BOUNDARY BEHAVIOR OF LARGE SOLUTIONS TO QUASILINEAR ELLIPTIC PROBLEMS WITH A NONLINEAR GRADIENT TERM

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Abstract. In this paper, we study the boundary behavior of solutions to boundary blow-up elliptic problems

\[
\begin{cases}
\text{div}(|\nabla u|^{m-2}\nabla u) \pm |\nabla u(x)|^{q(m-1)} = b(x)f(u), & x \in \Omega, \\
u > 0, & x \in \Omega, \\
u|_{\partial \Omega} = +\infty,
\end{cases}
\]

where \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N \), \( m > 1 \), \( q > 0 \), \( b \in C^\alpha(\overline{\Omega}) \), which is positive in \( \Omega \) and may be vanishing on the boundary and rapidly varying near the boundary, and \( f \) is rapidly varying or normalized regularly varying at infinity.

1. Introduction

In this paper, we plan to investigate the exact asymptotic behavior of solutions near the boundary for the following problems

\[
\begin{cases}
\text{div}(|\nabla u|^{m-2}\nabla u) \pm |\nabla u(x)|^{q(m-1)} = b(x)f(u), & x \in \Omega, \\
u > 0, & x \in \Omega, \\
u|_{\partial \Omega} = +\infty,
\end{cases}
\]

where the last condition means that \( u(x) \to +\infty \) as \( d(x) = \text{dist}(x, \partial \Omega) \to 0 \), and the solution is called “a large solution” or “an explosive solution”, \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N \) \( (N \geq 2) \), \( q > 0 \), \( m > 1 \). The function \( b \) satisfies:

- (b1) \( b \in C^\alpha(\Omega) \) for some \( \alpha \in (0,1) \), is non-negative in \( \Omega \);
- (b2) there exists \( k \in \Lambda \) such that

\[
0 < b_1 := \liminf_{d(x) \to 0} \frac{b(x)}{k^m(d(x))K^{m-2}(d(x))} \leq b_2 := \limsup_{d(x) \to 0} \frac{b(x)}{k^m(d(x))K^{m-2}(d(x))} < \infty,
\]

or


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(b3) there exists \( k \in \Lambda \) such that
\[
0 < b_{1q} := \liminf_{d(x) \to 0} \frac{b(x)}{kq(m-1)(d(x))} \leq b_{2q} := \limsup_{d(x) \to 0} \frac{b(x)}{kq(m-1)(d(x))} < \infty,
\]
where \( \Lambda \) denotes the set of all positive non-decreasing functions in \( C^1(0, \delta_0)(\delta_0 > 0) \) which satisfy
\[
\lim_{t \to 0^+} \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) = C_k \in [0, \infty), \quad K(t) = \int_{t}^{1} k(s)ds,
\]
and \( f \) satisfies:

(f1) \( f \in C^1[0, +\infty), f(0) = 0, f \) is increasing on \([0, +\infty)\);

(f2) \( \int_{1}^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)} < \infty \);

(f3) there exists \( C_f > 0 \) such that
\[
\lim_{s \to +\infty} \int_{s}^{\infty} f'(s) \int_{s}^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)} = C_f.
\]

We note that for each \( k \in \Lambda \), we have \( C_k \in [0, 1] \) and
\[
\lim_{t \to 0^+} \frac{K(t)}{k(t)} = 0, \quad \lim_{t \to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - \lim_{t \to 0^+} \frac{d}{dt} \frac{K(t)}{k(t)} = 1 - C_k. \tag{1.2}
\]

In fact, from (1.1), we can see \( \frac{d}{dt}(\frac{K(t)}{k(t)}) = C_k + \alpha(t) \), where \( \lim_{t \to 0^+} \alpha(t) = 0 \), so
\[
\frac{K(t)}{k(t)} = \int_{0}^{t} C_k dt + \int_{0}^{t} \alpha(t) dt = C_k t + \int_{0}^{t} \alpha(t) dt,
\]
since \( \lim_{t \to 0^+} \alpha(t) = 0 \), so \( \lim_{t \to 0^+} \frac{K(t)}{k(t)} = 0 \).

For example, \( k(t) = t^p, p > 0 \) or \( k(t) = e^{\sqrt{x}/\sqrt{x}} \) satisfies (1.1), we can conclude that they also satisfy (1.2).

The set \( \Lambda \) was first introduced by Cirstea and Rădulescu [3,4] in order to study the boundary behavior of solutions to the problem
\[
\Delta \omega = b(x)f(\omega), \quad x \in \Omega, \quad \omega|_{\partial \Omega} = \infty.
\]

Semilinear elliptic problems involving gradient term with boundary blow-up interested many authors. Namely Bandle and Giarrusso [1] developed existence and asymptotic behavior results for large solutions of
\[
\Delta u + |\nabla u(x)|^\alpha = f(u),
\]
in a bounded domain. In the case \( f(u) = p(x)u^\gamma, a > 0, \) and \( \gamma > \max(1, a) \), Lair and Wood [7-9] dealt with the above equation in bounded domain and the whole space, they
proved the existence of entire large solution under the condition \( \int_0^\infty r \max_{|x|=r} p(x) dr < \infty \) when the domain is \( \mathbb{R}^N \). Ghergu et al. [5] considered more general equation

\[
\Delta u + q(x)|\nabla u(x)|^a = p(x)f(u),
\]

where \( 0 \leq a \leq 2, \ p \) and \( q \) are Hölder continuous functions on \( (0, \infty) \). We note that the Keller-Osserman condition on \( f \) (see [2,16]) remains the key condition for the existence for their work. Ghergu and Radulescu [6] considered the following problem

\[
\begin{aligned}
\Delta u + |\nabla u(x)| = p(x)f(u), & \quad \text{in } \Omega, \\
|u| = 0, & \quad \text{in } \Omega,
\end{aligned}
\]

where \( \Omega \) is either a smooth bounded domain or the whole space and \( f \) is a nondecreasing function satisfying \( f \in C^0_{\text{loc}}(0, \infty), \ f(0) = 0, \ f > 0, \) on \( (0, \infty), \) and \( \\wedge = \sup_{x \geq 1} \frac{f(x)}{x} < \infty \). The authors studied the existence and nonexistence of large solutions under the assumption that

\[
\int_0^\infty r (\max_{|x|=r} p(x) - \min_{|x|=r} p(x)) \Psi(r) dr < +\infty,
\]

where \( \Psi(r) = \exp(\frac{\Lambda}{2} \int_0^r \min_{|x|=r} p(x) dr) \).

