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EXISTENCE OF SOLUTIONS FOR A COUPLED SYSTEM OF THREE–POINT FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we establish the existence of a solution for a coupled system of three-point fractional order boundary value problems by applying the Schauder fixed point theorem in a Banach space under suitable conditions.

1. Introduction

This paper is concerned with establishing the existence of a solution to the coupled system of three-point fractional order boundary value problems,

$$D_{0^+}^{\alpha}u(t) = f_1\Big(t, v(t), D_{0^+}^{p_1}v(t), D_{0^+}^{p_2}v(t)\Big),\ t \in (0,1), \tag{1}$$

$$D_{0^+}^{\beta}v(t) = f_2\Big(t, u(t), D_{0^+}^{q_1}u(t), D_{0^+}^{q_2}u(t)\Big),\ t \in (0,1), \tag{2}$$

$$u(0) = u'(0) = 0,\ u'(1) - \xi u'(\eta) = 0, \tag{3}$$

$$v(0) = v'(0) = 0,\ v'(1) - \xi v'(\eta) = 0, \tag{4}$$

where $2 < \alpha, \beta \leqslant 3, \eta \in (0,1), \xi \in \mathbb{R},\ f_1, f_2 : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are given functions and $D_{0^+}^{\alpha}, D_{0^+}^{\beta}, D_{0^+}^{p_i}, D_{0^+}^{q_i},\ i = 1,2$ are the standard Riemann–Liouville fractional order derivatives.

The study of fractional order differential equations has emerged as an important area of mathematics. It has wide range of applications in various fields of science and engineering such as physics, mechanics, control systems, flow in porous media, electromagnetics and viscoelasticity. For some of the recent developments in fractional calculus, we refer to Miller and Ross [13], Podlubny [15], Kilbas, Srivasthava and Trujillo [7], Kilbas and Trujillo [8], Lakshmikantham and Vatsala [11] and the references therein.

Recently, much interest has been created in establishing the existence of solutions for two-point and multi-point fractional order boundary value problems. To mention the related papers along these lines, we refer to Bai and Lü [2], Kauffman and Mboumi

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[6], Benchohra, Henderson, Ntoyuas and Ouahab [3], Su and Zhang [19] for fractional order differential equations, Su [18], Khan, Rehman and Henderson [9], Prasad and Krushna [16] for systems of fractional order differential equations.

Ahmad and Nieto [1] obtained an existence result for a coupled system of nonlinear fractional order boundary value problem,

$$D^{\alpha}u(t) = f(t,v(t),D^{p}v(t)),\ t \in (0,1),$$
$$D^{\beta}v(t) = g(t,u(t),D^{q}u(t)),\ t \in (0,1),$$
$$u(0) = 0, u(1) = \gamma u(\eta),\ v(0) = 0, v(1) = \gamma v(\eta),$$

where D is the standard Riemann–Liouville fractional order derivative, $1 < \alpha, \beta < 2$, p, q, η, γ satisfy certain conditions, by applying the Schauder fixed point theorem.

In [10], the authors studied the existence and uniqueness of solutions to the boundary value problem,

$$D^{\alpha}u(t) = f(t,u(t),u'(t)),\ t \in (0,1),$$
$$u(0) = 0,\ D^{p}u(1) = \delta D^{p}u(\eta),$$

where D is the Caputo fractional order derivative, $1 < \alpha \leqslant 2,\ 0 < \delta < p < 1,\ 0 < \eta \leqslant 1$, by using some fixed point theorem.

We assume that the following conditions hold throughout the paper:

(A1) $f_1, f_2 : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous,

(A2) $\alpha - q_i \geqslant 1, \beta - p_i \geqslant 1$, for $i = 1,2$,

(A3) $\xi \eta^{\alpha-2} < 1,\ \ \xi \eta^{\beta-2} < 1,\ \eta \in (0,1)$ and $\xi \in \mathbb{R}$,

(A4) there exists a nonnegative function $l(t) \in C(0,1)$ such that
$|f_1(t,x,y,z)| \leqslant l(t) + \varepsilon_1|x|^{\theta_1} + \varepsilon_2|y|^{\theta_2} + \varepsilon_3|z|^{\theta_3}$,
where $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $0 < \theta_1, \theta_2, \theta_3 < 1$,

(A5) there exists a nonnegative function $l^*(t) \in C(0,1)$ such that
$|f_2(t,x,y,z)| \leqslant l^*(t) + \delta_1|x|^{\psi_1} + \delta_2|y|^{\psi_2} + \varepsilon_3|z|^{\psi_3}$,
where $\delta_1, \delta_2, \delta_3 > 0$ and $0 < \psi_1, \psi_2, \psi_3 < 1$.

