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A GENERALIZATION OF DARBO'S FIXED POINT THEOREM AND LOCAL ATTRACTIVITY OF GENERALIZED NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS

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Abstract. We prove a generalization of a measure theoretic fixed point theorem of Darbo in Banach spaces which includes some well-known fixed point theorems of Dhage and Sadovskii as special cases. A generalized nonlinear functional integral equation is studied via Dhage fixed point theorem for attractivity of the solutions on unbounded intervals of real line. Finally the validity of our hypotheses imposed on the functional integral equation is also discussed with a numerical example.

1. Introduction

The concept of a measure of noncompactness plays a significant role in nonlinear functional analysis, especially in the study of metric and topological fixed point theory. It may be observed that several papers dealing with the existence and qualitative behaviour of solutions for different classes of nonlinear differential and integral equations employ the measures of noncompactness and fixed point theorems as the key tools in the work. In the present paper we prove some measure theoretic fixed point theorems along the lines of Darbo [7] and Dhage [10, 11] and discuss the local attractivity of the solutions for a wider class of nonlinear functional integral equations.

The first basic Kuratowski [19] measure of noncompactness is given by

$$\alpha(X) = \inf\Big\{\delta > 0 \,\Big|\, X = \bigcup_{i=1}^{n} X_i,\ \text{diam}\,(X_i) \leqslant \delta\Big\}$$

for bounded subsets X of a metric space E, where $\text{diam}(X)$ denotes the diameter of a set $X \subset E$, i.e., $\text{diam}\,(X_i) := \sup\{d(x,y) | x,y \in X_i\}$ and second important Hasudorff (or ball) measure of noncompactness is given by

$$\chi(X) = \inf\big\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } X\big\}.$$

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The above two basic measures of noncompactness are very much useful in nonlinear analysis and its applications. however, another axiomatic way of defining the measures of noncompactness is sometimes useful for applications to nonlinear problems of analysis. See Akhmerov *et al.* [4], Banas and Goebel [6], Deimling [8] and Väth [21]. Assume that E is a given Banach space with the norm $\|\cdot\|$. For any nonempty subset X of E, by $\overline{X}$ and $\overline{co}X$ we denote the closure and the convex closure of X respectively. We denote the standard algebraic operations on sets by the symbols λX for $\lambda \in \mathbb{R}$ and $X+Y$.

Further, let $\mathscr{P}_p(E)$ denote the class of all nonempty subsets of E with a property p. Here p may be $p =$closed (cl, in short), $p =$bounded (bd, in short), $p =$relatively compact (rcp, in short) etc. Thus, $\mathscr{P}_{cl}(E), \mathscr{P}_{bd}(E), \mathscr{P}_{cl,bd}$ and $\mathscr{P}_{rcp}(E)$ denote respectively the classes of closed, bounded, closed and bounded and relatively compact subsets of E.

The following axiomatic definition of a measure of noncompactness is adopted from Dhage [10, 11].

DEFINITION 1.1. A mapping $\mu : \mathscr{P}_{bd}(E) \to \mathbb{R}_+ = [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1^o $\emptyset \neq \mu^{-1}(\{0\}) \subset \mathscr{P}_{rcp}(E)$,

2^o $\mu(\overline{X}) = \mu(X)$,

3^o $\mu(\mathrm{co X}) = \mu(\mathrm{X})$,

4^o $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$,

5^o If $\{X_n\}$ is a sequence of closed chains of $\mathscr{P}_{bd}(E)$ such that $X_{n+1} \subset X_n$, $(n = 1,2,\ldots)$ and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then the intersection set $\overline{X}_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\mu^{-1}(\{0\})$ described in 1^o is called the kernel of the measure of noncompactness μ and denoted by $\ker\mu$. Furthermore, we observe that the intersection set X_∞ from axiom 5^o is a member of the kernel $\ker\mu$. As $\mu(X_\infty) \leqslant \mu(X_n)$ for any n, we have that $\mu(X_\infty) = 0$. This yields that $X_\infty \in \ker\mu$. This simple observation will be essential in our further investigations.

The measure μ of noncompactness is called **sublinear** if it satisfies

6^o $\mu(X_1 + X_2) \leqslant \mu(X_1) + \mu(X_2)$ for all $X_1, X_2 \in \mathscr{P}_{bd}(E)$ and

7^o $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

The following well-known result of Schauder plays a key role in the topological fixed point theory and applications (cf. [6, 18] and the references therein).

THEOREM 1.1. (Schauder) Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $\mathscr{T} : \Omega \to \Omega$ has at least one fixed point in the set Ω.

A first generalization of Schauder's fixed point is the following Darbo fixed point theorem in the setting of a Banach space.

THEOREM 1.2. (Darbo) Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathscr{T}:\Omega\to\Omega$ be a continuous mapping. Assume that there exists a constant $k\in[0,1)$ such that

$$\mu(\mathscr{T}X))\leqslant k\mu(X)$$

for any nonempty subset X of Ω, where μ is a measure of noncompactness defined in E. Then $\mathscr{T}$ has a fixed point.

Next, a generalization of fixed point theorem of Darbo was proved by Dhage [10]. Before going to the main results we recall the following useful definition introduced by Dhage [10].

DEFINITION 1.2. A mapping $\mathscr{T}:X\to X$ is called $\mathscr{D}$-set-Lipschitz if there exists an upper semi-continuous nondecreasing function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\mu(\mathscr{T}(A))\leqslant\varphi(\mu(A))$ for all $A\in\mathscr{P}_{bd}(X)$ with $\mathscr{T}(A)\in P_{bd}(X)$, where $\varphi(0)=0$. The function φ is sometimes called a $\mathscr{D}$-function of $\mathscr{T}$ on X. Especially when $\varphi(r)=kr, k>0$, $\mathscr{T}$ is called a k-set-Lipschitz mapping and if $k<1$, then $\mathscr{T}$ is called a k-set-contraction on X. Further, if $\varphi(r)<r$ for $r>0$, then $\mathscr{T}$ is called a nonlinear $\mathscr{D}$-set-contraction on X.

REMARK 1.1. (Dhage [11]) If $\phi,\psi\,\mathbb{R}_+\to\mathbb{R}_+$ are two $\mathscr{D}$-functions, then i) $\phi+\psi$, ii) $\lambda\phi$, $\lambda>0$, and iii) $\phi\circ\psi$ are also $\mathscr{D}$-functions on $\mathbb{R}_+$ and commonly used $\mathscr{D}$-functions are $\psi(r)=kr$, $\psi(r)=\frac{Lr}{K+r},L>0,K>0$, and $\psi(r)=\log(1+r)$ etc. A few details of $\mathscr{D}$-functions appear in Dhage [11] and the references cited therein.

LEMMA 1.1. (Dhage [10]) If φ is a $\mathscr{D}$-function on $\mathbb{R}_+$ into itself with $\varphi(r)<r$ for $r>0$, then $\lim_{n\to\infty}\varphi^n(t)=0$ for all $t\in[0,\infty)$ and vice-versa.

Using Lemma 1.1, Dhage [10] proved the following applicable measure theoretic fixed point result.

THEOREM 1.3. (Dhage [10]) Let C be a closed, convex and bounded subset of a Banach space X and let $\mathscr{T}:C\to C$ be a continuous and nonlinear $\mathscr{D}$-set-contraction. Then $\mathscr{T}$ has a fixed point.