Faten Toumi [19] extended the above result to the following problem

\[
\begin{aligned}
\Delta u + \lambda(|x|)|\nabla u(x)| = \varphi(x,u(x)), & \quad \text{in } \mathbb{R}^N, \\
|u| = 0, & \quad u \neq 0,
\end{aligned}
\]

where \( \lambda : [0, \infty) \to [0, \infty) \) is a continuous function and \( \varphi : \mathbb{R}^N \times [0, \infty) \to [0, \infty) \) is measurable, continuous with respect to the second variable.

Quasilinear elliptic problems or such problems involving gradient terms with boundary blow-up interested many authors, see [10-12,14,20,21].

As far as the authors know, however, there are less results which contain the exact asymptotic behaviour of solutions near the boundary to problem \( (P_\pm) \). In this paper, also applying Karamata regular variation theory (Karamata regular variation theory see [15–17,18]), perturbed method and constructing comparison functions, we show the asymptotic behaviour of solutions near the boundary to problem \( (P_\pm) \).

Our main results are as follows:

**Theorem 1.1.** Let \( q > 0, \ b \) satisfies \( (b_1) \ f \) satisfies \( (f_1), (f_2), (f_3) \) with \( C_f \geq 1 \) and the assumption that: \( (f_4) \) there exists \( \Gamma_f \in [0, \infty) \) such that

\[
\lim_{s \to +\infty} f(s) \left[ \int_s^\infty \frac{dv}{f_{m-1}(v)} \right]^{m-1} = \Gamma_f.
\]

(i) If \( \Gamma_f > 0, \ q > m/(m-1) \) and \( b \) satisfies \( (b_3) \), then every solution \( u_+ \) of problem \( (P_\pm) \) satisfies

\[
\lim_{d(x) \to 0} \frac{u_+(x)}{\Psi_1(K^q(d(x)))} = 1,
\]

(1.4)
where $K$ is in (1.1) with $k$ defined in $(b_3)$ and $\psi_1$ is uniquely determined by

$$\int_{\psi_1(t)}^{\infty} \frac{dv}{f^{m-1}(v)} = t, \quad t > 0; \quad (1.5)$$

(ii) if $b$ satisfies $(b_2)$ with $C_k > 0$ and $q \in (0, m/(m-1))$ and $\Gamma_f > 0$, then for every solution $u_{\pm}$ of problem $(P_{\pm})$

$$\lim_{d(x) \to 0} \frac{u_{\pm}(x)}{\psi_1(K^2(d(x)))} = 1, \quad (1.6)$$

where $K$ is in (1.1) with $k$ defined in $(b_2)$ and $\psi_1$ is uniquely determined by (1.5);

(iii) if $q = m/(m-1)$, $b$ satisfies $(b_2)$, and $2 - (m-1)C_k - 2\Gamma_f^{1/(m-1)} > 0$, then for every solution $u_{-}$ of problem $(P_-)$

$$\lim_{d(x) \to 0} \frac{u_{-}(x)}{\psi_1(K^2(d(x)))} = 1, \quad (1.7)$$

where $K$ is in (1.1) with $k$ defined in $(b_2)$ and $\psi_1$ is uniquely determined by (1.5);

(iv) if $q = m/(m-1)$, $b$ satisfies $(b_2)$, and $2 - (m-1)C_k + 2\Gamma_f^{1/(m-1)} > 0$, then every solution $u_{+}$ of problem $(P_+)$

$$\lim_{d(x) \to 0} \frac{u_{+}(x)}{\psi_1(K^2(d(x)))} = 1, \quad (1.8)$$

where $K$ is in (1.1) with $k$ defined in $(b_2)$ and $\psi_1$ is uniquely determined by (1.5).

2. Preliminaries

In this section, we present some bases of the theory which comes from Senta [18], Preliminaries in Resnick [17], Introductions and the appendix in Maric [15].

**Definition 2.1.** A positive measurable function $f$ defined on $[a, +\infty)$, for some $a > 0$, is called regularly varying at infinity with index $\rho$, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s \to +\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \quad (2.1)$$

For example, $f(s) = s^\rho$ is regularly varying at infinity. In particular, when $\rho = 0$, $f$ is called slowly varying at infinity. Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are:

1. every measure function on $[a, \infty)$ which has a positive limit at infinity;
2. $(\ln s)^\beta$ and $(\ln(\ln s))^\beta$, $\beta \in \mathbb{R}$;
3. $e^{(\ln s)^p}$, $0 < p < 1$. 
DEFINITION 2.2. A positive measurable function $f$ defined on $[a, +\infty)$, for some $a > 0$, is called rapidly varying at infinity if for each $p > 1$,

$$\lim_{s \to \infty} \frac{f(s)}{s^p} = \infty.$$  \hfill (2.2)

Clearly, if $f \in \text{RV}_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity. Some basic examples of rapidly varying functions at infinity are:

(1) $e^s$ and $e^{es}$;

(2) $e^{(\ln s)^p}$, $e^{s^\rho}$ and $e^{e^{s^\rho}}$, $p > 0$;

(3) $(\ln s)^\beta e^{s^\rho}$ and $s^\beta e^{s^\rho}$, $p > 0, \beta \in \mathbb{R}$;

(4) $s^\beta e^{(\ln s)^p}$ and $(\ln s)^\beta e^{(\ln s)^p}$, $p > 1, \beta \in \mathbb{R}$.

We also see that a positive measurable function $g$ defined on $(0, a)$ for some $a > 0$ is regularly varying at zero with index $\sigma$ (written as $g \in \text{RV}_{Z\sigma}$) if $t \to g(1/t)$ belongs to $\text{RV}_{-\sigma}$, $g$ is called rapidly varying at zero if $t \to g(1/t)$ is rapidly varying at infinity.

PROPOSITION 2.1. (Uniform convergence theorem) If $f \in \text{RV}_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form $(a_1, \infty)$ with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(a_1, \infty]$ provided $f$ is bounded on $(a_1, \infty]$ for all $a_1 > 0$.

PROPOSITION 2.2. (Representation theorem) A function $L$ is slowly varying at infinity if and only if it may be written in the form

$$L(s) = \varphi(s) \exp\left(\int_{a_1}^s \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a_1,$$  \hfill (2.3)

for some $a_1 > a$, where the functions $\varphi$ and $y$ are measurable and for $s \to \infty, y(s) \to 0$, and $\varphi(s) \to c_0$, with $c_0 > 0$.

We call that

$$\hat{L}(s) = c_0 \exp\left(\int_{a_1}^s \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a_1,$$  \hfill (2.4)

is normalized slowly varying at infinity and

$$f(s) = c_0 s^\rho \hat{L}(s), \quad s \geq a_1,$$  \hfill (2.5)

is normalized regularly varying at infinity with index $\rho$ (and written as $f \in \text{NRV}_\rho$).

