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OSCILLATION PROPERTIES OF HIGHER ORDER LINEAR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we devote to investigation of higher order impulsive delay differential equations. Some interesting results for oscillation properties of every bounded solution of equations are obtained. In addition, an example shows that impulses play an important role in the oscillation properties of the solutions.

1. Introduction

Impulsive delay differential equations are useful mathematical machinery in modelling many real processes and phenomena studied in optimal control , biology, mechanics, medicine, bio-technologies, electronics, physics, etc [7]. Recently, the oscillatory behavior of impulsive delay differential equations has attracted the attention of many researchers [5]-[15]. For instance, in [5], K. Gopalsamy and B., Zhang investigated oscillation of first order delay differential equations with impulses. [16] generalized the results of [5]. In [14], J. Yan established oscillation criteria for nonlinear several delays impulsive differential equations. In [6], G. Huang and J. Shen studied oscillation of second-order linear IDE with damping:

$$\begin{cases} x''(t)+a(t)x'(t)+p(t)x(t)=0,\ t \geqslant t_0,\ t \neq t_k, \\ x^{(j)}(t_k)-x^{(j)}\left(t_k^-\right)=d_k x^{(j)}\left(t_k^-\right),\ j=0,1. \end{cases} \tag{1.1}$$

In [15], J. Yan considered the delay effection to (1.1),

$$\begin{cases} x''(t)+a(t)x'(t)+\sum\limits_{i=1}^{n} p_i(t)x(g_i(t))=0,\ t \geqslant t_0,\ t \neq t_k, \\ x^{(j)}(t_k)-x^{(j)}\left(t_k^-\right)=d_k x^{(j)}\left(t_k^-\right),\ j=0,1 \end{cases} \tag{1.2}$$

and generalized the results in [6]. Next, oscillation and nonoscillation of even order impulsive differential equations were studied and some interesting results are obtained [9]-[13]. But papers devoted to the study of the oscillation and nonoscillation of higher

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order impulsive delay differential equations are quite rare. As we known, only X., Li consider the impulsive delay differential equation

$$\begin{cases} x^{(m)}(t)+a(t)x^{(m-1)}(t)+\sum\limits_{i=1}^{n} p_i(t)x(g_i(t))=0,\ t\geqslant t_0,\ t\neq t_k, \\ x^{(j)}(t_k)-x^{(j)}\left(t_k^-\right)=d_k x^{(j)}\left(t_k^-\right),\ j=0,1,\ldots,m-1. \end{cases} \tag{1.3}$$

Before stating the conditions on $d_k, a(t), p_i(t), g_i(t)$ in the above equations, let us first state what we like to present first. In this paper, we consider a kind of higher impulsive delay differential equation

$$\begin{cases} \left(r(t)x^{(m-1)}(t)\right)'+\sum\limits_{i=1}^{n} p_i(t)x(g_i(t))=0,\ t\geqslant t_0, t\neq t_k, \\ x^{(j)}(t_k)-x^{(j)}\left(t_k^-\right)=d_k x^{(j)}\left(t_k^-\right),\ j=0,1,\ldots,m-1, \end{cases} \tag{1.4}$$

where

$$x^{(j)}\left(t_k^+\right)=\lim_{h\to 0^+}\frac{x^{(j-1)}(t_k+h)-x^{(j-1)}(t_k)}{h}$$

$$x^{(j)}\left(t_k^-\right)=x^{(j)}(t_k)=\lim_{h\to 0^-}\frac{x^{(i-1)}(t_k+h)-x^{(j-1)}(t_k)}{h}$$

and the delay differential problem

$$\left(r(t)y^{(m-1)}(t)\right)'+\sum_{i=1}^{n} p_i(t)\prod_{g_i(t)<t_k\leqslant t}(1+d_k)^{-1}y(g_i(t))=0. \tag{1.5}$$

We assume the following conditions hold:

(A_1) $0\leqslant t_0<t_1<\cdots<t_k<\cdots$ are fixed points with $\lim\limits_{k\to\infty} t_k=\infty$;

(A_2) $p_i\in C[0,\infty)\to R,\ i=1,2,\cdots,n,$ are Lebesgue measurable and locally essentially bounded functions, $r\in C[0,\infty)\to R^+, R$ is the real axis;

(A_3) $g_i\in C[0,\infty)\to R,\ i=1,2,\cdots,n,$ are Lebesgue measurable functions and $g_i(t)\leqslant t$ satisfying $\lim\limits_{t\to\infty} g_i(t)=\infty$;

(A_4) $\{d_k\}$ is a sequence of constants for each j and $d_k>-1$.