REMARK 1.2. Let us denote by $\mathrm{Fix}(\mathscr{T})$ the set of all fixed points of the operator $\mathscr{T}$ which belong to C. It can be shown that the set $\mathrm{Fix}(\mathscr{T})$ existing in Theorem 2.1 belongs to the family $\ker\mu$. Indeed, if $\mathrm{Fix}(\mathscr{T})\notin\ker\mu$, then $\mu(\mathrm{Fix}(\mathscr{T}))>0$ and $\mathscr{T}(\mathrm{Fix}(\mathscr{T}))=\mathrm{Fix}(\mathscr{T})$. Now from nonlinear set-contractivity it follows that $\mu(\mathscr{T}(\mathrm{Fix}(\mathscr{T})))\leqslant\phi(\mu(\mathrm{Fix}(\mathscr{T})))$ which is a contradiction since $\phi(r)<r$ for $r>0$. Hence $\mathrm{Fix}\,(\mathscr{T})\in\ker\mu$. This particular property of the measures has been used in the study of attrativity of solutions for the nonlinear funcional integral equation in question.

Next, a slight generalization of above fixed point theorem of Dhage [10] recently obtained by Aghajani et al.[3] is as follows:

THEOREM 1.4. (Aghajani et al.[3]) Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathscr{T}:\Omega\to\Omega$ be a continuous operator satisfying the inequality

$$\mu(\mathscr{T}X))\leqslant\varphi(\mu(X)) \tag{1.1}$$

for every nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing functions such that $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t>0$. Then $\mathscr{T}$ has at least one fixed point in Ω.

Note that as mentioned in Dhage *et al.* [17], the condition $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t>0$, is very difficult to verify in actual practice, but when φ is an upper semi-continuous, it is equivalent to the condition $\varphi(t)<t$, $t>0$. So in this case, Theorem 1.4 is equivalent to Theorem 1.3 of Dhage [10]. In view of above facts, Theorem 1.4 is not applicable to nonlinear problems of differential and integral equations and the authors in [3] though not mentioned actually employed Theorem 1.3 and not Theorem 1.4 as a key tool in their study of nonlinear integral equations. Aghajani *et al.* [3] further proved the following generalization of Darbo's fixed point theorem using the same method.

THEOREM 1.5. Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathscr{T}:\Omega\to\Omega$ be a continuous operator such that

$$\psi(\mu(\mathscr{T}X))\leqslant\psi(\mu(X))-\phi(\mu(X)) \tag{1.2}$$

for every nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\phi,\psi:\mathbb{R}_+\to\mathbb{R}_+$ are given functions such that ψ is continuous and ϕ is lower semicontinuous on $\mathbb{R}_+$. Moreover, $\phi(0)=0$ and $\phi(t)>0$ for $t>0$. Then $\mathscr{T}$ has at least one fixed point in Ω.

After analyzing Theorem 1.5, one is convinced that the role of the function ψ in the proof is superficial and Theorem 1.5 is not in line with the Darbo fixed point theorem. Indeed, neither the function ψ has any role in the proof of the theorem nor it has any application potential in solving the problem of existence of solutions of any nonlinear functional integral equations for which the theorem was sought to focus. In this context, we prove in the following section some results related to useful generalizations of Darbo's fixed point and their interesting consequences.

2. Fixed Point Results

DEFINITION 2.1. Let $\mathscr{R}$ denote the class of those functions $\beta:\mathbb{R}_+\to\mathbb{R}_+$ which satisfy the conditions (i) β is continuous, and (ii) $\beta(t_n)\to 0$ implies that $t_n\to 0$.

It is not difficult to verify that the following functions satisfy the conditions given in Definition 2.1:

(a) $\beta(t) = \ln(1+t)$ for all $t \in \mathbb{R}_+$; and

(b) $\beta(t) = t - \ln(1+t)$ for all $t \in \mathbb{R}_+$.

Obviously, the identity mapping on $\mathbb{R}_+$ into itself also satisfies the requisite conditions of Definition 2.1.

We now state and prove our main result which can be considered as a generalization of Darbo's fixed point theorem and Theorem 1.4 of Aghajani et. al.[3]:

THEOREM 2.1. Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathscr{T} : \Omega \to \Omega$ be a continuous operator such that

$$\mu(\mathscr{T}X) \leqslant \mu(X) - \phi(\beta(\mu(X))) \tag{2.1}$$

for every nonempty subset X of Ω and each $\beta \in \mathscr{R}$, where μ is an arbitrary measure of noncompactness and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing function such that ϕ is a lower semicontinuous on $\mathbb{R}_+$ such that $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. Then $\mathscr{T}$ has at least one fixed point in Ω.

Proof. Define a sequence $\{\Omega_n\}$ as $\Omega_0 = \Omega$ and $\Omega_n = \overline{co}\,\mathscr{T}\Omega_{n-1}$ for $n = 1, 2, \cdots$. If there exists a natural number n_0 such that $\mu(\Omega_{n_0}) = 0$, then Ω_{n_0} is compact. By Theorem 1.2, $\mathscr{T}$ has a fixed point in Ω. Next, we assume that $\mu(\Omega_{n_0}) > 0$ for $n = 1, 2, \cdots$. Using (2.1), we get

$$\mu(\Omega_{n+1}) = \mu(\overline{co}\,\mathscr{T}\Omega_n) = \mu(\mathscr{T}\Omega_n) \leqslant \mu(\Omega_n) - \phi(\beta(\mu(\Omega_n))). \tag{2.2}$$

Now, taking into account that $\Omega_{n+1} \subset \Omega_n$, on the basis of axiom 2^o of Definition 1.1 the sequence $\{\mu(\Omega_n)\}$ is nonincreasing and nonnegative. From this we infer that $\mu(\Omega_n) \to r$ when $n \to \infty$, where $r \geqslant 0$ is a nonnegative real number. Since β is continuous, it follows that $\beta(\mu(\Omega_n)) \to \beta(r)$ as $n \to \infty$. Now, in view of (2.2) we obtain

$$\limsup_{n\to\infty} \mu(\Omega_{n+1}) \leqslant \limsup_{n\to\infty} \mu(\Omega_n) - \liminf_{n\to\infty} \phi(\beta(\mu(\Omega_n))).$$

This yields $r \leqslant r - \liminf_{n\to\infty} \phi(\beta(\mu(\Omega_n)))$. Since ϕ is nondecreasing, we obtain $\phi(\beta(r)) \leqslant \liminf_{n\to\infty} \phi(\beta(\mu(\Omega_n))) = 0$. From this, in view of the fact that $\phi(0) = 0$, we deduce that $\beta(\mu(\Omega_n)) \to 0$ as $n \to \infty$. By the definition of β we infer that $\mu(\Omega_n) \to 0$ as $n \to \infty$. Now, using axiom 6^o of Definition 1.1 we derive that the set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed, convex and $\Omega_\infty \subset \Omega$. Notice that $\mathscr{T}(\Omega_\infty) \subset \Omega$ i.e., $\mathscr{T}$ maps Ω_∞ into itself and $\Omega_\infty \in \ker \mu$. Now taking into account Schauder fixed point principle (cf. Theorem 1.2) we infer that the operator $\mathscr{T}$ has a fixed point x in the set Ω_∞. Since $\Omega_\infty \subset \Omega$, it follows that $x \in \Omega$. This completes the proof. □

Taking $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ to be an identity mapping on $\mathbb{R}_+$, then we obtain the following new fixed point result as a corollary with interesting consequences.

