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# ONE–DIMENSIONAL ATTRACTOR FOR A NON–AUTONOMOUS STRONGLY DAMPED LATTICE SYSTEM WITH PERIODIC DRIVING FORCE

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*Abstract.* In this paper, we consider one-dimensional attractor of a non-autonomous second order strongly damped lattice system with periodic driving force under Neumann boundary condition or periodic boundary condition. We obtain the existence of a global attractor and prove this attractor is homeomorphic to the circle.

## 1. Introduction

In this paper, we consider the following non-autonomous strongly damped lattice system

$$\begin{cases} \ddot{y}_i + k_i(A\dot{y})_i + (Ay)_i + \gamma_i\dot{y}_i + f(t,y_i) = 0, \\ y_i(0) = y_{i0}, \qquad \dot{y}_i(0) = y_{i,10}, \end{cases} \tag{1.1}$$

where $i = (i_1, i_2, ..., i_n) \in \mathbb{Z}_m^n = \mathbb{Z}^n \cap \{1 \leqslant i_1, i_2, ..., i_n \leqslant m\}$, $k_i \geqslant 0$, $\gamma_i > 0$, and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. $y = (y_i)_{i\in\mathbb{Z}_m^n}$ is a vector with the components $y_i$ and can be ordered as the following form of 1-dimensional vector in $\mathbb{R}^{m^n}$:

$$\begin{aligned} y &= (y_{(1,1,...,1)}, y_{(2,1,...,1)}, \cdots, y_{(m,1,...,1)}, \cdots, y_{(1,m,...m)}, y_{(2,m,...,m)}, \cdots, y_{(m,m,...,m)})^T \\ &= (y_1, y_2, \cdots, y_\nu, \cdots, y_{m^n})^T \in \mathbb{R}^{m^n}, \end{aligned}$$

where $\nu = i_1 + m(i_2 - 1) + \cdots + m^{n-1}(i_n - 1)$, $1 \leqslant i_1, i_2, ..., i_n \leqslant m$, $\dot{y} = (\dot{y}_i)_{i\in\mathbb{Z}_m^n}$.

$A$ is a nonnegative definite symmetric matrix on $\mathbb{R}^{m^n}$ with eigenvalues $\lambda_s (1 \leqslant s \leqslant m^n)$, and $0$ is the simple and minimal eigenvalue of $A$ with corresponding eigenvector $e = (1, ..., 1)^T \in \mathbb{R}^{m^n}$. $(A\dot{y})_i$, $(Ay)_i$ denote the ith component of $A\dot{y}$ and $Ay$, respectively. For convenience, write $\lambda_s$ as

$$0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_{m^n}.$$

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An example of $A$ is $A=-\triangle$, the negative discrete Laplace operator which is subject to Neumann boundary condition or periodic boundary condition: $\forall i=(i_1,i_2,\ldots,i_n)\in\mathbb{Z}_m^n, y_i=y_{(i_1,i_2,\ldots,i_n)}\in\mathbb{R}^{m^n}$,

$$(Ay)_i=2ny_{(i_1,i_2,\ldots,i_n)}-y_{(i_1-1,i_2,\ldots,i_n)}-y_{(i_1,i_2-1,\ldots,i_n)}-\cdots-y_{(i_1,i_2,\ldots,i_n-1)} \\ -y_{(i_1+1,i_2,\ldots,i_n)}-y_{(i_1,i_2+1,\ldots,i_n)}-\cdots-y_{(i_1,i_2,\ldots,i_n+1)}.$$

Equation (1.1) can be rewritten as

$$\begin{cases}\ddot{y}+kA\dot{y}+Ay+\gamma\dot{y}+F(t,y)=0,\\ y(0)=(y_{i0})_{i\in\mathbb{Z}_m^n}=y_0,\quad \dot{y}(0)=(y_{i,10})_{i\in\mathbb{Z}_m^n}=y_{10},\end{cases}\tag{1.2}$$

where $y=(y_i)_{i\in\mathbb{Z}_m^n}, \dot{y}=(\dot{y}_i)_{i\in\mathbb{Z}_m^n}, k\geqslant 0, \gamma>0, F(t,y)=(f(t,y_i))_{i\in\mathbb{Z}_m^n}$.

