alpha_m}(s)\,\Delta s \\ &> \phi^{-1}_{\alpha_m}\left[x^{[m]}(t)\right]\int_{t_1}^t r_m^{-1/\alpha_m}(s)\,\Delta s \\ &= \phi^{-1}_{\alpha_m}\left[x^{[m]}(t)\right] R_m(t,t_1). \end{aligned}$$

Noting that

$$\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]^\Delta = \frac{r_m^{-1/\alpha_m}(t)}{R_m(t,t_1)R_m(\sigma(t),t_1)}\left[R_m(t,t_1)\phi^{-1}_{\alpha_m}\left[x^{[m]}(t)\right] - x^{[m-1]}(t)\right],$$

we have

$$\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]^{\Delta} < 0 \qquad \text{for } t \in (t_1,\infty)_{\mathbb{T}}.$$

(b) By (2.1) and the fact that $x^{[m-1]}(t)/R_m(t,t_1)$ is decreasing on $(t_1,\infty)_{\mathbb{T}}$, we have for $t \in (t_1,\infty)_{\mathbb{T}}$

$$\begin{aligned} x^{[m-2]}(t) &> x^{[m-2]}(t) - x^{[m-2]}(t_1) = \int_{t_1}^{t} \left(x^{[m-2]}(s)\right)^{\Delta} \Delta s \\ &= \int_{t_1}^{t} \phi_{\alpha_{m-1}}^{-1} \left[\frac{x^{[m-1]}(s)}{R_m(s,t_1)}\right] \left[\frac{R_m(s,t_1)}{r_{m-1}(s)}\right]^{1/\alpha_{m-1}} \Delta s \\ &= \int_{t_1}^{t} \phi_{\alpha_{m-1}}^{-1} \left[\frac{x^{[m-1]}(s)}{R_m(s,t_1)}\right] \bar{R}_{m,m-1}(s,t_1)\Delta s \\ &> \phi_{\alpha_{m-1}}^{-1} \left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right] \int_{t_1}^{t} \bar{R}_{m,m-1}(s,t_1)\Delta s. \end{aligned}$$

This shows that (2.8) holds for $i = m-2$. Assume (2.8) holds for some $i \in \{1,\ldots,m-2\}$. Then for $t \in (t_1,\infty)_{\mathbb{T}}$

$$\begin{aligned} \left[x^{[i-1]}(t)\right]^{\Delta} &> \phi_{\alpha[i,m-1]}^{-1} \left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right] \left[\frac{1}{r_i(t)} \int_{t_1}^{t} \bar{R}_{m,i+1}(s,t_1)\Delta s\right]^{1/\alpha_i} \\ &= \phi_{\alpha[i,m-1]}^{-1} \left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right] \bar{R}_{m,i}(t,t_1). \end{aligned}$$

Replacing t by s in the above inequality and then integrating it from t_1 to $t \in (t_1,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} x^{[i-1]}(t) &> x^{[i-1]}(t) - x^{[i-1]}(t_1) \\ &\geqslant \int_{t_1}^{t} \phi_{\alpha[i,m-1]}^{-1} \left[\frac{x^{[m-1]}(s)}{R_m(s,t_1)}\right] \bar{R}_{m,i}(s,t_1)\Delta s \\ &> \phi_{\alpha[i,m-1]}^{-1} \left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right] \int_{t_1}^{t} \bar{R}_{m,i}(s,t_1)\Delta s. \end{aligned}$$

This shows that (2.8) holds for $i-1$. By induction, (2.8) holds for all $i = 0,1,\ldots,m-2$. □

THEOREM 2. *Assume Eq.* (1.1) *has an eventually positive solution* $x(t)$ *and* m *is given in Lemma* 1 *such that* $m \in \{1,\ldots,n-1\}$ *and* (2.1) *and* (2.2) *hold for* $t \geqslant t_1 \in [0,\infty)_{\mathbb{T}}$. *Then the following hold for* $t \in [t_1,\infty)_{\mathbb{T}}$*:*

(a) for $i = m, \ldots, n-1$, $\int_t^\infty p_{i+1}(s)\Delta s < \infty$ *and*

$$(-1)^{m+i} x^{[i]}(t) > \phi_{\alpha[1,i]}\left[x(g(t))\right] \int_t^\infty p_{i+1}(s)\Delta s; \tag{2.9}$$

(b) for $i = 0, 1, \ldots, m-1$,

$$x^{[i]}(t) > \phi^{-1}_{\alpha[i+1,m]}\left[x^{[m]}(t)\right] \int_{t_1}^t \bar{R}_{m,i+1}(s,t_1)\Delta s. \tag{2.10}$$

Proof. (a) Note that $m \in \{1, \ldots, n-1\}$ implies that $x^{[1]}(t) > 0$ and $x^{[n-1]}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. This implies that $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Replacing t by τ in Eq. (1.1), integrating from $t \geqslant t_1$ to $s \in [t, \infty)_{\mathbb{T}}$, and using the fact that g is nondecreasing, we have

$$\begin{aligned}
x^{[n-1]}(t) &> -x^{[n-1]}(s) + x^{[n-1]}(t) = \int_t^s p(\tau)\,\phi_{\alpha[1,n-1]}(x(g(\tau)))\Delta\tau \\
&\geqslant \phi_{\alpha[1,n-1]}(x(g(t))) \int_t^s p(\tau)\Delta\tau \\
&= \phi_{\alpha[1,n-1]}(x(g(t))) \int_t^s p_n(\tau)\Delta\tau.
\end{aligned}$$

Taking limits as $s \to \infty$ we obtain that

$$x^{[n-1]}(t) > \phi_{\alpha[1,n-1]}(x(g(t))) \int_t^\infty p_n(\tau)\Delta\tau.$$

This shows that $\int_t^\infty p_n(\tau)\Delta\tau < \infty$ and (2.9) holds for $i = n-1$. Assume $\int_t^\infty p_{i+1}(\tau)\Delta\tau < \infty$ and (2.9) holds for some $i \in \{m+1, \ldots, n-1\}$. Then

$$\begin{aligned}
(-1)^{m+i}\left[x^{[i-1]}(t)\right]^\Delta &> \phi^{-1}_{\alpha_i}\left[\phi_{\alpha[1,i]}(x(g(t)))\right]\left[\frac{1}{r_i(t)}\int_t^\infty p_{i+1}(\tau)\Delta\tau\right]^{1/\alpha_i} \\
&= \phi^{-1}_{\alpha_i}\left[\phi_{\alpha[1,i]}(x(g(t)))\right] p_i(t) \\
&= \phi_{\alpha[1,i-1]}(x(g(t)))\, p_i(t)
\end{aligned}$$

since

$$\phi^{-1}_{\alpha_i}(\phi_{\alpha[1,i]}(x(g(t)))) = \phi_{\alpha[1,i-1]}(x(g(t))).$$