COROLLARY 2.1. Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathscr{T}:\Omega\to\Omega$ be a continuous operator such that

$$\mu(\mathscr{T}X)\leqslant\mu(X)-\phi(\mu(X)) \tag{2.3}$$

for every nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ is a lower semicontinuous on $\mathbb{R}_+$ such that $\phi(0)=0$ and $\phi(t)>0$ for $t>0$. Then $\mathscr{T}$ has at least one fixed point in Ω.

When $\phi(r)=(1-k)r$, $0\leqslant k<1$, Corollary 2.1 reduces to Theorem 1.2 above due to Darbo [7]. Again, when $\mu(X)>0$, then from condition (2.3), we obtain the following Sadovskii's fixed point theorem for condensing mappings characterized by the inequality $\mu(\mathscr{T}X)<\mu(X)$. The mappings satisfying this contractive inequality are called condensing mappings on Banach spaces.

THEOREM 2.2. (Sadovskii [20]) Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathscr{T}:\Omega\to\Omega$ be a continuous and condensing mapping. Then T has at least one fixed point in Ω.

A slight variant of Theorem 2.1 can be formulated as given below.

THEOREM 2.3. Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathscr{T}:\Omega\to\Omega$ be a continuous operator such that

$$\mu(\mathscr{T}X)\leqslant\mu(X)-\phi(\mu(\mathscr{T}X)) \tag{2.4}$$

for every nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ is a nonincreasing lower semi-continuous function on $\mathbb{R}_+$ such that $\phi(0)=0$ and $\phi(t)>0$ for $t>0$. Then $\mathscr{T}$ has at least one fixed point in Ω.

Proof. The proof is similar to Theorem 2.1 and hence we omit the details.

In order to introduce further concepts used in this paper, let us assume that $E=BC(\mathbb{R}_+,\mathbb{R})$ and let Ω be a subset of E. Let $\mathscr{Q}:E\to E$ be an operator and consider the operator equation in E,

$$\mathscr{Q}x(t)=x(t) \text{ for all } t\in\mathbb{R}_+. \tag{2.5}$$

Below we give different characterizations of the solutions for the operator equation (2.5) on $\mathbb{R}_+$.

DEFINITION 2.2. We say that solutions of the equation (2.5) are locally attractive if there exists a closed ball $B[x_0,r_0]$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_0\in BC(\mathbb{R}_+,\mathbb{R})$ such that, for arbitrary solutions $x=x(t)$ and $y=y(t)$ of equation (2.5) belonging to $B[x_0,r_0]\cap\Omega$, we have

$$\lim_{t\to\infty}(x(t)-y(t))=0. \tag{2.6}$$

In case the limit (2.6) is uniform with respect to the set $B[x_0, r_0] \cap \Omega$, i.e., for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leqslant \varepsilon \tag{2.7}$$

for all solutions $x, y \in B[x_0, r_0] \cap \Omega$ of (2.5) and for $t \geqslant T$, we will then say that solutions of equation (2.5) are uniformly locally attractive on $\mathbb{R}_+$.

Let $X = BC(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous and bounded functions on $\mathbb{R}_+$ and define a norm $\|\cdot\|$ in X by

$$\|x\| = \sup\{|x(t)| : t \geqslant 0\}.$$

Clearly X is a Banach space with this supremum norm. Let us fix a bounded subset A of X and a positive real number T. For any $x \in A$ and $\varepsilon \geqslant 0$, denote by $\omega^T(x, \varepsilon)$, the modulus of continuity of x on the interval $[0, T]$ defined by

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leqslant \varepsilon\}.$$

Moreover, let

$$\omega^T(A, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in A\},$$

$$\omega_0^T(A) = \lim_{\varepsilon \longrightarrow 0} \omega^T(A, \varepsilon),$$

$$\omega_0(A) = \lim_{T \longrightarrow \infty} \omega_0^T(A).$$

By $A(t)$ we mean a set in $\mathbb{R}$ defined by $A(t) = \{x(t) \mid x \in A\}$ for $t \in \mathbb{R}_+$. We denote $\operatorname{diam}(A(t)) = \sup\{|x(t) - y(t)| : x, y \in A\}$. Finally we define a function μ on $\mathscr{P}_{bd}(X)$ by the formula

$$\mu(A) = \omega_0(A) + \limsup_{t \longrightarrow \infty} \operatorname{diam}(A(t)). \tag{2.8}$$

It is known that μ is a sublinear and therefore, a useful handy tool of measure of noncompactness in X for some practical applications.

3. Local Attractivity Result

In this section, as an application of our results in Section 2, we consider the following generalized nonlinear functional integral equation (in short GNFIE)

$$x(t) = F\left(t, x(\theta(t)), u(t, x(\alpha(t))), \int_0^{\beta(t)} f(t, s, x(\gamma(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\right), \tag{3.1}$$

for $t \in \mathbb{R}_+$, where $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $f, g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha, \beta, \gamma, \theta, \sigma, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions.

The functional integral equation (3.1) is "general" in the sense that it includes several classes of known integral equations discussed in the literature. See Dhage and Lakshmikantham [14], Dhage *et.al.* [13], Dhage and Ntouyas [15], Krasnoselskii [18], Väth [21], Dhage [9, 10] and the references therein. We now intend to obtain the solutions of GNFIE (3.1) in the space $BC(\mathbb{R}_+,\mathbb{R})$ of all bounded and continuous real-valued functions on $\mathbb{R}_+$.

We consider the following set of assumptions in what follows.

(H_0) The functions $\alpha,\beta,\gamma,\theta,\sigma,\eta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\theta(t) \geqslant t$ and $\alpha(t) \geqslant t$ for all $t \in \mathbb{R}_+$.

(H_1) There exists a $\mathscr{D}$-function φ and the constants $L_i > 0$, $i = 1,2,3$; such that

$$|F(t,x,x_1,x_2,x_3) - F(t,y,y_1,y_2,y_3)| \leqslant \varphi(|x-y|) + \sum_{i=1}^{3} L_i|x_i - y_i|$$

for all $t \in \mathbb{R}_+$ and $x,y,x_i,y_i \in \mathbb{R}$, $i = 1,2,3$. Moreover, the map $t \to F(t,0,0,0,0)$ is bounded with $F_0 = \sup\limits_{t\geqslant 0}|F(t,0,0,0,0)|$.

(H_2) There exist a $\mathscr{D}$-function φ_1 and a continuous function $k_1 \in BC(\mathbb{R}_+,\mathbb{R}_+)$ such that

$$|u(t,x) - u(t,y)| \leqslant k_1(t)\varphi_1(|x-y|),$$

for all $t \in \mathbb{R}_+$. Moreover, $\sup\limits_{t\geqslant 0} k_1(t) = K_1$.

(H_3) The function $t \to u(t,0) = u_0(t)$ is bounded and $C_0 = \sup\limits_{t\geqslant 0}|u(t,0)|$.

(H_4) The function $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist a continuous function $q : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and a $\mathscr{D}$-function φ_2 such that

$$|f(t,s,x) - f(t,s,y)| \leqslant q(t,s))\varphi_2(|x-y|)$$

for all $t,s \in \mathbb{R}_+$ and $x,y \in \mathbb{R}$. Moreover, $\lim\limits_{t\to\infty} \int_0^{\beta(t)} q(t,s)\,ds = 0$.

(H_5) The function $f_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $f_0(t) = \int_0^{\beta(t)} |f(t,s,0)|\,ds$ is bounded with $C_1 = \sup_{t\geqslant 0} f_0(t)$.

