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EXISTENCE OF A MILD SOLUTION FOR IMPULSIVE NEUTRAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In the present work, we investigate the existence of a mild solution of the fractional order differential equation with impulsive conditions in a Banach space. We establish the existence of a mild solution by using some fixed point theorems and resolvent operator theory. We present an example for showing the effectiveness of the main theory.

1. Introduction

In recent few decades, the fractional calculus has received much attention of researchers mainly due to its demonstrated applications in widespread fields of science and engineering, e.g., fluid flow, rheology dynamical, mechanics, electrical engineering, modelling of many physical phenomena and so on. Fractional calculus have been available and applicable to deal with real system characterized by power laws, anomalous diffusion process etc. The nonlinear oscillations of earthquake are one of such important models. The deficiency of continuum traffic flow can be characterized by the fractional derivative. Concerning this matter, we refer to the monographs [17, 24, 27, 29] and the references cited therein. Also, neutral differential equation arises in many areas of applied mathematics, science and engineering such as theory of aeroelasticity [18] and lossless transmission lines[14]. The theory of heat conduction in materials and the lumped control systems can be described by neutral differential equations. For more details on neutral functional differential equation, we refer to papers [7, 8, 9, 10, 11, 20] and references given therein.

The existence of the solution for the differential equations with nonlocal conditions has been investigated widely by many authors that the nonlocal conditions are more realistic than the classical initial condition such as in dealing with many physical problems. The differential equation with nonlocal conditions has been firstly considered by Byszewski [5]. In [21], authors have studied the existence of the mild solution to fractional integro-differential equations of Sobolev type with nonlocal conditions by

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using a fixed point theorem for condensing operators. In [30], authors have established the existence and uniqueness of the mild solutions for fractional differential equations with nonlocal conditions. The existence of solutions to semi-linear neutral fractional differential equations has been proved by the authors [1]. Recently, the approximate controllability to fractional neutral stochastic evolution equations involving nonlocal conditions has been investigated by the authors [11]. Concerning the developments in the study of nonlocal problems we refer to [2, 8, 9, 10, 11, 16, 20, 22, 30, 38, 39] and references given therein.

On the other hand, many real world processes and phenomena which are subjected during their development to short-term external influences can be modeled as impulsive differential equation with fractional order. Their duration is negligible compared to the total duration of the entire process or phenomena. Such process is investigated in various fields such as biology, physics, control theory, population dynamics, medicine and so on. Impulsive differential equations are an appropriate model to hereditary phenomena for which a delay argument arises in the modelling equations. For the general theory of such equations, we refer to the monographs [4],[19] and papers [1, 2, 12, 13, 16, 23, 25, 30, 31, 32, 34, 35, 36, 37, 38] and references given therein.

In our recent work [7]-[8], we have adopted the idea of [12] and studied the impulsive neutral fractional integro-differential equations in an arbitrary Banach space X involving single base point [13], [23], without assuming Lipschitz continuity of nonlinear function f and compactness of the solution operator $S_q(t)$, $t \geqslant 0$. In [7], we have obtained the existence of mild solution for the impulsive neutral fractional integro-differential equation with infinite delay by using Hausdorff's measure of noncompactness and Darbo fixed point theorem with analytic solution operator. In [8], we have considered the impulsive neutral fractional integro-differential equation with finite delay and nonlocal conditions. Utilizing Hausdorff's measure of noncompactness, we have established the existence results by mean of analytic solution operator and Darbo-Sadovskii fixed point theorem.

In this paper, our main concern is to establish the existence and uniqueness of a mild solution for the fractional order neutral differential equation with the multiple base points in a Banach space $(X, \|\cdot\|_X)$,

$$
\begin{aligned}
{}^cD^\eta[u(t)-F(t,u(h_1(t)))] &= A[u(t)-F(t,u(h_1(t)))]+G(t,u(h_2(t))), \\
& \qquad t\in[0,T],\ t\neq t_i\ 0<T<\infty, &(1.1)\\
\Delta u(t_i) &= I_i(u(t_i^-)),\quad i=1,2,\cdots,\delta,\ \ \delta\in\mathbb{N}, &(1.2)\\
u(0) &= u_0+g(u)\in X, &(1.3)
\end{aligned}
$$

where ${}^cD^\eta$ is the classical Caputo fractional derivative of order η, $0<\eta<1$ [27] and $A: X\supset D(A)\to X$ is a closed linear operator with dense domain $D(A)$ in a Banach space X and $I_i\in C(X,X)$, $0=t_0<t_1<\cdots<t_\delta<t_{\delta+1}=T$, $\Delta u|_{t=t_i}=u(t_i^+)-u(t_i^-)$ and $u(t_i^+)=\lim_{h\to 0+}u(t_i+h)$ and $u(t_i^-)=\lim_{h\to 0-}u(t_i+h)$ denote the right and left limits of $u(t)$ at $t=t_i$, respectively. The functions F, G, h_1 ,h_2 and g are appropriate continuous functions to be stated later and $h_j\in C(\mathscr{I},\mathscr{I})$, $j=1,2$.

In this study, we present two existence results for the mild solution to the system (1.1)-(1.3). Our first existence result is obtained by using Banach fixed point theorem and analytic semigroup with assuming Lipschitz continuity of f. Our second result is obtained by using Schaefer's fixed-point theorem and compact solution operator with the assumption that f does not satisfy the Lipschitz continuity.

The organization of the paper is as follows: Section 2 gives some basic definitions, lemmas and theorems as preliminaries as these are useful for proving our results. Section 3 focuses on proving the existence results for mild solutions to the system (1.1)-(1.3). Section 4 provides an example for illustrating the theory.

2. Preliminaries

In this section, we discuss some definitions and notations about sectorial operators, solution operator and analytic solution operators required for establishing our results.

Throughout this work, X is a complex Banach space equipped with the norm $\|\cdot\|_X$. The symbol $C(\mathscr{I};X)$ stands for the Banach space of all continuous functions from $\mathscr{I}=[0,T]$ into X with supremum norm i.e., $\|y\|_{\mathscr{I}}=\sup_{t\in\mathscr{I}}\|y(t)\|$. The notation $L(X,Y)$ denotes the Banach spaces of all bounded linear operators from X into Y with the operator norm denoted by $\|\cdot\|_{L(X,Y)}$ and when $X=Y$ then we write simply $L(X)$ and $\|\cdot\|_{L(X)}$. In addition, $PC(\mathscr{I},X)$ represents the Banach space of all the piecewise continuous functions from $\mathscr{I}$ into X with the norm

$$\|y\|_{PC}=\max\{\sup_{t\in\mathscr{I}}\|y(t+0)\|_X,\ \sup_{t\in\mathscr{I}}\|y(t-0)\|_X\},$$

and $B_r(x,X)$ denotes a closed ball with center at x and radius r in X.

To set the structure for our primary existence results, we recall the following definitions.