For $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function verifying the following periodicity conditions:

$$|f(t,y)|_0\leqslant c,\qquad f(t,y+\omega_0)=f(t+T,y)=f(t,y),\quad \text{for some } T>0,\tag{1.3}$$

where $c\geqslant 0, \omega_0>0$, $|\cdot|_0$ denotes the absolute value of number in $\mathbb{R}$.

For some special cases, Eq. (1.1) can be regarded as a discrete analogue of the initial-boundary value problem of the following continuous strongly damped wave equation (see, e.g., [18]):

$$y_{tt}-k\Delta y_t-\Delta y+\gamma y_t+f(t,y)=0.\tag{1.4}$$

When $k=0, f(t,y)=\sin y$, equation (1.1) reduces to a usual wave equation which arises as an evolutionary mathematical model in various systems (see [7, 14]), which has been studied widely by using of the concept of global attractors; see, for example, [1-3, 11, 13-14]. Under the periodic boundary conditions for any space dimension, Wang and Zhu [15] discussed the existence and Hausdorff dimension of the global attractor for discretization of the damped wave equations. Qian *et al* [12] showed that the discretized damped sine-Gordon equation under Neumann boundary condition has a one-dimensional global attractor, which is a restricted horizontal curve consisting of a running periodic solution. It is different from them, we here want to study the strongly damped lattice system under Neumann boundary condition or periodic boundary condition.

For a complex mathematical physics equations, we want to study its long-time behavior, so as to study the nature of its solution and its structure (see, e.g., [4-6, 9, 16-18]). But the understanding for the structure of the solutions is not enough. Li and Zhou [8] proved the existence of a global attractor of a second order strongly damped lattice system and the system has an unbounded one-dimensional global attractor. Martins [10] consider one-dimensional attractor for a dissipative system with a cylindrical phase space and give conditions for this attractor to be homeomorphic to the circle. Inspired by these two articles, we consider an attractor of a non-autonomous second order strongly damped lattice system with periodic driving force under Neumann boundary condition or periodic boundary condition. We obtain the existence of a global attractor and prove this attractor is homeomorphic to the circle. In particular, when $k=0$, the system (1.1) is the damped discretized wave equations, we prove the attractor is homeomorphic to the circle $\Gamma^1=\mathbb{R}/\omega_0\mathbb{Z}$ and this result is consistent with [10].

In the next section, we recall some notation and results regarding the attractor for second order strongly damped lattice system and prove the existence of global attractor. Finally, we prove the attractor is homeomorphic to the circle in Section 3.

## 2. Global attractor

Let $E=\mathbb{R}^{m^n}\times\mathbb{R}^{m^n}$, then system (1.2) is equivalent to the following initial value problem in E,

$$\begin{cases} \dot{Y}=CY+G(t,Y), \quad t>0, \\ Y(0)=Y_0=(y_0,y_{10})^T\in E, \end{cases} \tag{2.1}$$

where $Y=(y,\dot{y})^T$, $G(t,Y)=(0,-F(t,y))^T$, and

$$C=\begin{pmatrix} 0 & I \\ -A & -kA-\gamma I \end{pmatrix}.$$

By the assumptions (1.3), it is easy to check that the function $G(t,Y):E\to E$ is continuous differentiable and globally Lipschitz continuous with respect to $Y$. By the classical theory concerning the existence and uniqueness of the solutions of ordinary differential equations, we obtain the existence and uniqueness of solution $Y(t)$ for initial value problem (2.1) in $\mathbb{R}$.

For any $t\geqslant 0$, the mapping

$$U(t):Y_0=(y_0,y_{10})^T\to(y(t),\dot{y}(t))^T=Y(t,Y_0), \quad E\to E, \quad \forall t\geqslant 0,$$

where $Y(t,Y_0)$ is the solution of (2.1), then $\{U(t)|t\geqslant 0\}$ is the process in $E$.