(H_6) $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $a,b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,s,x)| \leqslant a(t)b(s)$$

for $t,s \in \mathbb{R}_+$, $t \geqslant s$ and $x \in \mathbb{R}$. Moreover, $\lim\limits_{t\to\infty} a(t) \int_0^{\sigma(t)} b(s)\,ds = 0$.

REMARK 3.1. Since the hypotheses (H_4) and (H_6) are held, we have that the functions

$$k_2(t) = \int_0^{\beta(t)} q(t,s)\,ds \quad \text{and} \quad v(t) = a(t)\int_0^{\sigma(t)} b(s)\,ds$$

are bounded on $\mathbb{R}_+$ and the positive numbers

$$K_2 = \sup_{t \geqslant 0} \int_0^{\beta(t)} q(t,s)\,ds \quad \text{and} \quad V = \sup_{t \geqslant 0} v(t)$$

exist.

THEOREM 3.1. Assume that the hypotheses (H_0) through (H_6) hold. Suppose that

$$\varphi(r) + L_1 K_1 \varphi_1(r) < r, \; r > 0, \tag{3.2}$$

and there exists a positive solution r_0 of the inequality

$$\varphi(r) + L_1 K_1 \varphi_1(r) + L_2 K_2 \varphi_2(r) + q \leqslant r, \tag{3.3}$$

where q is the constant defined by the equality

$$q = \{L_1 C_0 + L_2 C_1 + L_3 V + F_0\}.$$

Then the functional nonlinear integral equation (3.1) has a solution and the solutions are uniformly locally attractive on $\mathbb{R}_+$.

Proof. Consider the operator $\mathscr{Q}$ defined on the space $BC(\mathbb{R}_+,\mathbb{R})$ by the formula

$$\mathscr{Q}x(t) = F\left(t, x(\theta(t)), u(t,x(\alpha(t))), \int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\right), \tag{3.4}$$

for $t \in \mathbb{R}_+$. We shall show that the map $\mathscr{Q}$ satisfies all the conditions of Theorem 1.3 on X.

Step I: First we show that $\mathscr{Q}$ defines a mapping $\mathscr{Q}: X \longrightarrow X$. Since all the functions involved in (3.1) are continuous, $\mathscr{Q}x$ is continuous on $\mathbb{R}_+$ for each $x \in X$. Hence $\mathscr{Q}x$ is mapping from X into itself. As $\theta(\mathbb{R}_+) \subseteq \mathbb{R}_+$, we have $\max_{t \geqslant 0}|x(\theta(t))| \leqslant \max_{t \geqslant 0}|x(t)|$. On the other hand, hypotheses (H_0)-(H_5) imply that

$$\begin{aligned}
&|\mathscr{Q}x(t)| \\
&= \left|F\left(t, x(\theta(t)), u(t,x(\alpha(t))), \int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\right)\right| \\
&\leqslant \left|F\left(t, x(\theta(t)), u(t,x(\alpha(t))), \int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\right)\right. \\
&\qquad\qquad \left. - F(t,0,0,0,0)\right| + |F(t,0,0,0,0)|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \varphi(|x(\theta(t))|)+L_1|u(t,x(\alpha(t)))|+L_2\left|\int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds\right| \\
&+L_3\left|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\right|+|F(t,0,0,0,0)| \\
&\leqslant \varphi(|x(\theta(t))|)+L_1|u(t,x(t))-u(t,0)|+L_1|u(t,0)| \\
&+L_2\int_0^{\beta(t)} |f(t,s,x(\theta(s)))-f(t,s,0)|\,ds+L_2\int_0^{\beta(t)} |f(t,s,0)|\,ds \\
&+L_3\left|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\right|+F_0 \\
&\leqslant \varphi(|x(\theta(t))|)+L_1k_1(t)\varphi_1(|x(\alpha(t))|)+L_1|u(t,0)| \\
&\quad +L_2\int_0^{\beta(t)} q(t,s)\varphi_2(|x(\gamma(s))|)\,ds+L_2C_1+L_3a(t)\int_0^{\sigma(t)} b(s)\,ds+F_0 \\
&\leqslant \varphi(\|x\|)+L_1k_1(t)\varphi_1(\|x\|)+L_1C_0 \\
&\quad +L_2\int_0^{\beta(t)} q(t,s)\varphi_2(\|x\|)\,ds+L_2C_1+L_3v(t)+F_0 \\
&\leqslant \varphi(\|x\|)+L_1K_1\varphi_1(\|x\|)+L_2K_2\varphi_2(\|x\|) \\
&\quad +L_1C_0+L_2C_1+L_3V+F_0 \qquad (3.5)
\end{aligned}
$$

for all $t \in \mathbb{R}_+$. From (3.5), we deduce that $\mathscr{Q}x \in X$.

Step II: From (3.5) it follows that

$$
\|\mathscr{Q}x\| \leqslant \varphi(r)+L_1K_1\varphi_1(r)+L_2K_2\varphi_2(r)+q \leqslant r. \qquad (3.6)
$$

Now consider the closed ball $B[0,r_0] \subset C[0,T]$ in X centered at origin of radius r_0. Then $\mathscr{Q}$ defines a mapping $\mathscr{Q}: B[0,r_0] \to B[0,r_0]$. We show that $\mathscr{Q}$ is continuous on $B[0,r_0]$. Let $\varepsilon > 0$ be given and let $x,y \in B[0,r_0]$ be such that $\|x-y\| \leqslant \varepsilon$. Then by hypotheses (H_0)-(H_5),

$$
\begin{aligned}
&|\mathscr{Q}x(t)-\mathscr{Q}y(t)| \\
&\quad\leqslant \left|F\left(t,x(\theta(t)),u(t,x(\alpha(t))),\int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds,\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\right)\right. \\
&\quad\quad \left.-F\left(t,y(\theta(t)),u(t,y(\alpha(t))),\int_0^{\beta(t)} f(t,s,y(\gamma(s)))ds,\int_0^{\sigma(t)} g(t,s,y(\eta(s)))ds\right)\right| \\
&\quad\leqslant \varphi(|x(\theta(t))-y(\theta(t))|)+L_1k_1(t)\varphi_1(|x(\alpha(t))-y(\alpha(t))|) \\
&\quad\quad +L_2\left|\int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds-\int_0^{\beta(t)} f(t,s,y(\gamma(s)))ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&+L_3\left|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds-\int_0^{\sigma(t)} g(t,s,y(\eta(s)))ds\right| \\
&\leqslant \varphi(|x(\theta(t))-y(\theta(t))|)+L_1k_1(t)\varphi_1(|x(\alpha(t))-y(\alpha(t))|) \\
&\quad +L_2\int_0^{\beta(t)} |f(t,s,x(\gamma(s)))-f(t,s,y(\gamma(s)))|ds \\
&\quad +L_3\int_0^{\sigma(t)} |g(t,s,x(\eta(s)))-g(t,s,y(\eta(s)))|ds \\
&\leqslant \varphi(|x(\theta(t))-y(\theta(t))|)+L_1k_1(t)\varphi_1(|x(\alpha(t))-y(\alpha(t))|) \\
&\quad +L_2\int_0^{\beta(t)} q(t,s)\varphi_2(|x(\gamma(s))-y(\gamma(s))|)\,ds+L_3a(t)\int_0^{\sigma(t)} b(s)\,ds \\
&\leqslant \varphi(\|x-y\|)+L_1K_1\varphi_1(\|x-y\|)+L_2K_2\varphi_2(\|x-y\|)\,ds+L_3v(t) \\
&\leqslant \varphi(\varepsilon)+L_1K_1\varphi_1(\varepsilon)+L_2K_2\varphi_2(\varepsilon)\,ds+L_3v(t) \\
&\leqslant (1+L_1K_1+L_2K_2)\varepsilon+L_3v(t). \qquad (3.7)
\end{aligned}
$$