Replacing t by τ in the above inequality and then integrating it from $t \geqslant t_1$ to $s \in [t, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
(-1)^{m+i-1} x^{[i-1]}(t) &> (-1)^{m+i}(x^{[i-1]}(s) - x^{[i-1]}(t)) \\
&> \int_t^s \phi_{\alpha[1,i-1]}(x(g(\tau)))\, p_i(\tau)\Delta\tau.
\end{aligned}$$

Taking limits as $s \to \infty$ we obtain that

$$(-1)^{m+i-1} x^{[i-1]}(t) > \int_t^\infty \left[\phi_{\alpha[1,i-1]}(x(g(\tau)))\right] p_i(\tau)\Delta\tau$$

$$\geqslant \phi_{\alpha[1,i-1]}\left(x\left(g\left(t\right)\right)\right)\int_{t}^{\infty} p_i(\tau)\Delta\tau.$$

This shows that $\int_t^\infty p_i(\tau)\Delta\tau < \infty$ and (2.9) holds for $i-1$. Then the conclusion follows from induction.

(b) By (2.2), $x^{[m+1]}(t) < 0$ and hence $x^{[m]}(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Let $t \in [t_1,\infty)_{\mathbb{T}}$. Then

$$\begin{aligned}
x^{[m-1]}(t) &= x^{[m-1]}(t_1) + \int_{t_1}^{t} \left(x^{[m-1]}(s)\right)^{\Delta} \Delta s \\
&= x^{[m-1]}(t_1) + \int_{t_1}^{t} \phi_{\alpha_m}^{-1}\left[x^{[m]}(s)\right] r_m^{-1/\alpha_m}(s)\,\Delta s \\
&> \phi_{\alpha_m}^{-1}\left[x^{[m]}(t)\right] \int_{t_1}^{t} r_m^{-1/\alpha_m}(s)\,\Delta s \\
&= \phi_{\alpha_m}^{-1}\left[x^{[m]}(t)\right] \int_{t_1}^{t} \bar{R}_{m,m}(s,t_1)\,\Delta s.
\end{aligned}$$

This shows that (2.10) holds for $i = m-1$. Assume (2.10) holds for some $i \in \{1,\ldots,m-1\}$. Then

$$\begin{aligned}
\left(x^{[i-1]}(t)\right)^{\Delta} &> \phi_{\alpha[i,m]}^{-1}\left[x^{[m]}(t)\right]\left[\frac{1}{r_i(t)}\int_{t_1}^{t} \bar{R}_{m,i+1}(s,t_1)\,\Delta s\right]^{1/\alpha_i} \\
&= \phi_{\alpha[i,m]}^{-1}\left[x^{[m]}(t)\right] \bar{R}_{m,i}(t,t_1).
\end{aligned}$$

Replacing t by s in the above inequality and then integrating it for s from t_1 to t with $t \geqslant t_1$, we have

$$\begin{aligned}
x^{[i-1]}(t) &> x^{[i-1]}(t_1) + \int_{t_1}^{t} \phi_{\alpha[i,m]}^{-1}\left[x^{[m]}(s)\right] \bar{R}_{m,i}(\tau_1,t_1)\,\Delta s \\
&> \phi_{\alpha[i,m]}^{-1}\left[x^{[m]}(t)\right] \int_{t_1}^{t} \bar{R}_{m,i}(s,t_1)\,\Delta s.
\end{aligned}$$

This shows that (2.10) holds for $i-1$. By induction, (2.10) holds for all $i = 0,1,\ldots,m-1$. □

3. Oscillation Criteria for Even Order Equations

In this section, we establish oscillation criteria for Eq. (1.1) when n is even. It follows from Lemma 1 that there exists an odd $m \in \{1,\ldots,n-1\}$ such that (2.1) and (2.2) hold eventually. In the following, we denote $k_+ := \max\{k,0\}$ for any $k \in \mathbb{R}$.

The first result is a Fite-Wintner type oscillation criterion.

THEOREM 3. *Assume that*

$$\int_0^\infty p(t)\Delta t = \infty. \tag{3.1}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(g(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. As shown above, there exists an odd $m \in \{1,\dots,n-1\}$ such that (2.1) and (2.2) hold for $t \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then by Theorem 2, Part (a) we have that

$$x^{[n-1]}(t) > \phi_{\alpha[1,n-1]}[x(g(t))]\int_t^\infty p_n(s)\Delta s.$$

Note from (2.6) that $p_n(t) = p(t)$, this contradicts the assumption (3.1). □

In the following, we assume that p_i, $i = 2,\dots,n$, given by (2.6) are well defined.

THEOREM 4. *Assume there exists a $\rho \in C^1_{rd}([0,\infty)_{\mathbb{T}},(0,\infty))$ such that for every odd number $i \in \{1,\dots,n-1\}$,*

$$\limsup_{t\to\infty}\int_T^t \left[\rho(u)P_i(u,T) - \frac{((\rho^\Delta(u))_+)^{\alpha_i+1} r_i(u)}{(\alpha_i+1)^{\alpha_i+1}\rho^{\alpha_i}(u)}\right]\Delta u = \infty \tag{3.2}$$

for sufficiently large $T \in [0,\infty)_{\mathbb{T}}$, where

$$P_i(t,T) := p_{i+1}(t)\left[\int_T^{g(t)} \bar{R}_{i,1}(s,T)\Delta s\right]^{\alpha[1,i]} \Big/ R_i^{\alpha_i}(t,T). \tag{3.3}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(g(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. As shown above, there exists an odd $m \in \{1,\dots,n-1\}$ such that (2.1) and (2.2) hold for $t \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then by Theorem 2, Part (a) we have that for $i = m+1$

$$-x^{[m+1]}(t) > \phi_{\alpha[1,m+1]}[x(g(t))]\int_t^\infty p_{m+2}(s)\Delta s,$$

which, together with (2.6), implies that for $t \in [t_1,\infty)_{\mathbb{T}}$

$$-\left(x^{[m]}(t)\right)^\Delta > \phi_{\alpha[1,m]}[x(g(t))]\,p_{m+1}(t). \tag{3.4}$$