Similarly, $g$ is called normalized regularly varying at zero with index $\rho$, written as $g \in \text{NRV}_{Z\rho}$ if $t \to g(1/t)$ belongs to $\text{NRV}_\rho$. A function $f \in \text{RV}_\rho$ belongs to $\text{NRV}_\rho$ if and only if

$$f \in C^1[a_1, \infty), \text{ for some } a_1 > 0, \text{ and } \lim_{s \to \infty} \frac{s f'(s)}{f(s)} = \rho.$$  \hfill (2.6)
PROPOSITION 2.3. If functions $L, L_1$ are slowly varying at infinity, then
(i) $L^\sigma$ for every $\sigma \in \mathbb{R}$, $c_1 L + c_2 L_1$ $(c_1 \geq 0, c_2 \geq 0$ with $c_1 + c_2 > 0)$, $L \circ L_1$
(if $L_1(t) \to +\infty$ as $t \to +\infty$), are also slowly varying at infinity;
(ii) for every $\theta > 0$ and $t \to +\infty, \theta t^\theta L(t) \to +\infty$ and $t^{-\theta} L(t) \to 0$;
(iii) for $\rho \in \mathbb{R}$ and $t \to +\infty$, \( \frac{\ln(L(t))}{\ln t} \to 0 \) and \( \frac{\ln(\rho L(t))}{\ln t} \to \rho \).

PROPOSITION 2.4. (Asymptotic behavior) If a function $L$ is slowly varying at infinity, then for $a > 0$ and $t \to \infty$,
(i) $\int_{a}^{t} s^\beta L(s)ds \simeq (\beta + 1)^{-1} t^{1+\beta} L(t)$, for $\beta > -1$;
(ii) $\int_{a}^{t} s^\beta L(s)ds \simeq (-\beta - 1)^{-1} t^{1+\beta} L(t)$, for $\beta < -1$.

PROPOSITION 2.5. (Asymptotic behavior) If a function $H$ is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,
(i) $\int_{a}^{t} s^\beta H(s)ds \simeq (\beta + 1)^{-1} t^{1+\beta} H(t)$, for $\beta > -1$;
(ii) $\int_{a}^{t} s^\beta H(s)ds \simeq (-\beta - 1)^{-1} t^{1+\beta} H(t)$, for $\beta < -1$.

LEMMA 2.1. Let $k \in \Lambda$:
(i) if $C_k \in (0, 1)$, then $k \in \text{NRVZ}_{(1-C_k)/C_k}$;
(ii) if $C_k = 1$, then $k$ is normalized slowly varying at zero;
(iii) if $C_k = 0$, then $k$ is rapidly varying at zero.

Proof. By l’Hospital’s rule and (1.1), we have
\[
\lim_{t \to 0} \frac{K(t)}{tk(t)} = \lim_{t \to 0} \frac{K(t)}{t} = \lim_{t \to 0} \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) = C_k; \quad (2.7)
\]
(i)(ii) when $C_k > 0$, it follows by (1.2) that
\[
\lim_{t \to 0} \frac{tk'(t)}{k(t)} = \lim_{t \to 0} \frac{K(t)k'(t)}{k^2(t)} \lim_{t \to 0} \frac{tk(t)}{K(t)} = \frac{1 - C_k}{C_k}, \quad (2.8)
\]
i.e., $k \in \text{NRVZ}_{(1-C_k)/C_k}$ for $C_k \in (0, 1)$ and $k$ is normalized slowly varying at zero for $C_k = 1$;
(iii) when $C_k = 0$, for arbitrary $\gamma > 0$, it follows by (2.8) that $\lim_{t \to 0} \frac{tk'(t)}{k(t)} = +\infty$ and there exists $t_{0\gamma}$ such that
\[
\frac{k'(t)}{k(t)} > (\gamma + 1)t^{-1}, \quad \forall t \in (0, t_{0\gamma}]. \quad (2.9)
\]
Integrating (2.9) from $t$ to $t_{0\gamma}$, we obtain
\[
\ln(k(t_{0\gamma})) - \ln(k(t)) > (\gamma + 1)(\ln t_{0\gamma} - \ln t), \quad \forall t \in (0, t_{0\gamma}].
\]
i.e.,
\[
0 < \frac{k(t)}{t^\gamma} < \frac{k(t_0)}{t_0^{\gamma+1}} t, \quad t \in (0, t_0].
\]

Let \( t \to 0 \), we see by Definition 2.2 that \( k \) is rapidly varying at zero.

**Lemma 2.2.** If \( f \) satisfies \((f_1)\), \((f_2)\) and \((f_3)\), then

(i) \( C_f \in [1, +\infty) \);

(ii) if \((f_3)\) holds for \( C_f > 1 \), then \( f \in NRVC_{(C_f-1)} \);

(iii) when \( C_f = 1 \), then \( f \) is rapidly varying at infinity.

**Proof.** (i) Let
\[
J(s) = f_{m-1}^1(s) f'(s) \int_s^\infty \frac{dv}{f_{m-1}^1(v)}, \quad \forall \ s > 0.
\]
Integrating \( J(s) \) from \( a (a > 0) \) to \( t \) and integrate by parts, we obtain
\[
\int_a^t J(s) ds = f_{m-1}^1(t) \int_t^\infty \frac{dv}{f_{m-1}^1(v)} - f_{m-1}^1(a) \int_a^\infty \frac{dv}{f_{m-1}^1(v)} + t - a, \quad \forall \ t > a.
\]

It follows from the l’Hospital’s rule that
\[
0 \leq \lim_{t \to \infty} \frac{f_{m-1}^1(t) \int_t^\infty \frac{dv}{f_{m-1}^1(v)}}{t} = \lim_{t \to \infty} \frac{1}{t} \int_a^t J(s) ds - 1 = \lim_{t \to \infty} J(t) - 1 = C_f - 1,
\]
i.e., \( C_f \geq 1 \).

(ii) By (i), we see that
\[
\lim_{s \to +\infty} \frac{f(s)}{sf'(s)} = \lim_{s \to +\infty} \frac{f_{m-1}^1(s) \int_s^\infty \frac{dv}{f_{m-1}^1(v)}}{s J(s)} = \frac{1}{C_f} \lim_{s \to +\infty} \frac{f_{m-1}^1(s) \int_s^\infty \frac{dv}{f_{m-1}^1(v)}}{s} = \frac{C_f - 1}{C_f},
\]
i.e., \( f \in NRVC_{(C_f-1)} \) for \( C_f > 1 \).