Since $v(t)\to 0$ as $t\to\infty$, there exists $T>0$ such that $v(t)\leqslant\varepsilon$, for all $t>T$. Thus if $t>T$, then from (3.7) we have that

$$
|\mathscr{Q}x(t)-\mathscr{Q}y(t)|\leqslant(1+L_1K_1+L_2K_2+L_3)\varepsilon. \qquad (3.8)
$$

If $t<T$, then define a function $\omega=\omega(\varepsilon)$ by the formula

$$
\omega(\varepsilon)=\sup\{|g(t,s,x)-g(t,s,y)|: t,s\in[0,T], x,y\in[-r_0,r_0], |x-y|\leqslant\varepsilon\}. \qquad (3.9)
$$

Now $g(t,s,x)$ is a continuous and hence uniformly continuous on $[0,T]\times[0,T]\times[-r_0,r_0]$. As a result we have $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, from (3.7),

$$
|\mathscr{Q}x(t)-\mathscr{Q}y(t)|\leqslant(1+L_1K_1+L_2K_2)\varepsilon+L_3\omega(\varepsilon)
$$

for all $t\in\mathbb{R}_+$. Hence, it follows that

$$
\begin{aligned}
\|\mathscr{Q}x-\mathscr{Q}y\| &\leqslant \max\{(1+L_1K_1+L_2K_2+L_3)\varepsilon,(1+L_1K_1+L_2K_2)\varepsilon+L_3\omega(\varepsilon)\} \\
&\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

Hence $\mathscr{Q}$ is a continuous mapping from $B[0,r_0]$ into itself.

Step III: Here we show that $\mathscr{Q}$ is a nonlinear set-contraction on $B[0,r_0]$ in the sense of inequality (2.5). This will be done in the following two cases:

Case I : Let $A\subset B[0,r_0]$ be non-empty. Further fix the number $T>0$ and $\varepsilon>0$. Since the functions f and g are continuous on compact domain $[0,T]\times[0,T]\times[-r_0,r_0]$, there are constants $D_2>0$ and $D_3>0$ such that $|f(t,s,x)|\leqslant D_2$

and $|g(t,s,x)| \leqslant D_3$ for all $t,s \in [0,T]$ and $x \in [-r_0, r_0]$. Then choosing $t,\tau \in [0,T]$ such that $|t-\tau| \leqslant \varepsilon$ and taking into account our hypotheses, we obtain

$$
\begin{aligned}
&|\mathscr{Q}x(t) - \mathscr{Q}x(\tau)| \\
&\quad \leqslant \Big| F\Big(t, x(\theta(t)), u(t,x(\alpha(t))), \int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\Big) \\
&\quad - F\Big(\tau, x(\theta(\tau)), u(\tau,x(\alpha(\tau))), \int_0^{\beta(\tau)} f(\tau,s,x(\gamma(s)))ds, \int_0^{\sigma(\tau)} g(t,s,x(\eta(s)))ds\Big)\Big| \\
&\quad \leqslant \varphi(|x(\theta(t)) - x(\theta(\tau))|) + L_1|u(t,x(\alpha(t))) - u(\tau,x(\alpha(\tau)))| \\
&\quad\quad + L_2\Big|\int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds - \int_0^{\beta(\tau)} f(\tau,s,x(\gamma(s)))ds\Big| \\
&\quad\quad + L_3\Big|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds - \int_0^{\sigma(\tau)} g(\tau,s,x(\eta(s)))ds\Big| \\
&\quad\quad \leqslant \varphi(|x(\theta(t)) - x(\theta(\tau))|) + L_1|u(t,x(\alpha(t))) - u(\tau,x(\alpha(\tau)))| \\
&\quad\quad + L_2\Big|\int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds - \int_0^{\beta(t)} f(\tau,s,x(\gamma(s)))ds\Big| \\
&\quad\quad + L_2\Big|\int_0^{\beta(t)} f(\tau,s,x(\gamma(s)))ds - \int_0^{\beta(\tau)} f(\tau,s,x(\gamma(s)))ds\Big| \\
&\quad\quad + L_3\Big|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds - \int_0^{\sigma(\tau)} g(\tau,s,x(\eta(s)))ds\Big| \\
&\quad\quad + L_3\Big|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds - \int_0^{\sigma(\tau)} g(\tau,s,x(\eta(s)))ds\Big| \\
&\quad \leqslant \varphi(|x(\theta(t)) - x(\theta(\tau))|) + L_1|u(t,x(\alpha(t))) - u(\tau,x(\alpha(\tau)))| \\
&\quad\quad + L_2\int_0^{\beta(t)} |f(t,s,x(\theta(s))) - f(\tau,s,x(\theta(s)))|ds + L_2\Big|\int_{\beta(\tau)}^{\beta(t)} |f(\tau,s,x(\gamma(s)))|ds\Big| \\
&\quad\quad + L_3\Big|\int_{\sigma(\tau)}^{\sigma(t)} |g(\tau,s,x(\theta(s)))|ds\Big| + L_3\int_0^{\sigma(t)} |g(t,s,x(\eta(s))) - g(\tau,s,x(\eta(s)))|ds \\
&\quad\quad \leqslant \varphi(|x(\theta(t)) - x(\theta(\tau))|) + L_1\omega^T(u,\varepsilon) + L_2\beta_T\omega^T(f,\varepsilon) + L_2D_2\omega^T(\beta,\varepsilon) \\
&\quad\quad + L_3D_3\omega^T(\sigma,\varepsilon) + L_3\sigma_T\omega^T(g,\varepsilon)
\end{aligned} \tag{3.10}
$$

where,

$$\beta_T = \sup\{\beta(t) : t \in [0,T]\},$$

$$\begin{aligned}\sigma_T &= \sup\{\sigma(t) : t \in [0,T]\},\\ \omega^T(\beta,\varepsilon) &= \sup\{|\beta(t)-\beta(\tau)| : t,\tau \in [0,T], |t-\tau| \leqslant \varepsilon\},\\ \omega^T(\sigma,\varepsilon) &= \sup\{|\sigma(t)-\sigma(\tau)| : t,\tau \in [0,T], |t-\tau| \leqslant \varepsilon\},\end{aligned}$$

and

$$\begin{aligned}\omega^T(u,\varepsilon) &= \sup\{|u(t,x)-u(\tau,x)| : t,\tau \in [0,T], |t-\tau| \leqslant \varepsilon, |x| \leqslant r_0\}\\ \omega^T(f,\varepsilon) &= \sup\{|f(t,s,x)-f(\tau,s,x)| : t,\tau \in [0,T], |t-\tau| \leqslant \varepsilon, |x| \leqslant r_0\},\\ \omega^T(g,\varepsilon) &= \sup\{|g(t,s,x)-g(\tau,s,x)| : t,\tau \in [0,T], |t-\tau| \leqslant \varepsilon, |x| \leqslant r_0\}.\end{aligned}$$