(i) Assume $m = 1$. In this case, by (2.4) and (2.5) we see that

$$R_1(g(t),t_1) = \int_{t_1}^{g(t)} r_1^{-1/\alpha_1}(s)\Delta s \quad \text{and} \quad \bar{R}_{m,1}(t,t_1) = r_1^{-1/\alpha_1}(t).$$

From (3.4) and Theorem 1, Part (a) we have for $g(t) \in (t_1,\infty)_{\mathbb{T}}$

$$-\left(x^{[1]}(t)\right)^\Delta > \phi_{\alpha_1}[x(g(t))]\,p_2(t) = \phi_{\alpha_1}\left[\frac{x(g(t))}{R_1(g(t),t_1)}\right] R_1^{\alpha_1}(g(t),t_1)p_2(t)$$

$$\begin{aligned}&\geqslant \phi_{\alpha_1}\left[\frac{x(t)}{R_1(t,t_1)}\right] R_1^{\alpha_1}(g(t),t_1)p_2(t)\\&=\phi_{\alpha_1}[x(t)]\left[\frac{R_1(g(t),t_1)}{R_1(t,t_1)}\right]^{\alpha_1} p_2(t)\\&= p_2(t)\phi_{\alpha_1}[x(t)]\left[\int_{t_1}^{g(t)}\bar{R}_{m,1}(s,t_1)\Delta s\right]^{\alpha_1}\Big/ R_1^{\alpha_1}(t,t_1).\end{aligned}$$

(ii) Assume $m\geqslant 3$. By Theorem 1, Part (b) with $i=0$, we get for $t\in(t_1,\infty)_{\mathbb{T}}$

$$x(t)>\phi^{-1}_{\alpha[1,m-1]}\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]\int_{t_1}^{t}\bar{R}_{m,1}(s,t_1)\Delta s. \tag{3.5}$$

Then by Theorem 1, Part (a) we see that for $g(t)\in(t_1,\infty)_{\mathbb{T}}$

$$\begin{aligned}x(g(t))&>\phi^{-1}_{\alpha[1,m-1]}\left[\frac{x^{[m-1]}(g(t))}{R_m(g(t),t_1)}\right]\int_{t_1}^{g(t)}\bar{R}_{m,1}(s,t_1)\Delta s\\&\geqslant\phi^{-1}_{\alpha[1,m-1]}\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]\int_{t_1}^{g(t)}\bar{R}_{m,1}(s,t_1)\Delta s.\end{aligned}\tag{3.6}$$

Substituting (3.6) into (3.4) and noting that $\phi_{\alpha[1,m]}\phi^{-1}_{\alpha[1,m-1]}=\phi_{\alpha_m}$, we obtain that for $g(t)\in(t_1,\infty)_{\mathbb{T}}$

$$-\left(x^{[m]}(t)\right)^{\Delta}>p_{m+1}(t)\,\phi_{\alpha_m}\left[x^{[m-1]}(t)\right]\ \left[\int_{t_1}^{g(t)}\bar{R}_{m,1}(s,t_1)\Delta s\right]^{\alpha[1,m]}\Big/ R_m^{\alpha_m}(t,t_1).$$

Combining cases (i) and (ii) we see that for $g(t)\in(t_1,\infty)_{\mathbb{T}}$

$$-\left(x^{[m]}(t)\right)^{\Delta}>\phi_{\alpha_m}\left[x^{[m-1]}(t)\right]P_m(t,t_1),\ t\in[t_2,\infty)_{\mathbb{T}}. \tag{3.7}$$

Define

$$z_m(t):=\frac{\rho(t)x^{[m]}(t)}{\left[x^{[m-1]}(t)\right]^{\alpha_m}}.$$

By the product rule and the quotient rule, we have

$$\begin{aligned}z_m^{\Delta}(t)&=\frac{\rho(t)}{\left[x^{[m-1]}(t)\right]^{\alpha_m}}\left(x^{[m]}(t)\right)^{\Delta}+\left[\frac{\rho(t)}{\left[x^{[m-1]}(t)\right]^{\alpha_m}}\right]^{\Delta}x^{[m]}(\sigma(t))\\&=\rho(t)\frac{\left(x^{[m]}(t)\right)^{\Delta}}{\left[x^{[m-1]}(t)\right]^{\alpha_m}}\\&\quad+\left[\frac{\rho^{\Delta}(t)}{\left[x^{[m-1]}(\sigma(t))\right]^{\alpha_m}}-\frac{\rho(t)\left[\left[x^{[m-1]}(t)\right]^{\alpha_m}\right]^{\Delta}}{\left[x^{[m-1]}(t)\right]^{\alpha_m}\left[x^{[m-1]}(\sigma(t))\right]^{\alpha_m}}\right]x^{[m]}(\sigma(t))\end{aligned}$$

$$= \rho(t)\frac{\left(x^{[m]}(t)\right)^{\Delta}}{\left[x^{[m-1]}(t)\right]^{\alpha_m}} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} z_m(\sigma(t))$$
$$- \frac{\rho(t)}{\rho(\sigma(t))} \frac{\left[\left[x^{[m-1]}(t)\right]^{\alpha_m}\right]^{\Delta}}{\left[x^{[m-1]}(t)\right]^{\alpha_m}} z_m(\sigma(t)). \tag{3.8}$$

From (3.7) we get

$$z_m^{\Delta}(t) \leqslant -\rho(t)P_m(t,t_1) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} z_m(\sigma(t))$$
$$- \frac{\rho(t)}{\rho(\sigma(t))} \frac{\left[\left[x^{[m-1]}(t)\right]^{\alpha_m}\right]^{\Delta}}{\left[x^{[m-1]}(t)\right]^{\alpha_m}} z_m(\sigma(t)).$$

By the Pötzsche chain rule ([6, Theorem 1.90]) we obtain

$$\begin{aligned}
&\left[\left(x^{[m-1]}(t)\right)^{\alpha_m}\right]^{\Delta} \\
&\quad = \alpha_m \int_0^1 \left[x^{[m-1]}(t) + h\mu(t)(x^{[m-1]}(t))^{\Delta}\right]^{\alpha_m - 1} dh \left(x^{[m-1]}(t)\right)^{\Delta} \\
&\quad = \alpha_m \int_0^1 \left[(1-h)x^{[m-1]}(t) + hx^{[m-1]}(\sigma(t))\right]^{\alpha_m - 1} dh \left(x^{[m-1]}(t)\right)^{\Delta} \\
&\quad \geqslant \begin{cases} \alpha_m \left(x^{[m-1]}(\sigma(t))\right)^{\alpha_m - 1} \left(x^{[m-1]}(t)\right)^{\Delta}, & 0 < \alpha_m \leqslant 1, \\ \alpha_m \left[x^{[m-1]}(t)\right]^{\alpha_m - 1} \left(x^{[m-1]}(t)\right)^{\Delta}, & \alpha_m \geqslant 1. \end{cases}
\end{aligned}$$

