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SINGLE–POINT BLOW–UP FOR A SEMILINEAR REACTION–DIFFUSION SYSTEM

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Abstract. In this paper, we consider positive solutions of the system

$$u_t - \Delta u = u^r v^p, \quad v_t - \Delta v = u^q v^s$$

$t \in (0,T)$, $x \in \mathbf{B}(0,R) = \{x \in \mathbb{R}^n \,|\, |x| < R\}$ or $x \in \mathbb{R}^n$ and $p,q,r,s > 1$. We prove single-point blow-up if $r < q+1$ and $s < p+1$ and for a large class of radial decreasing solutions. This extends the result of Friedman and Giga for this basic system known only for $p = q = r = s$. We also obtain lower pointwise estimates for the blow-up profiles.

1. Introduction and main results

Let us consider the following reaction-diffusion system:

$$\begin{cases} u_t - \Delta u = |u|^{r-1}u\,|v|^{p-1}v, & x \in \Omega, t > 0, \\ v_t - \Delta v = |u|^{q-1}u\,|v|^{s-1}v, & x \in \Omega, t > 0, \\ u = v = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $p,q,r,s > 1$, $\Omega = \mathbf{B}(0,R) = \{x \in \mathbb{R}^n \,|\, |x| < R\}$ with $R > 0$ or $\Omega = \mathbb{R}^n$, $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$ and u_0, v_0 are positive, radially symmetric and radially nonincreasing. It is known that (1.1) has a unique *positive*, *radially symmetric* and *radially nonincreasing* maximal solution on $[0,T^*) \times \Omega$, *classical* for $t > 0$, i.e. $u, v > 0$, u, v depend only on $\rho = |x|$ at a given t (we shall identify $(u(t,x), v(t,x))$ and $(u(t,\rho), v(t,\rho))$, $u_\rho, v_\rho \leqslant 0$, where u_ρ, v_ρ denote the derivatives of u, v with respect to ρ, and u, $v \in C((0,T^*) \times [0,R]) \cap C^{1,2}((0,T^*) \times (0,R))$. This follows by standard contraction mapping argument. Moreover, if $T^* < \infty$, then

$$\limsup_{t \to T^*} (\|u(t)\| + \|v(t)\|_\infty) = +\infty, \tag{1.2}$$

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we say that the solution *blows up* in finite time with *blow-up time* T^*. See [1, 4, 12]. We also known that, if

$$r < q+1 \quad \text{and} \quad s < p+1$$

then the blow-up is *simultaneous*, i.e.

$$\limsup_{t\to T^*} \|u(t)\|_\infty = \limsup_{t\to T^*} \|v(t)\|_\infty = +\infty.$$

See [7, Theorem 4.1].

Throughout the paper, we assume that

$$pq-(r-1)(s-1) \neq 0 \tag{1.3}$$

and we shall use the notation

$$\alpha = \frac{p+1-s}{pq-(r-1)(s-1)}, \quad \beta = \frac{q+1-r}{pq-(r-1)(s-1)}. \tag{1.4}$$

By (1.3) and $p, q, r, s > 1$, for $r < q+1$ and $s < p+1$, we have $\alpha, \beta > 0$.

The purpose of this paper is to localize the blow-up points for (u, v) the solution of the system (1.1). Let us mention that Friedman and Giga ([5]) proved that the solution of the following system

$$\begin{cases} u_t - \Delta u = |v|^{p-1}v, \ x \in \Omega, t > 0, \\ v_t - \Delta v = |u|^{q-1}u, \ x \in \Omega, t > 0, \end{cases} \tag{1.5}$$

blows up at the single point $x = 0$ for a symmetric decreasing initial data, $n = 1$ and under the very restrictive condition $p = q$. Moreover, they extended their result to the system

$$\begin{cases} u_t - \Delta u = f(u, v), \quad x \in \Omega, t > 0, \\ v_t - \Delta v = g(u, v), \quad x \in \Omega, t > 0, \end{cases}$$

where the functions f and g satisfy some hypotheses, see [5, pp. 75-76]. Under these hypotheses, the solution of our system blows-up at the single point $x = 0$ only under the condition $p = q = r = s$. More recently, Souplet proved that single-point blow-up occurs for the system (1.5) for a large class of radial decreasing solutions, in a ball or in the whole space and without the condition $p = q$, see [10]. Note that for the system (1.5) only simultaneous blow-up occurs.

It is therefore natural to ask whether and under which conditions $x = 0$ is the single-point blow-up point for system (1.1). An answer to this question is given by the following theorem, which is the main result of this paper.

THEOREM 1. *Let* $\Omega = \mathbf{B}(0, R)$ *and* $p, q, r, s > 1$ *be such that*

$$r < q+1 \quad \textit{and} \quad s < p+1. \tag{1.6}$$

Let α, β be given by (1.4). Let (u,v) be a positive, radially symmetric and classical solution of (1.1) such that u_ρ, $v_\rho \leqslant 0$ and $T^* < \infty$. Assume that (u,v) satisfies the upper blow-up estimates:

$$\sup_{0<t<T^*} (T^*-t)^{\alpha}\|u(t)\|_\infty < \infty, \qquad \sup_{0<t<T^*} (T^*-t)^{\beta}\|v(t)\|_\infty < \infty. \tag{1.7}$$

Then blow-up occurs only at the origin, i.e.

$$\sup_{0<t<T^*} (u(t,\rho)+v(t,\rho)) < \infty, \quad \textit{for all } \rho \in (0,R). \tag{1.8}$$

REMARKS 1. (i) The upper blow-up estimates (1.7) are known to be true if we assume in addition that

$$u_t, v_t \geqslant 0 \quad \text{on } (0,T^*)\times \mathbf{B}_R,$$

where u_t, v_t denote the derivatives of u, v with respect to t, and if there exists a positive constant ε such that

$$\begin{cases} \Delta u_0 + (1-\varepsilon)u_0^r v_0^p \geqslant 0, \\ \Delta v_0 + (1-\varepsilon)u_0^q v_0^s \geqslant 0. \end{cases}$$

We note that the existence of a positive, radially symmetric and classical solution of (1.1) such that $u_\rho, v_\rho \leqslant 0$, $u_t, v_t \geqslant 0$ and $T^* < \infty$, can be obtained for initial data $(\lambda u_0, \lambda v_0)$ with $\lambda > 0$ large enough, whenever

$$\begin{cases} u_0,\, v_0 \quad \text{are positive, radially symmetric and nonincreasing}, \\ \Delta u_0 + u_0^r v_0^p \geqslant 0, \\ \Delta v_0 + u_0^q v_0^s \geqslant 0. \end{cases}$$

See [7]. We also refer to [2, 11] for other results related to properties (1.7).

(ii) The result of Theorem 1 remains true for the Cauchy problem (1.1) (that is, $R=\infty$) provided u_0, v_0 are not both constant. This follows from straightforward modifications of the proof.

Our second aim is to establish pointwise lower bounds on the blow-up profiles.

THEOREM 2. *Let $\Omega = \mathbf{B}(0,R)$ and $p,q,r,s > 1$ be such that*

$$r < q+1 \quad \textit{and} \quad s < p+1.$$

Let (u,v) be a positive, radially symmetric and classical solution of (1.1) such that $u_\rho, v_\rho \leqslant 0$, $u_t, v_t \geqslant 0$, $T^* < \infty$ and satisfies the upper blow-up estimates (1.7). Then we have the estimates

$$|x|^{2\alpha} u(T^*,x) \geqslant c_1, \quad 0 < |x| < \eta$$

and

$$|x|^{2\beta} v(T^*,x) \geqslant c_2, \quad 0 < |x| < \eta,$$

for some c_1, c_2, $\eta > 0$.

The organization of this paper is as follows. In the second section, we prove asymptotic comparison properties between the components of the solution (u,v) near blow-up points. Next, we use them to prove Theorem 1 in the third section. Finally, in Section 4, we establish the pointwise lower bounds on the blow-up profiles, which proves Theorem 2.

2. Asymptotic comparison of components

As in [5] (and cf. [6]) the basic idea for proving single-point blow-up is to apply the maximum principle to a couple (G,J) of functions of the form

$$G(t,\rho) = u_\rho + \varepsilon c(\rho) u^\gamma, \quad J(t,\rho) = v_\rho + \varepsilon d(\rho) v^{\overline{\gamma}}.$$

However, this turns out to require good comparison properties between u and v. Due to the general comparison properties used in [5], the result there for system (1.1) imposes the severe restriction $p=q=r=s$ (see Remark 1 below). To overcome this, we follow the strategy in [10]. Namely, instead of looking for comparison properties valid everywhere, we assume for contradiction that single-point blow-up fails (i.e., that blow-up occurs everywhere in a ball near the origin), and we prove *asymptotic comparison properties between components near blow-up points*. It turns out that they can be obtained without making any extra assumption on the exponents p,q,r,s and they are sufficient to handle the system satisfied by suitable functions of the form G,J.

This section is devoted to the derivation of such comparison properties. They are given by the following lemma.

LEMMA 1. *Let $\Omega = \mathbf{B}(0,R)$ and $p,q,r,s>1$ be such that*

$$r < q+1, \quad s < p+1.$$

Let (u,v) be a positive, radially symmetric and classical solution of (1.1), such that u_ρ, $v_\rho \leqslant 0$ and $T^* < \infty$. Assume that (u,v) satisfies the upper estimates (1.7). If there exists $\rho_0 \in (0,R)$ such that

$$\limsup_{t\to T^*} \big(u(t,\rho_0) + v(t,\rho_0)\big) = \infty,$$

then for all $0 \leqslant \rho < \rho_0$, we have

$$\lim_{t\to T^*} (T^*-t)^\alpha u(t,\rho) = A_0, \quad \lim_{t\to T^*} (T^*-t)^\beta v(t,\rho) = B_0, \tag{2.1}$$

where

$$A_0 = \left(\beta^p \alpha^{1-s}\right)^{\frac{1}{pq-(s-1)(r-1)}}, \quad B_0 = \left(\alpha^q \beta^{1-r}\right)^{\frac{1}{pq-(s-1)(r-1)}}. \tag{2.2}$$

In particular,

$$\lim_{t\to T^*} \left[\frac{u^{q+1-r}}{v^{p+1-s}}\right](t,\rho) = A_0^{q+1-r} B_0^{-(p+1)+s},$$

$$\textit{uniformly on } [0,\rho_1], \textit{ for each } \rho_1 \in (0,\rho_0). \tag{2.3}$$

This lemma will be proved in Subsection 2.3. As in [10], we prepare a number of preliminaries and of auxiliary results that will be given in Subsections 2.1 and 2.2. The main idea of the proof is to identify suitable space limits of rescaled solutions in terms of an ODE system, and of a criterion for excluding blow-up at a given point.