1.1. Preliminaries

In this section, we present some definitions and lemmas that are useful in the proof of our main results.

DEFINITION 1. [15] The Riemann–Liouville fractional integral of order $p > 0$ of a function $f : [0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau)\, d\tau,$$

provided the right-hand side is defined.

DEFINITION 2. [15] The Riemann–Liouville fractional derivative of order $p > 0$ of a function $f : [0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{p} f(t) = \frac{1}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_0^t (t-\tau)^{k-p-1} f(\tau) d\tau, \ (k-1 \leqslant p < k),$$

provided the right-hand side is defined.

LEMMA 1. [15] *If* $f(t) \in L_1(0,T)$, *then the equation*

$$D_{0^+}^{\sigma_n} y(t) = f(t)$$

has the unique solution $y(t) \in L_1(0,T)$, *which satisfies the initial conditions*

$$\left[D_{0^+}^{\sigma_{k-1}} y(t)\right]_{t=0} = b_k, \ k = 1,2,\cdots,n,$$

is

$$y(t) = \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau) d\tau + \sum_{t=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1}.$$

The rest of the paper is organized as follows. In Section 2, the Green's function for the homogeneous fractional order boundary value problem is constructed and sufficient lemmas are estimated. In Section 3, the existence of a solution for a coupled system of fractional order boundary value problem (1)-(4) is established using the Schauder fixed point theorem. In Section 4, as an application, the results are demonstrated with an example.

2. Green's Function and Lemmas

In this section, the Green's function for the homogeneous fractional order boundary value problem is constructed and sufficient lemmas are estimated, which are needed to establish the main results.

Let

$$X = \left\{u : \ u, D_{0^+}^{q_1} u, D_{0^+}^{q_2} u \in C[0,1]\right\}$$

be the Banach space equipped with the norm,

$$\|u\|_X = \max_{t \in I} |u(t)| + \max_{t \in I} \left|D_{0^+}^{q_1} u(t)\right| + \max_{t \in I} \left|D_{0^+}^{q_2} u(t)\right|,$$

and

$$Y = \left\{v : v, D_{0^+}^{p_1} v, D_{0^+}^{p_2} v \in C[0,1]\right\}$$

be the Banach space equipped with the norm,

$$\|v\|_Y = \max_{t \in I} |v(t)| + \max_{t \in I} \left|D_{0^+}^{p_1} v(t)\right| + \max_{t \in I} \left|D_{0^+}^{p_2} v(t)\right|.$$

LEMMA 2. *Let* $d = (1-\xi\eta^{\alpha-2})\Gamma(\alpha) \neq 0$. *If* $h(t) \in C[0,1]$, *then the fractional order differential equation,*

$$D_{0+}^{\alpha}u(t) = h(t),\ t \in (0,1), \tag{5}$$

with (3) *has a unique solution,*

$$u(t) = \int_0^1 G_1(t,s)h(s)ds,$$

where $G_1(t,s)$ *is the Green's function for the boundary value problem* (5),(3) *and is given by*

$$G_1(t,s) = \begin{cases} \underset{t\in[0,\eta]}{G_1(t,s)} = \begin{cases} G_{11}(t,s),\ 0 < s < t \leqslant \eta < 1, \\ G_{12}(t,s),\ 0 \leqslant t < s < \eta < 1, \\ G_{13}(t,s),\ 0 \leqslant t < \eta < s < 1, \end{cases} \\ \\ \underset{t\in[\eta,1]}{G_1(t,s)} = \begin{cases} G_{14}(t,s),\ 0 < \eta < s < t \leqslant 1, \\ G_{15}(t,s),\ 0 < \eta \leqslant t < s < 1, \\ G_{16}(t,s),\ 0 \leqslant s < \eta < t < 1, \end{cases} \end{cases} \tag{6}$$