If $0 < \alpha_m \leqslant 1,$ we have

$$z_m^{\Delta}(t) \leqslant -\rho(t)P_m(t,t_1) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} z_m(\sigma(t))$$
$$- \frac{\alpha_m \rho(t)}{\rho(\sigma(t))} \frac{\left(x^{[m-1]}(t)\right)^{\Delta}}{x^{[m-1]}(\sigma(t))} \left[\frac{x^{[m-1]}(\sigma(t))}{x^{[m-1]}(t)}\right]^{\alpha_m} z_m(\sigma(t));$$

and if $\alpha_m \geqslant 1$, we have

$$z_m^{\Delta}(t) \leqslant -\rho(t)P_m(t,t_1) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} z_m(\sigma(t))$$
$$- \frac{\alpha_m \rho(t)}{\rho(\sigma(t))} \frac{\left(x^{[m-1]}(t)\right)^{\Delta}}{x^{[m-1]}(\sigma(t))} \frac{x^{[m-1]}(\sigma(t))}{x^{[m-1]}(t)} z_m(\sigma(t)).$$

Using the fact that $\left(x^{[m-1]}(t)\right)^{\Delta} > 0$ on $[t_2,\infty)_{\mathbb{T}}$, we get that for any $\alpha_m > 0$,

$$z_m^{\Delta}(t) \leqslant -\rho(t)P_m(t,t_1) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} z_m(\sigma(t)) - \frac{\alpha_m \rho(t)}{\rho(\sigma(t))} \frac{\left(x^{[m-1]}(t)\right)^{\Delta}}{x^{[m-1]}(\sigma(t))} z_m(\sigma(t)).$$

Since $x^{[m]} = r_m \phi_{\alpha_m}\left[\left(x^{[m-1]}\right)^{\Delta}\right]$ is strictly decreasing,

$$(x^{[m-1]}(t))^{\Delta} = \left(\frac{x^{[m]}((t))}{r_m(t)}\right)^{\frac{1}{\alpha_m}} \geqslant \left(\frac{x^{[m]}(\sigma(t))}{r_m(t)}\right)^{\frac{1}{\alpha_m}}. \tag{3.9}$$

Then

$$\begin{aligned} z_m^{\Delta}(t) &\leqslant -\rho(t)P_m(t,t_1) + \frac{\left(\rho^{\Delta}(t)\right)_+}{\rho(\sigma(t))} z_m(\sigma(t)) \\ &\qquad - \frac{\alpha_m \rho(t)}{\rho(\sigma(t)) r_m^{1/\alpha_m}(t)} \left(\frac{x^{[m]}(\sigma(t))}{x^{[m-1]}(\sigma(t))}\right)^{\frac{1}{\alpha_m}} z_m(\sigma(t)) \\ &= -\rho(t)P_m(t,t_1) + \frac{\left(\rho^{\Delta}(t)\right)_+}{\rho(\sigma(t))} z_m(\sigma(t)) - \frac{\alpha_m \rho(t)}{\rho^{\lambda}(\sigma(t)) r_m^{1/\alpha_m}(t)} (z_m(\sigma(t)))^{\lambda} \\ &\leqslant -\rho(t)P_m(t,t_2) + \frac{\left(\rho^{\Delta}(t)\right)_+}{\rho(\sigma(t))} z_m(\sigma(t)) - \frac{\alpha_m \rho(t)}{\rho^{\lambda}(\sigma(t)) r_m^{1/\alpha_m}(t)} (z_m(\sigma(t)))^{\lambda}, \end{aligned} \tag{3.10}$$

where $\lambda := \frac{\alpha_m+1}{\alpha_m}$. Define

$$A^{\lambda} := \frac{\alpha_m \rho(t)}{\rho^{\lambda}(\sigma(t)) r_m^{1/\alpha_m}(t)} (z_m(\sigma(t)))^{\lambda} \quad \text{and} \quad B^{\lambda-1} := \frac{\left(\rho^{\Delta}(t)\right)_+ r_m^{1/(\alpha_m+1)}(t)}{\lambda\,(\alpha_m \rho(t))^{1/\lambda}}.$$

Then by the inequality (see [30])

$$\lambda A B^{\lambda-1} - A^{\lambda} \leqslant (\lambda-1) B^{\lambda}, \tag{3.11}$$

we get that

$$\frac{\left(\rho^{\Delta}(t)\right)_+}{\rho(\sigma(t))} z_m(\sigma(t)) - \frac{\alpha_m \rho(t)}{\rho^{\lambda}(\sigma(t)) r_m^{1/\alpha_m}(t)} (z_m(\sigma(t)))^{\lambda} \leqslant \frac{\left(\left(\rho^{\Delta}(t)\right)_+\right)^{\alpha_m+1} r_m(t)}{(\alpha_m+1)^{\alpha_m+1} \rho^{\alpha_m}(t)}.$$

From this and (3.10) we have

$$z_m^{\Delta}(t) \leqslant -\rho(t)P_m(t,t_2) + \frac{\left(\left(\rho^{\Delta}(t)\right)_+\right)^{\alpha_m+1} r_m(t)}{(\alpha_m+1)^{\alpha_m+1} \rho^{\alpha_m}(t)}.$$