REMARK 1. Let us mention that in [5], Friedman and Giga prove single-point blow-up for positive solutions of the following system

$$u_t - \Delta u = f(u,v), \quad v_t - \Delta v = g(u,v)$$

under the hypotheses $u \leqslant C(v+1)$ and $v \leqslant C(v+1)$, and the functions f and g satisfy the following conditions:

$$f, g \in C^1(R^2); \tag{2.4}$$
$$f(u,v),\ g(u,v) > 0,\ f_u, f_v, g_u, g_v \geqslant 0 \quad \text{if } u > 0, v > 0; \tag{2.5}$$
$$c_1 f \leqslant u f_u + v f_v, \quad \text{if} \quad v > M,\ u > \frac{v}{C} - 1 \text{ and for some constants } M,\ c_1 > 1; \tag{2.6}$$
$$c_2 g \leqslant u g_u + v g_v, \quad \text{if} \quad u > M,\ v > \frac{u}{C} - 1 \text{ and for some constants } M,\ c_2 > 1; \tag{2.7}$$
$$g(y+1, \varepsilon y) \geqslant \varepsilon f(y+1, \varepsilon y), \text{ for some small } \varepsilon > 0 \text{ and for all } y \geqslant 0; \tag{2.8}$$
$$f(y+1, \varepsilon y) \geqslant \varepsilon g(y+1, \varepsilon y), \text{ for some small } \varepsilon > 0 \text{ and for all } y \geqslant 0. \tag{2.9}$$

Then, for the particular case $f(u,v) = v^p u^r$ and $g(u,v) = u^q v^s$, the result of Friedman and Giga for the system (1.1) is true only in the very restrictive case $p = q = r = s$. Indeed, let $f(u,v) = v^p u^r$ and $g(u,v) = u^q v^s$, we obtain

$$u f_u + v f_v = p v^p u^r + r v^p u^r \geqslant c_1(v^p u^r), \quad \text{where } c_1 = \min(p,r)$$
$$u g_u + v g_v = q u^q v^s + s v^s u^q \geqslant c_2(v^s u^q), \quad \text{where } c_2 = \min(q,s).$$

Moreover, the hypotheses (2.8) and (2.9) imply that

$$\varepsilon^s (y+1)^q y^s \geqslant \varepsilon^{p+1} (y+1)^r y^p,$$
$$\varepsilon^p (y+1)^r y^p \geqslant \varepsilon^{s+1} (y+1)^q y^s.$$

Then, taking y in the neighborhood of 0 and then in neighborhood of ∞, we obtain

$$y^{s-p} \geqslant \varepsilon^{p+1-s},$$
$$y^{p-s} \geqslant \varepsilon^{s+1-p},$$
$$y^{q+s-(p+r)} \geqslant \varepsilon^{p+1-s},$$
$$y^{p+r-(q+s)} \geqslant \varepsilon^{s+1-p}.$$

So that these inequalities are valid, it necessary to choose p, q, r and s such that $p = q = r = s$.

2.1. Local criterion for excluding blow-up

As in [10], in this section, we allow Ω to be an arbitrary smooth domain in $\mathbb{R}^n$. We also allow sign-changing solutions of (1.1). Let $b \in \Omega$, we define the similarity variables around (T^*, b) by

$$\sigma = -\log(T^* - t), \quad y = \frac{x-b}{\sqrt{T^*-t}} = e^{\sigma/2}(x-b),$$

and the rescaled solution by

$$\begin{cases} w = w_b(\sigma, y) = (T^* - t)^{\alpha} u(t,x), \\ z = z_b(\sigma, y) = (T^* - t)^{\beta} v(t,x), \end{cases}$$

on

$$D := \{(\sigma, y) \mid b + ye^{-\sigma/2} \in \Omega, \sigma_0 < \sigma < \infty\}, \quad \sigma_0 = -\log T^*.$$

In similarity variables, the partial differential equations in system (1.1) read

$$\begin{cases} w_\sigma - \mathscr{L} w = |w|^{r-1} w |z|^{p-1} z - \alpha w, & (\sigma, y) \in D, \\ z_\sigma - \mathscr{L} z = |w|^{q-1} w |z|^{s-1} z - \beta z, & (\sigma, y) \in D, \end{cases}$$

where

$$\mathscr{L} = \Delta - \frac{y}{2}.\nabla = K^{-1}\nabla.(K\nabla), \quad K(y) = (4\pi)^{-n/2} e^{-|y|^2/4}, \quad \forall\, y \in \mathbb{R}^n.$$

We denote by $(T(\sigma))_{\sigma \geqslant 0}$ the semigroup associated with $\mathscr{L}$. More precisely, for each $\phi \in L^\infty(\mathbb{R}^n)$, we set $T(\sigma)\phi := w(\sigma, .)$, where w is the unique solution of

$$\begin{cases} w_\sigma = \mathscr{L} w, & y \in \mathbb{R}^n,\ \sigma > 0, \\ w(0,y) = \phi(y), & y \in \mathbb{R}^n. \end{cases}$$

For any $\phi \in L^\infty(\mathbb{R}^n)$, we put

$$\|\phi\|_{L^l_K} = \left(\int_{\mathbb{R}^n} |\phi(y)|^l K(y) dy \right)^{1/l}, \quad 1 \leqslant l < \infty.$$

Let $1 \leqslant k < l < \infty$, then

$$\|\phi\|_{L^k_K} \leqslant c(k,l,n) \|\phi\|_{L^l_K}, \quad 1 \leqslant k < l < \infty. \tag{2.10}$$

If the function ϕ is defined only on a subdomain of $\mathbb{R}^n$, then $\|\phi\|_{L^r_K}$ denotes the norm of the extension of ϕ by 0 on $\mathbb{R}^n$.

The semigroup $(T(\sigma))_{\sigma \geqslant 0}$ has the following two properties (see e.g. [10]):

(1) (*Contraction*) For any $1 \leqslant l < \infty$,

$$\|T(\sigma)\phi\|_{L^l_K} \leqslant \|\phi\|_{L^l_K}, \quad \sigma \geqslant 0, \phi \in L^\infty(\mathbb{R}^n). \tag{2.11}$$

(2) (*Delayed regularizing effect*) For any $1 \leqslant m < l < \infty$, there exist $C_0,\ \sigma^* > 0$ such that

$$\|T(\sigma)\phi\|_{L^l_K} \leqslant C_0\|\phi\|_{L^m_K}, \quad \sigma \geqslant \sigma^*, \phi \in L^\infty(\mathbb{R}^n). \tag{2.12}$$

REMARK 2. Let $\Omega = \mathbf{B}_R$ and u, v radially symmetric, then write $u(t,x) = U(t,\rho)$, $v(t,x) = V(t,\rho)$ with $\rho = |x|$. We set

$$\begin{cases} W = W_b(\sigma,\theta) = (T^* - t)^\alpha U(t,\rho), \\ Z = Z_b(\sigma,\theta) = (T^* - t)^\beta V(t,\rho), \\ \theta = \frac{\rho - |b|}{\sqrt{T^*-t}}. \end{cases}$$

Then (W,Z) is a solution of

$$\begin{cases} W_\sigma - W_{\theta\theta} + \frac{\theta}{2}W_\theta = |W|^{r-1}W\,|Z|^{p-1}Z - \alpha W + \frac{(n-1)e^{-\sigma/2}}{|b|+e^{-\sigma/2}\theta}W_\theta, & (\sigma,\theta) \in \widetilde{D}, \\ Z_\sigma - Z_{\theta\theta} + \frac{\theta}{2}Z_\theta = |W|^{q-1}W\,|Z|^{s-1}Z - \beta Z + \frac{(n-1)e^{-\sigma/2}}{|b|+e^{-\sigma/2}\theta}Z_\theta, & (\sigma,\theta) \in \widetilde{D}, \end{cases} \tag{2.13}$$

with

$$\widetilde{D} := \left\{(\sigma,\theta) \mid -|b|e^{\sigma/2} < \theta < (R-|b|)e^{\sigma/2},\ \sigma_0 < \sigma < \infty\right\},\ \sigma_0 = -\log(T^*).$$

We also note that

$$w_b(\sigma,y) = W_b\big(\sigma,\ |be^{\sigma/2} + y| - |b|e^{\sigma/2}\big). \tag{2.14}$$

And a similar relation holds for Z_b.

PROPOSITION 1. *Let $M > 0$ and (u,v) be a classical solution of (1.1) such that $T^* < \infty$ and*

$$(T^* - t)^\alpha\|u(t)\|_\infty \leqslant M, \quad (T^* - t)^\beta\|v(t)\|_\infty \leqslant M, \quad 0 < t < T^*. \tag{2.15}$$

Let $a \in \Omega$ and let (w,z) be the rescaled solution by similarity variables around (T^*,a). There exists $\varepsilon = \varepsilon(n,p,q,s,r,M)$ such that, if

$$\|w(\sigma_1)\|_{L^1_K} + \|z(\sigma_1)\|_{L^1_K} < \varepsilon \tag{2.16}$$

for some $\sigma_1 \geqslant \sigma_0 = -\log(T^*)$, then a is not a blow-up point of (u,v), i.e. (u,v) is uniformly bounded in the neighborhood of (T^*,a).

Proof. For given $\sigma_2 \geqslant \sigma_0$, we denote respectively by $\overline{w}$ and $\overline{z}$ the solution of

$$\begin{cases} \overline{w}_\sigma - \mathscr{L}\overline{w} = |\widetilde{w}|^r\,|\widetilde{z}|^p - \alpha\overline{w}, & y \in \mathbb{R}^n, \sigma > \sigma_2, \\ \overline{w}(\sigma_2,y) = |\widetilde{w}(\sigma_2,y)|, & y \in \mathbb{R}^n, \end{cases}$$

and

$$\begin{cases} \overline{z}_\sigma - \mathscr{L}\overline{z} = |\widetilde{w}|^q\,|\widetilde{z}|^s - \beta\overline{z}, & y\in\mathbb{R}^n, \sigma>\sigma_2,\\ \overline{z}(\sigma_2,y) = |\widetilde{z}(\sigma_2,y)|, & y\in\mathbb{R}^n,\end{cases}$$

where $\widetilde{w}$ and $\widetilde{z}$ denote the extensions by 0 of w and z to the whole of $\mathbb{R}^n$. We note that $(\overline{w},\overline{z})$ exists globally. We denote by

$$\begin{cases} \overline{u} = (T^*-t)^{-\alpha}\overline{w}\Big(-\log(T^*-t), \frac{x-b}{\sqrt{T^*-t}}\Big), & (t,x)\in(t_0,T^*)\times\Omega,\\ \overline{v} = (T^*-t)^{-\beta}\overline{z}\Big(-\log(T^*-t), \frac{x-b}{\sqrt{T^*-t}}\Big), & (t,x)\in(t_0,T^*)\times\Omega.\end{cases}$$

$(\overline{u},\overline{v})$ is a solution of the following system:

$$\begin{cases} \overline{u}_t - \Delta\overline{u} = |u|^r\,|v|^p, & x\in\Omega, t>t_0,\\ \overline{v}_t - \Delta\overline{v} = |u|^q\,|v|^s, & x\in\Omega, t>t_0,\\ \overline{u}(t_0,x) = |u(t_0,x)|, \ x\in\Omega,\\ \overline{v}(t_0,x) = |v(t_0,x)|, \ x\in\Omega.\end{cases}$$

By the maximum principle, we obtain $|u|\leqslant\overline{u}$ and $|v|\leqslant\overline{v}$ for all $t\geqslant t_0$. Return to $(\overline{w},\overline{z})$, we obtain that $|\widetilde{w}|\leqslant\overline{w}$ and $|\widetilde{z}|\leqslant\overline{z}$ for any $\sigma\geqslant\sigma_2$. By the variation constants formula, we deduce that

$$\begin{aligned} &|\widetilde{w}(\sigma_2+\sigma)| \leqslant \overline{w}(\sigma_2+\sigma)\\ &\quad = e^{-\alpha\sigma}T(\sigma)|\widetilde{w}(\sigma_2)| + \int_0^\sigma e^{-\alpha(\sigma-\tau)}T(\sigma-\tau)\Big(|\widetilde{w}(\sigma_2+\tau)|^r|\widetilde{z}(\sigma_2+\tau)|^p\Big)d\tau\\ &\quad = e^{-\alpha\sigma}T(\sigma)|\widetilde{w}(\sigma_2)| + \int_0^\sigma e^{-\alpha\sigma}e^{\alpha\tau}T(\sigma-\tau)\Big(|\widetilde{w}(\sigma_2+\tau)|^r|\widetilde{z}(\sigma_2+\tau)|^p\Big)d\tau.\end{aligned}$$

Then, we have

$$\begin{aligned} &e^{\alpha\sigma}|\widetilde{w}(\sigma_2+\sigma)|\\ &\quad \leqslant T(\sigma)|\widetilde{w}(\sigma_2)| + \int_0^\sigma e^{\alpha\tau}T(\sigma-\tau)\Big(|\widetilde{w}(\sigma_2+\tau)|^r|\widetilde{z}(\sigma_2+\tau)|^p\Big)d\tau. \end{aligned} \tag{2.17}$$