For any $\tau_0\geqslant 0$, let $\tau_0^-=\min\limits_{1\leqslant i\leqslant n}\inf\limits_{t\geqslant\tau_0} g_i(t)$. Let Ψ denote the set of functions $\phi:\left[\tau_0^-,\tau_0\right]\to R$,which are bounded and Lebesgue measurable on $\left[\tau_0^-,\tau_0\right]$.

DEFINITION 1. For any $\tau_0\geqslant 0$ and $\phi\in\Psi$, a function $x:\left[\tau_0^-,\infty\right)\to R$ is said to be a solution of (1.4) on $\left[\tau_0^-,\infty\right)$ satisfying the initial value condition

$$x(t)=\phi(t),\ \phi(\tau_0)>0,\quad t\in\left[\tau_0^-,\tau_0\right], \tag{1.6}$$

if the following conditions are satisfied:

(i) $x(t)$ satisfies (1.6);

(ii) $x^{(j)}(t)$ is absolutely continuous in each interval $\left(\tau_0,t_{k_0}\right),(t_k,t_{k+1}), k\geqslant k_0, k_0=\min\{k\,|t_k>\tau_0\}, x^{(j)}\left(t_k^+\right), x^{(j)}\left(t_k^-\right)$ exist and $x^{(j)}\left(t_k^-\right)=x^{(j)}(t_k)$, the second equality in (1.4) holds;

(iii) $x(t)$ satisfies the first equality in (1.4) almost everywhere in $\left(\tau_0^-,\infty\right)$.

DEFINITION 2. A solution of (1.4) is said to be non-oscillatory if it is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

By a solution y of of (1.5) on $\left[\tau_0^-,\infty\right)$, we mean a function which has an absolutely continuous derivative y' on $\left[\tau_0^-,\infty\right)$ satisfies (1.5) a.e. on $\left[\tau_0^-,\infty\right)$ and satisfies (1.6) on $\left[\tau_0^-,\tau_0\right]$. In this paper, we always suppose $\tau_0^- = t_0^-, \tau_0 = t_0$.

Our plan is the following. First, we prove the oscillation properties of (1.4) equality to (1.5) (Theorem 1-3), and obtain some sufficient conditons for ensure all bounded solutions of (1.4) to be nonoscillatory and oscillatory. Next, applying our Theorem 4 and Theorem 5, we state two results (Theorem 6 and Theorem 7) for (1.3). When $m=2$, Theorem 6 generalizes and improves Theorem 4 in [15]; when $m \geqslant 3$, Theorem 7 show that the boundedness and differentiability condition on $r(t)$ on Theorem 2.8 in [9] can be canceled. At last, an example is provided to illustrate the use of our results.

2. Main results

In this section we will establish theorems which enable us to reduce the oscillation and nonoscillation of (1.4) to the corresponding problem (1.5).

THEOREM 1. *Assume that (A_1)-(A_4) hold.*

(i) If y is a solution of (1.5) on $\left[t_0^-,\infty\right)$, then $x(t) = \prod_{t_0<t_k\leqslant t}(1+d_k)y(t)$ is a solution of (1.4) on $\left[t_0^-,\infty\right)$.

(ii) If x is a solution of (1.4) on $\left[t_0^-,\infty\right)$, then $y(t) = \prod_{t_0<t_k\leqslant t}(1+d_k)^{-1}x(t)$ is a solution of (1.5) on $\left[t_0^-,\infty\right)$.

Proof. First we shall prove (i). Let y be a solution of (1.5) on $\left[t_0^-,\infty\right)$, then $x(t) = \prod_{t_0<t_k\leqslant t}(1+d_k)y(t)$ has an absolutely continuous derivative x' on $\left(t_0^-,t_0\right), [t_k,t_{k+1}), k \geqslant 0$. For any $t \neq t_k, t > t_0^-$, it is easy to prove that

$$r(t)x^{(j)}(t) = \prod_{t_0<t_k\leqslant t}(1+d_k)r(t)y^{(j)}(t), j = 0,1,2,\cdots,m-1$$

then

$$\begin{aligned}\left[r(t)x^{(m-1)}(t)\right]' &= \prod_{t_0<t_k\leqslant t}(1+d_k)\left[r(t)y^{(m-1)}(t)\right]'\\ &= -\prod_{t_0<t_k\leqslant t}(1+d_k)\left(\sum_{i=1}^{n}p_i(t)\prod_{g_i(t)<t_k\leqslant t}(1+d_k)^{-1}y(g_i(t))\right)\\ &= -\left(\sum_{i=1}^{n}p_i(t)\prod_{t_0<t_k\leqslant g_i(t)}(1+d_k)y(g_i(t))\right)\\ &= -\left(\sum_{i=1}^{n}p_i(t)x(g_i(t))\right).\end{aligned}$$

