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BOUNDARY BEHAVIOR OF LARGE SOLUTIONS TO QUASILINEAR ELLIPTIC PROBLEMS WITH A NONLINEAR GRADIENT TERM

CHUNLIAN LIU AND ZUODONG YANG

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Abstract. In this paper,we study the boundary behavior of solutions to boundary blow-up elliptic problems

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) \pm |\nabla u(x)|^{q(m-1)} = b(x)f(u), & x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $m > 1$, $q > 0$, $b \in C^{\alpha}(\overline{\Omega})$, which is positive in Ω and may be vanishing on the boundary and rapidly varying near the boundary, and f is rapidly varying or normalized regularly varying at infinity.

1. Introduction

In this paper, we plan to investigate the exact asymptotic behavior of solutions near the boundary for the following problems

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) \pm |\nabla u(x)|^{q(m-1)} = b(x)f(u), & x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases} \tag{$P_\pm$}$$

where the last condition means that $u(x) \to +\infty$ as $d(x) = \operatorname{dist}(x,\partial\Omega) \to 0$, and the solution is called "a large solution" or "an explosive solution", Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geqslant 2)$, $q > 0$, $m > 1$. The function b satisfies:

(b_1) $b \in C^{\alpha}(\Omega)$ for some $\alpha \in (0,1)$, is non-negative in Ω;

(b_2) there exists $k \in \Lambda$ such that

$$0 < b_1 := \liminf_{d(x)\to 0} \frac{b(x)}{k^m(d(x))K^{m-2}(d(x))} \leqslant b_2 := \limsup_{d(x)\to 0} \frac{b(x)}{k^m(d(x))K^{m-2}(d(x))} < \infty,$$

or

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(b_3) there exists $k \in \Lambda$ such that

$$0 < b_{1q} := \liminf_{d(x)\to 0} \frac{b(x)}{k^{q(m-1)}(d(x))} \leqslant b_{2q} := \limsup_{d(x)\to 0} \frac{b(x)}{k^{q(m-1)}(d(x))} < \infty,$$

where Λ denotes the set of all positive non-decreasing functions in $C^1(0,\delta_0)(\delta_0 > 0)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) := C_k \in [0,\infty), \qquad K(t) = \int_0^t k(s)ds, \tag{1.1}$$

and f satisfies:

(f_1) $f \in C^1[0,+\infty), f(0) = 0, f$ is increasing on $[0,+\infty)$;

(f_2) $\int_1^\infty \frac{dv}{f^{\frac{1}{m-1}}(v)} < \infty$;

(f_3) there exists $C_f > 0$ such that

$$\lim_{s\to+\infty} f^{\frac{1}{m-1}-1}(s) f'(s) \int_s^\infty \frac{dv}{f^{\frac{1}{m-1}}(v)} = C_f.$$

We note that for each $k \in \Lambda$, we have $C_k \in [0,1]$ and

$$\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0, \quad \lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - \lim_{t\to 0^+} \frac{d}{dt}\frac{K(t)}{k(t)} = 1 - C_k. \tag{1.2}$$

In fact, from (1.1), we can see $\frac{d}{dt}(\frac{K(t)}{k(t)}) = C_k + \alpha(t)$, where $\lim_{t\to 0^+} \alpha(t) = 0$, so

$$\frac{K(t)}{k(t)} = \int_0^t C_k dt + \int_0^t \alpha(t)dt = C_k t + \int_0^t \alpha(t)dt,$$

since $\lim_{t\to 0^+} \alpha(t) = 0$, so $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$.

For example, $k(t) = t^p$, $p > 0$ or $k(t) = e^{\sqrt{x}}/\sqrt{x}$ satisfies (1.1),we can conclude that they also satisfy (1.2).

The set Λ was first introduced by Cirstea and Rădulescu [3,4] in order to study the boundary behavior of solutions to the problem

$$\Delta\omega = b(x)f(\omega), \quad x \in \Omega, \quad \omega|_{\partial\Omega} = \infty.$$

Semilinear elliptic problems involving gradient term with boundary blow-up interested many authors. Namely Bandle and Giarrusso [1] developed existence and asymptotic behavior results for large solutions of

$$\Delta u + |\nabla u(x)|^a = f(u),$$

in a bounded domain. In the case $f(u) = p(x)u^\gamma$, $a > 0$, and $\gamma > \max(1,a)$, Lair and Wood [7-9] dealt with the above equation in bounded domain and the whole space, they proved the existence of entire large solution under the condition $\int_0^\infty r\max_{|x|=r} p(x)dr < \infty$ when the domain is $\mathbb{R}^N$. Ghergu et al.[5] considered more general equation

$$\Delta u + q(x)|\nabla u(x)|^a = p(x)f(u),$$

where $0 \leqslant a \leqslant 2$, p and q are Hölder continuous functions on $(0,\infty)$. We note that the Keller-Osserman condition on f (see[2,16]) remains the key condition for the existence for their work. Ghergu and Radulescu [6] considered the following problem

$$\begin{cases} \Delta u + |\nabla u(x)| = p(x)f(u), & \text{in } \Omega, \\ u \geqslant 0, \quad \text{in } \Omega, \end{cases}$$

where Ω is either a smooth bounded domain or the whole space and f is a nondecreasing function satisfying $f \in C^{0,\alpha}_{loc}(0,\infty)$, $f(0) = 0$, $f > 0$, on $(0,\infty)$, and $\wedge = \sup_{x\geqslant 1} \frac{f(x)}{x} < \infty$. The authors studied the existence and nonexistence of large solutions under the assumption that

$$\int_0^\infty r(\max_{|x|=r} p(x) - \min_{|x|=r} p(x))\Psi(r)dr < +\infty,$$

where $\Psi(r) = \exp(\frac{\Lambda}{N-2}\int_0^\infty r\min_{|x|=r} p(x)dr)$.

Faten Toumi [19] extended the above result to the following problem

$$\begin{cases} \Delta u + \lambda(|x|)|\nabla u(x)| = \varphi(x,u(x)), & \text{in } \mathbb{R}^N, \\ u \geqslant 0, \quad u \neq 0, \\ \lim_{|x|\to+\infty} u(x) = +\infty, \end{cases}$$

where $\lambda : [0,\infty) \to [0,\infty)$ is a continuous function and $\varphi : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is measurable,continuous with respect to the second variable.

Quasilinear elliptic problems or such problems involving gradient terms with boundary blow-up interested many authors, see [10-12,14,20,21]

As far as the authors know, however, there are less results which contain the exact asymptotic behaviour of solutions near the boundary to problem $(P_\pm)$. In this paper, also applying Karamata regular variation theory (Karamata regular variation theory see [15–17,18]), perturbed method and constructing comparison functions, we show the asymptotic behaviour of solutions near the boundary to problem $(P_\pm)$.