(iii) When \( C_f = 1 \), we see by the proof of (ii) that
\[
\lim_{s \to +\infty} \frac{f(s)}{sf'(s)} = 0.
\]
Consequently, for arbitrary \( p > 1 \), there exists \( S_0 > 0 \) such that
\[
\frac{f'(s)}{f(s)} > (p + 1)s^{-1}, \quad \forall \ s > S_0,
\]
Integrating the above inequality from $S_0$ to $s$, we obtain

$$\ln(f(s)) - \ln(f(S_0)) > (p + 1)(\ln s - \ln S_0), \quad \forall \ s \geq S_0,$$

Letting $s \to +\infty$, we see by Definition 2.2 that $f$ is rapidly varying at infinity.

**Lemma 2.3.** Let $f$ satisfy $(f_1), (f_2), (f_3)$ and let $\psi_1$ be the solution to the problem

$$\int_1^\infty \frac{ds}{f^{-\frac{1}{m-1}}(s)} = t, \quad \forall \ t > 0.$$

Then:

(i) $-\psi_1'(t) = f^{-\frac{1}{m-1}}(\psi_1(t)), \quad \psi_1(t) > 0, \ t > 0, \quad \psi_1(0) := \lim_{t \to 0^+} \psi_1(t) = +\infty$, and

$$\psi''_1(t) = \frac{1}{m-1}f^{-\frac{2}{m-1}}(\psi_1(t))f'(\psi_1(t)), \quad t > 0;$$

(ii) $\psi_1 \in NRVZ_{-(C_f - 1)}$;

(iii) $-\psi_1' = f^{-\frac{1}{m-1}} \circ \psi_1 \in NRVZ_{-C_f/(m-1)}$.

**Proof.** By the definition of $\psi_1$ and a direct calculation, we show that (i) holds. (ii) It follows from the proof of Lemma 2.1 that

$$\lim_{t \to 0^+} t\frac{t\psi_1'(t)}{\psi_1(t)} = \lim_{t \to 0^+} t\frac{tf^{-\frac{1}{m-1}}(\psi_1(t))}{\psi_1(t)} = \lim_{s \to +\infty} \frac{f^{-\frac{1}{m-1}}(s) \int_s^{\infty} \frac{dv}{f^{-\frac{1}{m-1}}(v)}}{s} = -(C_f - 1),$$

i.e., $\psi_1 \in NRVZ_{-(C_f - 1)}$.

(iii) $(f_3)$ implies

$$\lim_{t \to 0^+} t\frac{t\psi_1''(t)}{\psi_1(t)} = \lim_{t \to 0^+} -\frac{t}{m-1}f^{-\frac{1}{m-1}}(\psi_1(t))f'(\psi_1(t))$$

$$= \lim_{s \to +\infty} \frac{1}{m-1}f^{-\frac{2}{m-1}}(s)f'(s) \int_s^{\infty} \frac{dv}{f^{-\frac{1}{m-1}}(v)} = -C_f/(m-1).$$

### 3. Proofs of the main results

**Lemma 3.1.** (Weak comparison principle) Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial \Omega$ and $\varphi : (0, a) \to (0, a)$ be continuous and non-decreasing, let $u_1, u_2 \in W^{1,m}(\Omega)$ satisfy

$$\int_{\Omega} |\nabla u_1|^{m-2} \nabla u_1 \nabla \psi dx + \int_{\Omega} \varphi u_1 \psi dx \leq \int_{\Omega} |\nabla u_2|^{m-2} \nabla u_2 \nabla \psi dx + \int_{\Omega} \varphi u_2 \psi dx,$$
For all non-negative $\psi \in W_{0}^{1,m}(\Omega)$. Then the inequality

$$u_1 \leq u_2, \text{ on } \partial \Omega,$$

implies that

$$u_1 \leq u_2, \text{ in } \Omega.$$

For any $\delta > 0$, we define

$$\Omega_{\delta} = \{x \in \Omega : 0 < d(x) < \delta\}.$$

Since $\Omega$ is smooth, there exists $\delta_0 > 0$ such that $d \in C^2(\Omega_{\delta_0})$ and

$$|\nabla d(x)| = 1, \ \forall x \in \Omega_{\delta_0}.$$

**PROOF OF THEOREM 1.1.**

(i) $q > m/(m-1), m > 1$ and $\Gamma_f > 0$, let $\varepsilon \in (0,b_{1q}/4)$ and

$$\xi_{01} = \left( \frac{b_{1q}}{q^{q(m-1)}\Gamma_f^{q-1}} \right)^{\frac{1}{m-1}}, \quad \xi_{02} = \left( \frac{b_{2q}}{q^{q(m-1)}\Gamma_f^{q-1}} \right)^{\frac{1}{m-1}};$$

$$\xi_1 = \xi_{01} \left( 1 - \frac{2\varepsilon}{b_{1q}} \right)^{\frac{1}{m-1}}, \quad \xi_2 = \xi_{02} \left( 1 + \frac{2\varepsilon}{b_{2q}} \right)^{\frac{1}{m-1}}.$$

It follows that

$$\frac{\xi_{01}^{2/(m-1)}}{< \xi_1 < \xi_2 < 2^{1/(m-1)}\xi_{02}}.$$

By $(b_1), (b_2), (1.2), (2.11)$ and $(f_4)$, we see that there is $\delta_{\varepsilon} \in (0,\delta_0/2)$ (which is corresponding to $\varepsilon$) sufficiently small that:

$$(b_{1q} - \varepsilon)k^q(d(x) - \rho) \leq (b_{1q} - \varepsilon)k^q(d(x)) < b(x), \ x \in D_{\rho}^{-} = \Omega_{2\delta_{\varepsilon}}/\bar{\Omega}_{\rho}$$

and

$$b(x) < (b_{2q} + \varepsilon)k^q(d(x)) \leq (b_{2q} + \varepsilon)k^q(d(x) + \rho), \ x \in D_{\rho}^{+} = \Omega_{2\delta_{\varepsilon}-\rho},$$

where $\rho \in (0,\delta_{\varepsilon})$.