So we get

$$\left[r(t)x^{(m-1)}(t)\right]' + \left(\sum_{i=1}^{n} p_i(t)x(g_i(t))\right) = 0, \quad t \geqslant t_0, t \neq t_k,$$

which implies that x is a solution of (1.4). On the other hand,

$$x^{(j)}(t_m) = \prod_{t_0 < t_k \leqslant t_m} (1+d_k)\, y^{(j)}(t_m)$$

and

$$x^{(j)}\left(t_m^-\right) = \prod_{t_0 < t_k \leqslant t_{m-1}} (1+d_k)\, y^{(j)}(t_m)$$

that is

$$x^{(j)}(t_m) = (1+d_k)\, x^{(j)}\left(t_m^-\right),$$

which implies that x solves the second condition in (1.4). So

$$x(t) = \prod_{t_0 < t_k \leqslant t} (1+d_k)\, y(t)$$

is a solution of (1.4) on $\left[t_0^-, \infty\right)$.

Next we prove (ii). Let x be a solution of (1.4). We shall prove that

$$y(t) = \prod_{t_0 < t_k \leqslant t} (1+d_k)^{-1} x(t)$$

is a solution of (1.5) on $\left[t_0^-, \infty\right)$. For any $t \neq t_k, t > t_0^-$,

$$\begin{aligned}
\left[r(t)y^{(m-1)}(t)\right]' &= \prod_{t_0 < t_k \leqslant t} (1+d_k)^{-1} \left[r(t)x^{(m-1)}(t)\right]' \\
&= -\prod_{t_0 < t_k \leqslant t} (1+d_k)^{-1} \left(\sum_{i=1}^{n} p_i(t)x(g_i(t))\right) \\
&= -\sum_{i=1}^{n} p_i(t) \prod_{t_0 < t_k \leqslant t} (1+d_k)^{-1} x(g_i(t)) \\
&= -\sum_{i=1}^{n} p_i(t) \prod_{t_0 < t_k \leqslant g_i(t)} (1+d_k)^{-1} \prod_{g_i(t) < t_k \leqslant t} (1+d_k)^{-1} x(g_i(t)) \\
&= -\sum_{i=1}^{n} p_i(t) \prod_{g_i(t) < t_k \leqslant t} (1+d_k)^{-1} y(g_i(t))
\end{aligned}$$

and

$$y^{(j)}(t_m) = \prod_{t_0 < t_k \leqslant t_m} (1+d_k)^{-1} x^{(j)}(t_m)$$

$$y^{(j)}\left(t_m^-\right) = \prod_{t_0 < t_k \leqslant t_{m-1}} (1+d_k)^{-1} x^{(j)}\left(t_m^-\right)$$

$$= \prod_{t_0<t_k\leqslant t_{m-1}} (1+d_k)^{-1} (1+d_m)^{-1} x^{(j)}(t_m)$$
$$= \prod_{t_0<t_k\leqslant t_m} (1+d_k)^{-1} x^{(j)}(t_m) = y^{(j)}(t_m).$$

So $y(t) = \prod_{t_0<t_k\leqslant t} (1+d_k)^{-1} x(t)$ is a solution of (1.5) on $\left[t_0^-,\infty\right)$. The proof is complete. $\square$

Applying Theorem 1, we obtain the following results.

THEOREM 2. *Assume that* (A_1)*-*(A_4) *hold. Then all solutions of* (1.4) *are oscillatory (nonoscillatory) if only if all solutions of* (1.5) *are oscillatory (nonoscillatory).*

THEOREM 3. *Assume that* (A_1)*-*(A_4) *hold and* $\prod_{t_0<t_k\leqslant t}(1+d_k)$ *is bounded. Then all solutions of* (1.4) *asymptotically approach to zero if only if all solutions of* (1.5) *asymptotically approach to zero.*

THEOREM 4. *Assume that* (A_1)*-*(A_4) *hold. Moreover, suppose that*

(A_5) $\prod_{t_0<t_k\leqslant t}(1+d_k)$ *is bounded and* $\liminf_{t\to\infty} \prod_{t_0<t_k\leqslant t}(1+d_k) > 0$;

(A_6) $p_i(t) \geqslant 0, i = 1,2,\cdots,n$;

(A_7) $\int_{t_0}^{t}\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{r(s)}\int_s^\infty \sum_{i=1}^{n} p_i(u) \prod_{g_i(u)<t_k\leqslant u}(1+d_k)^{-1}du\cdots d\sigma_{m-3}d\sigma_{m-2}<\infty.$

Then (1.4) *has a bounded nonoscillatory solution* x *with* $\liminf_{t\to\infty}|x(t)|>0$.