The above inequality further implies that

$$\begin{aligned}\omega^T(\mathscr{Q}x,\varepsilon) \leqslant \varphi(\omega^T(x,\varepsilon)) + L_1\omega^T(u,\varepsilon) + L_2\beta_T\omega^T(f,\varepsilon) + L_2D_2\omega^T(\beta,\varepsilon)\\ + L_3D_3\omega^T(\sigma,\varepsilon) + L_3\sigma_T\omega^T(g,\varepsilon).\end{aligned} \tag{3.11}$$

Since by hypotheses, the functions β,σ,φ,u and f,g are continuous respectively on $[0,T]$, $[0,T]\times[-r_0,r_0]$ and $[0,T]\times[0,T]\times[-r_0,r_0]$, we infer that they are uniformly continuous there. Hence we deduce that $\varphi(\omega^T(x,\varepsilon)) \to 0$, $\omega^T(u,\varepsilon) \to 0$, $\omega^T(\beta,\varepsilon) \to 0$, $\omega^T(\sigma,\varepsilon) \to 0$, $\omega^T(f,\varepsilon) \to 0$, $\omega^T(g,\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence from the above estimate (3.11), we obtain

$$\omega_0^T(\mathscr{Q}(A)) = 0,$$

and consequently

$$\omega_0(\mathscr{Q}(A)) = 0. \tag{3.12}$$

Case II: Now for any $x,y \in A$, one has

$$\begin{aligned}&|\mathscr{Q}x(t)-\mathscr{Q}y(t)|\\ &\leqslant \left|F\left(t,x(\theta(t)),u(t,x(\alpha(t))),\int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds,\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\right)\right.\\ &\left.-F\left(t,y(\theta(t)),u(t,y(\alpha(t))),\int_0^{\beta(t)} f(t,s,y(\gamma(s)))ds,\int_0^{\sigma(t)} g(t,s,y(\eta(s)))ds\right)\right|\\ &\leqslant \varphi(|x(\theta(t))-y(\theta(t))|) + L_1k_1(t)\varphi_1(|x(\alpha(t))-y(\alpha(t))|)\\ &\quad + L_2\left|\int_0^{\beta(t)} f(t,s,x(\gamma(s)))ds - \int_0^{\beta(t)} f(t,s,y(\gamma(s)))ds\right|\\ &\quad + L_3\left|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds - \int_0^{\sigma(t)} g(t,s,y(\eta(s)))ds\right|\\ &\leqslant \varphi(|x(\theta(t))-y(\theta(t))|) + L_1k_1(t)\varphi_1(|x(\alpha(t))-y(\alpha(t))|)\\ &\quad + L_2\int_0^{\beta(t)} |f(t,s,x(\gamma(s)))-f(t,s,y(\gamma(s)))|ds\end{aligned}$$

$$
\begin{aligned}
&+L_3\int_0^{\sigma(t)}|g(t,s,x(\eta(s)))-g(t,s,y(\eta(s)))|\,ds\\
&\leqslant \varphi(|x(\theta(t))-y(\theta(t))|)+L_1k_1(t)\varphi_1(|x(\alpha(t))-y(\alpha(t))|)\\
&+L_2\int_0^{\beta(t)}q(t,s)\varphi_2(|x(\gamma(s))-y(\gamma(s))|)\,ds+2L_3a(t)\int_0^{\sigma(t)}b(s)\,ds\\
&\leqslant \varphi(\operatorname{diam}(A(\theta(t))))+L_1k_1(t)\varphi_1(\operatorname{diam}(A(\alpha(t))))\\
&+L_2\int_0^{\beta(t)}q(t,s)\varphi_2(\operatorname{diam}(A(\gamma(\beta(s)))))\,ds+2L_3v(t)\\
&\leqslant \varphi(\operatorname{diam}(A(\theta(t))))+L_1k_1(t)\varphi_1(\operatorname{diam}(A(\alpha(t))))\\
&+L_2\int_0^{\beta(t)}q(t,s)\varphi_2(\operatorname{diam}(A))\,ds+2L_3v(t).
\end{aligned}
$$

for all $t\in\mathbb{R}_+$. Further, we also notice that $A\subset B[0,r_0]$ implies $\operatorname{diam}(A)\leqslant 2r_0$. Again, since $\theta(t)\geqslant t$ and $\alpha(t)\geqslant t$ we have that $\operatorname{diam}(A(\theta(t))\leqslant\operatorname{diam}(A(t))$ and $\operatorname{diam}(A(\alpha(t))\leqslant\operatorname{diam}(A(t))$ for all $t\in\mathbb{R}_+$. Therefore, as a result of above inequality, we obtain

$$
\begin{aligned}
\operatorname{diam}(\mathscr{Q}(A(t)))\leqslant{}& \varphi(\operatorname{diam}(A(t)))+L_1k_1(t)\varphi_1(\operatorname{diam}(A(t)))\\
&+L_2\int_0^{\beta(t)}q(t,s)\varphi_2(2r_0)\,ds+2L_3v(t)
\end{aligned}
\tag{3.13}
$$

for all $t\in\mathbb{R}_+$. Taking the limit superior in the above inequality yields

$$
\begin{aligned}
&\limsup_{t\to\infty}\operatorname{diam}(\mathscr{Q}(A(t)))\\
&\qquad\leqslant \limsup_{t\to\infty}\varphi(\operatorname{diam}(A(t)))+L_1K_1\limsup_{t\to\infty}\varphi_1(\operatorname{diam}(A(t)))\\
&\qquad\quad+L_2\varphi_2(2r_0)\limsup_{t\to\infty}\int_0^{\beta(t)}q(t,s)\,ds+2L_3\limsup_{t\to\infty}v(t)\\
&\qquad\leqslant \limsup_{t\to\infty}\varphi(\operatorname{diam}(A(t)))+L_1K_1\limsup_{t\to\infty}\varphi_1(\operatorname{diam}(A(t))))\\
&\qquad\leqslant \varphi\left(\limsup_{t\to\infty}\operatorname{diam}(A(t))\right)+L_1K_1\varphi_1\left(\limsup_{t\to\infty}\operatorname{diam}(A(t))\right)\\
&\qquad\leqslant \psi\left(\limsup_{t\to\infty}\operatorname{diam}(A(t))\right)
\end{aligned}
\tag{3.14}
$$

where, ψ is again a $\mathscr{D}$-function in view of Remark 1.1 defined by $\psi(r)=\phi(r)+L_1K_1\phi_1(r)$ and $\psi(r)<r$ for $r>0$.

Now from the inequalities (3.12) and (3.14) it follows that

$$
\mu(\mathscr{Q}(A))=\omega_0(\mathscr{Q}(A))+\limsup_{t\to\infty}\operatorname{diam}(\mathscr{Q}(A(t)))
$$

$$\leqslant \psi\left(0+\limsup_{t\to\infty} \operatorname{diam}(A(t))\right)$$

$$\leqslant \psi\left(\omega_0(A)+\limsup_{t\to\infty} \operatorname{diam}(A(t))\right)$$

$$\leqslant \psi(\mu((A))), \tag{3.15}$$

where μ is the measure of noncompactness defined in the space $BC(\mathbb{R}_+,\mathbb{R})$. This shows that $\mathscr{Q}$ is a nonlinear $\mathscr{D}$-set-contraction on $B[0,r_0]$ in the sense of Definition 1.2. Thus, the map $\mathscr{Q}$ satisfies all the conditions of Theorem 1.3 with $C = B[0,r_0]$ and an application of it yields that $\mathscr{Q}$ has a fixed point in $B[0,r_0]$. This further by definition of $\mathscr{Q}$ which implies that the GNFIE (3.1) has a solution in $B[0,r_0]$. Moreover, taking into account that the image of $B[0,r_0]$ under the operator $\mathscr{Q}$ which is again contained in the ball $B[0,r_0]$ we infer that the set $Fix(\mathscr{Q})$ of all fixed points of $\mathscr{Q}$ is contained in $B[0,r_0]$. If the set $Fix(\mathscr{Q})$ contains all solutions of the equation (3.1), then we conclude from Remark 1.2 that the set $Fix(\mathscr{Q})$ belongs to the family $\ker\mu$. Now, taking into account the description of sets belonging to $\ker\mu$ (given in Section 2) we deduce that all solutions of the equation (3.1) are uniformly locally attractive on $\mathbb{R}_+$. This completes the proof. □

4. Special Cases

As mentioned earlier, the GNIE (3.1) is more general in the literature on the theory of nonlinear integral equations and includes other several classes of well-known nonlinear integral equations studied earlier by different authors. Below we list some of our main observations in this direction.