Integrating both sides from t_2 to t we get

$$\int_{t_2}^{t}\left[\rho(u)P_m(u,t_2)-\frac{\left(\left(\rho^{\Delta}(u)\right)_+\right)^{\alpha_m+1}r_m(u)}{(\alpha_m+1)^{\alpha_m+1}\rho^{\alpha_m}(u)}\right]\Delta u\leqslant z_m(t_2)-z_m(t)\leqslant z_m(t_2),$$

which contradicts (3.2). This completes the proof. □

THEOREM 5. *Assume there exists a $\rho\in C^1_{rd}([0,\infty)_{\mathbb{T}},(0,\infty))$ such that for every odd number $i\in\{1,\ldots,n-1\}$,*

$$\limsup_{t\to\infty}\int_T^t\left[\rho(u)p_{i+1}(u)-\frac{\left[(\rho^{\Delta}(u))_+\right]^{\gamma_i+1}}{(\gamma_i+1)^{\gamma_i+1}\left[\rho(u)g^{\Delta}(u)\bar{R}_{i,1}(g(u),T)\right]^{\gamma_i}}\right]\Delta u=\infty \quad (3.12)$$

for sufficiently large $T\in[0,\infty)_{\mathbb{T}}$, where $\gamma_i:=\alpha[1,i]$. Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $t_0\in[0,\infty)_{\mathbb{T}}$ such that $x(g(t))>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. As shown at the beginning of this section, there exists an odd $m\in\{1,\ldots,n-1\}$ such that (2.1) and (2.2) hold for $t\geqslant t_1\in[t_0,\infty)_{\mathbb{T}}$. Define

$$w_m(t):=\frac{\rho(t)x^{[m]}(t)}{[x(g(t))]^{\gamma_m}}. \quad (3.13)$$

Similar to that in the proof of Theorem 4, we have that for $t\in[t_1,\infty)_{\mathbb{T}}$

$$w_m^{\Delta}(t)=\rho(t)\frac{\left(x^{[m]}(t)\right)^{\Delta}}{[x(g(t))]^{\gamma_m}}+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}w_m(\sigma(t))-\frac{\rho(t)\left[x^{\gamma_m}(g(t))\right]^{\Delta}}{\rho(\sigma(t))\left[x(g(t))\right]^{\gamma_m}}w_m(\sigma(t)) \quad (3.14)$$

and

$$-\left(x^{[m]}(t)\right)^{\Delta}\geqslant\phi_{\gamma_m}\left[x(g(t))\right]p_{m+1}(t). \quad (3.15)$$

Combining (3.14) and (3.15) we get

$$w_m^{\Delta}(t)\leqslant-\rho(t)p_{m+1}(t)+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}w_m(\sigma(t))-\frac{\rho(t)\left[x^{\gamma_m}(g(t))\right]^{\Delta}}{\rho(\sigma(t))\left[x(g(t))\right]^{\gamma_m}}w_m(\sigma(t)). \quad (3.16)$$

Since x and g are differentiable and g is nondecreasing, we have that

$$(x(g(t)))^{\Delta} = x^{\Delta}(g(t))\, g^{\Delta}(t).$$

Then by the Pötzsche chain rule ([6, Theorem 1.90]) we obtain

$$\begin{aligned}
[x^{\gamma_m}(g(t))]^{\Delta} &= \gamma_m \left(\int_0^1 \left[x(g(t)) + h\mu(t)(x(g(t)))^{\Delta} \right]^{\gamma_m - 1} dh \right) (x(g(t)))^{\Delta} \\
&= \gamma_m \left(\int_0^1 [(1-h)x(g(t)) + h\, x(g(\sigma(t)))]^{\gamma_m - 1} dh \right) x^{\Delta}(g(t))\, g^{\Delta}(t) \\
&\geqslant \begin{cases} \gamma_m\, [x(g(\sigma(t)))]^{\gamma_m - 1} x^{\Delta}(g(t))\, g^{\Delta}(t), & 0 < \gamma_m \leqslant 1, \\ \gamma_m\, [x(g(t))]^{\gamma_m - 1} x^{\Delta}(g(t))\, g^{\Delta}(t), & \gamma_m \geqslant 1. \end{cases}
\end{aligned}$$

If $0 < \gamma_m \leqslant 1,$ we have

$$\begin{aligned}
w_m^{\Delta}(t) \leqslant & -\rho(t)p_{m+1}(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} w_m(\sigma(t)) \\
& - \frac{\gamma_m \rho(t) g^{\Delta}(t)}{\rho(\sigma(t))} \frac{x^{\Delta}(g(t))}{x(g(\sigma(t)))} \left[\frac{x(g(\sigma(t)))}{x(g(t))} \right]^{\gamma_m} w_m(\sigma(t));
\end{aligned}$$

and if $\gamma_m \geqslant 1$, we have

$$\begin{aligned}
w_m^{\Delta}(t) \leqslant & -\rho(t)p_{m+1}(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} w_m(\sigma(t)) \\
& - \frac{\gamma_m \rho(t) g^{\Delta}(t)}{\rho(\sigma(t))} \frac{x^{\Delta}(g(t))}{x(g(\sigma(t)))} \frac{x(g(\sigma(t)))}{x(g(t))} w_m(\sigma(t)).
\end{aligned}$$

Using the fact that $x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ we see that for $\gamma_m > 0$

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\rho(t)p_{m+1}(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} w_m(\sigma(t)) \\
& - \frac{\gamma_m \rho(t) g^{\Delta}(t)}{\rho(\sigma(t))} \frac{x^{\Delta}(g(t))}{x(g(\sigma(t)))} w_m(\sigma(t)).
\end{aligned} \tag{3.17}$$

Now by (2.10) with $i = 1$ we have

$$x^{[1]}(t) \geqslant \phi^{-1}_{\alpha[2,m]} \left[x^{[m]}(t) \right] \int_{t_1}^{t} \bar{R}_{m,2}(s, t_1)\, \Delta s$$

which implies

$$x^{\Delta}(t) \geqslant \phi^{-1}_{\alpha[1,m]} \left[x^{[m]}(t) \right] \bar{R}_{m,1}(t, t_1) \quad \text{for } t \geqslant t_1.$$

Since $g(t) \leqslant \sigma(t)$, from (2.2) for $g(t) \geqslant t_1$

$$x^{\Delta}(g(t)) \geqslant \phi^{-1}_{\alpha[1,m]} \left[x^{[m]}(g(t)) \right] \bar{R}_{m,1}(g(t), t_1)$$

$$\begin{aligned}
&\geqslant \phi^{-1}_{\alpha[1,m]}\left[x^{[m]}(\sigma(t))\right]\bar{R}_{m,1}(g(t),t_1)\\
&=\frac{\phi^{-1}_{\alpha[1,m]}[w_m(\sigma(t))]}{\phi^{-1}_{\alpha[1,m]}[\rho(\sigma(t))]}x(g(\sigma(t)))\bar{R}_{m,1}(g(t),t_1)\\
&=\frac{\phi^{-1}_{\gamma_m}[w_m(\sigma(t))]}{\phi^{-1}_{\gamma_m}[\rho(\sigma(t))]}x(g(\sigma(t)))\bar{R}_{m,1}(g(t),t_1). \qquad (3.18)
\end{aligned}$$