Proof. We only need to prove that (1.5) has a bounded nonoscillatory solution y. From (A_7), there exists $T>0$ such that for all $t\geqslant T, g_i(t)\geqslant T_0>0, i=1,2,\cdots,n$, and for all $t\geqslant T$,

$$\int_T^t\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{r(s)}\times \int_s^\infty\sum_{i=1}^{n}p_i(u)\prod_{g_i(u)<t_k\leqslant u}(1+d_k)^{-1}du\cdots d\sigma_{m-3}d\sigma_{m-2}<\frac{1}{4}. \quad (2.1)$$

Let Y be denote the locally convex space of all continuous functions $y\in C([T_0,\infty),R)$ with the topology of convergence on compact subintervals of $[T_0,\infty)$. Let

$$\Gamma = \left\{y\in Y : \frac{\gamma}{2}\leqslant y(t)\leqslant\frac{2\gamma}{3}, t\geqslant T_0\right\},$$

where $\gamma>0$ is an arbitrary given constant. We note that Γ is a closed and convex subset of Y and it is nonempty.

Now, define map $F:\Gamma\to Y$ by

$$(Fy)(t) = \begin{cases} \frac{\gamma}{2}+(\Omega y)(t), & t>T,\\ \frac{\gamma}{2}, & T_0\leqslant t<T,\end{cases}$$

where

$$(\Omega y)(t)=\int_T^t\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{r(s)}\times \int_s^\infty\sum_{i=1}^n p_i(u)\prod_{g_i(u)<t_k\leqslant u}(1+d_k)^{-1}y(g_i(u))\,du\cdots d\sigma_{m-3}d\sigma_{m-2}. \tag{2.2}$$

First we verify $F\Gamma\subset\Gamma$. For all $y\in\Gamma$, it is obvious that $Fy\subset\Gamma$ for $T_0\leqslant t<T$. When $t>T$, combining (2.1) we get

$$\begin{aligned}
Fy&=\frac{\gamma}{2}+(\Omega y)(t)\\
&=\frac{\gamma}{2}+\int_T^t\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{r(s)}\times\\
&\quad\int_s^\infty\sum_{i=1}^n p_i(u)\prod_{g_i(u)<t_k\leqslant u}(1+d_k)^{-1}y(g_i(u))\,du\cdots d\sigma_{m-3}d\sigma_{m-2}\\
&\leqslant\frac{\gamma}{2}+\frac{2\gamma}{3}\int_T^t\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{r(s)}\times\\
&\quad\int_s^\infty\sum_{i=1}^n p_i(u)\prod_{g_i(u)<t_k\leqslant u}(1+d_k)^{-1}du\cdots d\sigma_{m-3}d\sigma_{m-2}\\
&\leqslant\frac{\gamma}{2}+\frac{2\gamma}{3}\cdot\frac{1}{4}=\frac{2\gamma}{3}.
\end{aligned}$$

So F maps Γ into Γ. On the other hand, $\{Fy\}$ is uniformly bounded. The continuity of $F\Gamma\to\Gamma$ is verified as follows: Let $y_n\in\Gamma, y\in\Gamma$. For any $\varepsilon>0$, there exists a postive integer N_ε such that $|y_n-y|<4\varepsilon$ for any $n>N_\varepsilon$. In particular

$$|y_n(g_i(t))-y(g_i(t))|<4\varepsilon,\ \ n>N_\varepsilon, t>T_0.$$

Hence

$$\begin{aligned}
|y_n(t)-y(t)|&\leqslant\int_T^t\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{r(s)}\int_s^\infty\sum_{i=1}^n p_i(u)\times\\
&\quad\prod_{g_i(u)<t_k\leqslant u}\left(1+\alpha_k^{[j]}\right)^{-1}|y_n(g_i(u))-y(g_i(u))|\,du\cdots d\sigma_{m-3}d\sigma_{m-2}\\
&\leqslant 4\varepsilon\int_T^t\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{r(s)}\times\\
&\quad\int_s^\infty\sum_{i=1}^n p_i(u)\prod_{g_i(u)<t_k\leqslant u}(1+d_k)^{-1}du\cdots d\sigma_{m-3}d\sigma_{m-2}\\
&\leqslant 4\varepsilon\cdot\frac{1}{4}=\varepsilon.
\end{aligned}$$