For $i = 1,2$,

$$2(\xi_{02}q)^{m-1} \left( \frac{K(t)}{k(t)} \right)^{(q-1)(m-1)-1}$$

$$\times \left[ q \left| \xi_i K^q(t) f^{\frac{1}{m-1}} \left( \psi_1(\xi_i K^q(t)) \right) f' \left( \psi_1(\xi_i K^q(t)) \right) \right| \right.$$

$$\left. + \left| (q-1)(m-1) + \frac{(m-1)K(t)k'(t)}{k^2(t)} + \frac{K(t)}{k(t)}|\Delta d(x)| \right| \right]$$
\[142\] CHUNLIAN LIU AND ZUODONG YANG

and by a direct calculation, it follows that, for \(x \in D^-\)

\[
\text{div}(\nabla \overline{u}_\varepsilon) - b(x) f(\overline{u}_\varepsilon(x)) = (m-1) (\psi_1(\xi_1 K^q(d_1(x))))^{m-2} \psi''(\xi_1 K^q(d_1(x))) (\xi_1 q)^{m(q-1)}(d_1(x)) k^m(d_1(x)) \\
+ (\psi_1(\xi_1 K^q(d_1(x)))) (\xi_1 q)^{m(q-1)}(q-1)(m-1)K^{q-1}(d_1(x)) k^m(d_1(x)) \\
+ (\psi_1(\xi_1 K^q(d_1(x))))(\xi_1 q)^{m(q-1)}(d_1(x))(m-1)k^{m-2}(d_1(x)) k'(d_1(x)) \\
+ (\psi_1(\xi_1 K^q(d_1(x))))(\xi_1 q)^{m(q-1)}(d_1(x)) k^{m-1}(d_1(x)) \Delta(d_1(x)) \\
- b(x) f(\psi_1(\xi_1 K^q(d_1(x)))) + \left[\xi_1 K^q(d_1(x)) k(d_1(x)) \psi_1'(\xi_1 K^q(d_1(x))) \right]^{q(m-1)} \\
= (-1)^m f(\psi_1(\xi_1 K^q(d_1(x)))) k^{m(q-1)}(d_1(x)) \left\{ (\xi_1 q)^{m(q-1)}(d_1(x)) \frac{K(d_1(x))}{k(d_1(x))} \right\}^{(q-1)(m-1)-1} \\
\times \left[ q\xi_1 K^q(d_1(x)) f^{q-1}(\psi_1(\xi_1 K^q(d_1(x)))) f'(\psi_1(\xi_1 K^q(d_1(x)))) \right]^{(q-1)(m-1)-1} \\
- (q-1)(m-1) \frac{(m-1)K(d_1(x))k'(d_1(x))}{k^2(d_1(x))} - \frac{K(d_1(x))}{k(d_1(x))} \Delta d(x) \\
- \left( \frac{b(x)}{K^{q(m-1)}(d_1(x))} - b_{1q} \right) - b_{1q} + (\xi_1 q)^{m-1} \Gamma_f^{-1} \\
+ (\xi_1 q)^{m-1} \left[ (\xi_1 K^q(t))^{m-1} f(\psi_1(\xi_1 K^q(t))) \right]^{q-1} - \Gamma_f^{-1} \right] \\
\leq 0, \]

i.e., \(\overline{u}_\varepsilon\) is a supersolution of problem \((P_+)\) in \(D^-\).

In a similar way, for \(x \in D^+\), we can show that \(\underline{u}_\varepsilon\) is a subsolution of problem \((P_+)\) in \(D^+\).

Now let \(u_+\) be an arbitrary solution of problem \((P_+)\) and

\[ C_1(\delta_\varepsilon) := \max_{d(x) \geq \delta_\varepsilon} u_+(x). \]

We see that

\[ u_+ \leq C_1(\delta_\varepsilon) + \overline{u}_\varepsilon, \quad \text{on } \partial D^-_. \]
Since $\psi_1$ is decreasing, see Lemma 2.3, and $\xi_{02} < \xi_2$, we have that

$$u_\varepsilon \leq \psi_1(\xi_{02}K^q(2\delta_2\varepsilon)) := C_2(\delta_2),$$

whenever $d(x) = 2\delta_2 - \rho$ and $u_\varepsilon \leq u_+ + C_2(\delta_\varepsilon)$ on $\partial D^+_\rho$.

It follows by $(f_1)$ and Lemma 3.1 that

$$u_+ \leq C_1(\delta_\varepsilon) + \bar{u}_\varepsilon \quad \text{on} \quad D^-_{\rho}, \quad u_\varepsilon \leq u_+ + C_2(\delta_\varepsilon) \quad \text{on} \quad D^+_\rho.$$ 

Hence by letting $\rho \to 0$, we have for $x \in D^-_{\rho} \cap D^+_\rho$,

$$1 - \frac{C_2(\delta_\varepsilon)}{\psi_1(\xi_{02}K^q(d(x)))} \leq \frac{u_+(x)}{\psi_1(\xi_{02}K^q(d(x)))}$$

and

$$\frac{u_+(x)}{\psi_1(\xi_{01}K^q(d(x)))} \leq 1 + \frac{C_1(\delta_\varepsilon)}{\psi_1(\xi_{01}K^q(d(x)))}.$$ 

Consequently,

$$1 \leq \liminf_{d(x) \to 0} \frac{u_+(x)}{\psi_1(\xi_{02}K^q(d(x)))}$$

and

$$\limsup_{d(x) \to 0} \frac{u_+(x)}{\psi_1(\xi_{01}K^q(d(x)))} \leq 1.$$ 

Thus by letting $\varepsilon \to 0$, we obtain

$$1 \leq \liminf_{d(x) \to 0} \frac{u_+(x)}{\psi_1(\xi_{02}K^q(d(x)))}$$

and

$$\limsup_{d(x) \to 0} \frac{u_+(x)}{\psi_1(\xi_{01}K^q(d(x)))} \leq 1.$$ 

By Lemma 2.3 (ii) and Proposition 2.1, we have

$$\limsup_{d(x) \to 0} \frac{\psi_1(\xi_{02}K^q(d(x)))}{\psi_1(K^q(d(x)))} = \limsup_{d(x) \to 0} \frac{\psi_1(\xi_{01}K^q(d(x)))}{\psi_1(K^q(d(x)))} = 1.$$ 

Thus

$$\limsup_{d(x) \to 0} \frac{u_+(x)}{\psi_1(K^q(d(x)))} = 1.$$ 

(ii) When $b$ satisfies $(b_2)$ with $C_k > 0$, either $q \in (0,m/(m-1)), m > 1$ and $\Gamma_f > 0$.

Let $\varepsilon \in (0,b_1/4)$ and

$$\xi_{03} = \frac{1}{2} \left( \frac{b_1}{2 - (m-1)(2-C_k)} \right)^{1/m-1},$$
\[
\xi_{04} = \frac{1}{2} \left( \frac{b_2}{2 - (m - 1)(2 - C_k)} \right)^{\frac{1}{m-1}},
\]
\[
(\xi_3)^{m-1} = (\xi_{03})^{m-1} - \frac{\varepsilon}{2 - (m - 1)(2 - C_k)},
\]
\[
(\xi_4)^{m-1} = (\xi_{04})^{m-1} + \frac{\varepsilon}{2 - (m - 1)(2 - C_k)}.
\]

It follows that

\[
\xi_{03}/\sqrt{2} < \xi_3 < \xi_4 < \sqrt{2}\xi_{04}.
\]

By \((b_1), (b_2), (1.2), (2.11)\) and \((f_4)\), we see that there is \(\delta_\varepsilon \in (0, \delta_0/2)\) (which is corresponding to \(\varepsilon\)) sufficiently small that:

\[
(b_1 - \varepsilon)k^m(d(x) - \rho)K^{m-2}(d(x) - \rho)
\]

\[
\leq (b_1 - \varepsilon)k^m(d(x))K^{m-2}(d(x)) < b(x), \quad x \in D_\rho^- = \Omega_2\delta_\varepsilon/\tilde{\Omega}_\rho,
\]

and

\[
b(x) < (b_2 + \varepsilon)k^m(d(x))K^{m-2}(d(x) - \rho)
\]

\[
\leq (b_2 + \varepsilon)k^m(d(x) + \rho)K^{m-2}(d(x) + \rho), \quad x \in D_\rho^+ = \Omega_2\delta_\varepsilon - \rho,
\]

where \(\rho \in (0, \delta_\varepsilon)\).