DEFINITION 1. The definition of one parameter **Mittag-Leffler** function is given by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},$$

and two parameter function of Mittag-Leffler type is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi i}\int_C\frac{\mu^{\alpha-\beta}e^{\mu}}{\mu^a-z}d\mu,\quad 0<\alpha,\beta,z\in\mathbb{C},$$

here C is a contour which start and ends at $-\infty$ and encircles the disc $|\mu|\leqslant|z|^{1/2}$ counter clockwise. The Laplace transform of the Mittag-Leffler is defined as

$$L(t^{\beta-1}E_{\alpha,\beta}(-\rho^\alpha t^\alpha))=\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha+\rho^\alpha},\quad \mathrm{Re}\,\lambda>\rho^{1/\alpha},\ \rho>0.$$

For more details, we refer to [27].

DEFINITION 2. [27] The Riemann-Liouville fractional integral operator $\mathscr{J}$ of order $\eta > 0$ is defined by

$$\mathscr{J}^{\eta}F(t) = \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}F(s)ds, \tag{2.1}$$

where $F \in L^1((0,T);X)$.

DEFINITION 3. [27] The Riemann-Liouville fractional derivative is given by

$$D^{\eta}F(t) = D_t^{\delta}\mathscr{J}^{\delta-\eta}F(t),\ \delta-1<\eta<\delta, \tag{2.2}$$

where $D_t^{\delta} = \frac{d^{\delta}}{dt^{\delta}}$, $F \in L^1((0,T);X)$, $\mathscr{J}^{\delta-\eta}F \in W^{\delta,1}((0,T);X)$. Here the notation $W^{\delta,1}((0,T);X)$ stands for the Sobolev space defined by

$$W^{\delta,1}((0,T);X) = \Big\{ v \in X : \exists z \in L^1((0,T);X) : \\ v(t) = \sum_{k=0}^{\delta-1} d_k \frac{t^k}{k!} + \frac{t^{\delta-1}}{(\delta-1)!} * z(t),\ t \in (0,T) \Big\}. \tag{2.3}$$

Note that $z(t) = v^{\delta}(t)$, $d_k = v^k(0)$.

DEFINITION 4. [27] The Caputo fractional derivative is given by

$${}^cD^{\eta}F(t) = \frac{1}{\Gamma(\delta-\eta)}\int_0^t (t-s)^{\delta-\eta-1}F^{\delta}(s)ds,\ \ \delta-1<\eta<\delta, \tag{2.4}$$

where $F \in C^{\delta-1}((0,T);X) \cap L^1((0,T);X)$. The Laplace transform of the Caputo derivative of order $\eta > 0$ is given by

$$L[{}^cD^{\eta}u(t);\lambda] = \lambda^{\eta}L[u(t)] - \sum_{k=0}^{\delta-1}\lambda^{\eta-k-1}u^k(0),\ \ \delta-1<\eta<\delta. \tag{2.5}$$

DEFINITION 5. [31] An operator A which is closed and linear, is called sectorial operator if there are constants $\omega \in \mathbb{R}$, $\theta \in [\pi/2, \pi]$, $M > 0$ such that the following two conditions are satisfied:

(1) $\rho(A) \supset \Sigma_{(\theta,\,\omega)} = \{\lambda \in C : \lambda \neq \omega,\ |\arg(\lambda-\omega)| < \theta\}$,

(2) $\|R(\lambda,A)\|_{L(X)} \leqslant \frac{M}{|\lambda-\omega|}$, $\omega \in \Sigma_{(\theta,\,\omega)}$,

where $\rho(A)$ be the resolvent set of A.

For more details, we refer to [3]. Consider the following Cauchy problem for the fractional evolution equation

$${}^cD^{\eta}u(t) = Au(t),\ \ t>0;\ \ u(0)=x,\ u^k(0)=0,\ k=1,\cdots,\delta-1, \tag{2.6}$$

where $\eta > 0$ and $\delta = \lceil \eta \rceil$.

DEFINITION 6. [3] A family $\{S_\eta(t)\}_{t\geqslant 0}\subset L(X)$ is called a solution operator for (2.6) if the following conditions are satisfied:

(a) $S_\eta(t)$ is strongly continuous for $t\geqslant 0$ and $S_\eta(0)=I$;

(b) $S_\eta(t)D(A)\subset D(A)$ and $AS_\eta(t)x=S_\eta(t)Ax$ for all $x\in D(A)$, $t\geqslant 0$;

(c) $S_\eta(t)x$ is a solution of (2.6) for all $x\in D(A)$, $t\geqslant 0$.

The solution operator $S_\eta(t)$ of (2.6) is also defined by (see [3])

$$\lambda^{\eta-1}(\lambda^\eta I-A)^{-1}x=\int_0^\infty e^{-\lambda t}S_\eta(t)xdt,\ \ \mathrm{Re}\,\lambda>\omega,\ x\in X, \tag{2.7}$$

where $\omega\geqslant 0$ and $\{\lambda^\eta : \mathrm{Re}\lambda>\omega\}\subset\rho(A)$.

An operator A is said to belong to $\mathscr{C}^\eta(X;M,\omega)$, or $\mathscr{C}^\eta(M,\omega)$ if the problem (2.6) has a solution operator $S_\eta(t)$ satisfying $\|S_\eta(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$. Denote $\mathscr{C}^\eta(\omega)=\bigcup\{\mathscr{C}^\eta(M,\omega);\ M\geqslant 1\}$, or $\mathscr{C}^\eta=\bigcup\{\mathscr{C}^\eta(\omega;\ \omega\geqslant 0)\}$ (Bazhlekova, [3]).

DEFINITION 7. [3] A solution operator $S_\eta(t)$ of (2.6) is said to be analytic if $S_\eta(t)$ admits an analytic extension to a sector $\sum_{\theta_0}$ for some $\theta_0\in(0,\pi/2]$.

An analytic solution operator $S_\eta(t)$ is said to be of analyticity type (θ_0,ω_0) if for each $\theta<\theta_0$ and $\omega>\omega_0$ there exists a positive constant $M=M(\theta,\omega)$ such that $\|S_\eta(t)\|\leqslant Me^{\omega\,\mathrm{Re}\,t}$, for $t\in\sum_\theta=\{t\in C/\{0\}:|\arg t|<\theta\}$. Denote $\mathscr{A}^\eta(\theta_0,\omega_0)=\{\mathscr{A}\in\mathscr{C}^\eta;\ \mathscr{A}\ \text{generates analytic solution operator}\ S_\eta(t)\ \text{of type}\ (\theta_0,\omega_0)\}$.

LEMMA 1. [3, 28] *Let $\eta\in(0,2)$. A linear closed densely defined operator A belongs to $\mathscr{A}^\eta(\theta_0,\omega_0)$ if and only if $\lambda^\eta\in\rho(A)$ for each $\lambda\in\sum_{\theta_0+\pi/2}(\omega_0)$, and for any $\omega>\omega_0$, $\theta<\theta_0$, there exists a constant $C=C(\theta,\omega)$ such that*

$$\|\lambda^{\eta-1}R(\lambda^\eta,A)\|\leqslant\frac{C}{|\lambda-\omega|},\ \lambda\in\sum_{\theta+\pi/2}(\omega). \tag{2.8}$$

Now, we have following result for mild solution of the non-homogenous Cauchy problem of fractional order.