1. If we define the function F as

   $$F(t,x_1,x_2,x_3,x_4) = q(t)+x_3+x_4,$$

   then the GNIE (3.1) reduces to the following nonolinear function integral equation (NFIE),

   $$x(t) = q(t) + \int_0^{\beta(t)} f(t,s,x(\gamma(s)))\,ds + \int_0^{\sigma(t)} g(t,s,x(\eta(s)))\,ds, \tag{4.1}$$

   for all $t \in \mathbb{R}_+$. The NFIE (4.1) has been studied in Dhage [10] and includes the well-known Volterra, Fredholm as well as integral equations of mixed type as special ases by choosing the functions β and σ appropriately.

2. On taking $\sigma(t) = \infty$ for all $t \in \mathbb{R}_+$ and

   $$F(t,x_1,x_2,x_3,x_4) = f(t,x_1,x_3,x_4),$$

we obtain the following integral equation studied in Agarwal *et.al.* [1],

$$x(t) = f\left(t, x(\theta(t)), \int_0^{\beta(t)} f(t,s,x(\gamma(s)))\,ds, \int_0^{\infty} g(t,s,x(\eta(s)))\,ds\right), \tag{4.2}$$

which again includes other several classes of known integral equations as special cases (cf. Agarwal *et.al* [2] and the references therein).

3. On taking $F(t,x_1,x_2,x_3,x_4) = f(t,x_2,x_4)$, we obtain the following nonlinear integral equation,

$$x(t) = f\left(t, u(t,x(\alpha(t))), \int_0^{\sigma(t)} g(t,s,x(\eta(s)))\,ds\right), \tag{4.3}$$

for all $t \in \mathbb{R}_+$. The nonlinear integral equation (4.2) has been studied in Dhage and Lakshmikantham [14] for the global existence and attractivity results for the solutions defined on $\mathbb{R}_+$.

4. When $F(t,x_1,x_2,x_3,x_4) = p(t,x_1) + x_2 x_4$, where $p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function, then the GNIE (3.1) reduces to the following nonlinear quadratic functional integral equation,

$$x(t) = p(t,x(\theta(t)) + \big[u(t,x(\alpha(t)))\big]\left(\int_0^{\sigma(t)} g(t,s,x(\eta(s)))\,ds\right) \tag{4.4}$$

for all $t \in \mathbb{R}_+$. The quadratic integral equation (4.2) again includes several other classes of quadratic integral equations as the special cases given in Dhage *et.al.* [13], Dhage and Ntouyas [16] and the references cited therein.

5. On taking $F(t,x_1,x_2,x_3,x_4) = x_1 + p(x_3,x_4)$, where $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, we obtain the following functional integral equation recently studied in Dhage *et.al.* [17],

$$x(t) = u(t,x(\alpha(t))) + p\left(\int_0^{\beta(t)} f(t,s,x(\gamma(s)))\,ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))\,ds\right), \tag{4.5}$$

which further yields the integral equation

$$x(t) = u(t,x(\alpha(t))) + \int_0^{\sigma(t)} g(t,s,x(\eta(s)))\,ds, \tag{4.6}$$

for all $t \in \mathbb{R}_+$ provided $p(x_3,x_4) = x_4$. The integral equation (4.5) is discussed in Banas and Dhage [5] and Aghajani *et.al.* [3] for existence and asymptotic stability of the solutions.

In a nutshell, the GNIE (3.1) is a wider class of nonlinear integral equations and covers more than a dozen of well-known different classes of integral equations. Therefore, our main existence result formulated in Theorem 3.1 automatically yields the existence as well as local uniform attractivity of the solutions defined on $\mathbb{R}_+$ for the integral equations (4.1) through (4.6) and many others not mentioned here with the corresponding hypotheses on the functions involved in the equations.

5. A Numerical Example

Consider the following nonlinear functional integral equation

$$\begin{aligned} x(t) = {} & \frac{15}{16}\ln\Big(1+|x(t^2+1)|\Big)+\frac{3}{4}\Big(\frac{1+t}{6+7t^2}\Big)\ln\Big(1+\frac{1}{2}|x(t+1)|\Big) \\ & +\frac{2}{5}\int_0^{\frac{t}{t^2+1}} \frac{1}{(1+t)}\frac{1}{1+s}\ln\Big(1+\frac{1}{3}|x(s^2+1)|\Big)\,ds \\ & +\frac{7}{3}\int_0^{\frac{t}{t^3+1}} t^3\exp(-t^5)\frac{1}{(1+s^2)}\frac{|\cos x(s^2+3)|}{1+|\sin x(s^2+3)|}\,ds \end{aligned} \tag{5.1}$$

for all $t,s\in\mathbb{R}_+$.

Let

$$F(t,x,x_1,x_2,x_3) = \frac{15}{16}\ln(1+|x|)+\frac{3}{4}x_1+\frac{2}{5}x_2+\frac{7}{3}x_3,$$

$$\varphi(t) = \frac{15}{16}\ln(1+t),\varphi_1(t) = \ln\Big(1+\frac{1}{2}t\Big),\varphi_2(t) = \ln\Big(1+\frac{1}{3}t\Big),$$

$$\theta(t) = t^2+1,\alpha(t) = t+1,\gamma(s) = s^2+1,\eta(s) = s^2+3,\beta(t) = \frac{t}{t^2+1},\sigma(t) = \frac{t}{t^3+1}$$

$$u(t,x) = \frac{1+t}{6+7t^2}\ln\Big(1+\frac{1}{2}|x(t)|\Big),v(t) = t^3\exp(-t^5),b(s) = \frac{1}{(1+s^2)}$$

for all $t,s\in\mathbb{R}_+$, and

$$f(t,s,x(\gamma(s))) = \frac{1}{(1+t)(1+s)}\ln\Big(1+\frac{1}{3}|x(s^2+1)|\Big)$$

$$g(t,s,x(\eta(s))) = t^3\exp(-t^5)\frac{1}{(1+s^2)}\frac{|\cos x(s^2+3)|}{1+|\sin x(s^2+3)|}$$

for all $t,s\in\mathbb{R}_+$ and $x\in\mathbb{R}$. Notice that:

(i) The functions $\alpha,\beta,\gamma,\theta,\sigma$ and η are obviously continuous. We also observe that $\theta(t)=t^2+1\geqslant 1\geqslant t$ for $0\leqslant t\leqslant 1$ and $\theta(t)=t^2+1>t+1>t$ for $t>1$. Thus, $\theta(t)\geqslant t$ for all $t\in\mathbb{R}_+$. Similarly, $\alpha(t)=t+1\geqslant t$ for all $t\in\mathbb{R}_+$. Hence (H_0) is satisfied.