Our main results are as follows:

THEOREM 1.1. *Let $q > 0$, b satisfies (b_1) f satisfies $(f_1),(f_2),(f_3)$ with $C_f \geqslant 1$ and the assumption that: (f_4) there exists $\Gamma_f \in [0,\infty)$ such that*

$$\lim_{s\to+\infty} f(s)\left[\int_s^\infty \frac{dv}{f^{\frac{1}{m-1}}(v)}\right]^{m-1} = \Gamma_f.$$

(i) *If $\Gamma_f > 0$, $q > m/(m-1)$ and b satisfies (b_3), then every solution u_+ of problem (P_+) satisfies*

$$\lim_{d(x)\to 0} \frac{u_+(x)}{\psi_1(K^q(d(x)))} = 1, \tag{1.4}$$

where K is in (1.1) with k defined in (b_3) and ψ_1 is uniquely determined by

$$\int_{\psi_1(t)}^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)} = t, \quad t > 0; \tag{1.5}$$

(ii) *if b satisfies (b_2) with $C_k > 0$ and $q \in (0, m/(m-1))$ and $\Gamma_f > 0$, then for every solution $u_\pm$ of problem $(P_\pm)$*

$$\lim_{d(x)\to 0} \frac{u_\pm(x)}{\psi_1(K^2(d(x)))} = 1, \tag{1.6}$$

where K is in (1.1) with k defined in (b_2) and ψ_1 is uniquely determined by (1.5);

(iii) *if $q = m/(m-1)$, b satisfies (b_2), and $2-(m-1)C_k - 2\Gamma_f^{1/(m-1)} > 0$, then for every solution u_- of problem (P_-)*

$$\lim_{d(x)\to 0} \frac{u_-(x)}{\psi_1(K^2(d(x)))} = 1, \tag{1.7}$$

where K is in (1.1) with k defined in (b_2) and ψ_1 is uniquely determined by (1.5);

(iv) *if $q = m/(m-1)$, b satisfies (b_2), and $2-(m-1)C_k + 2\Gamma_f^{1/(m-1)} > 0$, then every solution u_+ of problem (P_+)*

$$\lim_{d(x)\to 0} \frac{u_+(x)}{\psi_1(K^2(d(x)))} = 1, \tag{1.8}$$

where K is in (1.1) with k defined in (b_2) and ψ_1 is uniquely determined by (1.5).

2. Preliminaries

In this section, we present some bases of the theory which comes from Senta [18], Preliminaries in Resnick [17], Introductions and the appendix in Maric [15].

DEFINITION 2.1. A positive measurable function f defined on $[a, +\infty)$, for some $a > 0$, is called *regularly varying at infinity* with index ρ, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

For example, $f(s) = s^\rho$ is regularly varying at infinity. In particular, when $\rho = 0$, f is called *slowly varying at infinity*. Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are:

(1) every measure function on $[a, \infty)$ which has a positive limit at infinity;

(2) $(\ln s)^\beta$ and $(\ln(\ln s))^\beta$, $\beta \in \mathbb{R}$;

(3) $e^{(\ln s)^p}$, $0 < p < 1$.

DEFINITION 2.2. A positive measurable function f defined on $[a,+\infty)$, for some $a>0$, is called *rapidly varying at infinity* if for each $p>1$,

$$\lim_{s\to\infty}\frac{f(s)}{s^p}=\infty. \tag{2.2}$$

Clearly, if $f\in RV_\rho$, then $L(s):=f(s)/s^\rho$ is slowly varying at infinity. Some basic examples of rapidly varying functions at infinity are:

(1) e^s and e^{e^s};

(2) $e^{e^{(\ln s)^p}}$, e^{s^p} and $e^{e^{s^p}}$, $p>0$;

(3) $(\ln s)^\beta e^{s^p}$ and $s^\beta e^{s^p}$, $p>0,\beta\in\mathbb{R}$;

(4) $s^\beta e^{(\ln s)^p}$ and $(\ln s)^\beta e^{(\ln s)^p}$, $p>1,\beta\in\mathbb{R}$.

We also see that a positive measurable function g defined on $(0,a)$ for some $a>0$ is regularly varying at zero with index σ (written as $g\in RVZ_\sigma$) if $t\to g(1/t)$ belongs to $RV_{-\sigma}$, g is called rapidly varying at zero if $t\to g(1/t)$ is rapidly varying at infinity.

PROPOSITION 2.1. (Uniform convergence theorem) *If $f\in RV_\rho$, then (2.1) holds uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$. Moreover, if $\rho<0$, then uniform convergence holds on intervals of the form (a_1,∞) with $a_1>0$; if $\rho>0$, then uniform convergence holds on intervals $(a_1,\infty]$ provided f is bounded on $(a_1,\infty]$ for all $a_1>0$.*

PROPOSITION 2.2. (Representation theorem) *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(s)=\varphi(s)\exp\Big(\int_{a_1}^{s}\frac{y(\tau)}{\tau}d\tau\Big),\quad s\geqslant a_1, \tag{2.3}$$

for some $a_1>a$, where the functions φ and y are measurable and for $s\to\infty, y(s)\to 0$, and $\varphi(s)\to c_0$, with $c_0>0$.

We call that

$$\hat{L}(s)=c_0\exp\Big(\int_{a_1}^{s}\frac{y(\tau)}{\tau}d\tau\Big),\quad s\geqslant a_1, \tag{2.4}$$

is normalized slowly varying at infinity and

$$f(s)=c_0 s^\rho \hat{L}(s),\quad s\geqslant a_1, \tag{2.5}$$

is normalized regularly varying at infinity with index ρ (and written as $f\in NRV_\rho$).

Similarly, g is called normalized regularly varying at zero with index ρ, written as $g\in NRVZ_\rho$ if $t\to g(1/t)$ belongs to NRV_ρ. A function $f\in RV_\rho$ belongs to NRV_ρ if and only if

$$f\in C^1[a_1,\infty),\ \text{for some } a_1>0,\ \text{and } \lim_{s\to\infty}\frac{sf'(s)}{f(s)}=\rho. \tag{2.6}$$

PROPOSITION 2.3. *If functions L, L_1 are slowly varying at infinity, then*

(i) L^σ *for every* $\sigma \in \mathbb{R}$, $c_1L + c_2L_1$ $(c_1 \geqslant 0, c_2 \geqslant 0 with c_1 + c_2 > 0)$, $L \circ L_1$ *(if $L_1(t) \to +\infty$ as $t \to +\infty$), are also slowly varying at infinity;*

(ii) *for every* $\theta > 0$ *and* $t \to +\infty, t^\theta L(t) \to +\infty$ *and* $t^{-\theta}L(t) \to 0$;

(iii) *for* $\rho \in \mathbb{R}$ *and* $t \to +\infty$, $\frac{\ln(L(t))}{\ln t} \to 0$ *and* $\frac{\ln(t^\rho L(t))}{\ln t} \to \rho$.