By exchanging the roles of $\widetilde{w}$, p and r and $\widetilde{z}$, q and s in (2.17), we obtain

$$\begin{aligned} &e^{\beta\sigma}|\widetilde{z}(\sigma_2+\sigma)|\\ &\quad \leqslant T(\sigma)|\widetilde{z}(\sigma_2)| + \int_0^\sigma e^{\beta\tau}T(\sigma-\tau)\Big(|\widetilde{w}(\sigma_2+\tau)|^q|\widetilde{z}(\sigma_2+\tau)|^s\Big)d\tau, \end{aligned} \tag{2.18}$$

for all $\sigma_2\geqslant\sigma_0$, $\sigma>0$. By (2.15), we have

$$\|w(\sigma)\|_\infty = (T^*-t)^\alpha\|u(t)\|_\infty \leqslant M$$

and $\|z(\sigma)\|_\infty = (T^*-t)^\beta\|v(t)\|_\infty\leqslant M$, for all $\sigma\geqslant\sigma_0$. Then the function $h := \overline{w}+\overline{z}$ satisfies

$$h_\sigma - \mathscr{L}h = \overline{w}_\sigma + \overline{z}_\sigma - \mathscr{L}\overline{w} - \mathscr{L}\overline{z} = |\widetilde{w}|^r|\widetilde{z}|^p + |\widetilde{w}|^q|\widetilde{z}|^s - \alpha\overline{w} - \beta\overline{z}$$

$$
\begin{aligned}
&\leqslant M^{p-1+r}|\widetilde{z}| + M^{q-1+s}|\widetilde{w}| \\
&\leqslant C_1(|\widetilde{w}| + |\widetilde{z}|) \\
&\leqslant C_1(\overline{w} + \overline{z}) \\
&= C_1 h,
\end{aligned}
$$

with $C_1 = \max(M^{q-1+s}, M^{p-1+r})$. Therefore

$$h(\sigma) \leqslant e^{C_1\sigma} T(\sigma) h(\sigma_2), \quad \text{for all } \sigma_2 \geqslant \sigma_0,\ \sigma > 0.$$

Therefore

$$
\begin{aligned}
|\widetilde{w}(\sigma_2+\sigma)| + |\widetilde{z}(\sigma_2+\sigma)| &\leqslant \overline{w}(\sigma_2+\sigma) + \overline{z}(\sigma_2+\sigma) \\
&\leqslant e^{C_1\sigma} T(\sigma)(|\widetilde{w}(\sigma_2)| + |\widetilde{z}(\sigma_2)|), \text{ for all } \sigma_2 \geqslant \sigma_0,\ \sigma > 0. \qquad (2.19)
\end{aligned}
$$

Fix a finite l such that

$$l > \max(2,n)\max(p,q,r,s) \qquad (2.20)$$

and let σ^* be given by (2.12), with $m = 1$. For σ_1 given by (2.16), then by (2.19) and (2.11), we obtain

$$
\begin{aligned}
\|\widetilde{w}(\sigma_1+\sigma)\|_{L^1_K} + \|\widetilde{z}(\sigma_1+\sigma)\|_{L^1_K} &= \|(|\widetilde{w}| + |\widetilde{z}|)(\sigma_1+\sigma)\|_{L^1_K} \\
&\leqslant \|e^{C_1\sigma} T(\sigma)(|\widetilde{w}(\sigma_1)| + |\widetilde{z}(\sigma_1)|)\|_{L^1_K} \\
&\leqslant e^{C_1\sigma} \||\widetilde{w}(\sigma_1)| + |\widetilde{z}(\sigma_1)|\|_{L^1_K} \\
&< e^{C_1\sigma}\varepsilon.
\end{aligned}
$$

Therefore

$$\|\widetilde{w}(\sigma_1+\sigma)\|_{L^1_K} + \|\widetilde{z}(\sigma_1+\sigma)\|_{L^1_K} \leqslant C_2\varepsilon, \quad 0 < \sigma \leqslant \sigma^*,\ C_2 = e^{C_1\sigma^*}. \qquad (2.21)$$

Let now

$$
\begin{aligned}
A_\sigma = \Big\{ \sigma > 0 \mid\ & e^{\alpha\tau}\|\widetilde{w}(\sigma_1+\sigma^*+\tau)\|_{L^1_K} + e^{\beta\tau}\|\widetilde{z}(\sigma_1+\sigma^*+\tau)\|_{L^1_K} \\
&\leqslant 2C_2\varepsilon,\ \tau \in [0,\sigma] \Big\}.
\end{aligned}
$$

By (2.21), we have

$$\|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K} \leqslant C_2\varepsilon,$$

on the other hand the function

$$\sigma \mapsto e^{\alpha\sigma}\|\widetilde{w}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K} + e^{\beta\sigma}\|\widetilde{z}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K}$$

is continuous in $(0,\infty)$ and

$$\lim_{\sigma\to 0} f(\sigma) = \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K},$$

then $A_\sigma \neq \emptyset$. We denote by $T_0 = \sup A_\sigma$, note that $T_0 > 0$. We assume by contradiction that $T_0 < \infty$. Assuming that $p \leqslant q$, without loss of generality, hence $\alpha \leqslant \beta$, then by (2.21), we obtain

$$\|\widetilde{w}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K} + \|\widetilde{z}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K} \leqslant 2C_2\varepsilon e^{-\alpha\sigma}, \quad -\sigma^* \leqslant \sigma \leqslant T_0. \tag{2.22}$$

Let $0 \leqslant \tau \leqslant T_0$, by (2.19), (2.12) and (2.22), we have

$$\begin{aligned}
&\|\widetilde{w}(\sigma_1+\sigma^*+\tau)\|_{L^l_K} + \|\widetilde{z}(\sigma_1+\sigma^*+\tau)\|_{L^l_K} \\
&\qquad \leqslant 2e^{C_1\sigma^*}\|T(\sigma^*)(|\widetilde{w}(\sigma_1+\tau)| + |\widetilde{z}(\sigma_1+\tau)|)\|_{L^l_K} \\
&\qquad \leqslant 2e^{C_1\sigma^*}\|T(\sigma^*)|\widetilde{w}(\sigma_1+\tau)|\|_{L^l_K} + \|T(\sigma^*)(|\widetilde{z}(\sigma_1+\tau)|)\|_{L^l_K} \\
&\qquad \leqslant 2e^{C_1\sigma^*}C_0(\|\widetilde{w}(\sigma_1+\tau)|\|_{L^1_K} + \|\widetilde{z}(\sigma_1+\tau)\|_{L^1_K}) \\
&\qquad \leqslant 4C_2C_0e^{C_1\sigma^*}\varepsilon e^{-\alpha(\tau-\sigma^*)}.
\end{aligned}$$

Therefore

$$\|\widetilde{w}(\sigma_1+\sigma^*+\tau)\|_{L^l_K} + \|\widetilde{z}(\sigma_1+\sigma^*+\tau)\|_{L^l_K} \leqslant C_3\varepsilon e^{-\alpha\tau}, \tag{2.23}$$

for all $0 \leqslant \tau \leqslant T_0$, with $C_3 = 4C_2C_0e^{(\alpha+C_1)\sigma^*}$.

Using (2.17), with $\sigma_2 = \sigma_1+\sigma^*$, then by (2.11), (2.10) and (2.23), we obtain

$$\begin{aligned}
&e^{\alpha\sigma}\|\widetilde{w}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K} \\
&\quad \leqslant \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} \\
&\qquad + \int_0^\sigma e^{\alpha\tau}\|T(\sigma-\tau)(|\widetilde{w}|^r(\sigma_1+\sigma^*+\tau)|\widetilde{z}|^p(\sigma_1+\sigma^*+\tau))\|_{L^1_K}d\tau \\
&\quad \leqslant \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + \int_0^\sigma e^{\alpha\tau}\||\widetilde{w}|^r(\sigma_1+\sigma^*+\tau)|\widetilde{z}|^p(\sigma_1+\sigma^*+\tau)\|_{L^1_K}d\tau \\
&\quad \leqslant \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + \int_0^\sigma e^{\alpha\tau}\|\widetilde{w}(\sigma_1+\sigma^*+\tau)\|^r_\infty\|\widetilde{z}(\sigma_1+\sigma^*+\tau)\|^p_{L^p_K}d\tau \\
&\quad \leqslant \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + M^rC^p\int_0^\sigma e^{\alpha\tau}\|\widetilde{z}(\sigma_1+\sigma^*+\tau)\|^p_{L^l_K}d\tau \\
&\quad \leqslant \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + (CC_3\varepsilon)^pM^r\int_0^\sigma e^{\alpha\tau}e^{-\alpha p\tau}d\tau.
\end{aligned}$$

Therefore

$$e^{\alpha\sigma}\|\widetilde{w}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K} \leqslant \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + M^r(CC_3)^p[1/\alpha(p-1)]\varepsilon^p. \tag{2.24}$$

By exchanging the roles of $\widetilde{w}$, p and r and $\widetilde{z}$, q and s and by using (2.18) with $\sigma_2 = \sigma_1 + \sigma^*$, since $\alpha q > \beta$, we obtain

$$\begin{aligned}
e^{\beta\sigma}&\|\widetilde{z}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K}\\
&\leqslant \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K}\\
&\quad + \int_0^\sigma e^{\beta\tau}\|T(\sigma-\tau)\widetilde{w}^q(\sigma_1+\sigma^*+\tau)\widetilde{z}^s(\sigma_1+\sigma^*+\tau)\|_{L^1_K}d\tau\\
&\leqslant \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K} + \int_0^\sigma e^{\beta\tau}\|\widetilde{w}^q(\sigma_1+\sigma^*+\tau)\widetilde{z}^s(\sigma_1+\sigma^*+\tau)\|_{L^1_K}d\tau\\
&\leqslant \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K} + \int_0^\sigma e^{\beta\tau}\|\widetilde{w}(\sigma_1+\sigma^*+\tau)\|^q_{L^q_K}\|\widetilde{z}(\sigma_1+\sigma^*+\tau)\|^s_\infty d\tau\\
&\leqslant \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K} + M^sC^q\int_0^\sigma e^{\beta\tau}\|\widetilde{w}(\sigma_1+\sigma^*+\tau)\|^q_{L^l_K}d\tau\\
&\leqslant \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K} + M^s(CC_3\varepsilon)^q\int_0^\sigma e^{\beta\tau}e^{-\alpha q\tau}d\tau.
\end{aligned}$$

Therefore

$$e^{\beta\sigma}\|\widetilde{z}(\sigma_1+\sigma^*+\sigma)\|_{L^1_K} \leqslant \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K} + M^s(CC_3)^q[1/(\alpha q-\beta)]\varepsilon^q. \tag{2.25}$$

For $\sigma = T_0$ in (2.24) and (2.25). By definition of T_0 and by using (2.21) with $\sigma = \sigma^*$, we obtain

$$\begin{aligned}
2C_2\varepsilon &\leqslant e^{\beta T_0}\|\widetilde{z}(\sigma_1+\sigma^*+T_0)\|_{L^1_K} + e^{\alpha T_0}\|\widetilde{w}(\sigma_1+\sigma^*+T_0)\|_{L^1_K}\\
&\leqslant \|\widetilde{w}(\sigma_1+\sigma^*)\|_{L^1_K} + M^r(CC_3)^p[1/\alpha(p-1)]\varepsilon^p\\
&\quad + \|\widetilde{z}(\sigma_1+\sigma^*)\|_{L^1_K} + M^s(CC_3)^q[1/(\alpha q-\beta)]\varepsilon^q\\
&\leqslant C_2\varepsilon + M^r(CC_3)^p[1/\alpha(p-1)]\varepsilon^p + M^s(CC_3)^q[1/(\alpha q-\beta)]\varepsilon^q.
\end{aligned}$$

Therefore

$$\begin{aligned}
C_2 &\leqslant M^r(CC_3)^p[1/\alpha(p-1)]\varepsilon^{p-1} + M^s(CC_3)^q[1/(\alpha q-\beta)]\varepsilon^{q-1}\\
&\leqslant C\,\varepsilon^{\min(p-1,q-1)},
\end{aligned}$$

which is impossible for $\varepsilon > 0$ sufficiently small, because $C_2 = e^{C_1\sigma^*} > 0$. Consequently, $T_0 = \infty$. It follows in particular from (2.23) that

$$\|\widetilde{w}(\sigma_1+\sigma^*+\tau)\|_{L^l_K} \leqslant C_3\varepsilon e^{-\alpha\tau}, \quad \tau \geqslant 0.$$