THEOREM 1. *Suppose A is a sectorial operator and f satisfies the uniform Hölder condition with exponent $\beta\in(0,1]$, then*

$$u(t)=S_\eta(t)x_0+\int_0^t T_\eta(t-\varsigma)f(\varsigma)d\varsigma,\ \ t\in[0,T], \tag{2.9}$$

where

$$S_\eta(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\eta-1}R(\lambda^\eta,A)d\lambda, \tag{2.10}$$

$$T_\eta(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda^\eta, A)d\lambda,$$

is the mild solution for the following fractional Cauchy problem

$$^cD^\eta u(t) = Au(t) + f(t), \ 0 < \eta < 1, \ t \in \mathscr{I}, \tag{2.11}$$

$$u(0) = x_0 \in X, \tag{2.12}$$

where Γ *is a suitable path lying on* $\sum_{\theta,\,\omega}$. *For* $0 < \eta < 1$, $T_\eta(t)$ *is the* η*-resolvent family and* $S_\eta(t)$ *is the solution operator, generated by* A.

For more details about solution operators, we refer to [3, 28], and references cited in these papers.

Consider the set of functions

$$\begin{aligned} PC(\mathscr{I}, X) = \{z : \mathscr{I} \to X : z \in C((t_i, t_{i+1}], X), i = 0, 1, \cdots, \delta \text{ and } z(t_i^+) \\ \text{and } z(t_i^-) \text{ exist with } z(t_i^-) = z(t_i)\} \end{aligned} \tag{2.13}$$

equipped with the norm

$$\|z\|_{PC} = \sup_{t \in \mathscr{I}} \|z(t)\|_X,$$

which is a Banach space $(PC(\mathscr{I}, X), \|\cdot\|_{PC})$.

According to the Theorem 2.4. in [31], we present the following definition of a mild solution for equation (1.1).

DEFINITION 8. The function $u : \mathscr{I} \to X$ is said to be a mild solution of equation (1.1) if $u(\cdot) \in PC(\mathscr{I}, X)$ satisfies the following integral equation

$$u(t) = \begin{cases} S_\eta(t)[u_0 + g(u) - F(0, u(h_1(0)))] + F(t, u(h_1(t))) \\ + \int_0^t T_\eta(t-\varsigma)G(\varsigma, u(h_2(\varsigma)))d\varsigma, & t \in [0, t_1] \\ S_\eta(t-t_1)[u(t_1^-) + I_1(u(t_1^-)) - F(t_1, u(h_1(t_1^+)))] + F(t, u(h_1(t))) \\ + \int_{t_1}^t T_\eta(t-\varsigma)G(\varsigma, u(h_2(\varsigma)))d\varsigma, & t \in (t_1, t_2] \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ S_\eta(t-t_\delta)[u(t_\delta^-) + I_\delta(u(t_\delta^-)) - F(t_\delta, u(h_1(t_\delta^+)))] + F(t, u(h_1(t))) \\ + \int_{t_\delta}^t T_\eta(t-\varsigma)G(\varsigma, u(h_2(\varsigma)))d\varsigma, & t \in (t_\delta, T], \end{cases} \tag{2.14}$$

and also satisfies the following impulsive conditions $\triangle u|_{t=t_i} = I_i(u(t_i^-)), \ \ i = 1, \cdots, \delta$.

LEMMA 2. [3, 31] *If* $\eta \in (0,1)$ *and* $A \in A^\eta(\theta_0, \omega_0)$, *then for any* $x \in X$ *and* $t > 0$, *we have* $S_\eta(t)x \in D(A)$ *and*

$$\|S_\eta(t)\|_{L(X)} \leqslant Me^{\omega t}, \ \ \|T_\eta(t)\|_{L(X)} \leqslant Ce^{\omega t}(1 + t^{\eta-1}), \ \ t > 0, \ \ \omega > \omega_0.$$

Thus, we have

$$\|S_\eta(t)\|_{L(X)} \leqslant \widetilde{M}_S, \ \ \ \|T_\eta(t)\|_{L(X)} \leqslant t^{\eta-1}\widetilde{M}_T,$$

where $\widetilde{M}_S = \sup_{0 \leqslant t \leqslant T} \|S_\eta(t)\|_{L(X)}$, $\widetilde{M}_T = \sup_{0 \leqslant t \leqslant T} Ce^{\omega t}(1 + t^{1-\eta})$.

3. Existence Results

In this section, we provide two existence results for the mild solution of system (1.1)-(1.3). Our first existence result is based on Banach fixed point theorem which gives the existence and uniqueness of the mild solution to the system (1.1)-(1.3). The second existence result is established by using Schaefer's fixed point theorem and compact resolvent operator which provides the existence of a mild solution to system (1.1)-(1.3).

3.1. First Existence Result

To prove our first existence result, we make following assumptions on F, G, g and I_i.

Assumptions for the first existence result:

(A1) The function $F : \mathscr{I} \times X \to X$ is continuous and there exists a constant $L_F > 0$ such that

$$\|F(s_1,w_1) - F(s_2,w_2)\| \leqslant L_F[|s_1 - s_2| + \|w_1 - w_2\|_X], \tag{3.1}$$

and

$$\| F(t,w)\| \leqslant L_1\| w\| + L_2, \tag{3.2}$$

for every w, w_1, $w_2 \in X$ and t, s_1, $s_2 \in \mathscr{I}$ and L_1 and L_2 are positive constants.

(A2) The function $G : \mathscr{I} \times X \to X$ is continuous and there exists a constant $L_G > 0$ such that

$$\|G(t,w_1) - G(t,w_2)\| \leqslant L_G\|w_1 - w_2\|_X, \tag{3.3}$$

for every w_1, $w_2 \in X$ and $t, \in \mathscr{I}$.

(A3) $I_i : X \to X$, where $i = 1, \cdots, \delta$ are continuous functions and there exists a constant $L_I > 0$ such that

$$\|I_i(w_1) - I_i(w_2)\| \leqslant L_I\|w_1 - w_2\|_X \tag{3.4}$$

for each w_1, $w_2 \in X$.

(A4) The map $g : C(\mathscr{I};X) \to C(\mathscr{I};X)$ is a Lipschitz continuous function and there exists a constant $L_g > 0$ such that

$$\|g(w_1) - g(w_2)\| \leqslant L_g\|w_1 - w_2\|, \tag{3.5}$$

and

$$\|g(w)\| \leqslant \mathscr{D}_1\|w\| + \mathscr{D}_2, \tag{3.6}$$

for each $w, w_1, w_2 \in X$.

THEOREM 2. *Let us assume that the conditions* $(A1)-(A4)$ *are satisfied and*

$$\mathscr{L}=\widetilde{M}_S(L_g+1)+\widetilde{M}_SL_I+(\widetilde{M}_S+1)L_F+\widetilde{M}_TL_G\frac{T^\eta}{\eta}<1. \tag{3.7}$$

Then, the impulsive problem (1.1) has a unique mild solution $u\in X$ *on* $\mathscr{I}$.