Let $\widetilde{E}=e^{\perp}$ is the the orthogonal complement of $span\{e\}$ in $\mathbb{R}^{m^n}$, which is an invariant subspace of the linear operator $A$. Introducing an orthogonal projector $\widetilde{P}:\mathbb{R}^{m^n}\to\widetilde{E}$. Let $y$, $y^{(1)}$, $y^{(2)}\in\mathbb{R}^{m^n}$ with the components $y_i$, $y_i^{(1)}$, $y_i^{(2)}$ for $i=(i_1,i_2,\dots,i_n)\in\mathbb{Z}_m^n$, respectively, we define the weighted inner products and norms as follows:

$$(y^{(1)},y^{(2)})=\frac{1}{m^n}\sum_{i\in\mathbb{Z}_m^n}y_i^{(1)}y_i^{(2)}, \quad |y|=(y,y)^{1/2}=\left(\frac{1}{m^n}\sum_{i\in\mathbb{Z}_m^n}y_i^2\right)^{1/2}, \quad \|y\|=(Ay,y)^{1/2}.$$

It is easy to see that $|\cdot|$ is a norm in $\mathbb{R}^{m^n}$, and $\|\cdot\|$ is a norm in $\widetilde{E}$.
Let

$$E_0=(\widetilde{E},|\cdot|), \quad E_1=(\widetilde{E},\|\cdot\|),$$

and

$$V_0=(E_1\times\Gamma^1)\times(E_0\times\mathbb{R}), \quad V_1=E_1\times E_0,$$

where $\Gamma^1 = \mathbb{R}/\omega_0\mathbb{Z}$ is the one-dimensional torus. For any $y = (y_i)_{i\in\mathbb{Z}_m^n} \in \mathbb{R}^{m^n}$, write $\overline{y} = \widetilde{P}y$, then

$$(\overline{y}, e) = (y, e) - \left(\frac{1}{m^n}\sum_{i\in\mathbb{Z}_m^n} y_i\right)(e, e) = 0. \tag{2.2}$$

Then the solutions $(y(t), \dot{y}(t))^T$ of the system (2.1) can be decomposed into

$$y(t) = \overline{y}(t) + m(t)e, \qquad \dot{y}(t) = \dot{\overline{y}}(t) + \dot{m}(t)e,$$

where $m(t) = \frac{1}{m^n}\sum_{i\in\mathbb{Z}_m^n} y_i$ and

$$\overline{y}(t) = \widetilde{P}y(t) = y(t) - m(t)e \in \widetilde{E}, \qquad \dot{\overline{y}}(t) = \widetilde{P}\dot{y}(t) = \dot{y}(t) - \dot{m}(t)e \in \widetilde{E},$$

from (1.2), we have

$$\begin{cases} \ddot{\overline{y}} + kA\dot{\overline{y}} + A\overline{y} + \gamma\dot{\overline{y}} + \overline{F}(t, y) = 0, \\ \overline{y}(0) = \overline{y_0}, \qquad \dot{\overline{y}}(0) = \overline{y_{10}}, \end{cases} \tag{2.3}$$

where

$$\overline{F}(t, y) = F(t, y) - \left(\frac{1}{m^n}\sum_{i\in\mathbb{Z}_m^n} f(t, y_i)\right)e,$$

and

$$\overline{y}(0) = y(0) - \left(\frac{1}{m^n}\sum_{i\in\mathbb{Z}_m^n} y_i(0)\right)e, \qquad \dot{\overline{y}}(0) = \dot{y}(0) - \left(\frac{1}{m^n}\sum_{i\in\mathbb{Z}_m^n} \dot{y}_i(0)\right)e.$$

Let $Z = (\overline{y}, \overline{z})^T, \overline{z} = \dot{\overline{y}} + \varepsilon\overline{y}$, where $\varepsilon$ is

$$\varepsilon = \frac{k\lambda_1 + \gamma}{4 + 2(k\lambda_1 + \gamma)k + \gamma^2/\lambda_1}, \tag{2.4}$$

then the system (2.3) can be written as

$$Z_t + H_1(Z) = F_1(t, Z), \qquad Z(0) = (\overline{y_0}, \overline{y_{10}} + \varepsilon\overline{y_0})^T, \qquad t \geqslant 0, \tag{2.5}$$

where

$$F_1(t, Z) = \begin{pmatrix} 0 \\ -\overline{F}(t, y) \end{pmatrix}, \qquad H_1(Z) = \begin{pmatrix} \varepsilon\overline{y} - \overline{z} \\ A\overline{y} - \varepsilon(kA - \varepsilon)\overline{y} + (kA - \varepsilon)\overline{z} + \gamma(\overline{z} - \varepsilon\overline{y}) \end{pmatrix}. \tag{2.6}$$