So we know that F maps Γ continuous into a compact subset of Γ. Therefore, by Schauder-Tychonov's fixed point theorem, F has a fixed point y in Γ. It is easy to check that the fixed point y is a solution of (1.5). So (1.5) has a bounded nonoscillatory solution y. By Theorem 1, $x(t) = \prod_{t_0<t_k\leqslant t}(1+d_k)y(t)$ is a bounded nonoscillatory solution of (1.4). Using condition (A_5), we eventually get

$$\liminf_{t\to\infty}|x(t)| > 0.$$

The proof is complete. $\square$

Next we shall give an oscillation criterion for (1.4). Suppose m is a given even number and $m \geqslant 4$. First we give some lemmas which can be used to prove the following main Theorems.

LEMMA 1. (Lakshmikantham et al [7]) *Assume that*
(H_0) $m \in PC'(R^+, R)$ and $m(t)$ is left-continuous at $t_k, k = 1, 2, \ldots$,
(H_1) For $t_k, k = 1, 2, \ldots$ and $t \geqslant t_0$,

$$\begin{aligned} m'(t) &\leqslant p(t)m(t) + q(t),\ t \neq t_k, \\ m\left(t_k^+\right) &\leqslant d_k m(t_k) + b_k, \end{aligned}$$

where $p, q \in PC(R^+, R)$, $d_k \geqslant 0$ and b_k are real constants. Then for $t \geqslant t_0$,

$$\begin{aligned} m(t) \leqslant m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)\,ds\right) \\ + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)\,ds\right)\right) b_k \\ + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(\sigma)\,d\sigma\right) q(s)\,ds. \quad (2.3) \end{aligned}$$

LEMMA 2. *Let y be a given solution of (1.5). Assume that (A_1)-(A_4) and (A_6) hold and*

$$(A_8)\ \int_{t_0}^{\infty} \frac{1}{r(s)}\,ds = \infty,$$

Suppose that there exists $T \geqslant t_0$ such that $x(t) > 0$ for $t \geqslant T$, then there exists $T' \geqslant T$ and $N \in \{1, 3, \ldots, m-1\}$ such that for $t \geqslant T'$,

$$\begin{cases} x^{(i)}(t) > 0,\ i = 0, 1, \ldots, N; \\ (-1)^{(i-1)} x^{(i)}(t) > 0,\ i = N+1, \ldots, m-1, \\ \left(r(t)x^{(m-1)}(t)\right)' < 0. \end{cases} \quad (2.4)$$

The proofs are omitted, because their proofs are similar to [11] but without impulses.

THEOREM 5. *Assume that (A_1)-(A_4), (A_6) and (A_8) hold. Moreover suppose that*

(A_9) g_i has an absolutely continuous derivable g_i' on $\left(t_0^-,\infty\right)$, and $g_i' \geqslant 0$;

(A_{10}) $\int_{t_0}^{\infty} s^{m-3} ds \int_s^{\infty} \frac{1}{r(v)} dv \int_v^{\infty} \sum\limits_{i=1}^{n} p_i(w) \prod_{g_i(w)<t_k\leqslant w} (1+d_k)^{-1} dw = \infty$.

Then all bounded solutions of (1.4) are oscillatory.

Proof. We only need to prove that all the bounded solutions of (1.5) are oscillatory. Suppose that the assertion is not true. Without loss of generality, we may suppose that there exists $T>0$ such that $y(t)>0$ for $t \geqslant T$.

Firstly we consider the case when $N=1$. By Lemma 2, we get

$$y(t)>0,\ y'(t)>0,\ y''(t)<0,\ y'''(t)>0,\cdots,y^{(m-1)}(t)>0,\ t\geqslant T'\geqslant T.$$

So $(y(g_i(t)))' = y'(g_i(t))g_i'(t)>0$, which implies $y(g_i(t))$ is increasing in t for $t>T'$. Therefore, for $t>T'$,

$$\begin{aligned}\left(r(t)y^{(m-1)}(t)\right)' &= -\sum_{i=1}^{n} p_i(t) \prod_{g_i(t)<t_k\leqslant t} (1+d_k)^{-1} y(g_i(t)) \\ &\leqslant -\sum_{i=1}^{n} p_i(t) \prod_{g_i(t)<t_k\leqslant t} (1+d_k)^{-1} y\left(g_i\left(T'\right)\right) \\ &\leqslant -M\sum_{i=1}^{n} p_i(t) \prod_{g_i(t)<t_k\leqslant t} (1+d_k)^{-1}, \end{aligned} \tag{2.5}$$