$$\begin{aligned}
G_{11}(t,s) =& \frac{1}{d}\Big[(t-s)^{\alpha-1}(1-\xi\eta^{\alpha-2}) - t^{\alpha-1}(1-s)^{\alpha-2} + \xi t^{\alpha-1}(\eta-s)^{\alpha-2}\Big], \\
G_{12}(t,s) =& \frac{1}{d}\Big[\xi t^{\alpha-1}(\eta-s)^{\alpha-2} - t^{\alpha-1}(1-s)^{\alpha-2}\Big], \\
G_{13}(t,s) =& \frac{1}{d}\Big[-t^{\alpha-1}(1-s)^{\alpha-2}\Big], \\
G_{14}(t,s) =& \frac{1}{d}\Big[(t-s)^{\alpha-1}(1-\xi\eta^{\alpha-2}) - t^{\alpha-1}(1-s)^{\alpha-2} + \xi t^{\alpha-1}(\eta-s)^{\alpha-2}\Big], \\
G_{15}(t,s) =& \frac{1}{d}\Big[-t^{\alpha-1}(1-s)^{\alpha-2}\Big], \\
G_{16}(t,s) =& \frac{1}{d}\Big[(t-s)^{\alpha-1}(1-\xi\eta^{\alpha-2}) - t^{\alpha-1}(1-s)^{\alpha-2}\Big].
\end{aligned}$$

Proof. Let $u(t) \in C[0,1]$ be the solution of fractional order boundary value problem (5), (3) and is uniquely expressed as

$$\begin{aligned}
u(t) =& \frac{1}{d}\Bigg\{\int_0^t \Big[(1-\xi\eta^{\alpha-2})(t-s)^{\alpha-1} + \Big(\xi(\eta-s)^{\alpha-2} - (1-s)^{\alpha-2}\Big)t^{\alpha-1}\Big]h(s)ds \\
&+ \int_t^{\eta} \Big[\xi t^{\alpha-1}(\eta-s)^{\alpha-2} - t^{\alpha-1}(1-s)^{\alpha-2}\Big]h(s)ds \\
&- \int_{\eta}^{1} t^{\alpha-1}(1-s)^{\alpha-2}h(s)ds\Bigg\} \\
=& \int_0^1 G_1(t,s)h(s)ds.
\end{aligned}$$

In a similar manner, we construct the Green's function $G_2(t,s)$ for the homogeneous fractional order boundary value problem corresponding to (2) and (4).

Let $T : X \times Y \to X \times Y$ be the operator defined as

$$T(u,v)(t) = \Big(T_1 v(t), T_2 u(t)\Big), \tag{7}$$

where

$$T_1 v(t) = \int_0^1 G_1(t,s) f_1\Big(s, v(s), D_{0^+}^{p_1} v(s), D_{0^+}^{p_2} v(s)\Big) ds,$$

$$T_2 u(t) = \int_0^1 G_2(t,s) f_2\Big(s, u(s), D_{0^+}^{q_1} u(s), D_{0^+}^{q_2} u(s)\Big) ds.$$

In view of the continuity of G_1, G_2, f_1 and f_2, it follows that T is continuous.

Let $B = X \times Y$ be the Banach space defined as

$$B = \Big\{(u,v) : \|(u,v)\| \leqslant R,\ t \in I\Big\}, \tag{8}$$

where

$$\|(u,v)\| = \max\Big\{\|u\|_X, \|v\|_Y\Big\},$$

and

$$R \geqslant \max\Bigg\{\Big(4\Lambda_1\varepsilon_1\Big)^{\frac{1}{1-\theta_1}}, \Big(4\Lambda_1\varepsilon_2\Big)^{\frac{1}{1-\theta_2}}, \Big(4\Lambda_1\varepsilon_3\Big)^{\frac{1}{1-\theta_3}}, 4\gamma,$$
$$\Big(4\Lambda_2\delta_1\Big)^{\frac{1}{1-\psi_1}}, \Big(4\Lambda_2\delta_2\Big)^{\frac{1}{1-\psi_2}}, \Big(4\Lambda_2\delta_3\Big)^{\frac{1}{1-\psi_3}}, 4\nu\Bigg\}.$$

LEMMA 3. [1] *Assume that* $(A1)$ *is satisfied. Then* $(u,v) \in B$ *is a solution of the fractional order boundary value problem* (1)-(4) *if and only if* $(u,v) \in B$ *is a solution of* (7).