We define a new weighted inner product and norm in $V_1$ as

$$(Z_1, Z_2)_{V_1} = \alpha(A\overline{y}_1, \overline{y}_2) + (\overline{z}_1, \overline{z}_2), \qquad |Z|_{V_1} = (Z, Z)_{V_1}^{1/2}, \tag{2.7}$$

for any $Z_1 = (\overline{y}_1, \overline{z}_1)^T, Z_2 = (\overline{y}_2, \overline{z}_2)^T \in V_1$, where $\alpha$ is

$$\alpha = \frac{4 + (k\lambda_1 + \gamma)k + \gamma^2/\lambda_1}{4 + 2(k\lambda_1 + \gamma)k + \gamma^2/\lambda_1} \in (\frac{1}{2}, 1). \tag{2.8}$$

Obviously, the norm $|\cdot|_{V_1}$ in (2.7) is equivalent to the usual norm $\|\cdot\|$.

LEMMA 1. *For any $Z=(\overline{y},\overline{z})^T\in V_1$, we have*

$$(H_1(Z),Z)_{V_1}\geqslant \delta|Z|^2_{V_1}+\frac{k}{2}\|\overline{z}\|^2+\frac{\gamma}{2}|\overline{z}|^2\geqslant \delta|Z|^2_{V_1}+\frac{k\lambda_1+\gamma}{2}|\overline{z}|^2, \tag{2.9}$$

*where*

$$\delta=\frac{k\lambda_1+\gamma}{\delta_1+\sqrt{\delta_1\delta_2}},\qquad \delta_1=4+(k\lambda_1+\gamma)k+\frac{\gamma^2}{\lambda_1},\qquad \delta_2=(k\lambda_1+\gamma)k+\frac{\gamma^2}{\lambda_1}. \tag{2.10}$$

*Proof.* Let $h(y)=\gamma y$ , the proof is similar to Lemma 2.1 in [8]. □

LEMMA 2. *Assume that $\mathscr{B}\subset V_0$ is such that $\widetilde{P}\mathscr{B}$ is bounded in $V_0$, and $\{y_0,w(0)=y_{10}\}$ is given in $\mathscr{B}$. Then there exists $t_0=t_0(\mathscr{B},\mathscr{B}_0)>0$ depending on $\mathscr{B}$ and $\mathscr{B}_0$ such that for $t\geqslant t_0, \widetilde{P}\{y(t),\frac{dy}{dt}(t)\}\subset\mathscr{B}_0$.*

*Proof.* Let $Z=(\overline{y},\overline{z})^T\in V_1$ be a solution of (2.5). Take the inner product $(\cdot,\cdot)_{V_1}$ of (2.5) with $Z$, we have

$$\frac{1}{2}\frac{d}{dt}|Z|^2_{V_1}=-(H_1(Z),Z)_{V_1}+(F_1(t,Z),Z)_{V_1}. \tag{2.11}$$

By (2.6) and (2.9),

$$-2(H_1(Z),Z)_{V_1}\leqslant -2\delta|Z|^2_{V_1}-(k\lambda_1+\gamma)|\overline{z}|^2, \tag{2.12}$$

for the second term on the right-hand side of (2.11), we have

$$\begin{aligned}2(F_1(t,Z),Z)_{V_1} &= -2\left(F(t,y)-\left(\frac{1}{m^n}\sum_{i\in\mathbb{Z}^n_m}f(t,y_i)\right)e,\overline{z}\right)\\ &\leqslant 2|\overline{z}|\left(\frac{1}{m^n}\sum_{j\in\mathbb{Z}^n_m}\left(f(t,y_j)-\frac{1}{m^n}\sum_{i\in\mathbb{Z}^n_m}f(t,y_i)\right)^2\right)^{1/2}\\ &\leqslant 4c|\overline{z}| \leqslant (k\lambda_1+\gamma)|\overline{z}|^2+\frac{4c^2}{k\lambda_1+\gamma},\end{aligned} \tag{2.13}$$

where $c$ is given by (1.3). By (2.11)-(2.13), we have

$$\frac{d}{dt}|Z|^2_{V_1}\leqslant -2\delta|Z|^2_{V_1}+\frac{4c^2}{k\lambda_1+\gamma}. \tag{2.14}$$