Then

$$\|\widetilde{w}(\sigma)\|_{L^l_K} \leqslant C_4\varepsilon e^{-\alpha\sigma}, \quad \sigma \geqslant \sigma_1+\sigma^*,\ C_4 = C_3e^{\alpha(\sigma_1+\sigma^*)}.$$

Then, there exists $\delta > 0$ such that

$$\|\widetilde{w}(\sigma)\|_{L^l_K} \leqslant C_4\varepsilon(T^*-t)^\alpha, \quad T^*-\delta < t < T^*. \tag{2.26}$$

Now, by continuity of (w,z), there exists $\eta>0$ small such that (2.16) and hence (2.26) is still to be true when the point a is replaced by any $b\in\Omega$ such that $|b-a|<\eta$ (note that C_4 and ε are independent of a). Then, by (2.26) we have

$$\begin{aligned}
\|\widetilde{w}(\sigma)\|_{L^l_K} &= \left(\int_{\mathbb{R}^n}|\widetilde{w}(\sigma,y)|^l e^{-|y|^2/4}dy\right)^{1/l}\\
&= \left(\int_{\mathbb{R}^n}(T^*-t)^{\alpha l}|u(t,b+y\sqrt{T^*-t})|^l e^{-|y|^2/4}dy\right)^{1/l}\\
&\leqslant C_4\varepsilon(T^*-t)^{\alpha}, \quad \text{for all } T^*-\delta<t<T^*.
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^n}|u(t,b+y\sqrt{T^*-t})|^l e^{-|y|^2/4}dy\leqslant C, \quad T^*-\delta<t<T^*, |b-a|<\eta. \tag{2.27}$$

For $\delta<\eta/2$, it follows from Fubini-Tonelli's theorem that

$$\begin{aligned}
&\int_{|z-a|<\eta/2}|u(t,z)|^l dz\\
&\qquad\leqslant C(n)\delta^{-n/2}\int_{|b-a|<\eta}\int_{|y|<\delta^{1/2}}|u(t,b+y\sqrt{T^*-t})|^l e^{-|y|^2/4}e^{\delta/4}dydb\\
&\qquad\leqslant C, \qquad T^*-\delta<t<T^*.
\end{aligned}$$

Therefore

$$u\in L^\infty\big((T^*-\delta,T^*),L^l(\mathrm{B}(a,\eta))\big)$$

and similarly, we get also

$$v\in L^\infty\big((T^*-\delta,T^*),L^l(\mathrm{B}(a,\eta))\big).$$

Now set $f_1=u_t-\Delta u=u^r v^p$ and $f_2=v_t-\Delta v=v^s u^q$. By Hölder's inequality and (2.20), it follows that

$$f_1,f_2\in L^{k_1}\big((T^*-\delta,T^*),L^{k_2}(\mathrm{B}(a,\eta))\big)$$

with $k_1=\infty$ and $k_2>\max(1,n/2)$, hence $\frac{1}{k_1}+\frac{n}{2k_2}<1$. We deduce from standard local parabolic regularity [8] that

$$u,v\in L^\infty\big((T^*-\delta/2,T^*),L^\infty(\mathrm{B}(a,\eta/2))\big),$$

hence u and v are bounded around (T^*,a). $\square$

2.2. Properties of the rescaled ODE systems

In this section, we study the nonnegative and bounded solutions of the ODE system

$$\begin{cases} w' = w^r z^p - \alpha w, \\ z' = w^q z^s - \beta z, \end{cases} \tag{2.28}$$

where $p,\ q,\ r,\ s > 1,\ pq-(r-1)(s-1) \neq 0,\ \alpha,\ \beta > 0$ and $' = d/d\sigma$. The only nonnegative constant solutions of the system (2.28) are given by $(w,z) = (0,0)$ and $(w,z) = (A_0, B_0)$, where

$$A_0 = \left(\beta^p \alpha^{1-s}\right)^{\frac{1}{pq-(s-1)(r-1)}}, \quad B_0 = \left(\alpha^q \beta^{1-r}\right)^{\frac{1}{pq-(s-1)(r-1)}}.$$

Let us mention that, $(0,0)$ and (A_0,B_0) are also the only nonnegative constant solutions of the system

$$\begin{cases} w' = A_0^r z^p - \alpha w, \\ z' = B_0^s w^q - \beta z. \end{cases} \tag{2.29}$$

PROPOSITION 2. (a) *Let $\sigma_0 \in \mathbb{R}$ and (w,z) be a nonnegative, bounded and global solution of either (2.28) or (2.29) for $\sigma \geqslant \sigma_0$. Then for all $\sigma \geqslant \sigma_0$, one of the following holds:*

(i) $(w,z) = (0,0)$*;*

(ii) $(w,z) = (A_0,B_0)$*;*

(iii) $w'z' < 0$ *for all* $\sigma \geqslant \sigma_0$ *and* $\lim_{\sigma\to\infty} (w(\sigma), z(\sigma)) = (A_0,B_0)$*;*

(iv) *There exists $\overline{\sigma} \geqslant \sigma_0$ such that $w'z' < 0$ on $[\sigma_0, \overline{\sigma})$ and $w',\ z' < 0$ on $(\overline{\sigma}, \infty)$. Moreover, $\lim_{\sigma\to\infty} (w(\sigma), z(\sigma)) = (0,0)$.*

(b) *Let $\sigma_0 \in \mathbb{R}$. Then, the problems (2.28) and (2.29), with*

$$w(\sigma_0) \geqslant A_0, \quad z(\sigma_0) \geqslant B_0 \quad \text{and} \quad (w(\sigma_0), z(\sigma_0)) \neq (A_0,B_0)$$

have no nonnegative, bounded and global solutions for $\sigma \geqslant \sigma_0$.

(c) *Let (w,z) be a nonnegative, bounded and global solution of (2.28) or (2.29), for all $\sigma \in \mathbb{R}$. Then either:*

(i) $(w,z) = (0,0)$*;*

(ii) $(w,z) = (A_0,B_0)$*;*

(iii) $\lim_{\sigma\to-\infty} (w(\sigma), z(\sigma)) = (A_0,B_0)$ *and* $\lim_{\sigma\to\infty} (w(\sigma), z(\sigma)) = (0,0)$.

Proof. (*a*) *Step* 1 : Let $\mathscr{R} = [0,\infty[^2 \setminus \{(0,0);(A_0,B_0)\}$. We claim that the regions

$$\mathscr{R}_1 := \{(X,Y) \in \mathscr{R} \,|\, X^r Y^p - \alpha X \geqslant 0;\ X^q Y^s - \beta Y \geqslant 0\}$$
$$\mathscr{R}_2 := \{(X,Y) \in \mathscr{R} \,|\, X^r Y^p - \alpha X \leqslant 0;\ X^q Y^s - \beta Y \leqslant 0\}$$

are positively invariant for the system (2.28). Indeed, let (w,z) be a solution of (2.28) such that $(w(\sigma_0),z(\sigma_0)) \in \mathscr{R}_2$. By contradiction, we assume that (w,z) leaves $\mathscr{R}_2$ at some time $\sigma_1 \geqslant \sigma_0$. In particular, this implies $w(\sigma_1) > 0$ and $z(\sigma_1) > 0$ (by definition of the region), since (w,z) is not constant, then either $w'(\sigma_1) < 0$ and $z'(\sigma_1) = 0$ or $w'(\sigma_1) = 0$ and $z'(\sigma_1) < 0$. But we have $z''(\sigma_1) = (qw^{q-1}z^s w' + sz^{s-1}w^q z' - \beta z')(\sigma_1) = qw^{q-1}z^s w'(\sigma_1) < 0$ or $w''(\sigma_1) = (pz^{p-1}w^r z' + rw^{r-1}z^s w' - \alpha w')(\sigma_1) = pz^{p-1}w^r z'(\sigma_1) < 0$, therefore for $\sigma > \sigma_1$ with $\sigma - \sigma_1$ small, we have $w'(\sigma),\ z'(\sigma) < 0$, consequently $(w,z) \in \mathscr{R}_2$; a contradiction.

Similarly, we prove that the region $\mathscr{R}_1$ is positively invariant for the system (2.28) and the regions

$$\mathscr{R}_1 := \{(X,Y) \in \mathscr{R} \,|\, A_0^r Y^p - \alpha X \geqslant 0;\, B_0^s X^q - \beta Y \geqslant 0\}$$
$$\mathscr{R}_2 := \{(X,Y) \in \mathscr{R} \,|\, A_0^r Y^p - \alpha X \leqslant 0;\, B_0^s X^q - \beta Y \leqslant 0\}$$

are positively invariant for the system (2.29).

Moreover, we note that if, say, $(w(\sigma_0),z(\sigma_0)) \in \mathscr{R}_2$, then $w'(\sigma) < 0, z'(\sigma) < 0$ for all $\sigma > \sigma_0$. Indeed, assume on the contrary that there exists $\sigma_1 > \sigma_0$ such that $w'(\sigma_1) < 0$ and $z'(\sigma_1) = 0$ or $w'(\sigma_1) = 0$ and $z'(\sigma_1) < 0$. Then we have $z''(\sigma_1) = (qw^{q-1}z^s w' - \beta z' + sz^{s-1}w^r z')(\sigma_1) = qw^{q-1}z^s w'(\sigma_1) < 0$ or $w''(\sigma_1) = (pz^{p-1}w^r z' - \alpha w' + rw^{r-1}z^s w')(\sigma_1) = pz^{p-1}w^r z'(\sigma_1) < 0$. Therefore, for $\sigma_1 > \sigma$ with $\sigma_1 - \sigma$ small, we have $w'(\sigma) > 0$ or $z'(\sigma) > 0$; a contradiction with the positivre invariance of $\mathscr{R}_2$.

Step 2: Let (w,z) be a nonnegative, bounded, global and nonconstant solution of (2.28) or (2.29), then we have either:

(1) $(w(\sigma),z(\sigma)) \in \mathscr{R}\backslash\{\mathscr{R}_1 \cup \mathscr{R}_2\}$, i.e. $w'z' < 0$ for all $\sigma > \sigma_0$;

(2) There exists $\overline{\sigma} \geqslant \sigma_0$ such that $(w(\overline{\sigma}),z(\overline{\sigma})) \in \mathscr{R}_1$, then $w',\ z' > 0$ on $(\overline{\sigma},\infty)$ and $w'z' < 0$ on $[\sigma_0,\overline{\sigma})$;

(3) There exists $\overline{\sigma} \geqslant \sigma_0$ such that $(w(\overline{\sigma}),z(\overline{\sigma})) \in \mathscr{R}_2$, then $w',\ z' < 0$ on $(\overline{\sigma},\infty)$ and $w'z' < 0$ on $[\sigma_0,\overline{\sigma})$.

Indeed, by Step 1, if $(w(\overline{\sigma}),z(\overline{\sigma})) \in \mathscr{R}_1$, respectively $\mathscr{R}_2$, then $(w(\sigma),z(\sigma)) \in \mathscr{R}_1$, respectively $\mathscr{R}_2$ for all $\sigma > \overline{\sigma}$, which proves the existence of one of the later three cases.

In the first case, we have $w' < 0$ and $z' > 0$ or $w' > 0$ and $z' < 0$ for all $\sigma \geqslant \sigma_0$. Since (w,z) is bounded, then (w,z) must converge to an equilibrium (nonzero). This yields assertion (iii). In the second and the third cases, (w,z) must converge again to an equilibrium. Since $\mathscr{R}_2 \subset \{X \leqslant A_0 \text{ or } Y \leqslant B_0\}$ and $w',\ z' < 0$ on $(\overline{\sigma},\infty)$ then (u,v) must be converge to $(0,0)$. Finally, since $\mathscr{R}_1 \subset \{X \geqslant A_0 \text{ or } Y \geqslant B_0\}$ and $w',\ z' > 0$ on $(\overline{\sigma},\infty)$ then the second case cannot occur, which implies (iv).