(ii) (H_1) is satisfied with $\varphi(t) = \frac{15}{16}\ln(1+t)$, $L_1 = \frac{3}{4}, L_2 = \frac{2}{5}, L_3 = \frac{7}{3}$. Moreover, the map $t \mapsto F(t,0,0,0,0)$ is bounded with the following estimate:

$$\begin{aligned} F_0 &= \sup_{t \geqslant 0} |F(t,0,0,0,0)| \\ &= \sup_{t \geqslant 0} t^3 \exp(-t^5) \int_0^{\frac{t}{t^3+1}} \frac{1}{1+s^2}\, ds \\ &= \sup_{t \geqslant 0} \arctan\left(\frac{t}{t^3+1}\right) t^3 \exp(-t^5) \\ &\approx 0.0337. \end{aligned}$$

(iii) Since $u(t,x) = \frac{1+t}{6+7t^2}\ln\left(1 + \frac{1}{2}|x(t)|\right)$, we have that

$$\begin{aligned} |u(t,x) - u(t,y)| &= \frac{1+t}{6+7t^2}\ln\frac{1+\frac{1}{2}|x|}{1+\frac{1}{2}|y|} \\ &= \frac{1+t}{6+7t^2}\ln\left(1 + \frac{|x|/2 - |y|/2}{1+|y|/2}\right) \\ &\leqslant \frac{1+t}{6+7t^2}\ln\left(1 + \frac{1}{2}|x-y|\right) \\ &= k_1(t)\varphi_1(|x(t) - y(t)|), \end{aligned}$$

where $k_1(t) = \frac{1+t}{6+7t^2}$, i.e., (H_2) is satisfied with $\varphi_1(r) = \ln(1+\frac{1}{2}r)$, we see that $\varphi_1(r) < r$ for $r > 0$. Obviously the function φ is nondecreasing and continuous on $\mathbb{R}_+$. Moreover,

$$\lim_{n\to\infty} k_1(t) = \lim_{n\to\infty} \frac{1+t}{6+7t^2} = 0 \quad \text{and} \quad K_1 = \sup_{t \geqslant 0} \frac{1+t}{6+7t^2} = \frac{1}{6}.$$

(iv) (H_3) is satisfied, since the function $t \mapsto u(t,0) = u_0(t)$ is bounded with $C_0 = \sup_{t \geqslant 0} |u(t,0)| = 0$.

(v) For arbitrary but fixed $x, y \in \mathbb{R}$ such that $|x| \geqslant |y|$ and for $t > 0$ we obtain

$$\begin{aligned} |f(t,s,x) - f(t,s,y)| &= \frac{1}{(1+t)(1+s)}\ln\frac{1+\frac{1}{3}|x|}{1+\frac{1}{3}|y|} \\ &\leqslant \frac{1}{(1+t)(1+s)}\ln\left(1 + \frac{|x|/3 - |y|/3}{1+|y|/3}\right) \\ &\leqslant \frac{1}{(1+t)(1+s)}\ln\left(1 + \frac{1}{3}|x-y|\right) \\ &= \frac{1}{(1+t)(1+s)}\varphi_2(|x-y|) \\ &= q(t,s)\ \varphi_2(|x-y|), \end{aligned}$$

where $q(t,s) = \frac{1}{(1+t)(1+s)}$. The case is similar when $|y| \geqslant |x|$. Furthermore, we obtain

$$\begin{aligned} K_2 &= \sup_{t\geqslant 0} \int_0^{\frac{t}{t^2+1}} q(t,s)\,ds \\ &= \sup_{t\geqslant 0} \frac{1}{(1+t)} \int_0^{\frac{t}{t^2+1}} \frac{1}{1+s}\,ds \\ &= \sup_{t\geqslant 0} \frac{1}{(1+t)} \ln\Big(1+\frac{t}{t^2+1}\Big) \\ &\approx 0.153. \end{aligned}$$

Clearly, (H_4) is satisfied since

$$\lim_{t\to\infty} k_2(t) = \lim_{t\to\infty} \int_0^{\frac{t}{t^2+1}} q(t,s)\,ds = \lim_{t\to\infty} \frac{1}{(1+t)} \ln\left(1+\frac{t}{t^2+1}\right) = 0.$$

(vi) The function $f_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by

$$f_0(t) = \int_0^{\frac{t}{t^2+1}} |f(t,s,0)|\,ds$$

is bounded with $C_1 = \sup_{t\geqslant 0} f_0(t) = 0$ and so, (H_5) is satisfied.

(vi) (H_6) is satisfied since the function g acts continuously from the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$. Moreover, we have

$$|g(t,s,x)| \leqslant t^3 \exp(-t^5)\, \frac{1}{1+s^2} = a(t)b(s)$$

for all $t,s \in \mathbb{R}_+$ and $x \in \mathbb{R}$, then hypotheses (H_6) is satisfied. Indeed, we have

$$\begin{aligned} \lim_{t\to\infty} a(t) \int_0^{\frac{t}{t^3+1}} b(s)\,ds &= \lim_{t\to\infty} t^3 \exp(-t^5) \int_0^{\frac{t}{t^3+1}} \frac{1}{1+s^2}\,ds \\ &= \lim_{t\to\infty} \arctan\left(\frac{t}{t^3+1}\right) t^3 \exp(-t^5) = 0. \end{aligned}$$

Also we find that

$$V = \sup_{t\geqslant 0} v(t) = \sup_{t\geqslant 0} \arctan\left(\frac{t}{t^3+1}\right) t^3 \exp(-t^5) \approx 0.0337.$$

(vii) We now compute the value of q given by

$$q = L_1C_0 + L_2C_1 + L_3V + F_0.$$

Now,

$$\begin{aligned} q &= L_1C_0 + L_2C_1 + L_3V + F_0 \\ &= \frac{3}{4} \times 0 + \frac{2}{5} \times 0 + \frac{7}{3} \times (0.0337) + 0.0337 \\ &= 0.1123. \end{aligned}$$

(viii) Next, we consider the inequality

$$\frac{15}{16}\varphi(r)+L_1K_1\varphi_1(r)+L_2K_2\varphi_2(r)+q\leqslant r.$$

Here,

$$\frac{15}{16}\ln(1+r)+\frac{3}{4}\times\frac{1}{6}\ln\left(1+\frac{1}{2}r\right)+\frac{2}{5}\times(0.153)\ln\left(1+\frac{1}{3}r\right)+0.1123\leqslant r,$$

or, equivalently,

$$\frac{15}{16}\ln(1+r)+(0.125)\ln\left(1+\frac{1}{2}r\right)+(0.0612)\ln\left(1+\frac{1}{3}r\right)+0.1123\leqslant r.$$

It is easily seen that each number $r\geqslant 0.7$ satisfies the above inequality. Thus, as the number r_0 we can take $r_0=0.7$. Note that this estimate of r_0 can be improved.

Thus, the functions $\alpha,\beta,\gamma,\theta,\sigma,\eta,\varphi,\varphi_1,\varphi_2,u,f$ and g involved in (3.1) satisfy all the conditions of Theorem 3.1 and hence the GNFIE (3.1) has at least one solution in the space $BC(\mathbb{R}_+,\mathbb{R})$ and the solutions are locally uniformly ultimately attractive on $\mathbb{R}_+$ located in the ball $B[0,0.7]$.

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