PROPOSITION 2.4. (Asymptotic behavior) *If a function L is slowly varying at infinity, then for $a > 0$ and $t \to \infty$,*

(i) $\int_a^t s^\beta L(s)ds \cong (\beta+1)^{-1}t^{1+\beta}L(t)$, *for* $\beta > -1$;

(ii) $\int_t^\infty s^\beta L(s)ds \cong (-\beta-1)^{-1}t^{1+\beta}L(t)$, *for* $\beta < -1$.

PROPOSITION 2.5. (Asymptotic behavior) *If a function H is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,*

(i) $\int_a^t s^\beta H(s)ds \cong (\beta+1)^{-1}t^{1+\beta}H(t)$, *for* $\beta > -1$;

(ii) $\int_t^\infty s^\beta H(s)ds \cong (-\beta-1)^{-1}t^{1+\beta}H(t)$, *for* $\beta < -1$.

LEMMA 2.1. *Let $k \in \Lambda$:*

(i) *if* $C_k \in (0,1)$, *then* $k \in NRVZ_{(1-C_k)/C_k}$;

(ii) *if* $C_k = 1$, *then k is normalized slowly varying at zero;*

(iii) *if* $C_k = 0$, *then k is rapidly varying at zero.*

Proof. By l'Hospital's rule and (1.1), we have

$$\lim_{t\to 0}\frac{K(t)}{tk(t)} = \lim_{t\to 0}\frac{\frac{K(t)}{k(t)}}{t} = \lim_{t\to 0}\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = C_k; \tag{2.7}$$

(i)(ii) when $C_k > 0$, it follows by (1.2) that

$$\lim_{t\to 0}\frac{tk'(t)}{k(t)} = \lim_{t\to 0}\frac{K(t)k'(t)}{k^2(t)}\lim_{t\to 0}\frac{tk(t)}{K(t)} = \frac{1-C_k}{C_k}, \tag{2.8}$$

i.e.,$k \in NRVZ_{(1-C_k)/C_k}$ for $C_k \in (0,1)$ and k is normalized slowly varying at zero for $C_k = 1$;

(iii) when $C_k = 0$, for arbitrary $\gamma > 0$, it follows by (2.8) that $\lim_{t\to 0}\frac{tk'(t)}{k(t)} = +\infty$ and there exists t_{0_γ} such that

$$\frac{k'(t)}{k(t)} > (\gamma+1)t^{-1}, \quad \forall t \in (0, t_{0_\gamma}]. \tag{2.9}$$

Integrating (2.9) from t to t_{0_γ}, we obtain

$$\ln(k(t_{0_\gamma})) - \ln(k(t)) > (\gamma+1)(\ln t_{0_\gamma} - \ln t), \quad \forall t \in (0, t_{0_\gamma}],$$

i.e.,

$$0<\frac{k(t)}{t^{\gamma}}<\frac{k(t_{0_\gamma})}{t_0^{\gamma+1}}t,\quad t\in(0,t_{0_\gamma}].$$

Let $t\to 0$, we see by Definition 2.2 that k is rapidly varying at zero.

LEMMA 2.2. *If f satisfies (f_1), (f_2) and (f_3), then*
(i) $C_f\in[1,+\infty)$*;*
(ii) *if (f_3) holds for $C_f>1$, then $f\in NRV_{C_f/(C_f-1)}$;*
(iii) *when $C_f=1$, then f is rapidly varying at infinity.*

Proof. (i) Let

$$J(s)=f^{\frac{1}{m-1}-1}(s)f'(s)\int_s^{\infty}\frac{dv}{f^{\frac{1}{m-1}}(v)},\ \forall\ s>0.$$

Integrating $J(s)$ from $a(a>0)$ to t and integrate by parts, we obtain

$$\int_a^t J(s)ds=f^{\frac{1}{m-1}}(t)\int_t^{\infty}\frac{dv}{f^{\frac{1}{m-1}}(v)}-f^{\frac{1}{m-1}}(a)\int_a^{\infty}\frac{dv}{f^{\frac{1}{m-1}}(v)}+t-a,\ \forall\ \ t>a.$$

It follows from the l'Hospital's rule that

$$0\leqslant\lim_{t\to\infty}\frac{f^{\frac{1}{m-1}}(t)\int_t^{\infty}\frac{dv}{f^{\frac{1}{m-1}}(v)}}{t}=\lim_{t\to\infty}\frac{1}{t}\int_a^t J(s)ds-1=\lim_{t\to\infty}J(t)-1=C_f-1,$$

i.e.,$C_f\geqslant 1$.

(ii) By (i), we see that

$$\begin{aligned}\lim_{s\to+\infty}\frac{f(s)}{sf'(s)}&=\lim_{s\to+\infty}\frac{f^{\frac{1}{m-1}}(s)\int_s^{\infty}\frac{dv}{f^{\frac{1}{m-1}}(v)}}{sJ(s)}\\&=\frac{1}{C_f}\lim_{s\to+\infty}\frac{f^{\frac{1}{m-1}}(s)\int_s^{\infty}\frac{dv}{f^{\frac{1}{m-1}}(v)}}{s}\\&=\frac{C_f-1}{C_f},\end{aligned}$$

i.e., $f\in NRV_{C_f/(C_f-1)}$ for $C_f>1$.

(iii) When $C_f=1$, we see by the proof of (ii) that

$$\lim_{s\to+\infty}\frac{f(s)}{sf'(s)}=0.$$

Consequently, for arbitrary $p>1$, there exists $S_0>0$ such that

$$\frac{f'(s)}{f(s)}>(p+1)s^{-1},\ \forall\, s\geqslant S_0,$$

Integrating the above inequality from S_0 to s, we obtain

$$\ln(f(s)) - \ln(f(S_0)) > (p+1)(\ln s - \ln S_0),\ \ \forall\, s \geqslant S_0,$$

letting $s \to +\infty$, we see by Definition 2.2 that f is rapidly varying at infinity.