3. Existence of a solution

In this section, we establish the existence of a solution for a coupled system of nonlinear fractional order boundary value problem (1)-(4) by applying the Schauder fixed point theorem.

For our convenience, we denote

$$\Lambda_1 = \frac{1 - \xi\eta^{\alpha-2} + \alpha\xi(\eta^{\alpha-2} - \eta^{\alpha-1})}{(\alpha-1)(1-\xi\eta^{\alpha-2})\Gamma(\alpha+1)} + \frac{(\alpha-1)(2-\xi\eta^{\alpha-2}) + \xi\eta^{\alpha-1}(\alpha-q_1)}{(\alpha-1)(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_1+1)}$$
$$+ \frac{(\alpha-1)(2-\xi\eta^{\alpha-2}) + \xi\eta^{\alpha-1}(\alpha-q_2)}{(\alpha-1)(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_2+1)},$$

$$\Lambda_2 = \frac{1 - \xi\eta^{\beta-2} + \alpha\xi(\eta^{\beta-2} - \eta^{\alpha-1})}{(\beta-1)(1-\xi\eta^{\beta-2})\Gamma(\beta+1)} + \frac{(\beta-1)(2-\xi\eta^{\beta-2}) + \xi\eta^{\beta-1}(\beta-p_1)}{(\beta-1)(1-\xi\eta^{\beta-2})\Gamma(\beta-p_1+1)}$$

$$\times \frac{(\beta-1)(2-\xi\eta^{\beta-2})+\xi\eta^{\beta-1}(\beta-p_2)}{(\beta-1)(1-\xi\eta^{\beta-2})\Gamma(\beta-p_2+1)},$$

$$\begin{aligned}\gamma = &\max_{t\in[0,1]} \int_0^1 \left|G_1(t,s)l(s)ds\right| \\ &+ \frac{2-\xi\eta^{\alpha-2}}{1-\xi\eta^{\alpha-2}}\left\{\int_0^1 \left[\frac{(1-s)^{\alpha-q_1-1}}{\Gamma(\alpha-q_1)}+\frac{(1-s)^{\alpha-q_2-1}}{\Gamma(\alpha-q_2)}\right]l(s)ds\right\} \\ &+ \frac{1}{(1-\xi\eta^{\alpha-2})}\left[\frac{1}{\Gamma(\alpha-q_1)}+\frac{1}{\Gamma(\alpha-q_2)}\right]\int_0^\eta (\eta-s)^{\alpha-2}l(s)ds,\end{aligned}$$

and

$$\begin{aligned}\nu = &\max_{t\in[0,1]} \int_0^1 \left|G_2(t,s)l^*(s)ds\right| \\ &+ \frac{2-\xi\eta^{\beta-2}}{1-\xi\eta^{\beta-2}}\left\{\int_0^1 \frac{(1-s)^{\beta-p_1-1}}{\Gamma(\beta-p_1)}l^*(s)ds+\int_0^1 \frac{(1-s)^{\beta-p_2-1}}{\Gamma(\beta-p_2)}l^*(s)ds\right\} \\ &+ \frac{1}{(1-\xi\eta^{\beta-2})}\left[\frac{1}{\Gamma(\beta-p_1)}+\frac{1}{\Gamma(\beta-p_2)}\right]\int_0^\eta (\eta-s)^{\beta-2}l^*(s)ds.\end{aligned}$$

To establish the existence of a solution to the fractional order boundary value problem (1)-(4) by utilizing the Schauder fixed point theorem.

LEMMA 4. (Schauder fixed point theorem) *Let E be a Banach space and Ω be any nonempty convex and closed subset of E. If M is a continuous mapping of Ω into itself and $M\Omega$ is relatively compact, then the mapping M has at least one fixed point.*

THEOREM 1. *Assume that the conditions $(A1)$-$(A5)$ hold. Then there exists a solution for the coupled system of fractional order boundary value problem* (1)-(4).