Then, from (3.17) and (3.18), we get for $g(t)\geqslant t_1$

$$w^{\Delta}(t)\leqslant -\rho(t)p_{m+1}(t)+\frac{(\rho^{\Delta}(t))_+}{\rho(\sigma(t))}w_m(\sigma(t)) - \gamma_m\rho(t)g^{\Delta}(t)\bar{R}_{m,1}(g(t),t_1)\left[\frac{w_m(\sigma(t))}{\rho(\sigma(t))}\right]^{\lambda},$$

where $\lambda:=\frac{\gamma_m+1}{\gamma_m}$. Define

$$A^{\lambda}:=\gamma_m\rho(t)g^{\Delta}(t)\bar{R}_{m,1}(g(t),t_1)\left[\frac{w_m(\sigma(t))}{\rho(\sigma(t))}\right]^{\lambda}$$

and

$$B^{\lambda-1}:=\frac{(\rho^{\Delta}(t))_+}{\lambda\left(\gamma_m\rho(t)g^{\Delta}(t)\bar{R}_{m,1}(g(t),t_1)\right)^{1/\lambda}}.$$

Then from (3.11),

$$\frac{(\rho^{\Delta}(t))_+}{\rho(\sigma(t))}w_m(\sigma(t))-\gamma_m\rho(t)g^{\Delta}(t)\bar{R}_{m,1}(g(t),t_1)\left[\frac{w_m(\sigma(t))}{\rho(\sigma(t))}\right]^{\lambda} \leqslant \frac{\left[(\rho^{\Delta}(t))_+\right]^{\gamma_m+1}}{(\gamma_m+1)^{\gamma_m+1}\left[\rho(t)g^{\Delta}(t)\bar{R}_{m,1}(g(t),t_1)\right]^{\gamma_m}}.$$

The rest of the proof is similar to that of Theorem 4 and hence is omitted. □

As direct consequences of Theorems 3-5, we obtain oscillation criteria for Eq. (1.1) with $n=2$, namely, for the equation

$$\left(r_1(t)\phi_{\alpha_1}\left(x^{\Delta}(t)\right)\right)^{\Delta}+p(t)\phi_{\alpha_1}(x(g(t)))=0. \qquad (3.19)$$

COROLLARY 1. *Every solution of Eq. (3.19) is oscillatory provided one of the following conditions is satisfied:*

(a) $\int_0^{\infty}p(t)\Delta t=\infty$;

(b) there exists a $\rho\in C^1_{rd}([0,\infty)_{\mathbb{T}},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left[\rho(u)P_1(u,T)-\frac{((\rho^{\Delta}(u))_+)^{\alpha_1+1}r_1(u)}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(u)}\right]\Delta u=\infty$$

for sufficiently large $T \in [0,\infty)_{\mathbb{T}}$, *where*

$$P_1(t,T) := p(t)\, R_1^{\alpha_1}(g(t),T) \big/ R_1^{\alpha_1}(t,T);$$

(c) there exists a $\rho \in C_{rd}^1([0,\infty)_{\mathbb{T}},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[\rho(u)p(u) - \frac{\left[(\rho^\Delta(u))_+\right]^{\alpha_1+1} r_1(g(u))}{(\alpha_1+1)^{\alpha_1+1}\left[\rho(u)g^\Delta(u)\right]^{\alpha_1}}\right]\Delta u = \infty$$

for sufficiently large $T \in [0,\infty)_{\mathbb{T}}$.

REMARK 1. 1. Let $\alpha_1 > 1$ be an odd number and $g(t) = t$ on $[0,\infty)_{\mathbb{T}}$. Then Corollary 1 with condition (b) reduces to Theorem 3.1 in Saker [45].

2. Let $\alpha_1 > 0$ be a quotient of odd numbers and $g(t) \leqslant t$ on $[0,\infty)_{\mathbb{T}}$. Then Corollary 1 with condition (b) reduces to Theorem 2.1 in Erbe, Hassan and Peterson [20].

For Eq. (1.1) with an even $n \geqslant 4$, we have further criteria for oscillation as shown below.

THEOREM 6. *Assume*

$$\text{either} \quad \int_0^\infty p_{n-1}(t)\Delta t = \infty \quad \text{or} \quad \int_0^\infty p_{n-2}(t)\Delta t = \infty. \tag{3.20}$$

Suppose that there exists a $\rho \in C_{rd}^1([0,\infty)_{\mathbb{T}},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[\rho(u)P_{n-1}(u,T) - \frac{\left((\rho^\Delta(u))_+\right)^{\alpha_{n-1}+1} r_{n-1}(u)}{(\alpha_{n-1}+1)^{\alpha_{n-1}+1}\rho^{\alpha_{n-1}}(u)}\right]\Delta u = \infty \tag{3.21}$$

for sufficiently large $T \in [0,\infty)_{\mathbb{T}}$, *where*

$$P_{n-1}(t,T) := p(t)\left[\int_T^{g(t)} \bar{R}_{n-1,1}(s,T)\Delta s\right]^{\alpha[1,n-1]} \Bigg/ R_{n-1}^{\alpha_{n-1}}(t,T).$$

Then every solution of Eq. (1.1) is oscillatory.

THEOREM 7. *Assume (3.20) holds. Suppose that there exists a*

$$\rho \in C_{rd}^1([0,\infty)_{\mathbb{T}},(0,\infty))$$

such that

$$\limsup_{t\to\infty}\int_T^t \Bigg[\rho(u)p(u) - \frac{\left[(\rho^\Delta(u))_+\right]^{\gamma_{n-1}+1}}{(\gamma_{n-1}+1)^{\gamma_{n-1}+1}\left[\rho(u)g^\Delta(u)\bar{R}_{n-1,1}(g(u),T)\right]^{\gamma_{n-1}}}\Bigg]\Delta u = \infty \tag{3.22}$$

for sufficiently large $T \in [0,\infty)_{\mathbb{T}}$, *where* $\gamma_{n-1} := \alpha[1,n-1]$. *Then every solution of Eq. (1.1) is oscillatory.*

Proofs of Theorems 6 and 7. Assume (1.1) has a nonoscillatory solution $x(t)$ on $[0,\infty)_{\mathbb{T}}$. Then without loss of generality, assume there is a $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(g(t)) > 0$, for $t \in [t_0,\infty)_{\mathbb{T}}$. As shown at the beginning of this section, there exists an odd $m \in \{1,\dots,n-1\}$ such that (2.1) and (2.2) hold for $t \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$.