LEMMA 2.3. *Let f satisfy (f_1), (f_2), (f_3) and let ψ_1 be the solution to the problem*

$$\int_{\psi_1(t)}^{\infty} \frac{ds}{f^{\frac{1}{m-1}}(s)} = t, \quad \forall\ t > 0.$$

Then:

(i) $-\psi_1'(t) = f^{\frac{1}{m-1}}(\psi_1(t))$, $\psi_1(t) > 0$, $t > 0$, $\psi_1(0) := \lim_{t\to 0^+} \psi_1(t) = +\infty$, *and*

$$\psi_1''(t) = \frac{1}{m-1} f^{\frac{2}{m-1}-1}(\psi_1(t)) f'(\psi_1(t)),\ \ t > 0;$$

(ii) $\psi_1 \in NRVZ_{-(C_f-1)}$;

(iii) $-\psi_1' = f^{\frac{1}{m-1}} \circ \psi_1 \in NRVZ_{-C_f/(m-1)}$.

Proof. By the definition of ψ_1 and a direct calculation, we show that (i) holds.

(ii) It follows from the proof of Lemma 2.1 that

$$\lim_{t\to 0^+} \frac{t\psi_1'(t)}{\psi_1(t)} = -\lim_{t\to 0^+} \frac{t f^{\frac{1}{m-1}}(\psi_1(t))}{\psi_1(t)} = -\lim_{s\to+\infty} \frac{f^{\frac{1}{m-1}}(s)\int_s^\infty \frac{dv}{f^{\frac{1}{m-1}}(v)}}{s} = -(C_f - 1), \tag{2.10}$$

i.e., $\psi_1 \in NRVZ_{-(C_f-1)}$.

(iii) (f_3) implies

$$\begin{aligned}\lim_{t\to 0^+} \frac{t\psi_1''(t)}{\psi_1'(t)} &= -\lim_{t\to 0^+} \frac{t}{m-1} f^{\frac{1}{m-1}-1}(\psi_1(t)) f'(\psi_1(t)) \\ &= -\lim_{s\to+\infty} \frac{1}{m-1} f^{\frac{1}{m-1}-1}(s) f'(s) \int_s^\infty \frac{dv}{f^{\frac{1}{m-1}}(v)} \\ &= -C_f/(m-1). \end{aligned} \tag{2.11}$$

3. Proofs of the main results

LEMMA 3.1. (Weak comparison principle) *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary $\partial\Omega$ and $\varphi : (0,a) \to (0,a)$ be continuous and non-decreasing, let u_1, $u_2 \in W^{1,m}(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{m-2} \nabla u_1 \nabla \psi dx + \int_\Omega \varphi u_1 \psi dx \leqslant \int_\Omega |\nabla u_2|^{m-2} \nabla u_2 \nabla \psi dx + \int_\Omega \varphi u_2 \psi dx,$$

For all non-negative $\psi \in W_0^{1,m}(\Omega)$. *Then the inequality*

$$u_1 \leqslant u_2, \quad on \ \partial\Omega,$$

implies that

$$u_1 \leqslant u_2, \quad in \ \Omega.$$

For any $\delta > 0$, we define

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}.$$

Since Ω is smooth, there exists $\delta_0 > 0$ such that $d \in C^2(\Omega_{\delta_0})$ and

$$|\nabla d(x)| = 1, \quad \forall x \in \Omega_{\delta_0}.$$

PROOF OF THEOREM 1.1.

(i) $q > m/(m-1), m > 1$ and $\Gamma_f > 0$, let $\varepsilon \in (0, b_{1q}/4)$ and

$$\xi_{01} = \left(\frac{b_{1q}}{q^{q(m-1)}\Gamma_f^{q-1}}\right)^{\frac{1}{m-1}}, \quad \xi_{02} = \left(\frac{b_{2q}}{q^{q(m-1)}\Gamma_f^{q-1}}\right)^{\frac{1}{m-1}};$$

$$\xi_1 = \xi_{01}\left(1 - \frac{2\varepsilon}{b_{1q}}\right)^{\frac{1}{m-1}}, \quad \xi_2 = \xi_{02}\left(1 + \frac{2\varepsilon}{b_{2q}}\right)^{\frac{1}{m-1}}.$$

It follows that

$$\frac{\xi_{01}}{2^{1/(m-1)}} < \xi_1 < \xi_2 < 2^{1/(m-1)}\xi_{02}.$$

By $(b_1), (b_2), (1.2), (2.11)$ and (f_4), we see that there is $\delta_\varepsilon \in (0, \delta_0/2)$ (which is corresponding to ε) sufficiently small that:

$$(b_{1q} - \varepsilon)k^q(d(x) - \rho) \leqslant (b_{1q} - \varepsilon)k^q(d(x)) < b(x), \; x \in D_\rho^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\rho$$

and

$$b(x) < (b_{2q} + \varepsilon)k^q(d(x)) \leqslant (b_{2q} + \varepsilon)k^q(d(x) + \rho), \;\; x \in D_\rho^+ = \Omega_{2\delta_\varepsilon - \rho},$$

where $\rho \in (0, \delta_\varepsilon)$.

For $i = 1, 2$,

$$\begin{aligned}
&2(\xi_{02}q)^{m-1}\left(\frac{K(t)}{k(t)}\right)^{(q-1)(m-1)-1} \\
&\qquad \times \left[q\left|\xi_i K^q(t) f^{\frac{1}{m-1}-1}\Big(\psi_1(\xi_i K^q(t))\Big) f'\Big(\psi_1(\xi_i K^q(t))\Big)\right|\right. \\
&\qquad \left. + \left|(q-1)(m-1) + \frac{(m-1)K(t)k'(t)}{k^2(t)}\right| + \frac{K(t)}{k(t)}|\Delta d(x)|\right]
\end{aligned}$$

$$+2(\xi_{02}q^q)^{m-1}\left|\left[(\xi_i K^q(t))^{m-1}f\Big(\psi_1(\xi_i K^q(t))\Big)\right]^{q-1}-\Gamma_f^{q-1}\right|$$
$$<\varepsilon,\ \forall (x,t)\in\Omega_{2\delta_\varepsilon}\times(0,2\delta_\varepsilon).$$

Let

$$d_1(x)=d(x)-\rho,\quad d_2(x)=d(x)+\rho,$$
$$\bar{u}_\varepsilon=\psi_1(\xi_1 K^q(d_1(x))),\quad x\in D_\rho^- \quad\text{and}\quad \underline{u}_\varepsilon=\psi_1(\xi_2 K^q(d_2(x))),\quad x\in D_\rho^+.$$