where $M = \min\limits_{1\leqslant i\leqslant n} y(g_i(T'))>0$. Let $\varphi(t) = r(t)y^{(m-1)}(t)$, then $u(t)\geqslant 0$ for $t>T'$. From (2.5), we have

$$\begin{aligned}\varphi'(t) &\leqslant -M\sum_{i=1}^{n} p_i(t) \prod_{g_i(t)<t_k\leqslant t} (1+d_k)^{-1}, \\ \varphi\left(t_k^+\right) &= \varphi(t_k).\end{aligned}$$

Using Lemma 1, we obtain for $t>T'$

$$\varphi(\infty) \leqslant \varphi(t) - M\int_t^{\infty} \sum_{i=1}^{n} p_i(s) \prod_{g_i(s)<t_k\leqslant s} (1+d_k)^{-1} ds.$$

That is

$$0 \leqslant r(\infty)y^{(m-1)}(\infty) \leqslant r(t)y^{(m-1)}(t) - M\int_t^{\infty} \sum_{i=1}^{n} p_i(s) \prod_{g_i(s)<t_k\leqslant s} (1+d_k)^{-1} ds.$$

So

$$y^{(m-1)}(t) \geqslant \frac{M}{r(t)} \int_t^{\infty} \sum_{i=1}^{n} p_i(s) \prod_{g_i(s)<t_k\leqslant s} (1+d_k)^{-1} ds. \tag{2.6}$$

Let $\psi(t) = -y^{(m-2)}(t)$, then $\psi(t) \geqslant 0$ and from (2.6), we have

$$\psi'(t) \leqslant -\frac{M}{r(t)} \int_t^\infty \sum_{i=1}^n p_i(s) \prod_{g_i(s)<t_k \leqslant s} (1+d_k)^{-1} ds,$$
$$\psi\left(t_k^+\right) = \psi(t_k).$$

Applying Lemma 1, we get

$$0 \leqslant \psi(\infty) \leqslant \psi(t) - M \int_t^\infty \frac{1}{r(s)} ds \int_s^\infty \sum_{i=1}^n p_i(v) \prod_{g_i(v)<t_k \leqslant v} (1+d_k)^{-1} dv.$$

It follows that

$$y^{(m-2)}(t) \leqslant -M \int_t^\infty \frac{1}{r(s)} ds \int_s^\infty \sum_{i=1}^n p_i(v) \prod_{g_i(v)<t_k \leqslant v} (1+d_k)^{-1} dv. \tag{2.7}$$

Multiplying (2.7) by t^{m-3} and integrate on $[T^*, t), T^* > T'$ to find

$$\int_{T^*}^t s^{m-3} y^{(m-2)}(s) ds$$
$$\leqslant -M \int_{T^*}^t s^{m-3} ds \int_s^\infty \frac{1}{r(v)} dv \int_v^\infty \sum_{i=1}^n p_i(w) \prod_{g_i(w)<t_k \leqslant w} (1+d_k)^{-1} dw. \tag{2.8}$$

On the other hand,

$$\begin{aligned}
\int_{T^*}^t s^{m-3} y^{(m-2)}(s) ds &= \int_{T^*}^t s^{m-3} dy^{(m-3)}(s) \\
&\geqslant s^{m-3} y^{(m-3)}(s) \big|_{T^*}^t - (m-3) \int_{T^*}^t s^{m-4} y^{(m-3)}(s) ds \\
&= s^{m-3} y^{(m-3)}(s) \big|_{T^*}^t - (m-3) \int_{T^*}^t s^{m-4} dy^{(m-4)}(s) \\
&= s^{m-3} y^{(m-3)}(s) \big|_{T^*}^t - (m-3) \times \\
&\quad \left\{ s^{m-4} y^{(m-4)}(s) \big|_{T^*}^t - (m-4) \int_{T^*}^t s^{m-5} dy^{(m-5)}(s) \right\} \\
&= s^{m-3} y^{(m-3)}(s) \big|_{T^*}^t - \Big\{ (m-3) s^{m-4} y^{(m-4)}(s) \big|_{T^*}^t - \\
&\quad (m-3)(m-4) s^{m-5} y^{(m-5)}(s) \big|_{T^*}^t + \\
&\quad (m-3)(m-4)(m-5) \int_{T^*}^t s^{m-6} dy^{(m-6)}(s) \Big\} \\
&= \cdots\cdots \\
&= s^{m-3} y^{(m-3)}(s) \big|_{T^*}^t + \sum_{i=0}^{m-4} t^i y^i(t) (-1)^{m+i+1} \frac{(m-3)!}{i!} + \\
&\quad \sum_{i=0}^{m-4} (T^*)^i y^i(T^*) (-1)^{m+i} \frac{(m-3)!}{i!}
\end{aligned}$$