(b) Let

$$\mathscr{R}_3 := \{(X,Y) \in \mathscr{R} \,|\, X \geqslant A_0;\, Y \geqslant B_0\}.$$

$\mathscr{R}_3$ is positively invariant for the system (2.28).

Indeed, we assume that (w,z) leaves $\mathscr{R}_3$ at some time $\sigma_1 \geqslant \sigma_0$, since $(w,z) \neq (A_0,B_0)$, then either $w(\sigma_1) = A_0$ and $z(\sigma_1) > B_0$ or $w(\sigma_1) > A_0$ and $z(\sigma_1) = B_0$, therefore $w'(\sigma_1) = (z^p w^r - \alpha w)(\sigma_1) > A_0^r B_0^p - \alpha A_0 = 0$ or $z'(\sigma_1) = (w^q z^s - \beta z)(\sigma_1) > A_0^q B_0^s -$

$\beta B_0 = 0$. Therefore $w > A_0$ and $z > B_0$ for $\sigma - \sigma_1 > 0$ small; a contradiction. Therefore $(w(\sigma), z(\sigma)) \in \mathscr{R}_3$ for all $\sigma \geqslant \sigma_0$. Similarly, we prove that the region $\mathscr{R}_3$ is positively invariant for the system (2.29).

If we assume that (w,z) is a global and bounded solution of (2.28) or (2.29), then we have a contradiction with (a), because (i), (ii) and (iv) cannot occur, because $w \geqslant A_0$, $z \geqslant B_0$ and $(w,z) \neq (A_0, B_0)$, and for the case (iii), we have either $w' < 0$ and $z' > 0$ or $w' > 0$ and $z' < 0$ for all $\sigma \geqslant \sigma_0$, then (w,z) cannot converge to (A_0, B_0) when σ tends to ∞.

(c) Let (w,z) be a nonnegative, bounded, global and nonconstant solution of (2.28) or (2.29), then by Step 2, we have either:

(1) $w' > 0,\ z' < 0$ for all $\sigma \in \mathbb{R}$ and $\lim\limits_{\sigma\to\infty}(w,z) = (A_0^-, B_0^+)$;

(2) $w' < 0,\ z' > 0$ for all $\sigma \in \mathbb{R}$ and $\lim\limits_{\sigma\to\infty}(w,z) = (A_0^+, B_0^-)$;

(3) There exists $\overline{\sigma} \in \mathbb{R} \cup \{-\infty\}$ such that $w'z' < 0$ on $(-\infty, \overline{\sigma})$, $w',\ z' < 0$ on $(\overline{\sigma}, \infty)$ and $\lim\limits_{\sigma\to\infty}(w,z) = (0,0)$.

In all cases, (w,z) is bounded and monotone as σ tends to $-\infty$. Therefore, it must converge to an equilibrium $(0,0)$ or (A_0, B_0) when $\sigma \to -\infty$. In cases (1) and (2) both limits are impossible. Indeed, if $\lim\limits_{\sigma\to\infty}(w,z) = (A_0^-, B_0^+)$, since $z' < 0$ then z cannot tend neither to 0 nor to B_0, as $\sigma \to -\infty$. Similarly, if $\lim\limits_{\sigma\to\infty}(w,z) = (A_0^+, B_0^-)$, since $w' < 0$ then w cannot tend neither to 0 nor to A_0, as $\sigma \to -\infty$.

In the third case, we have either $w' < 0$ on $\mathbb{R}$, or $z' < 0$ on $\mathbb{R}$. This rules out convergence to $(0,0)$ as $\sigma \to -\infty$ and we conclude that $\lim\limits_{\sigma\to-\infty}(w,z) = (A_0, B_0)$. □

We shall also need the following consequence of Proposition 2(b), concerning the system of differential inequalities corresponding to (2.29).

PROPOSITION 3. *Let $\sigma_0 \in \mathbb{R}$. Then the problem*

$$\begin{cases} w' \geqslant A_0^r z^p - \alpha w, & \sigma > \sigma_0, \\ z' \geqslant B_0^s w^q - \beta z, & \sigma > \sigma_0, \end{cases} \tag{2.30}$$

with

$$w(\sigma_0) \geqslant A_0, \quad z(\sigma_0) \geqslant B_0 \quad \textit{and} \quad (w(\sigma_0), z(\sigma_0)) \neq (A_0, B_0), \tag{2.31}$$

has no nonnegative, bounded and global solutions.

Proof. Let $(\underline{w}, \underline{z})$ be the unique maximal solution of (2.29), such that $\underline{w}(\sigma_0) = w(\sigma_0)$ and $\underline{z}(\sigma_0) = z(\sigma_0)$. We put $0 < T_1 \leqslant \infty$ its maximal existence time and $0 < T^* \leqslant \infty$ the maximal existence time of (w,z). We define the function f by

$$f(x,y) = \left(A_0^r x^p - \alpha y,\ B_0^s y^q - \beta x\right),$$

since f is a C^1 function on $\mathbb{R}^2$, then by a comparison principle of ODE, we have

$$\begin{cases} w \geqslant \underline{w}, \\ z \geqslant \underline{z}, \quad \text{for all } \sigma_0 \leqslant \sigma \leqslant \min(T_1, T^*). \end{cases}$$

Then, by Proposition 2(b), (w, z) cannot be a nonnegative, bounded and global solution of (2.30). □

2.3. Proof of the Comparison Lemma 1

In this subsection, by using the local criterion for excluding blow-up (Proposition 1) and the properties of the rescaled ODE system (Proposition 2 and Proposition 3), we prove Lemma 1. We distinguish two cases. In a first step, we prove (2.1) for $\rho \neq 0$, then for $\rho = 0$, in a second step.

Step 1. Let $\rho = |a| \in (0, \rho_0)$. Let (W, Z) be a radial rescaling of (u, v) by similarity variables around (T^*, a) defined in Remark 2 and let K and $\|\,.\,\|_{L^1_K}$ defined in the subsection 2.1. Fix a sequence $(\sigma_j)_j$ such that $\sigma_j \to \infty$. By (1.7), W and Z are bounded. By (2.13) and parabolic estimates, it follows that for some subsequence denoted also (σ_j), the sequence of translates (W_j, Z_j) defined by

$$W_j := W(\sigma + \sigma_j, \theta), \quad Z_j := Z(\sigma + \sigma_j, \theta)$$

converges in $W^{1,2;q}(Q)$ to some pair of functions (ϕ, ψ), for each compact Q of $\mathbb{R} \times \mathbb{R}$ and each $q \in (1, \infty)$. Consequently, (ϕ, ψ) is a bounded solution of

$$\begin{cases} \phi_\sigma - \phi_{\theta\theta} + \frac{\theta}{2}\phi_\theta = \phi^r \psi^p - \alpha\phi, & \theta, \sigma \in \mathbb{R}, \\ \psi_\sigma - \psi_{\theta\theta} + \frac{\theta}{2}\psi_\theta = \phi^q \psi^s - \beta\psi, & \theta, \sigma \in \mathbb{R}. \end{cases}$$

Moreover, since u_ρ, $v_\rho \leqslant 0$ then,

$$W_\theta = e^{-[\alpha+1/2]\sigma} u_\rho, Z_\theta = e^{-[\beta+1/2]\sigma} v_\rho \leqslant 0, \; \sigma_0 < \sigma < \infty,$$
$$-|a|e^{\sigma/2} < \theta < (R - |a|)e^{\sigma/2}.$$

Therefore

$$\phi_\theta, \psi_\theta \leqslant 0, \quad \theta, \sigma \in \mathbb{R}. \tag{2.32}$$

Since ϕ and ψ are bounded and nonincreasing, we may define

$$\phi_\pm(\sigma) = \lim_{\theta \to \pm\infty} \phi(\sigma, \theta), \quad \psi_\pm(\sigma) = \lim_{\theta \to \pm\infty} \psi(\sigma, \theta).$$

This gives that

$$\phi_+ \leqslant \phi_-, \quad \psi_+ \leqslant \psi_-.$$

Moreover, we have

$$\begin{cases} \phi_{\pm}' = \phi_{\pm}^r \psi_{\pm}^p - \alpha \phi_{\pm}, & \sigma \in \mathbb{R}, \\ \psi_{\pm}' = \phi_{\pm}^q \psi_{\pm}^s - \beta \psi_{\pm}, & \sigma \in \mathbb{R}. \end{cases} \tag{2.33}$$

Indeed, let

$$\begin{cases} U(t,x) := (T^* - t)^{-\alpha} \phi\big(-\log(T^* - t), \frac{x}{\sqrt{T^*-t}}\big), \\ V(t,x) := (T^* - t)^{-\beta} \psi\big(-\log(T^* - t), \frac{x}{\sqrt{T^*-t}}\big), \end{cases} \tag{2.34}$$

for $x \in \mathbb{R}$ and $-\infty < t < T^*$. We observe that

$$U_{\pm}(t) = \lim_{x \to \pm\infty} U(t,x) = (T^* - t)^{-\alpha} \phi_{\pm}(-\log(T^* - t)). \tag{2.35}$$

Similarly for V. Moreover, (U,V) is a solution of the system

$$\begin{cases} U_t - U_{xx} = U^r V^p, & x \in \mathbb{R}, \ -\infty < t < T^*, \\ V_t - V_{xx} = U^q V^s, & x \in \mathbb{R}, \ -\infty < t < T^*. \end{cases} \tag{2.36}$$

Fix $\chi \in \mathscr{D}(\mathbb{R})$, with $\int_{\mathbb{R}} \chi = 1$ and let $\xi \in \mathscr{D}(-\infty, T^*)$. Let $j \in \mathbb{N}$, we replace x by $x + j$ in (2.36), multiplying by $\chi(x)\xi(t)$ and integrating on $(-\infty, T^*) \times \mathbb{R}$, we obtain

$$\int_{-\infty}^{T^*} \int_{\mathbb{R}} U_t(t, x+j)\chi(x)\xi(t)dxdt = \int_{-\infty}^{T^*} \int_{\mathbb{R}} [V^p U^r + U_{xx}](t, x+j)\chi(x)\xi(t)dxdt.$$

Integrating by parts, we get

$$\begin{aligned} &\int_{-\infty}^{T^*} \int_{\mathbb{R}} U_t(t, x+j)\chi(x)\xi(t)dxdt \\ &\quad = -\int_{-\infty}^{T^*} \int_{\mathbb{R}} U(t, x+j)\chi(x)\xi'(t)dxdt \\ &\quad = \int_{-\infty}^{T^*} \int_{\mathbb{R}} U^r V^p(t, x+j)\chi(x)\xi(t)dxdt + \int_{-\infty}^{T^*} \int_{\mathbb{R}} U(t, x+j)\chi''(x)\xi(t)dxdt. \end{aligned}$$

We put

$$A_j = \int_{-\infty}^{T^*} \int_{\mathbb{R}} U(t, x+j)\chi(x)\xi'(t)dxdt,$$

since ϕ is bounded then

$$\begin{aligned} U(t,x)\chi(x)\xi'(t) &= (T^* - t)^{-\alpha} \phi \chi(x)\xi'(t) \\ &\leqslant C(T^* - t)^{-\alpha} \chi(x)\xi'(t) \in L^1((-\infty, T^*) \times \mathbb{R}), \end{aligned}$$

by (2.35) and dominated convergence theorem, we obtain

$$\lim_{j \to +\infty} A_j = \int_{-\infty}^{T^*} \Big(\int_{\mathbb{R}} \chi(x)dx \Big) U_+(t)\xi'(t)dt = \int_{-\infty}^{T^*} U_+(t)\xi'(t)dt. \tag{2.37}$$

Similarly, we obtain

$$\lim_{j\to+\infty}\int_{-\infty}^{T^*}\int_{\mathbb{R}} V^pU^r(t,x+j)\chi(x)\xi(t)dxdt = \int_{-\infty}^{T^*} V_+^pU_+^r(t)\xi(t)dt, \tag{2.38}$$

$$\lim_{j\to+\infty}\int_{-\infty}^{T^*}\int_{\mathbb{R}} U(t,x+j)\chi''(x)\xi(t)dxdt$$
$$= \int_{-\infty}^{T^*}\Big(\int_{\mathbb{R}}\chi''(x)dx\Big)U_+(t)\xi(t)dt = 0. \tag{2.39}$$

By (2.37), (2.38) and (2.39), we obtain

$$-\int_{-\infty}^{T^*} U_+(t)\xi'(t)dt = \int_{-\infty}^{T^*}(V_+^p(t)U_+^r(t))\xi(t)dt.$$

Therefore

$$\int_{-\infty}^{T^*}(U_+'(t) - V_+^p(t)U_+^r(t))\xi(t)dt = 0, \quad \text{for all } \xi\in\mathscr{D}(-\infty,T^*).$$

Then $U_+'(t) = V_+^p(t)U_+^r(t)$ on $(-\infty,T^*)$ in the distribution sense, since $U_+,\ V_+\in L^\infty_{loc}(-\infty,T^*)$ then the result is true in the classical sense.