By Gronwall inequality, we have the absorbing inequality in $V_1$:

$$|Z(t)|^2_{V_1}\leqslant |Z(0)|^2_{V_1}e^{-2\delta t}+\frac{2c^2}{\delta(k\lambda_1+\gamma)}(1-e^{-2\delta t}),$$

i.e.,

$$\limsup_{t\to+\infty}|Z(t)|^2_{V_1} \leqslant \frac{4c^2}{\delta(k\lambda_1+\gamma)} = \rho_0. \tag{2.15}$$

If $y = y(t)$ is a solution of (1.2), then $\overline{y} = \widetilde{P}y$, the orthogonal projection of $y \in \mathbb{R}^{m^n}$ into $\overline{y} \in \widetilde{E}$ satisfies (2.15). We have $y(t) = \overline{y}(t) + m(t)e$, and

$$\frac{d^2m}{dt^2}(t) + \gamma\frac{dm}{dt}(t) + \frac{1}{m^n}\sum_{i\in Z_m^n} f(t,y_i) = 0.$$

By (1.3), we have

$$\begin{aligned}\left|\frac{dm}{dt}(t)\right| &\leqslant \left|\frac{dm}{dt}(0)\right|e^{-\gamma t} + \frac{1}{m^n}\left|\int_0^t \sum_{i\in Z_m^n}(-f(t,y_i))e^{-\gamma(t-\tau)}d\tau\right| \\ &\leqslant \left|\frac{dm}{dt}(0)\right|e^{-\gamma t} + \frac{c}{\gamma}(1-e^{-\gamma t}),\end{aligned}$$

i.e.,

$$\limsup_{t\to+\infty}\left|m'(y(t))\right| \leqslant \frac{2c}{\gamma} = a_0. \tag{2.16}$$

We note that the projection $\widetilde{P}:\mathbb{R}^{m^n} \to \widetilde{E}$ induces a projection on $V_0$, denoted by $\widetilde{P}$ again:

$$\widetilde{P}:\psi = \{y = \overline{y}+m(y), w = \overline{w}+m(w)\} \to \widetilde{P}\psi = \{\overline{y}, w = \overline{w}+m(w)\}, \quad V_0 \to V_0, \tag{2.17}$$

and we select in $\widetilde{P}V_0$ a bounded set $\mathscr{B}_0$ defined by

$$\|\{\overline{y},\overline{w}+\varepsilon\overline{y}\}\|_{V_0} \leqslant \rho_0', \quad |m(\overline{w})| \leqslant a_0',$$

where $\rho_0' > \rho_0, a_0' > a_0$ are defined by (2.15) and (2.16), respectively. The proof is completed. □

We observe that the change of $y$ into $y+\omega_0$ leaves the system (1.2) unchanged and it is natural to consider the function $y$ modulo $\omega_0$. Let $V_i = \widetilde{P}V_i \times \mathbb{R}$ $(i=0,1)$, such that if $\psi = \{y = \overline{y}+m(y), w = \overline{w}+m(w)\} \in V_i$ $(i=0,1)$, then $\psi \equiv \{\widetilde{P}\psi, m(y)\}$, where $\widetilde{P}\psi$ is defined by (2.17). Then we consider $\widetilde{V}_i = \widetilde{P}V_i \times \Gamma^1$ $(i=0,1)$, where $\Gamma^1 = \mathbb{R}/\omega_0\mathbb{Z}$ is a torus. We find that the process $\{U(t)\}_{t\geqslant 0}$ induces a process $\{\widetilde{U}(t)\}_{t\geqslant 0}$ on $\widetilde{V}_i$:

$$\{\overline{y_0}, y_{10}, m(y_0)(mod\ \omega_0)\} \to \{\overline{y}(t), \dot{y}(t), m(y(t))(mod\ \omega_0)\}.$$