For \(i = 3, 4,\)

\[
4(2\xi_{04})^{m-1} |\xi_iK^2(t)f_{\frac{1}{m-1}}(\psi_1(\xi_iK^2(t)))f'(\psi_1(\xi_iK^2(t))) - 1|
\]

\[
+ (m - 1)(2\xi_{04})^{m-1} \left| \frac{K(t)k'(t)}{k^2(t)} - (1 - C_k) \right| + (2\xi_{04})^{m-1} \left| K(t) - \Delta d(x) \right|
\]

\[
+ 2(2\xi_{04})^{m-1} \left| \frac{K(t)}{k(t)} - 1 \right|^{(q-1)(m-1)} < \varepsilon, \quad \forall (x, t) \in \Omega_2\delta_\varepsilon \times (0, 2\delta_\varepsilon).
\]

Let

\[
d_1(x) = d(x) - \rho, \quad d_2(x) = d(x) + \rho,
\]

\[
\tilde{u}_\varepsilon = \psi_1(\xi_3K^2(d_1(x))), \quad x \in D_\rho^- \quad \text{and} \quad \bar{u}_\varepsilon = \psi_1(\xi_4K^2(d_2(x))) \quad x \in D_\rho^+.
\]

By using

\[
(2\xi_3)^{m-1}(2 - (m - 1)(2 - C_k)) = b_1,
\]

and by a direct calculation, it follows that, for \(x \in D_\rho^-\),

\[
\text{div}(|\nabla \bar{u}_\varepsilon|^{m-2}\nabla \bar{u}_\varepsilon) - b(x)f(\bar{u}_\varepsilon(x)) \pm |\bar{u}_\varepsilon(x)|^{q(m-1)}
\]

\[
= (m - 1) \left( \psi_1'(\xi_3K^2(d_1(x))) \right)^{m-2} \psi_1''(\xi_3K^2(d_1(x))) (2\xi_3)^{m}K^m(d_1(x))k^m(d_1(x))
\]

\[
+ 2(2\xi_{04})^{m-1} \left| \frac{K(t)}{k(t)} - 1 \right|^{(q-1)(m-1)} < \varepsilon, \quad \forall (x, t) \in \Omega_2\delta_\varepsilon \times (0, 2\delta_\varepsilon).
\]
\[+ \left( \psi'_{1}(\xi_{3}K^{2}(d_{1}(x))) \right)^{m-1} (2\xi_{3})^{m-1}(m-1)K^{m-2}(d_{1}(x))k^{m}(d_{1}(x))
\]
\[+ \left( \psi'_{1}(\xi_{3}K^{2}(d_{1}(x))) \right)^{m-1} (2\xi_{3})^{m-1}K^{m-1}(d_{1}(x))(m-1)K^{m-2}(d_{1}(x))k'(d_{1}(x))
\]
\[+ \left( \psi'_{1}(\xi_{3}K^{2}(d_{1}(x))) \right)^{m-1} (2\xi_{3})^{m-1}K^{m-1}(d_{1}(x))k^{m-1}(d_{1}(x))\Delta(d_{1}(x))
\]
\[- b(x)f(\psi_{1}(\xi_{3}K^{2}(d_{1}(x)))) \pm \left[ 2\xi_{3}K(d_{1}(x))k(d_{1}(x))\psi'_{1}(\xi_{3}K^{2}(d_{1}(x))) \right]^{q(m-1)}
\]
\[= (-1)^{m}f(\psi_{1}(\xi_{3}K^{2}(d_{1}(x))))k^{m}(d_{1}(x))K^{m-2}(d_{1}(x))
\]
\[\times \left\{ 2(2\xi_{3})^{m-1}\left( \xi_{3}K^{2}(d_{1}(x))f_{m-1}^{x}(\psi_{1}(\xi_{3}K^{2}(d_{1}(x))))f'(\psi_{1}(\xi_{3}K^{2}(d_{1}(x)))) - 1 \right) + 2(2\xi_{3})^{m-1} - (m-1)(2\xi_{3})^{m-1}
\]
\[- (m-1)(2\xi_{3})^{m-1}\left( k'(d_{1}(x))K(d_{1}(x)) \right) - (1 - C_{k})
\]
\[- (m-1)(2\xi_{3})^{m-1}(1 - C_{k}) - (2\xi_{3})^{m-1}\left( K(d_{1}(x)) \right) k'(d_{1}(x))\Delta d_{1}(x)
\]
\[- \left( \frac{(-1)^{m}b(x)}{k^{m}(d_{1}(x))K^{m-2}(d_{1}(x))} - b_{1} \right) - b_{1}
\]
\[\pm (2^{2}\xi_{3})^{m-1}\left( (\xi_{3}K^{2}(d_{1}(x)))^{m-1}f(\psi_{1}(\xi_{3}K^{2}(d_{1}(x)))) \right)^{q-1}
\]
\[\times \left( K(d_{1}(x)) \right)^{1-(q-1)(m-1)} \right\}
\]
\[\leq 0,
\]
i.e., \( \bar{u}_{e} \) is a supersolution of problem \((P_{\pm})\) in \( D_{\tilde{r}} \).

In a similar way, for \( x \in D_{\tilde{r}}^{+} \), we can show that \( u_{e} \) is a subsolution of the problem \((P_{\pm})\) in \( D_{\tilde{r}}^{-} \).

The last part of the proof is the same as that of (i).

(iii) When \( q = m/(m-1) \), \( b \) satisfies \((b_{2})\) and \( 2 - (m-1)C_{k} - 2\Gamma_{f}^{1/(m-1)} > 0 \) for problem \((P_{-})\).