We claim that (3.20) implies that $m = n-1$. In fact, if $1 \leqslant m \leqslant n-3$, then for $t \geqslant t_1$

$$\left(x^{[n-1]}(t)\right)^{\Delta} < 0,\ x^{[n-1]}(t) > 0,\ x^{[n-2]}(t) < 0,\ x^{[n-3]}(t) > 0. \tag{3.23}$$

Since $x^{\Delta}(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, then $x(t) > x(t_1) := c_1 > 0$ for $t \geqslant t_1$. Then there exists a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $x(g(t)) \geqslant c_1$ for $t \geqslant t_2$. It follows that for $t \in [t_2,\infty)_{\mathbb{T}}$

$$[x(g(t))]^{\alpha[1,n-1]} \geqslant c := c_1^{\alpha[1,n-1]} > 0.$$

Integrating (1.1) from t to $\tau \in [t,\infty)_{\mathbb{T}}$ and usung (3.23) we get that

$$\begin{aligned} x^{[n-1]}(t) &\geqslant -x^{[n-1]}(\tau) + x^{[n-1]}(t) \\ &= \int_t^{\tau} p(s)\,\phi_{\alpha[1,n-1]}(x(g(s)))\,\Delta s \geqslant c\int_t^{\tau} p(s)\,\Delta s. \end{aligned}$$

By taking limits as $\tau \to \infty$ we have

$$x^{[n-1]}(t) \geqslant c\int_t^{\infty} p(s)\,\Delta s.$$

It is known from Theorem 3 that $\int_t^{\infty} p(s)\,\Delta s < \infty$. Thus,

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geqslant c^{1/\alpha_{n-1}}\left[\frac{1}{r_{n-1}(t)}\int_t^{\infty} p(s)\,\Delta s\right]^{1/\alpha_{n-1}} = c^{1/\alpha_{n-1}} p_{n-1}(t). \tag{3.24}$$

Assume $\int_0^{\infty} p_{n-1}(t)\Delta t = \infty$. By integrating (3.24) from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$ we get

$$x^{[n-2]}(t) - x^{[n-2]}(t_2) \geqslant c^{1/\alpha_{n-1}}\int_{t_2}^{t} p_{n-1}(s)\,\Delta s.$$

As a result,

$$\lim_{t\to\infty} x^{[n-2]}(t) = \infty,$$

which contradicts the fact that $x^{[n-2]} < 0$ on $[t_2,\infty)_{\mathbb{T}}$.

Assume $\int_0^{\infty} p_{n-2}(t)\Delta t = \infty$. By integrating the inequality (3.24) from t to $\tau \in [t,\infty)_{\mathbb{T}}$ and then taking limits as $\tau \to \infty$ and using the fact $x^{[n-2]} < 0$ eventually, we get

$$-x^{[n-2]}(t) > c^{1/\alpha_{n-1}}\int_t^{\infty} p_{n-1}(s)\Delta s,$$

which implies

$$-\left(x^{[n-3]}(t)\right)^{\Delta} > c^{1/\alpha[n-2,n-1]}\left[\frac{1}{r_{n-2}(t)}\int_t^{\infty} p_{n-1}(s)\Delta s\right]^{1/\alpha_{n-2}}$$

$$= c^{1/\alpha[n-2,n-1]} p_{n-2}(t).$$

Again, integrating above inequality from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(t_2) - x^{[n-3]}(t) \geqslant c^{1/\alpha[n-2,n-1]} \int_{t_2}^{t} p_{n-2}(s)\Delta s.$$

As a result,

$$\lim_{t\to\infty} x^{[n-3]}(t) = -\infty,$$

which contradicts the fact that $x^{[n-3]} > 0$ on $[t_2,\infty)_{\mathbb{T}}$. This shows that if (3.20) holds, then $m = n-1$. The rest of proof of Theorems 6 and Theorem 7 are similar to the proof of Theorems 4 and 5 with $m = n-1$ respectively and hence can be omitted. □

4. Oscillation Criteria for Odd Order Equations

In this section we establish the oscillation criteria for Eq. (1.1) when n is odd. It follows from Lemma 1 that there exists an even $m \in \{0,\ldots,n-1\}$ such that (2.1) and (2.2) hold eventually.

THEOREM 8. *Assume* (3.1) *holds. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(g(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. As shown above, there exists an even $m \in \{0,\ldots,n-1\}$ such that (2.1) and (2.2) hold for $t \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$.

(i) Assume $m \geqslant 2$. Then the same argument as in the proof of Theorem 3 leads to a contradiction to the assumption (3.1).

(ii) We show that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k = 0,1,\ldots,n.$$

Since $x^{\Delta}(t) < 0$ on $[t_1,\infty)_{\mathbb{T}}$, then $\lim_{t\to\infty} x(t) = l_1 \geqslant 0$. Assume $l_1 > 0$. Then for sufficiently large $t_2 \in [t_1,\infty)_{\mathbb{T}}$, we have $x(g(t)) \geqslant l_1$ for $t \geqslant t_2$. It follows that

$$[x(g(t))]^{\alpha[1,n-1]} \geqslant l := l_1^{\alpha[1,n-1]} > 0 \ \text{ for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Integrating (1.1) from t to $s \in [t,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} x^{[n-1]}(t) &> -x^{[n-1]}(s) + x^{[n-1]}(t) \\ &= \int_t^s p(\tau)\,\phi_{\alpha[1,n-1]}(x(g(\tau)))\,\Delta\tau \geqslant l \int_t^s p(\tau)\,\Delta\tau. \end{aligned}$$

Letting $s \to \infty$ in the above we reach a contradiction to (3.1). This completes the proof. □

THEOREM 9. *Assume* (3.20) *and* (3.21) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

THEOREM 10. *Assume* (3.20) *and* (3.22) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Proofs of Theorems 9 *and* 10*.* Assume Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[0,\infty)_{\mathbb{T}}$. Then, without loss of generality, assume there is a $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(g(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. As shown at the beginning of this section, there exists an even $m \in \{0,\ldots,n-1\}$ such that (2.1) and (2.2) hold for $t \geqslant t_1 \in [t_0,\infty)_{\mathbb{T}}$.

(i) Assume $m \geqslant 2$. The same argument as in the proof of Theorems 6 and 7 and hence is omitted.