Then Lemma 2 imply

LEMMA 3. *The sets* $\mathscr{B}_0 \times \Gamma^1$ *and* $\mathscr{B} \times \Gamma^1$ *are the absorbing set for the process* $\{\widetilde{U}(t)\}_{t\geqslant 0}$ *in* $V_0$ *and* $V_1$*, respectively.*

THEOREM 1. *The strongly damped lattice system (1.1) associated with a process* $\{\widetilde{U}(t)\}_{t\geqslant 0}$ *which operates in*

$$\widetilde{V}_0 = \mathbb{R}^{m^n-1} \times \mathbb{R}^{m^n} \times \Gamma^1$$

*possesses a global attractor* $\mathscr{A}$ *which attracts the bounded set in* $\widetilde{V}_0$*.*

## 3. One-dimensional attractor

In this section, we will establish the condition that the global attractor is homeomorphic to the circle $\Gamma^1$ of the strongly damped lattice system (1.1).

Defining the equivalence relation in $\mathbb{R}^{m^n}$ by

$$a \sim b \Leftrightarrow a - b \in \omega_0 e\mathbb{Z}.$$

The set of equivalence classes will be denoted by $\mathscr{C}$ and $\widehat{a}$ will denote the class of $a \in \mathbb{R}^{m^n}$. The space $\mathscr{C}$ is a metric space with the distance

$$d(\widehat{a}, \widehat{b}) = \inf_{y \in \widehat{a-b}} \|y\|.$$

Defining the function

$$\pi : \mathscr{C} \to \Gamma^1 = \mathbb{R}/\omega_0\mathbb{Z}.$$

The following hypothesis clarifies the notion of dissipation.

$(\mathbf{H_1})$ There exists a non-empty compact set $B \subset \mathscr{C}$ such that for every compact set $B_0 \subset \mathscr{C}$, there exists $\tau_0 \in \mathbb{R}$ in such a way that if $y$ is a solution with $y(t_0) \in B_0$, then $\widehat{y(t)} \in B, \forall t > t_0 + \tau_0$.

Notice that $G(t, Y + R) = G(t, Y)$, where $R = (\omega_0 e, 0)^T \in \mathbb{R}^{m^n} \times \mathbb{R}^{m^n}$ and $R \in KerC$. From the section 2, we know that $A$ is symmetric, let $\{e, e_2, e_3, e_4, \cdots, e_{m^n}\}$ be the orthonormal basis of $\lambda_1 = 0, \lambda_2, \lambda_3, \cdots, \lambda_{m^n}$ of $A$. Elementary computation show that

$$(k\lambda_i + \gamma)^2 - 4\lambda_i > 0, \quad i = 1, 2, \cdots, m^n, \tag{3.1}$$

i.e., when $k\gamma > 1$, spectral sets of $C$ are $\mu_1 = 0, \mu_2, \cdots, \mu_{m^n}$.

THEOREM 2. (Theorem 5 in [10]) *Assume that $(H_1)$ hold, there exists a nonsingular matrix $M \in M_{m^n \times m^n}(\mathbb{R})$ such that*

$$MCM^{-1} = diag\{0, \mu_2, \mu_3, \ldots, \mu_{m^n}\},$$

*and the function $G(t, Y) = MJ(t, M^{-1}Y)$ is K-Lipschitz on the second variable with $K < -\frac{\mu_2}{2}$. Then $\pi_{/\mathscr{A}}$ is an homeomorphism from $\mathscr{A}$ onto $\Gamma^1$.*

If $Q_1 = (e|e_2|\cdots|e_{m^n})$ is $m^n \times m^n$ matrix whose columns are formed by the eigenvectors of $A$, then $Q_1^T = Q_1^{-1}$ and $Q_1^T A Q_1 = diag(0, \lambda_2, \cdots, \lambda_{m^n})$. On the other hand, if $Q_2 = \begin{pmatrix} Q_1^T & 0 \\ 0 & Q_1^T \end{pmatrix}$, then

$$Q_2 C Q_2^{-1} = \begin{pmatrix} 0 & I \\ -diag(0, \lambda_2, \cdots, \lambda_{m^n}) & -k\, diag(0, \lambda_2, \cdots, \lambda_{m^n}) - \gamma I \end{pmatrix}.$$