Let \( \varepsilon \in (0,b_{1}/4) \) and

\[\xi_{05} = \frac{1}{2} m^{-1} \frac{b_{1}}{2 - (m-1)C_{k} - 2\Gamma_{f}^{1/(m-1)}},\]
\[\xi_{06} = \frac{1}{2} m^{-1} \frac{b_{2}}{2 - (m-1)C_{k} - 2\Gamma_{f}^{1/(m-1)}},\]
\[\xi_{5} = \xi_{05} - \frac{2\varepsilon}{2 - (m-1)C_{k} - 2\Gamma_{f}^{1/(m-1)}},\]
\[ \xi_6 = \xi_{06} + \frac{2\epsilon}{2 - (m-1)C_k - 2\Gamma_f^{1/(m-1)}}, \]

It follows that
\[ \frac{m-1}{\sqrt{2}} < \xi_5 < \xi_6 < \frac{m-1}{\sqrt{2}}\xi_{06}. \]

By \((b_1), (b_2), (1.2), (2.11)\) and \((f_4)\), we see that there is \(\delta_\epsilon \in (0, \delta_0/2)\) (which is corresponding to \(\epsilon\)) sufficiently small that:
\[ (b_1 - \epsilon)k^m(d(x) - \rho)K^{m-2}(d(x) - \rho) \leq (b_1 - \epsilon)k^m(d(x))K^{m-2}(d(x)) < b(x), \ x \in D_\rho^- = \Omega_{2\delta_\epsilon}/\overline{\xi_\rho} \]

and
\[ b(x) < (b_2 + \epsilon)k^m(d(x))K^{m-2}(d(x) - \rho) \leq (b_2 + \epsilon)k^m(d(x) + \rho)K^{m-2}(d(x) + \rho), \ x \in D_\rho^+ = \Omega_{2\delta_\epsilon-\rho}, \]

where \(\rho \in (0, \delta_\epsilon)\).

And for \(i = 5, 6,\)
\[
4(2\xi_{06})^{m-1} \left| \xi_iK^2(t)f^{\frac{1}{m-1}}(\psi_1(\xi_iK^2(t)))f'(\psi_1(\xi_iK^2(t))) - 1 \right|
\]
\[
+ (m-1)(2\xi_{06})^{m-1} \left| \frac{K(t)k'(t)}{k^2(t)} - (1 - C_k) \right| + (2\xi_{06})^{m-1} \left| \frac{K(t)}{k(t)}|\Delta d(x)| \right|
\]
\[
+ 2(2\xi_{06})^{m-1} \left| \xi_iK^2(t)f^{\frac{1}{m-1}}(\psi_1(\xi_iK^2(t))) - \Gamma_f^{m-1} \right|
\]
\[ < \epsilon, \ \forall (x,t) \in \Omega_{2\delta_\epsilon} \times (0, 2\delta_\epsilon). \]

Let
\[ d_1(x) = d(x) - \rho, \quad d_2(x) = d(x) + \rho, \]
\[ \overline{u}_\epsilon = \psi_1(\xi_5K^2(d_1(x))), \ x \in D_\rho^- \quad \text{and} \quad \underline{u}_\epsilon = \psi_1(\xi_6K^2(d_2(x))) \ x \in D_\rho^+. \]

By using
\[ (2\xi_5)^{m-1}(2 - (m-1)C_k - 2\Gamma_f^{1/m-1}) = b_1, \]

and by a direct calculation, it follows that, for \(x \in D_\rho^-\),
\[
\text{div}(|\nabla \overline{u}_\epsilon|^{m-2} \nabla \overline{u}_\epsilon) - b(x)f(\overline{u}_\epsilon(x)) - |\overline{u}_\epsilon(x)|^m
\]
\[ = (m-1) \left( \psi'_1(\xi_5K^2(d_1(x))) \right)^{m-2} \psi''_1 \left( \xi_5K^2(d_1(x)) \right) (2\xi_5)^m \left( 2\xi_5 \right)^m (d_1(x)) K^m(d_1(x))
\]
\[ + \left( \psi'_1(\xi_5K^2(d_1(x))) \right)^{m-1} (2\xi_5)^{m-1} (m-1)K^{m-2}(d_1(x)) K^m(d_1(x))
\]
\[ + \left( \psi'_1(\xi_5K^2(d_1(x))) \right)^{m-1} (2\xi_5)^{m-1} (m-1)K^{m-2}(d_1(x)) K'(d_1(x))
\]
It follows that
\[
\psi_1'(\xi_5 K^2(d_1(x)))^{m-1} (2\xi_5)^{m-1} K^{m-1}(d_1(x)) k^{m-1}(d_1(x)) \Delta(d_1(x))
\]
\[
- b(x)f(\psi_1(\xi_5 K^2(d_1(x)))) - 2\xi_5 K(d_1(x)) k(d_1(x)) \psi_1'(\xi_5 K^2(d_1(x)))\]
\[
= (-1)^m f(\psi_1(\xi_5 K^{m-2}(d_1(x)))) k^m(d_1(x)) K^{m-2}(d_1(x))
\times \left\{ 2(2\xi_5)^{m-1} \left( \xi_5 K^2(d_1(x)) f_{m-1}^{-1}(\psi_1(\xi_5 K^2(d_1(x))) f'(\psi_1(\xi_5 K^{d_1(x)})) - 1 \right) 
+ 2(2\xi_5)^{m-1} - (m-1)(2\xi_5)^{m-1} 
- (m-1)(2\xi_5)^{m-1} \left( \frac{k'(d_1(x)) K(d_1(x))}{k^2(d_1(x))} - 1 - C_k \right) 
- (m-1)(2\xi_5)^{m-1} (1 - C_k) - (2\xi_5)^{m-1} \frac{K(d_1(x))}{k(d_1(x))} \Delta d_1(x) 
- \left( \frac{(-1)^m b(x)}{k^m(d_1(x)) K^{m-2}(d_1(x))} - b_1 \right) 
- b_1 - 2(2\xi_5)^{m-1} \left( \xi_5 K^2(d_1(x)) f_{m-1}^{-1} \psi_1(\xi_5 K^2(d_1(x))) - \Gamma_{m-1} \right) 
- 2(2\xi_5)^{m-1} \Gamma_{m-1} \right\}
\leq 0,
\]

i.e., \( \bar{u}_\varepsilon \) is a supersolution of problem \( (P_-) \) in \( D^-_\rho \).

In a similar way, for \( x \in D^+_\rho \), we can show that \( u_\varepsilon \) is a subsolution of problem \( (P_-) \) in \( D^+_\rho \). The last part of the proof is the same as that of (i).

(iv) When \( q = m/(m-1) \) \( b \) satisfies \( (b_2) \) and \( 2 - (m-1)C_k + 2\Gamma_{m-1}^{1/(m-1)} > 0 \) for problem \( (P_+) \).