By the same argument, we obtain $U_-' = V_-^pU_-^r$, $V_+' = U_+^qV_+^s$ and $V_-' = U_-^qV_-^s$. Converting back to $\phi_\pm$ et $\psi_\pm$, we obtain (2.33). Since $\phi_+\leqslant\phi_-$ and $\psi_+\leqslant\psi_-$ by Proposition 2(c), only two cases are possible:

Case 1. $(\phi_+,\psi_+)\equiv(A_0,B_0)$. Then $(\phi_-,\psi_-)\equiv(A_0,B_0)$, therefore $(\phi,\psi)\equiv(A_0,B_0)$;
Case 2. $\lim_{\sigma\to\infty}(\phi_+(\sigma),\psi_+(\sigma)) = (0,0)$.

In the first case, for all $t_j\to T^*$, there exists a subsequence such that

$$\lim_{j\to\infty}(T^*-t_j)^\alpha U(t_j,\rho) = \lim_{j\to\infty}W(\sigma_j,0) = A_0$$

and

$$\lim_{j\to\infty}(T^*-t_j)^\beta V(t_j,\rho) = \lim_{j\to\infty}Z(\sigma_j,0) = B_0.$$

Let us assume that case 2 occurs and show that this leads to a contradiction. Let b such that $|a|<|b|<\rho_0$ and let (w_b,z_b) and (W_b,Z_b) be respectively the rescaling and the radial rescaling of (u,v) by similarity variables around of (T^*,b) defined in the subsection 2.1. Then there exist $\varepsilon>0$ and $\sigma_1>\sigma_0$ such that

$$\|w_b(\sigma_1)\|_{L^1_K} + \|z_b(\sigma_1)\|_{L^1_K}\leqslant\varepsilon. \tag{2.40}$$

Indeed, by assumption, there exists $\overline{\sigma}$ such that

$$\phi_+,\psi_+<\varepsilon/8.$$

Then, by definition of ψ_+ and ϕ_+, there exists $\overline{\theta}$ such that

$$\phi(\overline{\sigma},\overline{\theta}),\ \psi(\overline{\sigma},\overline{\theta}) < \varepsilon/4.$$

Consequently, there exists j_0 such that, for all $j > j_0$ we have

$$W_a(\overline{\sigma}+\sigma_j,\overline{\theta}), Z_a(\overline{\sigma}+\sigma_j,\overline{\theta}) < \varepsilon/2.$$

For $\overline{\theta}_j = \overline{\theta} - (|b|-|a|)e^{(\overline{\sigma}+\sigma_j)/2}$, we have

$$W_b(\overline{\sigma}+\sigma_j,\overline{\theta}_j), Z_b(\overline{\sigma}+\sigma_j,\overline{\theta}_j) < \varepsilon/2.$$

Since

$$W_\theta = e^{-[\alpha+1/2]\sigma}u_\rho,\ \ Z_\theta = e^{-[\beta+1/2]\sigma}v_\rho \leqslant 0,\ \sigma_0 < \sigma < \infty,$$
$$-|a|e^{\sigma/2} < \theta < (R-|a|)e^{\sigma/2} \quad \text{and} \quad \overline{\theta}_j \geqslant -|b|e^{(\overline{\sigma}+\sigma_j)/2},$$

then

$$W_b(\overline{\sigma}+\sigma_j,\theta), Z_b(\overline{\sigma}+\sigma_j,\theta) < \varepsilon/2, \quad \theta > \overline{\theta}_j.$$

Using $\overline{\theta}_j \to -\infty$ as $j\to\infty$, along with (2.14), $w_b, z_b \leqslant M$ and $K \in L^1$, we infer that

$$\|w_b(\overline{\sigma}+\sigma_j)\|_{L^1_K} + \|z_b(\overline{\sigma}+\sigma_j)\|_{L^1_K} \leqslant \varepsilon \quad \text{for } j \text{ large}.$$

By Proposition 1, (2.40) implies that $u(t,b)$ and $v(t,b)$ are bounded when $t\to T^*$, contradicting $|b| < \rho_0$. This concludes the proof of Lemma 1, in the case when $\rho \in (0,\rho_0)$.

Step 2. By Step 1, we know that

$$\lim_{t\to T^*}(T^*-t)^\alpha u(t,\rho) = A_0, \quad \lim_{t\to T^*}(T^*-t)^\beta v(t,\rho) = B_0, \quad 0<\rho<\rho_0.$$

Since $u_\rho,\ v_\rho \leqslant 0$, then

$$(T^*-t)^\alpha u(t,0) \geqslant (T^*-t)^\alpha u(t,\rho), \quad (T^*-t)^\beta v(t,0) \geqslant (T^*-t)^\alpha v(t,\rho).$$

Therefore

$$\liminf_{t\to T^*}(T^*-t)^\alpha u(t,0) \geqslant A_0, \quad \liminf_{t\to T^*}(T^*-t)^\beta v(t,0) \geqslant B_0. \tag{2.41}$$

By contradiction, we assume that $\limsup_{t\to T^*}(T^*-t)^\alpha u(t,0) > A_0$, i.e. there exist a subsequence $(t_j)_j$ and $l > A_0$ such that

$$\limsup_{t\to T^*}(T^*-t)^\alpha u(t,0) = l. \tag{2.42}$$

Let $(w,z)=(w_0,z_0)$ be the rescaled solution given by similarity variables around $(T^*,0)$, there exists a subsequence $(\sigma_j)_j$ such that $(w(\sigma+\sigma_j,y),z(\sigma+\sigma_j,y))$ converges locally uniformly to a nonnegative and bounded solution (ϕ,ψ) of the system

$$\begin{cases} \phi_\sigma-\Delta\phi+\frac{y}{2}.\nabla\phi=\phi^r\psi^p-\alpha\phi, & y\in\mathbb{R}^n,\ \sigma\in\mathbb{R},\\ \psi_\sigma-\Delta\psi+\frac{y}{2}.\nabla\psi=\phi^q\psi^s-\beta\psi, & y\in\mathbb{R}^n,\ \sigma\in\mathbb{R}. \end{cases} \tag{2.43}$$

Moreover, by Step 1, $u_\rho, v_\rho\leqslant 0$ and $\lim_{t\to T^*}(T^*-t)^\alpha u(t,0)=l>A_0$, we obtain

$$\phi\geqslant A_0,\quad \psi\geqslant B_0\quad\text{and}\quad \phi(0,0)>A_0. \tag{2.44}$$

Multiplying (2.43) by K and integrating by parts, we obtain

$$\begin{cases} \frac{d}{d\sigma}\int_{\mathbb{R}^n}\phi K dy=\int_{\mathbb{R}^n}\phi^r\psi^p K dy-\alpha\int_{\mathbb{R}^n}\phi K dy,\\ \frac{d}{d\sigma}\int_{\mathbb{R}^n}\psi K dy=\int_{\mathbb{R}^n}\phi^q\psi^s K dy-\beta\int_{\mathbb{R}^n}\psi K dy. \end{cases}$$

Let $f(\sigma)=\int_{\mathbb{R}^n}\phi K dy$ and $g(\sigma)=\int_{\mathbb{R}^n}\psi K dy$, by Jensen's inequality and (2.44), we conclude that

$$\begin{cases} f'\geqslant A_0^r g^p-\alpha f,\ \sigma\in\mathbb{R},\\ g'\geqslant B_0^s f^q-\beta g,\ \sigma\in\mathbb{R}. \end{cases}$$

Moreover, $f(0)>A_0$ and $g(0)\geqslant B_0$, this contradicts the Proposition 3, since (f,g) is a nonnegative, bounded and global solution. Consequently,

$$\limsup_{t\to T^*}(T^*-t)^\alpha u(t,0)\leqslant A_0.$$

By exchanging the roles of u, A_0 and α and v, B_0 and β, we prove that

$$\limsup_{t\to T^*}(T^*-t)^\beta v(t,0)\leqslant B_0.$$

By (2.41), we conclude that

$$\lim_{t\to T^*}(T^*-t)^\alpha u(t,0)=A_0\quad\text{and}\quad \lim_{t\to T^*}(T^*-t)^\beta v(t,0)=B_0.$$

In particular, by the continuity of the function $x\mapsto x^p$ for $p>0$ on $\mathbb{R}_+$, we have

$$\lim_{t\to T^*}(T^*-t)^{\alpha(q+1-r)}u^{q+1-r}(t,\rho)=A_0^{q+1-r},\quad \text{for all } \rho\in[0,\rho_1].$$

$$\lim_{t\to T^*}(T^*-t)^{\beta(p+1-s)}v^{p+1-s}(t,\rho)=B_0^{p+1-s},\quad \text{for all } \rho\in[0,\rho_1].$$

Therefore

$$\lim_{t\to T^*}\frac{u^{q+1-r}(t,\rho')}{v^{p+1-s}(t,\rho)}=A_0^{q+1-r}B_0^{-(p+1)+s},\quad \text{for all } \rho,\rho'\in[0,\rho_1].$$

Since we have

$$\frac{u^{q+1-r}(t,\rho_1)}{v^{p+1-s}(t,0)} - A_0^{q+1-r}B_0^{-(p+1)+s}$$
$$\leqslant \frac{u^{q+1-r}}{v^{p+1-s}}(t,\rho) - A_0^{q+1-r}B_0^{-(p+1)+s}$$
$$\leqslant \frac{u^{q+1-r}(t,0)}{v^{p+1-s}(t,\rho_1)} - A_0^{q+1-r}B_0^{-(p+1)+s}.$$

Then

$$\left|\frac{u^{q+1-r}}{v^{p+1-s}}(t,\rho) - A_0^{q+1-r}B_0^{-(p+1)+s}\right|$$
$$\leqslant \left|\frac{u^{q+1-r}(t,\rho_1)}{v^{p+1-s}(t,0)} - A_0^{q+1-r}B_0^{-(p+1)+s}\right| + \left|\frac{u^{q+1-r}(t,0)}{v^{p+1-s}(t,\rho_1)} - A_0^{q+1-r}B_0^{-(p+1)+s}\right|$$
$$\underset{t\to T^*}{\longrightarrow} 0 .$$

Then, we conclude that u^{q+1-r}/v^{p+1-s} converges uniformly to $A_0^{q+1-r}B_0^{-(p+1)+s}$ on $[0,\rho_1]$.

3. Single-point blow-up

In this section, we are concerned with the proof of Theorem 1. As in [3, 5, 10], we consider the following modified functions

$$G(t,\rho) = u_\rho + \varepsilon c(\rho)u^\gamma, \quad J(t,\rho) = v_\rho + \varepsilon d(\rho)v^{\overline{\gamma}}, \tag{3.1}$$

with

$$c(\rho) = \sin^2(\pi\rho/a), \quad d(\rho) = Kc(\rho), \tag{3.2}$$

where γ, $\overline{\gamma} > 1$ and ε, K, $a > 0$ are to be fixed later. We note that $G,\ J \in C((0,T^*)\times[0,R]) \cap C^{1,2}((0,T^*)\times(0,R))$. We show that (G,J) satisfies a new system of parabolic inequalities (See Lemma 2 below), from a maximum principle we deduce that $G,\ J \leqslant 0$ on $[\tau,T^*)\times[0,\rho_0/2]$ for some $\tau\in(0,T^*)$. By integrating these inequalities, we obtain upper bounds on u and v, away from $\rho=0$, hence in particular single-point blow-up occurs.