$$
\begin{aligned}
&= t^{m-3}y^{(m-3)}(t) - (T^*)^{m-3}y^{(m-3)}(T^*) - y(t)(m-3)! + \\
&\quad \sum_{i=1}^{m-4} t^i y^i(t)(-1)^{m+i+1}\frac{(m-3)!}{i!} + \\
&\quad \sum_{i=0}^{m-4} (T^*)^i y^i(T^*)(-1)^{m+i}\frac{(m-3)!}{i!} \\
&\geqslant -(T^*)^{m-3}y^{(m-3)}(T^*) - y(t)(m-3)! + \\
&\quad \sum_{i=0}^{m-4} (T^*)^i y^i(T^*)(-1)^{m+i}\frac{(m-3)!}{i!}. \qquad (2.9)
\end{aligned}
$$

In view of (2.8), (2.9), we obtain

$$
\begin{aligned}
&-(T^*)^{m-3}y^{(m-3)}(T^*) - y(t)(m-3)! \\
&\qquad + \sum_{i=0}^{m-4}(T^*)^i y^i(T^*)(-1)^{m+i}\frac{(m-3)!}{i!} \\
&\leqslant -M\int_{T^*}^{t} s^{m-3}ds\int_s^{\infty}\frac{1}{r(v)}dv\int_v^{\infty}\sum_{i=1}^{n}p_i(w)\prod_{g_i(w)<t_k\leqslant w}(1+d_k)^{-1}dw. \qquad (2.10)
\end{aligned}
$$

Using (A_{10}), we obtain $y(t)\to\infty$ as $t\to\infty$, which is a contradiction.

Next we consider the case when $N>1$. By Lemma 2, we get $y'(t)>0$, $y''(t)>0$, $t>T'$, so y' is increasing in t for $t\in[T',\infty)$. We note

$$y(t) = y(T') + \int_{T'}^{t} y'(s)ds \geqslant y(T') + y'(T')(t-T').$$

So $y(t)\to\infty$, as $t\to\infty$, which is a contradiction. The proof is complete. $\square$

As appplication of the previous results, first of all, we point out that main results in [8] are valid even if the differentiability and boundness of $r(t)$ is removed. Indeed, multiplying $\exp\left(\int_{t_0}^{t}a(s)\,ds\right)$ on the first equality of (1.3), then (1.3) can be rewritten as

$$
\begin{cases}
\left[\exp\left(\int_{t_0}^{t}a(s)\,ds\right)x^{(m-1)}(t)\right]' \\
\quad +\exp\left(\int_{t_0}^{t}a(s)\,ds\right)\sum_{i=1}^{n}p_i(t)x(g_i(t)) = 0,\ t\geqslant t_0, t\neq t_k, \\
x^{(j)}(t_k) - x^{(j)}(t_k^-) = d_k x^{(j)}(t_k^-),\ j=0,1,\dots,m-1.
\end{cases} \qquad (2.11)
$$

It has the form of (1.4) by setting

$$\overline{r}(t) = \exp\left(\int_{t_0}^{t}a(s)\,ds\right),\ \ \overline{P}_i(t) = \exp\left(\int_{t_0}^{t}a(s)\,ds\right)p_i(t).$$

Our Theorems 4 and 5 directly lead us to the following results.

THEOREM 6. *Assume that (A_1)-(A_6) hold. Moreover, suppose that*

$$(A_7') \quad \int_{t_0}^{t}\int_{t_0}^{\sigma_{m-2}}\cdots\int_{t_0}^{\sigma_1}\frac{1}{\overline{r}(s)}\int_{s}^{\infty}\overline{r}(u)\sum_{i=1}^{n}p_i(u)\times \prod_{g_i(u)<t_k\leqslant u}(1+d_k)^{-1}du\cdots d\sigma_{m-3}d\sigma_{m-2}<\infty.$$

Then (1.3) *has a bounded nonoscillatory solution* x *with* $\liminf_{t\to\infty}|x(t)|>0$.

REMARK 1. When $m=2$, (1.3) deduces to (1.2), so Theorem 6 generalizes and improves Theorem 4 in [15].