By using

$$(\xi_1 q^q)^{m-1}\Gamma_f^{q-1}=b_{1q}-2\varepsilon,$$

and by a direct calculation, it follows that, for $x\in D_\rho^-$

$$\begin{aligned}
&\operatorname{div}(|\nabla\bar{u}_\varepsilon|^{m-2}\nabla\bar{u}_\varepsilon)-b(x)f(\bar{u}_\varepsilon(x))+|\bar{u}_\varepsilon(x)|^{q(m-1)}\\
&=(m-1)\big(\psi_1'(\xi_1K^q(d_1(x)))\big)^{m-2}\psi_1''\big(\xi_1K^q(d_1(x))\big)(\xi_1 q)^m K^{m(q-1)}(d_1(x))k^m(d_1(x))\\
&\quad+\big(\psi_1'(\xi_1K^q(d_1(x)))\big)^{m-1}(\xi_1 q)^{m-1}(q-1)(m-1)K^{(q-1)(m-1)-1}(d_1(x))k^m(d_1(x))\\
&\quad+\big(\psi_1'(\xi_1K^q(d_1(x)))\big)^{m-1}(\xi_1 q)^{m-1}K^{(q-1)(m-1)}(d_1(x))(m-1)k^{m-2}(d_1(x))k'(d_1(x))\\
&\quad+\big(\psi_1'(\xi_1K^q(d_1(x)))\big)^{m-1}(\xi_1 q)^{m-1}K^{(q-1)(m-1)}(d_1(x))k^{m-1}(d_1(x))\Delta(d_1(x))\\
&\quad-b(x)f(\psi_1(\xi_1K^q(d_1(x))))+\Big[\xi_1K^q(d_1(x))k(d_1(x))\psi_1'(\xi_1K^q(d_1(x)))\Big]^{q(m-1)}\\
&\quad=(-1)^m f(\psi_1(\xi_1K^q(d_1(x))))k^{q(m-1)}(d_1(x))\Big\{(\xi_1 q)^{m-1}(\frac{K(d_1(x))}{k(d_1(x))})^{(q-1)(m-1)-1}\\
&\quad\times\Big[q\xi_iK^q(d_1(x))f^{\frac{1}{m-1}-1}\Big(\psi_1(\xi_iK^q(d_1(x)))\Big)f'\Big(\psi_1(\xi_iK^q(d_1(x)))\Big)\\
&\qquad-(q-1)(m-1)-\frac{(m-1)K(d_1(x))k'(d_1(x))}{k^2(d_1(x))}-\frac{K(d_1(x))}{k(d_1(x))}\Delta d(x)\Big]\\
&\qquad-\Big(\frac{b(x)}{K^{q(m-1)}(d_1(x))}-b_{1q}\Big)-b_{1q}+(\xi_1q^q)^{m-1}\Gamma_f^{q-1}\\
&\qquad+(\xi_1q^q)^{m-1}\Big[\Big((\xi_1K^q(t))^{m-1}f(\psi_1(\xi_iK^q(t)))\Big)^{q-1}-\Gamma_f^{q-1}\Big]\Big\}\\
&\quad\leqslant 0,
\end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of problem (P_+) in D_ρ^-.

In a similar way, for $x\in D_\rho^+$,we can show that $\underline{u}_\varepsilon$ is a subsolution of of problem (P_+) in D_ρ^+.

Now let u_+ be an arbitrary solution of problem (P_+) and

$$C_1(\delta_\varepsilon):=\max_{d(x)\geqslant\delta_\varepsilon}u_+(x).$$

We see that

$$u_+\leqslant C_1(\delta_\varepsilon)+\bar{u}_\varepsilon,\ \text{ on }\partial D_\rho^-.$$

Since ψ_1 is decreasing, see Lemma 2.3, and $\xi_{02} < \xi_2$, we have that

$$\underline{u}_\varepsilon \leqslant \psi_1(\xi_{02}K^q(2\delta_{2\varepsilon})) := C_2(\delta_\varepsilon),$$

whenever $d(x) = 2\delta_{2\varepsilon} - \rho$ and $\underline{u}_\varepsilon \leqslant u_+ + C_2(\delta_\varepsilon)$ on ∂D_ρ^+.

It follows by (f_1) and Lemma 3.1 that

$$u_+ \leqslant C_1(\delta_\varepsilon) + \bar{u}_\varepsilon \text{ on } D_\rho^-, \qquad \underline{u}_\varepsilon \leqslant u_+ + C_2(\delta_\varepsilon) \text{ on } D_\rho^+.$$

Hence by letting $\rho \to 0$, we have for $x \in D_\rho^- \bigcap D_\rho^+$,

$$1 - \frac{C_2(\delta_\varepsilon)}{\psi_1(\xi_2 K^q(d(x)))} \leqslant \frac{u_+(x)}{\psi_1(\xi_2 K^q(d(x)))}$$

and

$$\frac{u_+(x)}{\psi_1(\xi_1 K^q(d(x)))} \leqslant 1 + \frac{C_1(\delta_\varepsilon)}{\psi_1(\xi_1 K^q(d(x)))}.$$

Consequently,

$$1 \leqslant \liminf_{d(x)\to 0} \frac{u_+(x)}{\psi_1(\xi_2 K^q(d(x)))}$$

and

$$\limsup_{d(x)\to 0} \frac{u_+(x)}{\psi_1(\xi_1 K^q(d(x)))} \leqslant 1.$$

Thus by letting $\varepsilon \to 0$, we obtain

$$1 \leqslant \liminf_{d(x)\to 0} \frac{u_+(x)}{\psi_1(\xi_{02} K^q(d(x)))}$$

and

$$\limsup_{d(x)\to 0} \frac{u_+(x)}{\psi_1(\xi_{01} K^q(d(x)))} \leqslant 1.$$

By Lemma 2.3 (ii) and Proposition 2.1, we have

$$\limsup_{d(x)\to 0} \frac{\psi_1(\xi_{02} K^q(d(x)))}{\psi_1(K^q(d(x)))} = \limsup_{d(x)\to 0} \frac{\psi_1(\xi_{01} K^q(d(x)))}{\psi_1(K^q(d(x)))} = 1.$$

Thus

$$\limsup_{d(x)\to 0} \frac{u_+(x)}{\psi_1(K^q(d(x)))} = 1.$$

(ii) When b satisfies (b_2) with $C_k > 0$, either $q \in (0, m/(m-1)), m > 1$ and $\Gamma_f > 0$.