Defining $Q_3$ :

$$Q_3 : \mathbb{R}^{2m^n} \to \mathbb{R}^{2m^n}$$

$$(y_1, y_2, \dots, y_{m^n}, \dot{y}_1, \dot{y}_2, \dots, \dot{y}_{m^n}) \to (y_1, \dot{y}_1, y_2, \dot{y}_2, \dots, y_{m^n}, \dot{y}_{m^n})$$

is such that

$$Q_3Q_2CQ_2^{-1}Q_3^{-1} = \begin{pmatrix} 0 & 1 & & & & \\ 0 & -\gamma & & & \cdots & 0 \\ & & 0 & 1 & & \\ & & -\lambda_2 & -k\lambda_2 - \gamma & & \\ \vdots & & & & \ddots & \vdots \\ & & & & 0 & 1 \\ 0 & & \cdots & & -\lambda_{m^n} & -k\lambda_{m^n} - \gamma \end{pmatrix}.$$

For every

$$A_i = \begin{pmatrix} 0 & 1 \\ -\lambda_i & -k\lambda_i - \gamma \end{pmatrix}, i = 1, \cdots, m^n,$$

assume that $k\gamma > 1$, the matrix

$$P_i = \begin{pmatrix} \frac{1}{\sqrt{2}\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}} & \frac{1}{\sqrt{2}\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}} \\ -\frac{k\lambda_i+\gamma}{2\sqrt{2}\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}} + \frac{1}{2\sqrt{2}} & -\frac{k\lambda_i+\gamma}{2\sqrt{2}\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}} - \frac{1}{2\sqrt{2}} \end{pmatrix}$$

$$P_i^{-1} = \begin{pmatrix} \frac{k\lambda_i+\gamma}{\sqrt{2}} + \frac{\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}}{\sqrt{2}} & \sqrt{2} \\ -\frac{k\lambda_i+\gamma}{\sqrt{2}} + \frac{\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}}{\sqrt{2}} & -\sqrt{2} \end{pmatrix}$$

are such that

$$P_i^{-1}A_iP_i = diag\left(-\frac{k\lambda_i+\gamma}{2} + \frac{\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}}{2}, -\frac{k\lambda_i+\gamma}{2} - \frac{\sqrt{(k\lambda_i+\gamma)^2-4\lambda_i}}{2}\right).$$

We conclude that

$$Q_4 = \begin{pmatrix} P_1^{-1} & & 0 \\ & \ddots & \\ 0 & & P_{m^n}^{-1} \end{pmatrix} \tag{3.2}$$

is such that $Q_4Q_3Q_2CQ_2^{-1}Q_3^{-1}Q_4^{-1}$ is diagonal and have on its diagonal entries the eigenvalues of $C$. Notice that if

$$k\gamma > 1, \;\; k > (2\sqrt{\lambda_{m^n}} - \alpha)/\lambda_{m^n},$$

then the non-null eigenvalues of $C$ are all real and negative. Moreover, the largest non-null eigenvalue of $C$ is

$$-\frac{k\lambda_2+\gamma}{2} + \frac{\sqrt{(k\lambda_2+\gamma)^2-4\lambda_2}}{2}.$$

In the following, we will prove the attractor $\mathscr{A}$ is homeomorphic to the circle $\Gamma^1 = \mathbb{R}/\omega_0\mathbb{Z}$ using Theorem 2.

LEMMA 4. *Assume that* $k\gamma > 1, k > (2\sqrt{\lambda_{m^n}} - \alpha)/\lambda_{m^n}$ *hold. If* $Y = (0,z)^T \in \mathbb{R}^{2m^n}$*, then* $\|Q_4Q_3Q_2Y\| = 2\|z\|$ *and for any* $Y = (y,z)^T \in \mathbb{R}^{2m^n}$*, we have*

$$\|Q_4Q_3Q_2Y\| \geqslant \sqrt{(k\lambda_{m^n}+\gamma)^2 - 4\lambda_{m^n}}\|y\|.$$