Proof. Let $u_0\in X$ be fixed. Define a mapping $Q:PC(\mathscr{I};X)\to PC(\mathscr{I};X)$ such that

$$(Qu)(t)=\begin{cases} S_\eta(t)[u_0+g(u)-F(0,u(h_1(0)))]+F(t,u(h_1(t))) & \\ +\int_0^t T_\eta(t-\varsigma)G(\varsigma,u(h_2(\varsigma)))d\varsigma, & t\in[0,t_1]\\ S_\eta(t-t_1)[u(t_1^-)+I_1(u(t_1^-))-F(t_1,u(h_1(t_1^+)))]+F(t,u(h_1(t))) & \\ +\int_{t_1}^t T_\eta(t-\varsigma)G(\varsigma,u(h_2(\varsigma)))d\varsigma, & t\in(t_1,t_2]\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\ S_\eta(t-t_\delta)[u(t_\delta^-)+I_\delta(u(t_\delta^-))-F(t_\delta,u(h_1(t_\delta^+)))]+F(t,u(h_1(t))) & \\ +\int_{t_\delta}^t T_\eta(t-\varsigma)G(\varsigma,u(h_2(\varsigma)))d\varsigma, & t\in(t_\delta,T]. \end{cases} \tag{3.8}$$

By the assumptions $(A1)$-$(A4)$, it can be easily shown that the map Q is well defined on $PC(\mathscr{I};X)$. Furthermore, for $u^*,u^{**}\in PC(\mathscr{I};X)$ and $t\in[0,t_1]$, we get

$$\begin{aligned} &\|Qu^*(t)-Qu^{**}(t)\|\\ &\quad\leqslant \| S_\eta(t)\|_{L(X)}\|g(u^*)-g(u^{**})\|\\ &\qquad+\| S_\eta(t)\|_{L(X)}\| F(0,u^*(h_1(0)))-F(0,u^{**}(h_1(0)))\|_X\\ &\qquad+\| F(t,u^*(h_1(t)))-F(t,u^{**}(h_1(t)))\|_X\\ &\qquad+\int_0^t \| T_\eta(t-\varsigma)\|\| G(\varsigma,u^*(h_2(\varsigma)))-G(\varsigma,u^{**}(h_2(\varsigma)))\|_X d\varsigma,\\ &\quad\leqslant \widetilde{M}_SL_g \sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\|+\widetilde{M}_SL_F\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\|\\ &\qquad+L_F\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\|+\widetilde{M}_TL_G\\ &\qquad\times\int_0^t(t-\varsigma)^{\eta-1}\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\|d\varsigma\\ &\quad\leqslant [\widetilde{M}_S(L_g+L_F)+L_F+\widetilde{M}_TL_G\frac{T^\eta}{\eta}]\| u^*-u^{**}\|_{PC}. \end{aligned} \tag{3.9}$$

Similarly, for $t\in(t_i,t_{i+1}]$, $i=1,\cdots,\delta$, we obtain

$$\begin{aligned} &\|Qu^*(t)-Qu^{**}(t)\|\\ &\quad\leqslant \|S_\eta(t-t_i)[u^*(t_i^-)-u^{**}(t_i^-)]\|+\|S_\eta(t-t_i)[I_i(u^*(t_i^-))-I_i(u^{**}(t_i^-))]\|\\ &\qquad+\|S_\eta(t-t_i)[F(t_i,u^*(h_i(t_i^+)))-F(t_i,u^{**}(h_i(t_i^+)))]\| \end{aligned}$$

$$
\begin{aligned}
&+\|F(t,u^*(h_1(t)))-F(t,u^{**}(h_1(t)))\| \\
&+\int_{t_i}^{t}\|T_\eta(t-\varsigma)[G(\varsigma,u^*(h_2(\varsigma)))-G(\varsigma,u^{**}(h_2(\varsigma)))]\|d\varsigma, \\
\leqslant\ &\widetilde{M}_S\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\|+\widetilde{M}_SL_I\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\| \\
&+\widetilde{M}_SL_F\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\|+L_F\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\| \\
&+\widetilde{M}_TL_G\int_{t_i}^{t}(t-\varsigma)^{\eta-1}\sup_{\varsigma\in[0,T]}\|u^*(\varsigma)-u^{**}(\varsigma)\|d\varsigma, \\
\leqslant\ &\Big[\widetilde{M}_S+\widetilde{M}_SL_I+(\widetilde{M}_S+1)L_F+\widetilde{M}_TL_G\frac{T^\eta}{\eta}\Big]\times\|\,u^*-u^{**}\|_{PC}. \qquad (3.10)
\end{aligned}
$$

Thus, for all $t\in\mathscr{I}$, we conclude

$$
\begin{aligned}
\|Qu^*(t)-Qu^{**}(t)\|\leqslant\ &[\widetilde{M}_S(L_g+1)+\widetilde{M}_SL_I+(\widetilde{M}_S+1)L_F+\widetilde{M}_TL_G\frac{T^\eta}{\eta}] \\
&\times\|\,u^*-u^{**}\|_{PC}. \qquad (3.11)
\end{aligned}
$$

Taking supremum over t, we get,

$$
\|Qu^*-Qu^{**}\|_{PC}\leqslant\mathscr{L}\|\,u^*-u^{**}\|_{PC}. \qquad (3.12)
$$

Since $\mathscr{L}=\widetilde{M}_S(L_g+1)+\widetilde{M}_SL_I+(\widetilde{M}_S+1)L_F+\widetilde{M}_TL_G\frac{T^\eta}{\eta}<1$. Thus, it gives that Q is a strictly contraction map i.e., there exists a unique fixed point of the map Q on $\mathscr{I}$ which is a unique mild solution to the problem (1.1)-(1.3).

3.2. Second Existence Result

Our next result based on the Schaefer's fixed-point theorem. The statement of the theorem is as follows:

THEOREM 3. *Let $Q:X\to X$ be a continuous and a compact map such that the set $\{x\in X: x=\lambda Qx$ for some $0\leqslant\lambda\leqslant1\}$ is bounded, then Q has a fixed point.*

For this second result, we need to assume a new set of assumptions on G, g, F and $I_i, i=1,\cdots,\delta$.

Assumptions for the second existence result:

$(B1)$ $G:\mathscr{I}\times X\to X$ is a continuous function and there exist a continuous function $m_G:\mathscr{I}\to(0,\infty)$ and continuous non-decreasing function $W:[0,\infty)\to(0,\infty)$ such that

$$
\|\,G(\tau,w_1)\|_X\leqslant m_G(t)W(\|\,w_1\|),\quad(\tau,w_1)\in\mathscr{I}\times X, \qquad (3.13)
$$

and $\int_0^\infty\frac{d\varsigma}{W(\varsigma)}<\infty$

$(B2)$ The function $F:\mathscr{I}\times X\to X$ is a completely continuous function satisfying assumption $(A1)$.

$(B3)$ The functions $I_i : X \to X$ are completely continuous and there exists $\Omega > 0$ such that

$$\Omega = \max_{1 \leqslant i \leqslant \delta,\, y \in X} \{\| I_i(y)\|_X\}. \tag{3.14}$$

$(B4)$ The operator families $S_\eta(t), t \geqslant 0$ and $T_\eta(t), t \geqslant 0$ are compacts.