Let $\varepsilon \in (0, b_1/4)$ and

$$\xi_{03} = \frac{1}{2}\left(\frac{b_1}{2-(m-1)(2-C_k)}\right)^{\frac{1}{m-1}},$$

$$\xi_{04} = \frac{1}{2}\left(\frac{b_2}{2-(m-1)(2-C_k)}\right)^{\frac{1}{m-1}},$$
$$(\xi_3)^{m-1} = (\xi_{03})^{m-1} - \frac{\varepsilon}{2-(m-1)(2-C_k)},$$
$$(\xi_4)^{m-1} = (\xi_{04})^{m-1} + \frac{\varepsilon}{2-(m-1)(2-C_k)}.$$

It follows that

$$\xi_{03}/\sqrt[m-1]{2} < \xi_3 < \xi_4 < \sqrt[m-1]{2}\xi_{04}.$$

By $(b_1),(b_2),(1.2),(2.11)$ and (f_4),we see that there is $\delta_\varepsilon \in (0,\delta_0/2)$(which is corresponding to ε) sufficiently small that:

$$\begin{aligned}&(b_1-\varepsilon)k^m(d(x)-\rho)K^{m-2}(d(x)-\rho)\\ &\qquad \leqslant (b_1-\varepsilon)k^m(d(x))K^{m-2}(d(x)) < b(x),\ x\in D_\rho^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\rho,\end{aligned}$$

and

$$\begin{aligned}&b(x) < (b_2+\varepsilon)k^m(d(x))K^{m-2}(d(x)-\rho)\\ &\qquad \leqslant (b_2+\varepsilon)k^m(d(x)+\rho)K^{m-2}(d(x)+\rho),\ x\in D_\rho^+ = \Omega_{2\delta_\varepsilon-\rho},\end{aligned}$$

where $\rho\in(0,\delta_\varepsilon)$.

For $i=3,4$,

$$\begin{aligned}&4(2\xi_{04})^{m-1}\left|\xi_i K^2(t) f^{\frac{1}{m-1}-1}(\psi_1(\xi_i K^2(t)))f'(\psi_1(\xi_i K^2(t)))-1\right|\\ &\quad +(m-1)(2\xi_{04})^{m-1}\left|\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right| + (2\xi_{04})^{m-1}\frac{K(t)}{k(t)}|\Delta d(x)|\\ &\quad + 2(2^q\xi_{04})^{m-1}\left|(\xi_i K^2(t))^{m-1} f(\psi_1(\xi_i K^2(t)))-\Gamma_f+\Gamma_f\right|^{q-1}\\ &\qquad \times\left(\frac{K(t)}{k(t)}\right)^{1-(q-1)(m-1)} < \varepsilon,\ \forall (x,t)\in\Omega_{2\delta_\varepsilon}\times(0,2\delta_\varepsilon).\end{aligned}$$

Let

$$d_1(x) = d(x)-\rho,\quad d_2(x) = d(x)+\rho,$$
$$\bar{u}_\varepsilon = \psi_1(\xi_3 K^2(d_1(x))),\quad x\in D_\rho^-\quad \text{and}\quad \underline{u}_\varepsilon = \psi_1(\xi_4 K^2(d_2(x)))\quad x\in D_\rho^+.$$

By using

$$(2\xi_3)^{m-1}(2-(m-1)(2-C_k)) = b_1,$$

and by a direct calculation, it follows that, for $x\in D_\rho^-$,

$$\begin{aligned}&\operatorname{div}(|\nabla\bar{u}_\varepsilon|^{m-2}\nabla\bar{u}_\varepsilon) - b(x)f(\bar{u}_\varepsilon(x)) \pm |\bar{u}_\varepsilon(x)|^{q(m-1)}\\ &\quad = (m-1)\Big(\psi_1'(\xi_3 K^2(d_1(x)))\Big)^{m-2}\psi_1''\Big(\xi_3 K^2(d_1(x))\Big)(2\xi_3)^m K^m(d_1(x))k^m(d_1(x))\end{aligned}$$

$$
\begin{aligned}
&+\Big(\psi_1'(\xi_3K^2(d_1(x)))\Big)^{m-1}(2\xi_3)^{m-1}(m-1)K^{m-2}(d_1(x))k^m(d_1(x))\\
&+\Big(\psi_1'(\xi_3K^2(d_1(x)))\Big)^{m-1}(2\xi_3)^{m-1}K^{m-1}(d_1(x))(m-1)k^{m-2}(d_1(x))k'(d_1(x))\\
&+\Big(\psi_1'(\xi_3K^2(d_1(x)))\Big)^{m-1}(2\xi_3)^{m-1}K^{m-1}(d_1(x))k^{m-1}(d_1(x))\Delta(d_1(x))\\
&-b(x)f(\psi_1(\xi_3K^2(d_1(x))))\pm\Big[2\xi_3K(d_1(x))k(d_1(x))\psi_1'(\xi_3K^2(d_1(x)))\Big]^{q(m-1)}\\
=\;&(-1)^m f(\psi_1(\xi_3K^2(d_1(x))))k^m(d_1(x))K^{m-2}(d_1(x))\\
\times&\Big\{2(2\xi_3)^{m-1}\Big(\xi_3K^2(d_1(x))f^{\frac{1}{m-1}-1}(\psi_1(\xi_3K^2(d_1(x))))f'(\psi_1(\xi_3K^(d_1(x))))-1\Big)\\
&+2(2\xi_3)^{m-1}-(m-1)(2\xi_3)^{m-1}\\
&-(m-1)(2\xi_3)^{m-1}\Big(\frac{k'(d_1(x))K(d_1(x))}{k^2(d_1(x))}-(1-C_k)\Big)\\
&-(m-1)(2\xi_3)^{m-1}(1-C_k)-(2\xi_3)^{m-1}\frac{K(d_1(x))}{k(d_1(x))}\Delta d_1(x)\\
&-\Big(\frac{(-1)^m b(x)}{k^m(d_1(x))K^{m-2}(d_1(x))}-b_1\Big)-b_1\\
&\pm(2^q\xi_3)^{m-1}\Big((\xi_3K^2(d_1(x)))^{m-1}f(\psi_1(\xi_3K^2(d_1(x))))\Big)^{q-1}\\
\times&\Big(\frac{K(d_1(x))}{k(d_1(x))}\Big)^{1-(q-1)(m-1)}\Big\}\\
\leqslant\;&0,
\end{aligned}
$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of problem $(P_\pm)$ in D_ρ^-.

In a similar way, for $x\in D_\rho^+$,we can show that $\underline{u}_\varepsilon$ is a subsolution of of problem $(P_\pm)$ in D_ρ^+.

The last part of the proof is the same as that of (i).

(iii) When $q=m/(m-1)$, b satisfies (b_2) and $2-(m-1)C_k-2\Gamma_f^{1/(m-1)}>0$ for problem (P_-).