(ii) We show that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k = 0,1,...,n.$$

Since $x^{\Delta} < 0$ on $[t_1,\infty)_{\mathbb{T}}$, then $\lim_{t\to\infty} x(t) = l_1 \geqslant 0$. Assume $l_1 > 0$. Then there exists a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $x(g(t)) \geqslant l_1$ for $t \geqslant t_2$. It follows that

$$[x(g(t))]^{\alpha[1,n-1]} \geqslant l := l_1^{\alpha[1,n-1]} > 0 \ \text{ for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Integrating (1.1) from t to $\tau \in [t,\infty)_{\mathbb{T}}$ and using (3.23) we get

$$\begin{aligned} x^{[n-1]}(t) &\geqslant -x^{[n-1]}(\tau) + x^{[n-1]}(t) \\ &= \int_t^{\tau} p(s)\,\phi_{\phi_{\alpha[1,n-1]}}(x(g(s)))\,\Delta s \geqslant l\int_t^{\tau} p(s)\,\Delta s. \end{aligned}$$

Hence by taking limits as $\tau \to \infty$ we have

$$x^{[n-1]}(t) \geqslant l\int_t^{\infty} p(s)\,\Delta s.$$

It is known that $\int_t^{\infty} p(s)\,\Delta s < \infty$. Thus,

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geqslant l^{1/\alpha_{n-1}}\left[\frac{1}{r_{n-1}(t)}\int_t^{\infty} p(s)\,\Delta s\right]^{1/\alpha_{n-1}} = l^{1/\alpha_{n-1}} p_{n-1}(t).$$

Assume $\int_0^{\infty} p_{n-1}(s)\Delta s = \infty$. By integrating the above inequality from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$ we get

$$x^{[n-2]}(t) - x^{[n-2]}(t_2) \geqslant l^{\frac{1}{\alpha_{n-1}}}\int_{t_2}^{t} p_{n-1}(s)\,\Delta s. \tag{4.1}$$

As a result,

$$\lim_{t\to\infty} x^{[n-2]}(t) = \infty,$$

which contradicts the fact that $x^{[n-2]} < 0$ on $[t_1,\infty)_{\mathbb{T}}$.

Assume $\int_0^\infty p_{n-2}(s)\Delta s = \infty$. By integrating the inequality (4.3) from t to ∞ and using the fact that $x^{[n-2]} < 0$ eventually, we get

$$-x^{[n-2]}(t) > l^{1/\alpha_{n-1}} \int_t^\infty p_{n-1}(s)\Delta s,$$

which implies

$$\begin{aligned} -\left(x^{[n-3]}(t)\right)^{\Delta} &> l^{1/\alpha[n-2,n-1]} \left[\frac{1}{r_{n-2}(t)} \int_t^\infty p_{n-1}(s)\Delta s\right]^{1/\alpha_{n-2}} \\ &= l^{1/\alpha[n-2,n-1]} p_{n-2}(t). \end{aligned}$$

Again integrating the above inequality from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(t_2) - x^{[n-3]}(t) \geqslant l^{1/\alpha[n-2,n-1]} \int_{t_2}^t p_{n-2}(s)\Delta s.$$

As a result,

$$\lim_{t\to\infty} x^{[n-3]}(t) = -\infty,$$

which contradicts the fact that $x^{[n-3]} > 0$ on $[t_2,\infty)_{\mathbb{T}}$. This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. □

As direct consequences of Theorems 8-10, we obtain oscillation criteria for Eq. (1.1) when $n = 3$, namely, for the equation

$$\left(r_2(t)\phi_{\alpha_2}\left(\left[r_1(t)\phi_{\alpha_1}\left(x^{\Delta}(t)\right)\right]^{\Delta}\right)\right)^{\Delta} + p(t)\,\phi_{\alpha_1\alpha_2}(x(g(t))) = 0. \tag{4.2}$$

COROLLARY 2. *Every solution of Eq.* (4.2) *is either oscillatory or tends to zero eventually provided one of the following conditions is satisfied:*

(a) $\int_0^\infty p(t)\Delta t = \infty$;

(b) either $\int_0^\infty p_1(t)\Delta t = \infty$ *or* $\int_0^\infty p_2(t)\Delta t = \infty$, *and there exists a* $\rho \in C_{rd}^1([0,\infty)_{\mathbb{T}},(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\rho(u)P_2(u,T) - \frac{\left(\left(\rho^{\Delta}(u)\right)_+\right)^{\alpha_2+1} r_2(u)}{(\alpha_2+1)^{\alpha_2+1}\rho^{\alpha_2}(u)}\right]\Delta u = \infty \tag{4.3}$$

for sufficiently large $T \in [0,\infty)_{\mathbb{T}}$, *where*

$$P_2(t,T) := p(t)\left[\int_T^{g(t)} \bar{R}_{2,1}(s,T)\Delta s\right]^{\alpha[1,2]} \Big/ R_2^{\alpha_2}(t,T);$$

(c) either $\int_0^\infty p_1(t)\Delta t = \infty$ *or* $\int_0^\infty p_2(t)\Delta t = \infty$*, and there exists a* $\rho \in C^1_{rd}([0,\infty)_{\mathbb{T}},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[\rho(u)p(u) - \frac{\left[(\rho^\Delta(u))_+\right]^{\gamma_2+1}}{(\gamma_2+1)^{\gamma_2+1}\left[\rho(u)g^\Delta(u)\bar{R}_{2,1}(g(u),T)\right]^{\gamma_2}}\right]\Delta u = \infty$$

for sufficiently large $T \in [0,\infty)_{\mathbb{T}}$.

REMARK 2. 1. Let $\alpha_1 = 1$, α_2 a quotient of odd integers, and $g(t) \leqslant t$ on $[0,\infty)_{\mathbb{T}}$. Then Corollary 2 with condition (a) reduces to Corollary 2.1 in Hassan [32]; and Corollary 2 with condition (c) reduces to Corollary 2.3 in Hassan [32].

2. Let $\alpha_1 = \alpha_2 = 1$ and $g(t) = t$ on $[0,\infty)_{\mathbb{T}}$. Then Corollary 2 with condition (c) reduces to Theorem 1 in Erbe, Peterson and Saker [17].

3. Let $\alpha_1 = 1$, $\alpha_2 \geqslant 1$ a quotient of odd integers, and $g(t) = t$ on $[0,\infty)_{\mathbb{T}}$. Then Corollary 2 with condition (c) reduces to Theorem 1 in Erbe, Peterson and Saker [18].

4. Let α_1 and α_2 be quotients of odd integers and $g(t) \leqslant t$ on $[0,\infty)_{\mathbb{T}}$. Then Corollary 2 with condition (c) reduces to Theorem 3.1 in Chen [8].

5. Let $g(t) = t$ on $[0,\infty)_{\mathbb{T}}$. Then Corollary 2 with condition (c) reduces to Theorem 2.1 in Yu and Wang [48].

6. Corollary 2 with condition (b) is new.

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