$(B5)$ $\frac{\widetilde{M_T} T^\eta}{\eta(1-\overline{M})} \int_0^T m_G(\varsigma)d\varsigma < \int_{\omega_1}^\infty \frac{d\varsigma}{W(\varsigma)}$, where $\omega_1 = \frac{M_*}{(1-\overline{M})}$,

$$M_* = \max\{\widetilde{M_S}\|u_0\| + (\widetilde{M_S}+1)L_2 + \widetilde{M_S}\mathscr{D}_2, \widetilde{M_S}\Omega + (\widetilde{M_S}+1)L_2\} \text{ and}$$
$$\overline{M} = \widetilde{M_S}(\mathscr{D}_1 + 1) + (\widetilde{M_S}+1)L_1 < 1.$$

THEOREM 4. *Assume that $(B1)-(B5)$ are satisfied. Then, there exists at least one mild solution of the impulsive problem (1.1)-(1.3) on $\mathscr{I}$.*

Proof. Consider the operator $Q : PC(\mathscr{I};X) \to PC(\mathscr{I};X)$ as in Theorem 2. It can be easily proved that map Q is well defined on $PC(\mathscr{I};X)$.

Step 1: The map Q is continuous.

To prove the continuity, let u_n be sequence in $PC(\mathscr{I};X)$ such that $\lim_{n\to\infty} u_n(t) = u(t)$ i.e. $u_n \to u$ as $n \to \infty$ in $PC(\mathscr{I};X)$. Since G and F are continuous. Therefore, by the continuity of G, F and g we deduce that for each $\tau \in \mathscr{I}$

$$G(\tau, u_n(h_2(\tau))) \to G(\tau, u(h_2(\tau))), \quad \text{as } n \to \infty, \tag{3.15}$$
$$F(\tau, u_n(h_1(\tau))) \to F(\tau, u(h_1(\tau))), \quad \text{as } n \to \infty, \tag{3.16}$$
$$g(u_n) \to g(u), \quad \text{as } n \to \infty. \tag{3.17}$$

Now for every $t \in [0,t_1]$, we have

$$\begin{aligned}
&\| (Qu_n)(t) - (Qu)(t)\| \\
&\quad \leqslant \|S_\eta(t)[g(u_n) - g(u)]\| + \|S_\eta(t)[F(0,u_n(h_1(0))) - F(0,u(h_1(0)))]\| \\
&\qquad + \| F(t,u_n(h_1(t))) - F(t,u(h_1(t)))\| \\
&\qquad + \int_0^t \| T_\eta(t-\varsigma)\| \cdot \| G(\varsigma,u_n(h_2(\varsigma))) - G(\varsigma,u(h_2(\varsigma)))\| d\varsigma, \\
&\quad \leqslant \widetilde{M_S}\|g(u_n) - g(u)\| + \widetilde{M_S}\|F(0,u_n(h_1(0))) - F(0,u(h_1(0)))\| \\
&\qquad + \| F(t,u_n(h_1(t))) - F(t,u(h_1(t)))\| \\
&\qquad + \widetilde{M}_T \int_0^t (t-\varsigma)^{\eta-1} \| G(\varsigma,u_n(h_2(\varsigma))) - G(\varsigma,u(h_2(\varsigma)))\| d\varsigma,
\end{aligned}$$

Thus, by the dominated convergence theorem, we get that

$$\| (Qu_n)(t) - (Qu)(t)\| \to 0, \quad \text{as} \quad n \to \infty, \tag{3.18}$$

i.e., $Qu_n(t)$ converges to $Qu(t)$ in X for each $t \in [0,t_1]$.
For $t \in (t_i, t_{i+1}]$, $i = 1,2,\cdots,\delta$, we get

$$\| (Qu_n)(t) - (Qu)(t)\|$$

$$\begin{aligned}
&\leqslant \|S_\eta(t-t_i)[u_n(t_i^-)-u(t_i^-)]\| + \|S_\eta(t-t_i)[I_i(u_n(t_i^-))-I_i(u(t_i^-))]\| \\
&\quad + \|S_\eta(t-t_i)[F(t_i,u_n(h_1(t_i^+)))-F(t_i,u(h_1(t_i^+)))]\| \\
&\quad + \|F(t,u_n(h_1(t)))-F(t,u_n(h_1(t)))\| \\
&\quad + \int_{t_i}^t \|T_\eta(t-\varsigma)[G(\varsigma,u_n(h_2(\varsigma)))-G(\varsigma,u(h_2(\varsigma)))]d\varsigma.
\end{aligned} \tag{3.19}$$

By the continuity of I_i, G, F and dominated convergence theorem, we have

$$\lim_{n\to\infty} \| (Qu_n)(t)-(Qu)(t)\|_{PC} = 0. \tag{3.20}$$

Therefore it implies that $Qu_n(t)$ converges to $Qu(t)$ in $PC(\mathscr{I};X)$. Hence, this proves the continuity of the map Q.

Step2. Secondly, we show that Q maps bounded sets into bounded sets in the space $PC(\mathscr{I};X)$. To prove the result, it is enough to show that for any $r>0$ there exists $\gamma>0$ such that $\| Qu\|_{PC} \leqslant \mathscr{H}$ for each $u\in B_r(PC)=\{u\in PC(\mathscr{I};X) : \| u\|_{PC}\leqslant r\}$. Let

$$G_1 = \sup_{t\in\mathscr{I},\, u\in B_r} \| G(t,u(h_2(t)))\|,$$

then for any $u\in B_r(PC)$, $t\in[0,t_1]$, we have

$$\begin{aligned}
\| Qu(t)\|_X &\leqslant \widetilde{M_S}[\| u_0\| + \mathscr{D}_1 r + \mathscr{D}_2] + (L_1 r + L_2)(1+\widetilde{M_S}) + \frac{\widetilde{M_T} T^\eta G_1}{\eta}, \\
&= \mathscr{H}_0,
\end{aligned}$$

For $t\in(t_i,t_{i+1}], i=1,\cdots,\delta$, we get

$$\| Qu(t)\| \leqslant \widetilde{M_S}[r+\Omega] + (L_1 r+L_2)(\widetilde{M_S}+1) + \frac{\widetilde{M_T} T^\eta G_1}{\eta} = \mathscr{H}_i.$$

Thus, for $t\in[0,T]$, we obtain

$$\| Qu(t)\| \leqslant \mathscr{H}, \tag{3.21}$$

where $\mathscr{H} = \max_{0\leqslant i\leqslant\delta} \mathscr{H}_i$.

Step 3. Q maps bounded sets into equicontinuous sets of $PC(\mathscr{I};X)$.

To this end, we show that $Q(B_r)$ is equicontinuous. Take $0\leqslant\zeta_1<\zeta_2\leqslant t_1$ and $u\in C([0,t_1];X)$, we have