Let $\varepsilon\in(0,b_1/4)$ and

$$\xi_{05}=\frac{1}{2}\sqrt[m-1]{\frac{b_1}{2-(m-1)C_k-2\Gamma_f^{1/(m-1)}}},$$

$$\xi_{06}=\frac{1}{2}\sqrt[m-1]{\frac{b_2}{2-(m-1)C_k-2\Gamma_f^{1/(m-1)}}},$$

$$\xi_5=\xi_{05}-\frac{2\varepsilon}{2-(m-1)C_k-2\Gamma_f^{1/(m-1)}},$$

$$\xi_6 = \xi_{06} + \frac{2\varepsilon}{2-(m-1)C_k - 2\Gamma_f^{1/(m-1)}}.$$

It follows that

$$\xi_{05}/\sqrt[m-1]{2} < \xi_5 < \xi_6 < \sqrt[m-1]{2}\xi_{06}.$$

By $(b_1),(b_2),(1.2),(2.11)$ and (f_4),we see that there is $\delta_\varepsilon \in (0,\delta_0/2)$ (which is corresponding to ε) sufficiently small that:

$$(b_1-\varepsilon)k^m(d(x)-\rho)K^{m-2}(d(x)-\rho)$$
$$\leqslant (b_1-\varepsilon)k^m(d(x))K^{m-2}(d(x)) < b(x),\ x\in D_\rho^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\rho$$

and

$$b(x) < (b_2+\varepsilon)k^m(d(x))K^{m-2}(d(x)-\rho)$$
$$\leqslant (b_2+\varepsilon)k^m(d(x)+\rho)K^{m-2}(d(x)+\rho),\ x\in D_\rho^+ = \Omega_{2\delta_\varepsilon-\rho},$$

where $\rho\in(0,\delta_\varepsilon)$.

And for $i=5,6$,

$$\begin{aligned}
&4(2\xi_{06})^{m-1}\left|\xi_i K^2(t) f^{\frac{1}{m-1}-1}(\psi_1(\xi_i K^2(t)))f'(\psi_1(\xi_i K^2(t)))-1\right| \\
&\quad + (m-1)(2\xi_{06})^{m-1}\left|\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right| + (2\xi_{06})^{m-1}\frac{K(t)}{k(t)}|\Delta d(x)| \\
&\quad + 2(2\xi_{06})^{m-1}\left|\xi_i K^2(t) f^{\frac{1}{m-1}}(\psi_1(\xi_i K^2(t))) - \Gamma_f^{\frac{1}{m-1}}\right| \\
&\quad < \varepsilon,\ \forall (x,t)\in \Omega_{2\delta_\varepsilon}\times(0,2\delta_\varepsilon).
\end{aligned}$$

Let

$$d_1(x) = d(x)-\rho, \quad d_2(x) = d(x)+\rho,$$

$$\bar{u}_\varepsilon = \psi_1(\xi_5 K^2(d_1(x))),\ \ x\in D_\rho^- \ \text{ and } \ \underline{u}_\varepsilon = \psi_1(\xi_6 K^2(d_2(x)))\ \ x\in D_\rho^+.$$

By using

$$(2\xi_5)^{m-1}(2-(m-1)C_k - 2\Gamma_f^{\frac{1}{m-1}}) = b_1,$$

and by a direct calculation, it follows that, for $x\in D_\rho^-$,

$$\begin{aligned}
&\operatorname{div}(|\nabla \bar{u}_\varepsilon|^{m-2}\nabla \bar{u}_\varepsilon) - b(x)f(\bar{u}_\varepsilon(x)) - |\bar{u}_\varepsilon(x)|^m \\
&= (m-1)\Big(\psi_1'(\xi_5 K^2(d_1(x)))\Big)^{m-2}\psi_1''\Big(\xi_5 K^2(d_1(x))\Big)(2\xi_5)^m K^m(d_1(x))k^m(d_1(x)) \\
&\quad + \Big(\psi_1'(\xi_5 K^2(d_1(x)))\Big)^{m-1}(2\xi_5)^{m-1}(m-1)K^{m-2}(d_1(x))k^m(d_1(x)) \\
&\quad + \Big(\psi_1'(\xi_5 K^2(d_1(x)))\Big)^{m-1}(2\xi_5)^{m-1}K^{m-1}(d_1(x))(m-1)k^{m-2}(d_1(x))k'(d_1(x))
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\psi_1'(\xi_5 K^2(d_1(x)))\right)^{m-1}(2\xi_5)^{m-1}K^{m-1}(d_1(x))k^{m-1}(d_1(x))\Delta(d_1(x)) \\
&\quad - b(x)f(\psi_1(\xi_5 K^2(d_1(x)))) - \left[2\xi_5 K(d_1(x))k(d_1(x))\psi_1'(\xi_5 K^2(d_1(x)))\right]^m \\
&= (-1)^m f(\psi_1(\xi_5 K^{m-2}(d_1(x))))k^m(d_1(x))K^{m-2}(d_1(x)) \\
&\times \Bigg\{2(2\xi_5)^{m-1}\left(\xi_5 K^2(d_1(x))f^{\frac{1}{m-1}-1}(\psi_1(\xi_5 K^2(d_1(x))))f'(\psi_1(\xi_5 K^(d_1(x)))) - 1\right) \\
&\quad + 2(2\xi_5)^{m-1} - (m-1)(2\xi_5)^{m-1} \\
&\quad - (m-1)(2\xi_5)^{m-1}\left(\frac{k'(d_1(x))K(d_1(x))}{k^2(d_1(x))} - (1-C_k)\right) \\
&\quad - (m-1)(2\xi_5)^{m-1}(1-C_k) - (2\xi_5)^{m-1}\frac{K(d_1(x))}{k(d_1(x))}\Delta d_1(x) \\
&\quad - \left(\frac{(-1)^m b(x)}{k^m(d_1(x))K^{m-2}(d_1(x))} - b_1\right) \\
&\quad - b_1 - 2(2\xi_5)^{m-1}\left(\xi_5 K^2(d_1(x))f^{\frac{1}{m-1}}(\psi_1(\xi_5 K^2(d_1(x)))) - \Gamma_f^{\frac{1}{m-1}}\right) \\
&\quad - 2(2\xi_5)^{m-1}\Gamma_f^{\frac{1}{m-1}}\Bigg\} \\
&\leqslant 0,
\end{aligned}
$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of problem (P_-) in D_ρ^-.

In a similar way, for $x \in D_\rho^+$, we can show that $\underline{u}_\varepsilon$ is a subsolution of of problem (P_-) in D_ρ^+. The last part of the proof is the same as that of (i).

(iv) When $q = m/(m-1)$ b satisfies (b_2) and $2-(m-1)C_k + 2\Gamma_f^{1/(m-1)} > 0$ for problem (P_+).