THEOREM 7. *When* m *is even and* $m\geqslant 4$, *assume that* (A_1)-(A_4), (A_6), (A_8) *and* (A_9) *hold. Moreover, suppose that*

$$(A_{10}') \quad \int_{t_0}^{\infty}s^{m-3}ds\int_{s}^{\infty}\frac{1}{\overline{r}(v)}\int_{v}^{\infty}\overline{r}(w)\sum_{i=1}^{n}p_i(w)\prod_{g_i(w)<t_k\leqslant w}(1+d_k)^{-1}dw=\infty.$$

Then all bounded solutions of (1.3) *are oscillatory.*

It can easy to see that let

$$\overline{r}(t)=\exp\left(\int_{t_0}^{t}a(s)\,ds\right),\ \overline{P}(t)=\overline{r}(t)\,p(t),\ \overline{P}_i(t)=\overline{r}(t)\,p_i(t).$$

Multiplying $\overline{r}(t)$ on the first equality of (1.1), then (1.1) reduces to the problem

$$\begin{cases}[\overline{r}(t)x'(t)]'+\overline{P}(t)x(t)=0,\ t\geqslant t_0, t\neq t_k,\\ x^{(j)}(t_k)-x^{(j)}\left(t_k^-\right)=d_kx^{(j)}\left(t_k^-\right),\ j=0,1.\end{cases}\tag{2.12}$$

Multiplying $\overline{r}(t)$ on the first equality of (1.2), then (1.2) reduces to the problem

$$\begin{cases}[\overline{r}(t)x'(t)]'+\sum\limits_{i=1}^{n}\overline{P}_i(t)x(g_i(t))=0,\ t\geqslant t_0, t\neq t_k,\\ x^{(j)}(t_k)-x^{(j)}\left(t_k^-\right)=d_kx^{(j)}\left(t_k^-\right),\ j=0,1.\end{cases}\tag{2.13}$$

REMARK 2. The above equations (2.12) and (2.13) are special form of (1.4), so our Theorems generalize and improve the relative results in [6, 15].

EXAMPLE 1. Consider the equation

$$\begin{cases}\left(\frac{1}{t}x'''(t)\right)'+\frac{21}{16}t^{-\frac{9}{2}}\left(t-\frac{1}{3}\right)^{-\frac{1}{2}}x\left(t-\frac{1}{3}\right)=0, t\geqslant\frac{1}{2}, t\neq k,\\ x^{(j)}(t_k)-x^{(j)}\left(t_k^-\right)=\left(\frac{k^j}{(k+1)^{j+2}}-1\right)x^{(j)}\left(t_k^-\right), j=0,1,2,\end{cases}\tag{2.14}$$

where $m=4$, $n=1$,

$$p_1(t)=\frac{21}{16}t^{-\frac{9}{2}}\left(t-\frac{1}{3}\right)^{-\frac{1}{2}},\ g_1(t)=t-\frac{1}{3},\ t_k=k,\ d_k=\frac{k^j}{(k+1)^{j+2}}-1.$$

It is easy to see that (A_1)-(A_4), (A_6), (A_8) and (A_9) satisfied, and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{t} sds \int_{s}^{\infty} \frac{1}{r(v)} dv \int_{v}^{\infty} \sum_{i=1}^{n} p_1(w) \prod_{g_1(w)<t_k \leqslant w} (1+d_k)^{-1} dw \\
&\quad = \int_{\frac{1}{2}}^{t} sds \int_{s}^{\infty} vdv \int_{v}^{\infty} \frac{21}{16} w^{-\frac{9}{2}} \left(w - \frac{1}{3}\right)^{-\frac{1}{2}} \prod_{g_1(w)<k \leqslant w} \frac{(k+1)^{j+2}}{k^j} dw \\
&\quad \geqslant \int_{\frac{1}{2}}^{t} sds \int_{s}^{\infty} vdv \int_{v}^{\infty} \frac{21}{16} w^{-\frac{9}{2}} \left(w - \frac{1}{3}\right)^{-\frac{1}{2}} w^2 dw \\
&\quad = \infty.
\end{aligned}$$

By Theorem 5, every bounded solution of (2.14) is oscillatory. But the delay differential equation

$$\left(\frac{1}{t} x'''(t)\right)' + \frac{21}{16} t^{-\frac{9}{2}} \left(t - \frac{1}{3}\right)^{-\frac{1}{2}} x\left(t - \frac{1}{3}\right) = 0,\ t \geqslant \frac{1}{2}$$

has a nonnegative solution $x = \sqrt{t}$. This example shows that impulses play an important role in the oscillatory properties of equations under perturbing impulses.

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