Let \( \varepsilon \in (0, b_1/4) \) and
\[
\xi_{07} = \frac{1}{2} m^{-1} \sqrt{\frac{b_1}{2 - (m-1)C_k + 2\Gamma_{m-1}^{1/(m-1)}},
\]
\[
\xi_{08} = \frac{1}{2} m^{-1} \sqrt{\frac{b_2}{2 - (m-1)C_k + 2\Gamma_{m-1}^{1/(m-1)}},
\]
\[
\xi_7 = \xi_{07} - \frac{2\varepsilon}{2 - (m-1)C_k + 2\Gamma_{m-1}^{1/(m-1)}},
\]
\[
\xi_8 = \xi_{08} + \frac{2\varepsilon}{2 - (m-1)C_k + 2\Gamma_{m-1}^{1/(m-1)}},
\]

It follows that
\[
\xi_{07} / \sqrt{2} < \xi_7 < \xi_8 < \sqrt{2} \xi_{08}.
\]
By \((b_1), (b_2), (1.2), (2.11)\) and \((f_4)\), we see that there is \(\delta_\varepsilon \in (0, \delta_0/2)\) (which is corresponding to \(\varepsilon\)) sufficiently small that:

\[
(b_1 - \varepsilon)k^m(d(x) - \rho)K^{m-2}(d(x) - \rho) \\
\leq (b_1 - \varepsilon)k^m(d(x))K^{m-2}(d(x)) < b(x), \ x \in D^-_\rho,
\]

and

\[
b(x) < (b_2 + \varepsilon)k^m(d(x))K^{m-2}(d(x) - \rho) \\
\leq (b_2 + \varepsilon)k^m(d(x) + \rho)K^{m-2}(d(x) + \rho), \ x \in D^+_\rho,
\]

where \(D^-_\rho = \Omega_{2\delta_\varepsilon}/\tilde{\Omega}_\rho, \ D^+_\rho = \Omega_2\delta_\varepsilon - \rho\) and \(\rho \in (0, \delta_\varepsilon)\).

And for \(i = 7, 8\),

\[
4(2\xi_{08})^{-m-1}\left|\xi_iK^2(t)f^{-\frac{1}{m-1}}(\psi_1(\xi_iK^2(t)))f'(\psi_1(\xi_iK^2(t))) - 1\right|
\]

\[
+ (m-1)(2\xi_{08})^{-m-1}\left|\frac{K(t)k'(t)}{k^2(t)} - (1 - C_k)\right| + (2\xi_{08})^{-m-1}\frac{K(t)}{k(t)}|\Delta d(x)|
\]

\[
+ 2(2\xi_{08})^{-m-1}\left|\xi_iK^2(t)f^{-\frac{1}{m-1}}(\psi_1(\xi_iK^2(t))) - \frac{1}{\mu_f}\right|
\]

\[
< \varepsilon, \ \forall (x, t) \in \Omega_2\delta_\varepsilon \times (0, 2\delta_\varepsilon).
\]

Let \(d_1(x) = d(x) - \rho, \ d_2(x) = d(x) + \rho\) and

\[
\bar{u}_e = \psi_1(\xi_7K^2(d_1(x))), \ x \in D^-_\rho \ \text{and} \ u_e = \psi_1(\xi_8K^2(d_2(x))) \ x \in D^+_\rho.
\]

By using

\[
(2\xi_7)^{-m-1}(2 - (m-1)C_k + 2\frac{1}{\mu_f}) = b_1,
\]

and by a direct calculation, it follows that, for \(x \in D^-_\rho\),

\[
\text{div}(|\nabla \bar{u}_e|^{m-2}\nabla \bar{u}_e) - b(x)f(\bar{u}_e(x)) + |\bar{u}_e(x)|^m
\]

\[
= (m-1)\left(\psi'_1(\xi_7K^2(d_1(x)))\right)^{m-2}\psi''_1\left(\xi_7K^2(d_1(x))\right)(2\xi_7)^{m-1}(d_1(x))K^m(d_1(x))
\]

\[
+ \left(\psi'_1(\xi_7K^2(d_1(x)))\right)^{m-1}\left(2\xi_7\right)^{m-1}(m-1)K^{m-2}(d_1(x))K^m(d_1(x))
\]

\[
+ \left(\psi'_1(\xi_7K^2(d_1(x)))\right)^{m-1}\left(2\xi_7\right)^{m-1}K^{m-1}(d_1(x))(m-1)K^{m-2}(d_1(x))k'(d_1(x))
\]

\[
+ \left(\psi'_1(\xi_7K^2(d_1(x)))\right)^{m-1}\left(2\xi_7\right)^{m-1}K^{m-1}(d_1(x))K^{m-1}(d_1(x))\Delta(d_1(x))
\]

\[
- b(x)f(\psi_1(\xi_7K^2(d_1(x)))) \left[2\xi_7K(d_1(x))k(d_1(x))\psi'_1(\xi_7K^2(d_1(x)))\right]^m
\]
\[ \begin{align*}
&= (-1)^{m} f(\psi_1(\xi_7 K^{m-2}(d_1(x)))) k^m(d_1(x)) K^{m-2}(d_1(x)) \\
&\times \left\{ 2(2 \xi_7)^{m-1} \left( \xi_7 K^2(d_1(x)) f \frac{1}{m-1} (\psi_1(\xi_7 K^2(d_1(x)))) f'(\psi_1(\xi_7 K^2(d_1(x)))) - 1 \right) \\
&+ 2(2 \xi_7)^{m-1} - (m-1)(2 \xi_7)^{m-1} \\
&- (m-1)(2 \xi_7)^{m-1} \left( \frac{k'(d_1(x)) K(d_1(x))}{k^2(d_1(x))} - (1 - C_k) \right) \\
&- (m-1)(2 \xi_7)^{m-1} \left( 1 - C_k \right) - (2 \xi_7)^{m-1} \frac{K(d_1(x))}{k(d_1(x))} \Delta d_1(x) \\
&- \left( \frac{(-1)^{m} b(x)}{k^m(d_1(x)) K^{m-2}(d_1(x))} - b_1 \right) \\
&- b_1 + 2(2 \xi_7)^{m-1} \left( \xi_7 K^2(d_1(x)) f \frac{1}{m-1} (\psi_1(\xi_7 K^2(d_1(x)))) - \Gamma_f^1 \right) \\
&+ 2(2 \xi_7)^{m-1} \Gamma_f^1 \right\}
\leq 0,
\end{align*} \]

i.e., \( \tilde{\pi}_e \) is a supersolution of problem \( (P_+) \) in \( D_{\tilde{\rho}}^- \).

In a similar way, for \( x \in D^+ \), we can show that \( \tilde{u}_e \) is a subsolution of problem \( (P_+) \) in \( D_{\tilde{\rho}}^- \). The last part of the proof is the same as that of (i).

The existence of solutions of Problem \( (P_{\pm}) \) is similar as that in references [12, 13].

REFERENCES


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