Let $\varepsilon \in (0, b_1/4)$ and

$$
\xi_{07} = \frac{1}{2}\sqrt[m-1]{\frac{b_1}{2-(m-1)C_k + 2\Gamma_f^{1/(m-1)}}},
$$

$$
\xi_{08} = \frac{1}{2}\sqrt[m-1]{\frac{b_2}{2-(m-1)C_k + 2\Gamma_f^{1/(m-1)}}},
$$

$$
\xi_7 = \xi_{07} - \frac{2\varepsilon}{2-(m-1)C_k + 2\Gamma_f^{1/(m-1)}},
$$

$$
\xi_8 = \xi_{08} + \frac{2\varepsilon}{2-(m-1)C_k + 2\Gamma_f^{1/(m-1)}}.
$$

It follows that

$$
\xi_{07}/\sqrt[m-1]{2} < \xi_7 < \xi_8 < \sqrt[m-1]{2}\xi_{08}.
$$

By $(b_1),(b_2),(1.2),(2.11)$ and (f_4), we see that there is $\delta_\varepsilon \in (0,\delta_0/2)$ (which is corresponding to ε) sufficiently small that:

$$(b_1-\varepsilon)k^m(d(x)-\rho)K^{m-2}(d(x)-\rho) \leqslant (b_1-\varepsilon)k^m(d(x))K^{m-2}(d(x)) < b(x),\ x\in D_\rho^-,$$

and

$$b(x) < (b_2+\varepsilon)k^m(d(x))K^{m-2}(d(x)-\rho) \leqslant (b_2+\varepsilon)k^m(d(x)+\rho)K^{m-2}(d(x)+\rho),\ \ x\in D_\rho^+,$$

where $D_\rho^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\rho$, $D_\rho^+ = \Omega_{2\delta_\varepsilon-\rho}$ and $\rho\in(0,\delta_\varepsilon)$.

And for $i=7,8$,

$$\begin{aligned}
&4(2\xi_{08})^{m-1}\left|\xi_i K^2(t)f^{\frac{1}{m-1}-1}(\psi_1(\xi_i K^2(t)))f'(\psi_1(\xi_i K^2(t)))-1\right| \\
&\quad +(m-1)(2\xi_{08})^{m-1}\left|\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right| + (2\xi_{08})^{m-1}\frac{K(t)}{k(t)}|\Delta d(x)| \\
&\quad + 2(2\xi_{08})^{m-1}\left|\xi_i K^2(t)f^{\frac{1}{m-1}}(\psi_1(\xi_i K^2(t)))-\Gamma_f^{\frac{1}{m-1}}\right| \\
&\quad < \varepsilon,\ \ \forall (x,t)\in\Omega_{2\delta_\varepsilon}\times(0,2\delta_\varepsilon).
\end{aligned}$$

Let $d_1(x)=d(x)-\rho$, $d_2(x)=d(x)+\rho$ and

$$\bar{u}_\varepsilon = \psi_1(\xi_7 K^2(d_1(x))),\ \ x\in D_\rho^- \ \text{ and } \ \underline{u}_\varepsilon = \psi_1(\xi_8 K^2(d_2(x)))\ \ x\in D_\rho^+.$$

By using

$$(2\xi_7)^{m-1}(2-(m-1)C_k+2\Gamma_f^{\frac{1}{m-1}}) = b_1,$$

and by a direct calculation, it follows that, for $x\in D_\rho^-$,

$$\begin{aligned}
&\operatorname{div}(|\nabla\bar{u}_\varepsilon|^{m-2}\nabla\bar{u}_\varepsilon) - b(x)f(\bar{u}_\varepsilon(x)) + |\bar{u}_\varepsilon(x)|^m \\
&= (m-1)\Big(\psi_1'(\xi_7K^2(d_1(x)))\Big)^{m-2}\psi_1''\Big(\xi_7K^2(d_1(x))\Big)(2\xi_7)^m K^m(d_1(x))k^m(d_1(x)) \\
&\quad + \Big(\psi_1'(\xi_7K^2(d_1(x)))\Big)^{m-1}(2\xi_7)^{m-1}(m-1)K^{m-2}(d_1(x))k^m(d_1(x)) \\
&\quad + \Big(\psi_1'(\xi_7K^2(d_1(x)))\Big)^{m-1}(2\xi_7)^{m-1}K^{m-1}(d_1(x))(m-1)k^{m-2}(d_1(x))k'(d_1(x)) \\
&\quad + \Big(\psi_1'(\xi_7K^2(d_1(x)))\Big)^{m-1}(2\xi_7)^{m-1}K^{m-1}(d_1(x))k^{m-1}(d_1(x))\Delta(d_1(x)) \\
&\quad - b(x)f(\psi_1(\xi_7K^2(d_1(x)))) - \Big[2\xi_7K(d_1(x))k(d_1(x))\psi_1'(\xi_7K^2(d_1(x)))\Big]^m
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^m f(\psi_1(\xi_7 K^{m-2}(d_1(x))))k^m(d_1(x))K^{m-2}(d_1(x)) \\
&\times \Big\{ 2(2\xi_7)^{m-1}\Big(\xi_7 K^2(d_1(x)) f^{\frac{1}{m-1}-1}(\psi_1(\xi_7 K^2(d_1(x))))f'(\psi_1(\xi_7 K^2(d_1(x)))) - 1\Big) \\
&\quad + 2(2\xi_7)^{m-1} - (m-1)(2\xi_7)^{m-1} \\
&\quad - (m-1)(2\xi_7)^{m-1}\Big(\frac{k'(d_1(x))K(d_1(x))}{k^2(d_1(x))} - (1-C_k)\Big) \\
&\quad - (m-1)(2\xi_7)^{m-1}(1-C_k) - (2\xi_5)^{m-1}\frac{K(d_1(x))}{k(d_1(x))}\Delta d_1(x) \\
&\quad - \Big(\frac{(-1)^m b(x)}{k^m(d_1(x))K^{m-2}(d_1(x))} - b_1\Big) \\
&\quad - b_1 + 2(2\xi_7)^{m-1}\Big(\xi_7 K^2(d_1(x)) f^{\frac{1}{m-1}}(\psi_1(\xi_7 K^2(d_1(x)))) - \Gamma_f^{\frac{1}{m-1}}\Big) \\
&\quad + 2(2\xi_7)^{m-1}\Gamma_f^{\frac{1}{m-1}} \Big\} \\
&\leqslant 0,
\end{aligned}
$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of problem (P_+) in D_ρ^-.

In a similar way, for $x \in D_\rho^+$,we can show that $\underline{u}_\varepsilon$ is a subsolution of of problem (P_+) in D_ρ^+. The last part of the proof is the same as that of (i).

The existence of solutions of Problem $(P_\pm)$ is similar as that in references [12, 13].

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Chunlian Liu
College of Xinglin
Nantong University
Jiangsu Nantong 226008
China
e-mail: liusisi_step@163.com

Zuodong Yang
Institute of Mathematics
School of Mathematics Science
Nanjing Normal University
Jiangsu Nanjing 210023
China
School of Teacher Education
Nanjing Normal University, Jiangsu Nanjing 210097
China
e-mail: zdyang_jin@263.net