$$\begin{aligned}
&\| Qu(\zeta_2)-Qu(\zeta_1)\| \\
&\quad \leqslant \| [S_\eta(\zeta_2)-S_\eta(\zeta_1)](u_0+g(u)-F(0,u(h_1(0))))\| \\
&\qquad + \| F(\zeta_2,u(h_1(\zeta_2)))-F(\zeta_1,u(h_1(\zeta_1)))\| \\
&\qquad + \| \int_0^{\zeta_2} T_\eta(\zeta_2-\varsigma)G(\varsigma,u(h_2(\varsigma)))d\varsigma - \int_0^{\zeta_1} T_\eta(\zeta_1-\varsigma)G(\varsigma,u(h_2(\varsigma)))d\varsigma\|, \\
&\quad \leqslant \| [S_\eta(\zeta_2)-S_\eta(\zeta_1)]\|[\| u_0\| + \|g(u)\| + \| F(0,u(h_1(0)))\|] + L_F(|\zeta_2-\zeta_1|) \\
&\qquad + \int_{\zeta_1}^{\zeta_2} \| T_\eta(\zeta_2-\varsigma)\|\| G(\varsigma,u(h_2(\varsigma)))\|d\varsigma,
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\zeta_1} \| T_\eta(\zeta_2-\varsigma)-T_\eta(\zeta_1-\varsigma)\| \| G(\varsigma,u(h_2(\varsigma)))\| d\varsigma,\\
&\leqslant \| [S_\eta(\zeta_2)-S_\eta(\zeta_1)]\| [\| u_0\| + \|g(u)\| + \| F(0,u(h_1(0)))\|] + L_F(|\zeta_2-\zeta_1|)\\
&+\widetilde{M_T}\int_{\zeta_1}^{\zeta_2} (\zeta_2-\varsigma)^{\eta-1}\| G(\varsigma,u(h_2(\varsigma)))\| d\varsigma,\\
&+\widetilde{M_T}\int_0^{\zeta_1} ((\zeta_2-\varsigma)^{\eta-1}-(\zeta_1-\varsigma)^{\eta-1})\| G(\varsigma,u(h_2(\varsigma)))\| d\varsigma. \qquad (3.22)
\end{aligned}
$$

Since $S_\eta(t)$ and $T_\eta(t)$ are compact, therefore $\| Qu(\zeta_2)-Qu(\zeta_1)\| \to 0$ as $\zeta_2 \to \zeta_1$. Therefore Q is equicontinuous on $[0,t_1]$.

For $\zeta_1,\zeta_2 \in (t_i,t_{i+1}]$ with $t_i < \zeta_1 < \zeta_2 \leqslant t_{i+1}$, where $i = 1,\cdots,\delta$,

$$
\begin{aligned}
&\|Qu(\zeta_2)-Qu(\zeta_1)\|\\
&\leqslant \| [S_\eta(\zeta_2-t_i)-S_\eta(\zeta_1-t_i)](u(t_i^-)+I_i(u(t_i^-))-F(t_i,u(h_1(t_i^+))))\|\\
&+\| F(\zeta_2,u(h_1(\zeta_2)))-F(\zeta_1,u(h_1(\zeta_1)))\|\\
&+\| \int_{t_i}^{\zeta_2} T_\eta(\zeta_2-\varsigma)G(\varsigma,u(h_2(\varsigma)))d\varsigma - \int_{t_i}^{\zeta_1} T_\eta(\zeta_1-\varsigma)G(\varsigma,u(h_2(\varsigma)))d\varsigma\|,\\
&\leqslant \| [S_\eta(\zeta_2-t_i)-S_\eta(\zeta_1-t_i)](u(t_i^-)+I_i(u(t_i^-))-F(t_i,u(h_1(t_i^+))))\|\\
&+L_F[|\zeta_2-\zeta_1| + \|u(h_1(\zeta_2))-u(h_1(\zeta_1))\|]\\
&+\int_{\zeta_1}^{\zeta_2} \| T_\eta(\zeta_2-\varsigma)\| \| G(\varsigma,u(h_2(\varsigma)))\| d\varsigma,\\
&+\int_0^{\zeta_1} \| T_\eta(\zeta_2-\varsigma)-T_\eta(\zeta_1-\varsigma)\| \| G(\varsigma,u(h_2(\varsigma)))\| d\varsigma,\\
&\leqslant \| [S_\eta(\zeta_2-t_i)-S_\eta(\zeta_1-t_i)](u(t_i^-)+I_i(u(t_i^-))-F(t_i,u(h_1(t_i^+))))\|\\
&+L_F[|\zeta_2-\zeta_1| + \|u(h_1(\zeta_2))-u(h_1(\zeta_1))\|]\\
&+\widetilde{M_T}\sup_{t\in[0,T]} \| G(t,u(h_2(t)))\| \frac{(\zeta_2-\zeta_1)^\eta}{\eta}\\
&+\widetilde{M_T}\int_0^{\zeta_1} ((\zeta_2-\varsigma)^{\eta-1}-(\zeta_1-\varsigma)^{\eta-1})\| G(\varsigma,u(h_2(\varsigma)))\| d\varsigma. \qquad (3.23)
\end{aligned}
$$

Since $S_\eta(t)$ and $T_\eta(t)$ are compact, therefore $\| Qu(\zeta_2)-Qu(\zeta_1)\| \to 0$ as $\zeta_2 \to \zeta_1$. Therefore Q is equicontinuous on $(t_i,t_{i+1}]$. Hence we conclude that $Q(B_r)$ is equicontinuous.

Step 4. Q maps B_r into a compact set in X.

For this, we decompose Q into Q_1 and Q_2, where

$$
Q_1(t) = \begin{cases}
F(t,u(h_1(t))) + \int_0^t T_\eta(t-\varsigma)G(\varsigma,u(h_2(u(\varsigma))))d\varsigma, & t\in[0,t_1],\\
F(t,u(h_1(t))) + \int_{t_1}^t T_\eta(t-\varsigma)G(\varsigma,u(h_2(u(\varsigma))))d\varsigma, & t\in(t_1,t_2],\\
\vdots \qquad \vdots \qquad \vdots & \\
F(t,u(h_1(t))) + \int_{t_\delta}^t T_\eta(t-\varsigma)G(\varsigma,u(h_2(u(\varsigma))))d\varsigma, & t\in(t_\delta,T],
\end{cases} \qquad (3.24)
$$

and

$$Q_2u(t)=\begin{cases} S_\eta(t)(u_0+g(u)), & t\in[0,t_1],\\ S_\eta(t-t_1)[u(t_1^-)+I_1(u(t_1^-))-F(t_1,u(h_1(t_1^+)))], & t\in(t_1,t_2]\\ \vdots \quad \vdots \quad \vdots & \\ S_\eta(t-t_\delta)[u(t_\delta^-)+I_\delta(u(t_\delta^-))-F(t_\delta,u(h_1(t_\delta^+)))], & t\in(t_\delta,T]. \end{cases} \tag{3.25}$$

We now prove that $\{Q_1u(t),\ u\in B_r\}$ is relatively compact on X for all $t\in\mathscr{I}$. It is obvious that the set $\{Q_1u(t),\ u\in B_r\}$ is relatively compact in X for $t=0$. Let $0<t\leqslant T$ be fixed and $0<\varepsilon<t$. For $u\in B_r$ and $t\in[0,t_1]$, we define an operator $Q_{1,\varepsilon}$ as

$$Q_{1,\varepsilon}u(t)=F(t,u(h_1(t)))+\int_0^{t-\varepsilon}T_\eta(t-\varsigma)G(\varsigma,u(h_2(u(\varsigma))))d\varsigma. \tag{3.26}$$

Similarly, for $t\in(t_i,t_{i+1}]$, $i=1,\cdots,\delta$. Let $t_i<t<s\leqslant t_{i+1}$ be fixed and ε be a real number satisfying $0<\varepsilon<t$. For $u\in B_r$, we define

$$Q_{1,\varepsilon}u(t)=F(t,u(h_1(t)))+\int_{t_i}^{t-\varepsilon}T_\eta(t-\varsigma)G(\varsigma,u(h_2(u(\varsigma))))d\varsigma. \tag{3.27}$$