As a starting point of our improvement, by using the comparison properties between u and v in Lemma 1, we get the following lemma:

LEMMA 2. *Let (u,v) be a positive classical solution of (1.1) and satisfy the assumptions of Lemma 1, let $a=\rho_0/2$, where ρ_0 is as in Lemma 1. Then there exist γ, $\overline{\gamma}>1$, $K>0$ and $\tau\in(0,T^*)$, such that for all $\varepsilon\in(0,1]$, G and J satisfy*

$$\begin{cases} G_t - G_{\rho\rho} - \frac{n-1}{\rho}G_\rho + \frac{n-1}{\rho^2}G \leqslant pv^{p-1}u^rJ_+ + \left[ru^{r-1}v^p - 2c'\gamma\varepsilon u^{\gamma-1}\right]G, \\ J_t - J_{\rho\rho} - \frac{n-1}{\rho}J_\rho + \frac{n-1}{\rho^2}J \leqslant qu^{q-1}v^sG_+ + \left[sv^{s-1}u^q - 2d'\overline{\gamma}\varepsilon v^{\overline{\gamma}-1}\right]J, \end{cases} \tag{3.3}$$

for all $t\in(\tau,T^*)$ and for all $\rho\in[0,a]$.

Proof. Let $F = u^\gamma$. By derivation of (3.1), we obtain

$$G_t - G_{\rho\rho} = (u_\rho)_t + \varepsilon c F_t - (u_{\rho\rho})_\rho - \varepsilon c'' F - 2\varepsilon c' F_\rho - \varepsilon c F_{\rho\rho}$$
$$= (u_t - u_{\rho\rho})_\rho + \varepsilon\Big(c\big(F_t - F_{\rho\rho}\big) - 2c' F_\rho - c'' F\Big).$$

Using

$$(u_t - u_{\rho\rho})_\rho = \Big(\frac{n-1}{\rho} u_\rho + u^r v^p\Big)_\rho$$
$$= \frac{n-1}{\rho} u_{\rho\rho} - \frac{n-1}{\rho^2} u_\rho + p v^{p-1} u^r v_\rho + r u^{r-1} v^p u_\rho,$$

$$F_t - F_{\rho\rho} = \gamma u^{\gamma-1} u_t - \gamma(\gamma-1) u^{\gamma-2} u_\rho^2 - \gamma u^{\gamma-1} u_{\rho\rho}$$
$$\leqslant \gamma u^{\gamma-1}\big(u_t - u_{\rho\rho}\big)$$
$$= \gamma u^{\gamma-1}\Big(\frac{n-1}{\rho} u_\rho + u^r v^p\Big),$$

$u_\rho = G - \varepsilon c u^\gamma$ and $v_\rho = J - \varepsilon d v^{\overline{\gamma}}$, we obtain

$$G_t - G_{\rho\rho} \leqslant \frac{n-1}{\rho}(G - \varepsilon c u^\gamma)_\rho - \frac{n-1}{\rho^2}(G - \varepsilon c u^\gamma) + p v^{p-1} u^r (J - \varepsilon d v^{\overline{\gamma}})$$
$$+ \; r u^{r-1} v^p (G - \varepsilon c u^\gamma) + \varepsilon u^{\gamma-1}\Big[\gamma c\Big(\frac{n-1}{\rho} u_\rho + u^r v^p\Big) - 2\gamma c' u_\rho - c'' u\Big]$$
$$= \frac{n-1}{\rho} G_\rho - \varepsilon \frac{n-1}{\rho} c' u^\gamma - \varepsilon \frac{n-1}{\rho} c\gamma u^{\gamma-1} u_\rho - \frac{n-1}{\rho^2} G$$
$$+ \; \varepsilon \frac{n-1}{\rho^2} c u^\gamma + p v^{p-1} u^r (J - \varepsilon d v^{\overline{\gamma}}) + r u^{r-1} v^p (G - \varepsilon c u^\gamma)$$
$$+ \; \varepsilon u^{\gamma-1}\Big[\gamma c\Big(\frac{n-1}{\rho} u_\rho + u^r v^p\Big) - 2\gamma c'(G - \varepsilon c u^\gamma) - c'' u\Big].$$

Consequently,

$$G_t - G_{\rho\rho} - \frac{n-1}{\rho} G_\rho + \frac{n-1}{\rho^2} G$$
$$\leqslant p v^{p-1} u^r J_+ + \big[r u^{r-1} v^p - 2c'\gamma\varepsilon u^{\gamma-1}\big] G + \varepsilon H_1, \qquad (3.4)$$

with

$$H_1 := -p d v^{p+\overline{\gamma}-1} u^r - r c u^{r+\gamma-1} v^p$$
$$+ \; u^{\gamma-1}\Big[\gamma c(u^r v^p) + 2\varepsilon\gamma c' c u^\gamma + u\Big(\frac{n-1}{\rho}\Big(\frac{c}{\rho} - c'\Big) - c''\Big)\Big].$$

We get

$$\begin{aligned}\widetilde{H}_1 &:= \frac{H_1}{c\,u^{r+\gamma-1}v^p}\\ &= -pK\frac{v^{\overline{\gamma}-1}}{u^{\gamma-1}} - r + \gamma + 2\varepsilon\gamma c'\frac{u^\gamma}{u^r v^p} + \frac{u}{u^r v^p}\Big(\frac{n-1}{\rho}\Big(\frac{1}{\rho}-\frac{c'}{c}\Big)-\frac{c''}{c}\Big). \end{aligned} \tag{3.5}$$

By exchanging the roles of $u,\ q,\ r,\ \gamma$ and c and $v,\ p,\ s,\ \overline{\gamma}$ and d, we have

$$\begin{aligned} J_t - J_{\rho\rho} - \frac{n-1}{\rho}J_\rho + \frac{n-1}{\rho^2}J \\ \leqslant qu^{q-1}v^s G_+ + \big[sv^{s-1}u^q - 2d'\overline{\gamma}\varepsilon v^{\overline{\gamma}-1}\big]J + \varepsilon H_2, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned}\widetilde{H}_2 &:= \frac{H_2}{dv^{\overline{\gamma}-1}(u^q v^s)}\\ &= -\frac{q}{K}\frac{u^{\gamma-1}}{v^{\overline{\gamma}-1}} - s + \overline{\gamma} + 2\varepsilon\overline{\gamma}d'\frac{v^{\overline{\gamma}}}{u^q v^s} + \frac{v}{u^q v^s}\Big(\frac{n-1}{\rho}\Big(\frac{1}{\rho}-\frac{d'}{d}\Big)-\frac{d''}{d}\Big). \end{aligned} \tag{3.7}$$

We choose γ and $\overline{\gamma}$ such that

$$1 < \gamma < p\frac{q+1-r}{p+1-s} + r, \quad 1 < \overline{\gamma} < q\frac{p+1-s}{q+1-r} + s \tag{3.8}$$

and

$$\gamma - 1 = \frac{q+1-r}{p+1-s}(\overline{\gamma}-1) \quad (\text{i. e. } \overline{\gamma} = \frac{\gamma(p+1-s)+q-p+s-r}{q+1-r}). \tag{3.9}$$

On the other hand, we have

$$\frac{1}{\rho} - \frac{c'}{c} \underset{0}{\backsim} \frac{1}{\rho} - 2\frac{1}{\rho} = -\frac{1}{\rho}.$$

Therefore

$$\frac{n-1}{\rho}\Big(\frac{1}{\rho}-\frac{c'}{c}\Big) \underset{0}{\backsim} -\frac{n-1}{\rho^2} \underset{\rho\to 0^+}{\longrightarrow} -\infty\,.$$

$$\begin{aligned} -\frac{c''}{c} &= -\frac{2\pi^2}{a^2}\cot^2\Big(\frac{\pi}{a}\rho\Big) + \frac{2\pi^2}{a^2} \\ &\underset{\rho\to 0^+}{\longrightarrow} -\infty\,. \end{aligned}$$

Then

$$\frac{n-1}{\rho}\Big(\frac{1}{\rho}-\frac{c'}{c}\Big) - \frac{c''}{c} \underset{\rho\to 0}{\longrightarrow} -\infty\,.$$

Moreover, when $\rho \to a^-$,

$$\frac{n-1}{\rho}\Big(\frac{1}{\rho}-\frac{c'}{c}\Big)-\frac{c''}{c}=\frac{n-1}{\rho^2}-\Big[\frac{2\pi}{a}\frac{n-1}{\rho}+\frac{2\pi^2}{a^2}\cot\frac{\pi}{a}\rho\Big]\cot\frac{\pi}{a}\rho+\frac{2\pi^2}{a^2}$$
$$\underset{\rho\to a^-}{\longrightarrow}-\infty.$$

Since $\frac{n-1}{\rho}\Big(\frac{1}{\rho}-\frac{c'}{c}\Big)-\frac{c''}{c}$ is continuous on $]0,a[$, then there exists $C>0$ (C denotes a positive constant which may vary from line to line) such that

$$\frac{n-1}{\rho}\Big(\frac{1}{\rho}-\frac{c'}{c}\Big)-\frac{c''}{c}\leqslant C,\quad \text{for all } \rho\in[0,a]. \tag{3.10}$$

By (2.3), there exist $C>0$ and $\tau_1\in(0,T^*)$ such that

$$u^{(q+1-r)/(p+1-s)}(t,\rho)\leqslant Cv(t,\rho)\quad \text{in } [\tau_1,T^*)\times[0,a].$$

Therefore

$$v^{-p}(t,\rho)\leqslant Cu^{\frac{-p(q+1-r)}{p+1-s}}(t,\rho)\quad \text{in } [\tau_1,T^*)\times[0,a]. \tag{3.11}$$

Then, by (3.5), (3.10) and (3.11), we obtain

$$\begin{aligned}\widetilde{H}_1&\leqslant -pK\frac{v^{\overline{\gamma}-1}}{u^{\gamma-1}}-r+\gamma+Cu^{\gamma-r-\frac{p(q+1-r)}{p+1-s}}+\frac{C}{u^{r-1}v^p}\\&\leqslant -pK\frac{v^{\overline{\gamma}-1}}{u^{\gamma-1}}-r+\gamma+C(u(t,a))^{\gamma-r-\frac{p(q+1-r)}{p+1-s}}+\frac{C}{u^{r-1}(t,a)v^p(t,a)}\end{aligned}$$

in $[\tau_1,T^*)\times[0,a]$, where we also used $u_\rho,v_\rho\leqslant 0$ and (3.8). As a consequence of (2.1), (3.9) and (3.8), the RHS of the last inequality converges uniformly on $[0,a]$ to

$$L:=-pK\frac{B_0^{\overline{\gamma}-1}}{A_0^{\gamma-1}}-r+\gamma,$$

as $t\to T^*$, where (A_0,B_0) is given by (2.2). Taking

$$K=\frac{A_0^{\gamma-1}}{B_0^{\overline{\gamma}-1}}\frac{q+1-r}{p+1-s},$$

it follows from (3.8) that $L<0$. Therefore, there exists $\tau\in[\tau_1,T^*)$ such that $H_1\leqslant 0$ in $Q:=[\tau,T^*)\times[0,a]$. By (3.4), we conclude that

$$G_t-G_{\rho\rho}-\frac{n-1}{\rho}G_\rho+\frac{n-1}{\rho^2}G\leqslant pv^{p-1}u^rJ_++\big[ru^{r-1}v^p-2c'\gamma\varepsilon u^{\gamma-1}\big]G\quad \text{in } Q.$$

Similarly, by (3.6), (3.7) and by exchanging the roles of u, q, r, γ and c and v, p, s, $\overline{\gamma}$ and d, we prove that

$$J_t-J_{\rho\rho}-\frac{n-1}{\rho}J_\rho+\frac{n-1}{\rho^2}J\leqslant qu^{q-1}v^sG_++\big[sv^{s-1}u^q-2d'\overline{\gamma}\varepsilon v^{\overline{\gamma}-1}\big]J\quad \text{in } Q.\ \square$$

With Lemma 1 and Lemma 2 at hand, we turn now to prove Theorem 1.