*Proof.* The first equality is a straightforward consequence by (3.2). Given $Y = (y,z)^T \in \mathbb{R}^{2m^n}$, because $\lambda_{m^n}$ is the largest of the eigenvalues of $A$, then

$$\begin{aligned}
\|Q_4Q_3Q_2Y\| &= \|Q_4(ey, ez, e_2y, e_2z, \cdots, e_{m^n}y, e_{m^n}z)\| \\
&= \|\left(\frac{k\lambda_1+\gamma}{\sqrt{2}} + \frac{\sqrt{(k\lambda_1+\gamma)^2-4\lambda_1}}{\sqrt{2}}\right)ey + \sqrt{2}ez, \\
&\qquad \left(-\frac{k\lambda_1+\gamma}{\sqrt{2}} + \frac{\sqrt{(k\lambda_1+\gamma)^2-4\lambda_1}}{\sqrt{2}}\right)ey - \sqrt{2}ez, \cdots\| \\
&= \sqrt{((k\lambda_1+\gamma)^2-4\lambda_1)(ey)^2 + 4(\frac{k\lambda_1+\gamma}{2}ey+ez)^2 + \cdots} \\
&\geqslant \sqrt{((k\lambda_1+\gamma)^2-4\lambda_1)(ey)^2 + \cdots + ((k\lambda_{m^n}+\gamma)^2-4\lambda_{m^n})(e_{m^n}y)^2} \\
&\geqslant \min_{i=1,\dots,m^n}\sqrt{((k\lambda_i+\gamma)^2-4\lambda_i)}\|Q_1^Ty\| = \sqrt{((k\lambda_{m^n}+\gamma)^2-4\lambda_{m^n}}\|y\|.
\end{aligned}$$

The proof is completed. □

THEOREM 3. *Suppose that* $k\gamma > 1$*, and* $f(t,y)$ *is K-Lipschitz on* $y$*, if*

$$K < \frac{\sqrt{(k\lambda_{m^n}+\gamma)^2-4\lambda_{m^n}}}{8}\left(k\lambda_2+\gamma-\sqrt{(k\lambda_2+\gamma)^2-4\lambda_2}\right),$$

*then* $\pi_{/\mathscr{A}}$ *is an homeomorphism from* $\mathscr{A}$ *onto* $\Gamma^1$.

*Proof.* By Theorem 2, we only need to show that the Lipschitz constant of

$$J(t,Y) = Q_4Q_3Q_2G(t,\ Q_2^{-1}Q_3^{-1}Q_4^{-1}Y)$$

on the second variable $y$ is less than

$$\frac{(k\lambda_2+\gamma)-\sqrt{(k\lambda_2+\gamma)^2-4\lambda_2}}{4}.$$

For any

$$Z = Q_4Q_3Q_2(y,z)^T, \quad Z' = Q_4Q_3Q_2(y',z')^T \in \mathbb{R}^{2m^n}$$

and $t \in \mathbb{R}$, from Lemma 4, we have

$$\begin{aligned}
\|J(t,Z) - J(t,Z')\| &= \|Q_4Q_3Q_2(0, -F(t,y)+F(t,y')\| \\
&= 2\|F(t,y)-F(t,y')\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2K\|y-y'\| \\
&\leqslant \frac{\sqrt{(k\lambda_{m^n}+\gamma)^2-4\lambda_{m^n}}}{4}\left(k\lambda_2+\gamma-\sqrt{(k\lambda_2+\gamma)^2-4\lambda_2}\right)\|y-y'\| \\
&\leqslant \frac{\left(k\lambda_2+\gamma-\sqrt{(k\lambda_2+\gamma)^2-4\lambda_2}\right)}{4}\|Z-Z'\|.
\end{aligned}
$$

The proof is completed. □

REMARK 1. When $k=0$, the system (1.1) is the damped discretized wave equations

$$
\begin{cases} \ddot{y}_i+(Ay)_i+\gamma_i\dot{y}_i+f(t,y_i)=0, \\ y_i(0)=y_{i0}, \qquad \dot{y}_i(0)=y_{i,10}, \end{cases}
$$

when $\lambda_{m^n}<\gamma^2/4$, the attractor is homeomorphic to the circle $\Gamma^1=\mathbb{R}/\omega_0\mathbb{Z}$, this result is consistent with [10].

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