By using the compactness of $T_\eta(t),t>0$ and completely continuity of F, we conclude that the set $\mathscr{U}_\varepsilon(t)=\{(Q_{1,\varepsilon}u)(t): u\in B_r\}$ is relatively compact in X for each ε, $0<\varepsilon<t$. Moreover, for every $u\in B_r$ and $t\in(t_i,t_{i+1}]$, $i=1,\cdots,\delta$, we have

$$\|\, Q_1u(t)-Q_{1,\varepsilon}u(t)\|\leqslant\int_{t-\varepsilon}^{t}\|\, T_\eta(t-\varsigma)\|\|\, G(\varsigma,u(h_2(\varsigma)))\|d\varsigma. \tag{3.28}$$

Therefore, taking $\varepsilon\to0$ we can see that there are relatively compact sets arbitrarily close to the set $\mathscr{U}(t)=\{Q_1u(t),\ u\in B_r\}$. Therefore, the set $\{Q_1u(t),\ u\in B_r\}$ is relatively compact in X. Hence, we conclude that Q_1 is compact for all $t\in\mathscr{I}$ by the Arzelá-Ascoli theorem.

Next we show that $\{Q_2u(t),\ u\in B_r\}$ is relatively compact in X for all $t\in\mathscr{I}$. For $t\in[0,t_1]$ we have $Q_2u(t)=S_\alpha(t)[u_0+g(u)-F(0,u(h_1(0)))]$, by the compactness of $S_\eta(t),t>0$, it follows that $\{Q_2u(t),\ u\in B_r\}$ is relatively compact subset of X for all $t\in[0,t_1]$. On the other hand, for $t\in(t_i,t_{i+1}]$, where $\delta\geqslant i\geqslant1$ and $u\in B_r$, there exist $\widehat{r}>0$ such that

$$\widehat{Q_2u(t)}_i\in\begin{cases} S_\eta(t-t_i)[u_n(t_i)+I_i(u_n(t_i))-F(t_i,u(h_1(t_i^+)))],\ t\in(t_i,t_{i+1}),\ u_n\in B_{\widehat{r}},\\ S_\eta(t_{i+1}-t_i)[u_n(t_i)+I_i(u_n(t_i))-F(t_i,u(h_1(t_i^+)))],\ t=t_{i+1},\ u_n\in B_{\widehat{r}},\\ u_n(t_i)+I_i(u_n(t_i))-F(t_i,u(h_1(t_i^+))),\ \ t=t_i,\quad u_n\in B_{\widehat{r}}, \end{cases} \tag{3.29}$$

where $B_{\widehat{r}}$ is an open ball of radius $\widehat{r}$. From $(A1)$ and $(A6)$-$(A8)$, it follows that $\widehat{Q_2u(t)}_i$ is relatively compact in X for all $t\in(t_i,t_{i+1}]$. Hence, by the assumptions $(B1)-(B4)$ and Arzela-Ascoli theorem, we conclude that Q_2 is compact for all $t\in\mathscr{I}$. Therefore $Q=Q_1+Q_2$ is compact.

Step 5. (A priori bounds) We prove that the set $E = \{u \in PC(\mathscr{I};X)$ such that $u = \lambda Qu$ for some $0 < \lambda < 1\}$ is bounded.

Let $u \in E$ with $u(t) = \lambda Qu(t)$ for some $0 < \lambda < 1$. Then for each $t \in [0,t_1]$,

$$\begin{aligned}\| u(t)\|_X &\leqslant \lambda[\widetilde{M_S}\| u_0\| + \widetilde{M_S}(\mathscr{D}_1\|u(t)\| + \mathscr{D}_2) + \widetilde{M_S}(L_1\| u(t)\| + L_2) + L_1\| u(t)\| + L_2 \\ &\quad + \widetilde{M_T}\int_0^t (t-\varsigma)^{\eta-1}\| G(\varsigma, u(h_2(\varsigma)))\| d\varsigma,] \\ &\leqslant \lambda[\widetilde{M_S}\| u_0\| + \widetilde{M_S}(\mathscr{D}_1\|u(t)\| + \mathscr{D}_2) + \widetilde{M_S}(L_1\| u(t)\| + L_2) + L_1\| u(t)\| + L_2 \\ &\quad + \widetilde{M_T}\int_0^t (t-\varsigma)^{\eta-1} m_G(\varsigma) W(\| u(\varsigma)\|)d\varsigma], \\ &\leqslant \lambda[\widetilde{M_S}\| u_0\| + \widetilde{M_S}(\mathscr{D}_1\|u(t)\| + \mathscr{D}_2) + L_1(\widetilde{M_S}+1)\| u(t)\| + L_2(1+\widetilde{M_S}) \\ &\quad + \frac{\widetilde{M_T}T^\eta}{\eta}\int_0^t m_G(\varsigma) W(\| u(\varsigma)\|)d\varsigma], \end{aligned} \tag{3.30}$$

Moreover, for $t \in (t_i, t_{i+1}]$, $i = 1,2,\cdots,\delta$,

$$\begin{aligned}\| u(t)\|_X &\leqslant \lambda[\|S_\eta(t-t_i)u(t_i^-)\| + \|S_\eta(t-t_i)I_i(u(t_i^-))\| + \|S_\eta(t-t_i)F(t_i, u(h_1(t_i^+))\| \\ &\quad + \|F(t, u(h_1(t)))\| + \int_{t_i}^t \|T_\eta(t-\varsigma)G(\varsigma, u(h_2(\varsigma)))d\varsigma\|] \\ &\leqslant \lambda[\widetilde{M_S}\sup_{\varsigma\in\mathscr{I}}\|u(\varsigma)\| + \widetilde{M_S}\Omega + \widetilde{M_S}(L_1\sup_{\varsigma\in\mathscr{I}}\|u(t)\| + L_2) + (L_1\sup_{\varsigma\in\mathscr{I}}\|u(t)\| + L_2) \\ &\quad + \frac{\widetilde{M_T}T^\eta}{\eta}\int_{t_i}^t m_G(\varsigma) W(\| u(\varsigma)\|)d\varsigma] \end{aligned} \tag{3.31}$$

Thus, for each $t \in \mathscr{I}$, we obtain

$$\begin{aligned}\| u(t)\|_X \leqslant M_* + \left[\widetilde{M_S}(\mathscr{D}_1+1) + (\widetilde{M_S}+1)L_1\right]\|u(t)\| \\ + \frac{\widetilde{M_T}T^\alpha}{\alpha}\int_0^t m_G(\varsigma) W(\| u(\varsigma)\|)d\varsigma,\end{aligned} \tag{3.32}$$

where $M_* = \max\{\widetilde{M_S}\|u_0\| + (\widetilde{M_S}+1)L_2 + \widetilde{M_S}\mathscr{D}_2, \widetilde{M_S}\Omega + (\widetilde{M_S}+1)L_2\}$. Therefore, for all $t \in \mathscr{I} = [0,T]$, by the Young inequality [[3], page 6], we get

$$\begin{aligned}\| u(t)\|_X &\leqslant \frac{M_*}{1-\overline{M}} + \frac{\widetilde{M_T}T^\eta}{\eta(1-\overline{M})}\int_0^t m_G(\varsigma) W(\| u(\varsigma)\|)d\varsigma, \\ &\leqslant \omega_1 + \frac{\widetilde{M_T}T^\eta}{\eta(1-\overline{M})}\int_0^t m_G(\varsigma) W(\| u(\varsigma)\|)d\varsigma,\end{aligned} \tag{3.33}$$

where $\omega_1 = \frac{M_*}{(1-\overline{M})}$ and $\overline{M} = \widetilde{M_S}(\mathscr{D}_1+1) + (\widetilde{M_S}+1)L_1$. Then for all $t \in \mathscr{I}$,

$$\| u(t)\| \leqslant \beta_\lambda(t) \triangleq \omega_1 + \frac{\widetilde{M_T}T^\eta}{\eta(1-\overline{M})}\int_0^t m_G(\varsigma) W(\| u(\varsigma)\|)d\varsigma.$$