Proof of Theorem 1. Let (u,v) be a solution of the system (1.1) satisfy the hypotheses of Theorem 1 and assume by contradiction that there exists $\rho_0 \in (0,R)$ such that

$$\limsup_{t\to T^*}(u(t,\rho_0)+v(t,\rho_0)) = \infty. \tag{3.12}$$

Since we have

$$u_\rho \leqslant 0 \quad \text{on } \{(0,T^*)\times\{0\}\}\cup\{(0,T^*)\times\{R\}\}\cup\{\{0\}\times[0,R]\},\ R<\infty$$

and

$$u_t - u_{\rho\rho} - \frac{n-1}{\rho}u_\rho = f(t,\rho) \quad \text{on } (0,T^*)\times(0,R),$$

with

$$f(t,\rho) = u^r v^p \quad \text{and} \quad f_\rho = pv^{p-1}u^r v_\rho + ru^{r-1}v^p u_r \leqslant 0,$$

then by using the maximum principle [9, Lemma 52.18], we have $u_\rho < 0$ on $(0,T^*)\times(0,R]$ and $u_{\rho\rho}(t,0) < 0$ on $(0,T^*)$. By exchanging the roles of u and v, we obtain $v_\rho < 0$ on $(0,T^*)\times(0,R]$ and $v_{\rho\rho}(t,0) < 0$ on $(0,T^*)$. (If $\Omega = \mathbb{R}^n$ $(0,R]$ can be replaced by $(0,\infty)$).

Let J, G and τ be given by Lemma 2. Taking ε sufficiently small, we have J, $G \leqslant 0$ on $\{(\tau,T^*)\times\{0\}\}\cup\{(\tau,T^*)\times\{a\}\}\cup\{\{\tau\}\times[0,a]\}$, with $a=\rho_0/2$, then by using the maximum principle (See Remark 3 below), we obtain $J,\ G\leqslant 0$ on $(\tau,T^*)\times[0,a]$.

Since $J\leqslant 0$, we obtain

$$-v_\rho \geqslant \varepsilon d\, v^{\overline{\gamma}}.$$

By integration, we deduce that

$$v^{1-\overline{\gamma}}(t,a) \geqslant C_5\int_0^a \sin^2(\pi\xi/a)d\xi > 0, \quad \text{for all } \tau\leqslant t<T^*.$$

Using $\bar{\gamma}>1$, it follows that $v(t,a)$, and similarly $u(t,a)$, is bounded on $[\tau,T^*)$. Since $u_\rho, v_\rho \leqslant 0$, this yields a contradiction with (3.12). The theorem is proved. □

REMARK 3. For completeness, we now prove the maximum principle for the system of parabolic inequalities (3.3). Fixing $T_1\in(\tau,T^*)$, multiplying (3.3) by $\rho^{n-1}G+\geqslant 0$ and integrating by parts over $(0,a)$. Since $G(t,0)$, $G(t,a)\leqslant 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^a \rho^{n-1}G_+^2 d\rho \\
&\qquad \leqslant p\int_0^a \rho^{n-1}v^{p-1}u^r J_+G_+d\rho + \int_0^a \rho^{n-1}\left[ru^{r-1}v^p - 2c'\gamma\varepsilon u^{\gamma-1}\right]G_+^2 d\rho
\end{aligned}$$

$$-\int_0^a \chi_{\{G>0\}}\rho^{n-1}G_\rho^2 d\rho$$
$$\leqslant C\int_0^a \rho^{n-1}(J_+^2+G_+^2)d\rho, \quad \text{for all } t\in(\tau,T_1).$$

Similarly, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^a \rho^{n-1}J_+^2 d\rho \leqslant C\int_0^a \rho^{n-1}(J_+^2+G_+^2)d\rho.$$

Then,

$$\frac{1}{2}\frac{d}{dt}\int_0^a \rho^{n-1}(J_+^2+G_+^2)d\rho \leqslant C\int_0^a \rho^{n-1}(J_+^2+G_+^2)d\rho.$$

Integrating over (τ,t). Since $G(\tau,.),J(\tau,.)\leqslant 0$, by Gronwall's lemma, we conclude that $J_+^2+G_+^2=0$. Then, $G,J\leqslant 0$ in $[\tau,T^*)\times[0,a]$.

4. Lower pointwise estimates

In this section we are concerned with the proof of Theorem 2. As a starting point we prepare the following lemma:

LEMMA 3. *Under the assumptions of Theorem 2, we obtain*

$$\|u(t)\|_\infty \geqslant C_1(T^*-t)^{-\alpha}, \quad \|v(t)\|_\infty \geqslant C_2(T^*-t)^{-\beta}, \quad T^*/2<t<T^*,$$

for some constants C_1, $C_2>0$ depending on u and v.

Proof. We put

$$U(t)=u(t,0), \quad V(t)=v(t,0).$$

Under the assumptions of Theorem 2, blow-up occurs only at the origin, then $U(T^*)=V(T^*)=\infty$. By (1.1), it follows that

$$U'(t)=u_t(t,0)=\Delta u(t,0)+u^r(t,0)v^p(t,0).$$

Since $\Delta u(t,0)\leqslant 0$, then $U'(t)\leqslant U^rV^p$. Similarly, we obtain $V'(t)\leqslant V^sU^q$. Therefore, by (1.7), $\|u(t)\|_\infty=u(t,0)$ and $\|v(t)\|_\infty=v(t,0)$, there exists C (C denotes a positive constant which may vary from line to line) such that

$$u(t,0)\leqslant Cv^{\frac{p+1-s}{q+1-r}}(t,0).$$

Therefore

$$u^r(t,0)\leqslant Cv^{r\frac{p+1-s}{q+1-r}}(t,0). \tag{4.1}$$

Similarly, we have

$$v^s(t,0) \leqslant Cu^{s\frac{q+1-r}{p+1-s}}(t,0). \tag{4.2}$$

Therefore,

$$U'(t) \leqslant CV^{p+r(p+1-s)/(q+1-r)} \quad \text{and} \quad V'(t) \leqslant CU^{q+s(q+1-r)/(p+1-s)}.$$

Then,

$$V'(t) \leqslant C\Big(U(0) + C\int_0^t V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma\Big)^{q+s\frac{q+1-r}{p+1-s}}, \quad \text{for all } t \in (0,T^*).$$

Therefore

$$V'(t) \leqslant C\Big(\int_0^t V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma\Big)^{q+s\frac{q+1-r}{p+1-s}}, \quad \text{for all } t \in (T^*/2,T^*).$$

Multiplying by $V^{p+r(p+1-s)/(q+1-r)}$ and integrating between 0 and t, we obtain

$$V'V^{p+r\frac{p+1-s}{q+1-r}} \leqslant C\Big(\int_0^t V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma\Big)^{q+s\frac{q+1-r}{p+1-s}} V^{p+r\frac{p+1-s}{q+1-r}},$$

$$V^{p+r\frac{p+1-s}{q+1-r}+1} \leqslant C\Big(\int_0^t V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma\Big)^{q+s\frac{q+1-r}{p+1-s}+1},$$

$$V^{p+r\frac{p+1-s}{q+1-r}} \leqslant C\Big(\int_0^t V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma\Big)^{[q+1+s\frac{q+1-r}{p+1-s}]\frac{p+r\frac{p+1-s}{q+1-r}}{p+1+r\frac{p+1-s}{q+1-r}}},$$

for all $t \in (T^*/2,T^*)$. Then, we put

$$f(t) = \int_0^t V^{p+r(p+1-s)/(q+1-r)}(\sigma)d\sigma$$

and

$$l = \Big[q+1+s\frac{q+1-r}{p+1-s}\Big]\frac{p+r\frac{p+1-s}{q+1-r}}{p+1+r\frac{p+1-s}{q+1-r}},$$

we have

$$f'(t) \leqslant Cf^l(t),$$

$$\int_t^{T^*} \frac{f'}{f^l}(\sigma)d\sigma \leqslant C(T^*-t),$$

$$\frac{1}{1-l}\Big(f^{1-l}(T^*) - f^{1-l}(t)\Big) \leqslant C(T^*-t), \quad \text{for all } t \in (T^*/2,T^*).$$

Since $-l+1 = -\frac{1}{\alpha} < 0$, then $f^{1-l}(T^*) = 0$. Therefore

$$\frac{1}{l-1}f^{1-l}(t) \leqslant C(T^*-t),$$

$$f(t)^{1-l} \leqslant C(l-1)(T^*-t), \quad \text{for all } t \in (T^*/2, T^*).$$

Therefore

$$\begin{aligned} f(t) &\geqslant C(T^*-t)^{-\alpha}, \\ \int_0^t V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma &\geqslant C(T^*-t)^{-\alpha}, \quad \text{for all } t \in (T^*/2, T^*). \end{aligned}$$

Let $\tau \in (T^*/2, T^*)$, by (1.7) and V was being nondecreasing, there exists $C' > 0$ such that

$$\begin{aligned} C(T^*-t)^{-\alpha} &\leqslant \int_0^\tau V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma + \int_\tau^t V^{p+r\frac{p+1-s}{q+1-r}}(\sigma)d\sigma \\ &\leqslant C'(T^*-\tau)^{-\beta\left(p+r\frac{p+1-s}{q+1-r}\right)+1} + (t-\tau)V^{p+r\frac{p+1-s}{q+1-r}}(t). \end{aligned}$$

Taking $\tau = T^* - \gamma(T^*-t)$, for $\gamma > \max(1, \left(2C'/C\right)^{1/\alpha})$, we obtain

$$V(t) \geqslant \frac{C}{2\gamma}(T^*-t)^{-\beta}.$$

The lower estimate on U follows similarly. $\square$

Proof of Theorem 2. Since $v_t \geqslant 0$, $u_\rho \leqslant 0$ and $v_\rho \leqslant 0$ then,

$$\begin{aligned} \frac{\partial}{\partial\rho}\left(\frac{1}{2}v_\rho^2 + v(u^q v^s)\right) &= (v_{\rho\rho} + u^q v^s)v_\rho + qu^{q-1}v^{s+1}u_\rho + su^q v^s v_\rho \\ &= \left(v_t - \frac{n-1}{\rho}v_\rho\right)v_\rho + qvu^{q-1}u_\rho + su^q v^s v_\rho \leqslant 0. \end{aligned}$$

Then

$$\begin{aligned} \left(\frac{1}{2}v_\rho^2 + v(u^q v^s)\right)(t,\rho) &\leqslant \left(\frac{1}{2}v_\rho^2 + v(u^q v^s)\right)(t,0) \\ &= v(u^q v^s)(t,0). \end{aligned}$$

On the other hand, by (1.7) and Lemma 3, there exists $C > 0$ such that

$$\begin{aligned} u(t,0) &\leqslant C(T^*-t)^{-\alpha} \\ &= C\left((T^*-t)^{-\beta}\right)^{\frac{\alpha}{\beta}} \\ &\leqslant Cv^{\frac{\alpha}{\beta}}(t,0), \quad \text{for all } t \in (T^*/2, T^*). \end{aligned}$$

Therefore, by (4.2), we obtain

$$\frac{1}{2}v_\rho^2 \leqslant Cv(t,0)^{s+1}v^{\frac{\alpha}{\beta}q}(t,0), \quad \text{for all } t \in (T^*/2, T^*).$$

Then

$$\|v_\rho(t)\|_\infty \leqslant C v^{(m+1)/2}(t,0), \quad \text{for all } t \in (T^*/2, T^*),$$

with $m = [q(p+1)+s(1-r)]/(q+1-r)$. As in [10, p. 187], there exists $\eta > 0$ such that

$$v(T^*, |x|) \geqslant C|x|^{-2\beta}, \quad \text{for all } x \in (0, \eta).$$

The estimate on u is obtained similarly. □

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