Calculating $\beta_\lambda'(t)$ for $t \in \mathscr{I}$, we obtain

$$\beta_\lambda'(t) \leqslant \frac{\widetilde{M_T} T^\eta}{\eta(1-\overline{M})} m_G(t) W(\| u(t)\|).$$

Thus we have

$$\frac{d\beta_\lambda(t)}{W(\| \beta_\lambda(t)\|)} \leqslant \frac{d\beta_\lambda(t)}{W(\| u(t)\|)} \leqslant \frac{\widetilde{M_T} T^\eta}{\eta(1-\overline{M})} m_G(t)dt. \tag{3.34}$$

Since $W(\varsigma)$ is positive and non-decreasing. Integrating both sides, we get

$$\int_0^{\beta_\lambda(t)} \frac{d\varsigma}{W(\varsigma)} \leqslant \frac{\widetilde{M_T} T^\eta}{\eta(1-\overline{M})} \int_0^T m_G(s)ds < \int_{\omega_1}^\infty \frac{d\varsigma}{W(\varsigma)}, \tag{3.35}$$

where we have, $\beta_\lambda(0) = \omega_1$, $\beta_\lambda(t)$ is positive and non-decreasing. Hence, from the above inequality, we obtain that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded. This implies that set $\{u \in PC(\mathscr{I};X) : \; u = \lambda Qu, \; 0 < \lambda < 1\}$ is bounded in X. Hence by Schaefer's fixed point theorem, we get that Q has a fixed point on $\mathscr{I} = [0,T]$. This completes the proof of the theorem.

4. Application

We consider the following fractional order impulsive partial functional differential system of the form

$$\begin{aligned}
&\frac{\partial^\eta}{\partial t^\eta}[z(t,x) + \int_0^t b(t,\xi,x)[z(\sin t,\xi) + \frac{\partial}{\partial \xi} z(\sin t,\xi)]d\xi] \\
&= \frac{\partial^2}{\partial x^2}[z(t,x) + \int_0^t b(t,\xi,x)[z(\sin t,\xi) + \frac{\partial}{\partial \xi} z(\sin t,\xi)]d\xi] + \chi(t, \frac{\partial}{\partial x} z(\sin t,x)), \\
&\quad t \in [0,1], \;\; \pi \geqslant x \geqslant 0,
\end{aligned} \tag{4.1}$$

$$z(t,0) = z(t,\pi) = 0, \tag{4.2}$$

$$z(0,x) = z_0(x), \;\; \pi \geqslant x \geqslant 0, \tag{4.3}$$

$$\Delta z|_{t_i} = z(t_{t_i}^+) - z(t_{t_i}^-) = I_i(z(t_{t_i}^-)), \; i = 1, \cdots, \delta, \tag{4.4}$$

where $0 < \eta < 1$ and $0 < t_1 < t_2 < \cdots < t_\delta < 1$ and $b \in C([0,1] \times [0,\pi] \times [0,\pi], \mathbb{R})$ and $\chi \in C([0,1] \times \mathbb{R}, \mathbb{R})$ are continuous function. Take $X = L^2[0,\pi]$ and let an operator A such that

$$Af = f'' \tag{4.5}$$

with the domain

$$D(A) = H^2([0,\pi]) = \{f(\cdot) \in X : f', f'' \in X \text{ and } f(0) = f(\pi) = 0\}. \tag{4.6}$$

It implies that A generates a strongly continuous semigroup $T(\cdot)$ which is analytic and semi-adjoint. The operator $T(t)$ is given by

$$T(t)f = \sum_{m=1}^{\infty} e^{-m^2 t}(f, z_m)z_m.$$

Also, A has a discrete spectrum, the eigenvalues are $-m^2$, $m \in \mathbb{N}$ with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. We have that if $f \in D(A)$, then $Af = \sum_{m=1}^{\infty} m^2(f, z_m)z_m$ for all $f \in X$ and $t > 0$.

Now we assume following assumptions:

(i) $b : [0,1] \times [0,\pi] \times [0,\pi] \to \mathbb{R}$ is continuously differentiable with $b(t,\xi,0) = b(t,\xi,\pi) = 0$.

(ii) The function b is measurable and

$$\sup_{0 \leqslant t \leqslant 1} \int_0^{\pi}\int_0^{\pi} b^2(t,\xi,x)d\xi dx < \infty. \tag{4.7}$$

and function $\frac{\partial^2}{\partial x^2}$ is measurable and

$$K_1 = \sup_{0 \leqslant t \leqslant 1} [\int_0^{\pi}\int_0^{\pi} (\frac{\partial^2}{\partial x^2} b(t,\xi,x))^2 d\xi dx]^{1/2} < \infty \tag{4.8}$$

(iii) $\chi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with respect to the second argument and there exist positive constant a_0 such that

$$\| \chi(t,x_1) - \chi(t,x_2)\| \leqslant a_0 \| x_1 - x_2\| \tag{4.9}$$

for $t \in [0,1]$, x_1, $x_2 \in \mathbb{R}$.

(iv) $I_i \in C(X,X)$, $i = 1,2,\cdots,\delta$ such that

$$\| I_i(x)\| \leqslant \psi_i(\| x\|), \tag{4.10}$$

for $x \in X$, where $\psi_i \in ([0,1],\mathbb{R}_+)$ is nondecreasing function.

Let $h_1(t) = h_2(t) = \sin t$, , $F(t,z)(x) = \int_0^{\pi} b(t,\xi,x)[z(\xi),z'(\xi)]d\xi$, and $G(t,z)(x) = \chi(t,z'(\xi))$.

Therefore the equation (4.1)-(4.2) can be reformulated as

$$\frac{d^{\eta}}{dt^{\eta}}[u(t) + F(t,u(h_1(t)))] = A[u(t) + F(t,u(h_1(t)))] + G(t,u(h_2(t))),$$
$$0 \leqslant t \leqslant 1, \tag{4.11}$$
$$u(0) = u_0, \tag{4.12}$$
$$\Delta u|_{t_i} = I_i(u(t_{t_i}^-)),\ i = 1,2,\cdots,\delta. \tag{4.13}$$

It is not difficult to verify that F and G satisfy the condition $(A1)$ and $(A2)$ respectively, and from (ii) it is clear that $F(t,z)$ is bounded linear operator on $\mathbb{R}$. Thus from Theorem 2, the system (4.11)-(4.13) admits a mild solution $[0,\ T]$ as well as (4.1)-(4.2).

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