Proof. Using the assumptions $(A1)$-$(A5)$ together with the results

$$D_{0^+}^{q_1} t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-q_1)} t^{\alpha-q_1-1} \text{ and } D_{0^+}^{q_2} t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-q_2)} t^{\alpha-q_2-1}.$$

Firstly, we prove that $T: B \to B$, where B is given in (8). We have,

$$\begin{aligned}\left|T_1 v(t)\right| = &\left|\int_0^1 G_1(t,s) f_1\left(s, v(s), D_{0^+}^{p_1} v(s), D_{0^+}^{p_2} v(s)\right) ds\right| \\ \leqslant &\int_0^1 \left|G_1(t,s)l(s)\right|ds + \sum_{i=1}^{3}\left(\varepsilon_i |R|^{\theta_i}\right)\int_0^1 \left|G_1(t,s)\right|ds \\ \leqslant &\int_0^1 \left|G_1(t,s)l(s)\right|ds + \sum_{i=1}^{3}\left(\varepsilon_i |R|^{\theta_i}\right)\left[-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.\end{aligned}$$

$$+\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{(1-\xi\eta^{\alpha-2})}ds - \frac{\xi t^{\alpha-1}}{(1-\xi\eta^{\alpha-2})}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha)}ds\Bigg]$$
$$\leqslant \int_0^1 \Big|G_1(t,s)l(s)\Big|ds + \sum_{i=1}^{3}\Big(\varepsilon_i|R|^{\theta_i}\Big)\left[\frac{1-\xi\eta^{\alpha-2}+\alpha\xi(\eta^{\alpha-2}-\eta^{\alpha-1})}{(\alpha-1)\Gamma(\alpha+1)(1-\xi\eta^{\alpha-2})}\right],$$

and

$$\begin{aligned}
\Big|D_{0^+}^{q_1}T_1v(t)\Big| &= \Bigg|D_{0^+}^{q_1}I_{0^+}^{\alpha}f_1\Big(t,v(t),D_{0^+}^{p_1}v(t),D_{0^+}^{p_2}v(t)\Big)\\
&\quad - \frac{1}{(1-\xi\eta^{\alpha-2})}\Big[I_{0^+}^{\alpha}f_1\Big(1,v(1),D_{0^+}^{p_1}v(1),D_{0^+}^{p_2}v(1)\Big)\\
&\quad - \xi I_{0^+}^{\alpha}f_1\big(\eta,v(\eta),D_{0^+}^{p_1}v(\eta),D_{0^+}^{p_2}v(\eta)\big)\Big]D_{0^+}^{q_1}t^{\alpha-1}\Bigg|\\
&= \Bigg|I_{0^+}^{\alpha-q_1}f_1\Big(t,v(t),D_{0^+}^{p_1}v(t),D_{0^+}^{p_2}v(t)\Big)\\
&\quad - \frac{1}{(1-\xi\eta^{\alpha-2})}\Bigg[I_{0^+}^{\alpha}f_1\Big(1,v(1),D_{0^+}^{p_1}v(1),D_{0^+}^{p_2}v(1)\Big)\\
&\quad - \xi I_{0^+}^{\alpha}f_1\Big(\eta,v(\eta),D_{0^+}^{p_1}v(\eta),D_{0^+}^{p_2}v(\eta)\Big)\Bigg]\frac{\Gamma\alpha}{\Gamma(\alpha-q_1)}t^{\alpha-q_1-1}\Bigg|\\
&\leqslant \frac{1}{\Gamma(\alpha-q_1)(1-\xi\eta^{\alpha-2})}\Bigg[(2-\xi\eta^{\alpha-2})\int_0^1(1-s)^{\alpha-q_1-1}l(s)ds\\
&\quad + \xi\int_0^\eta(\eta-s)^{\alpha-2}l(s)ds\Bigg] + \Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big)\\
&\quad \times\left[\frac{(\alpha-1)(2-\xi\eta^{\alpha-2})+\xi\eta^{\alpha-1}(\alpha-q_1)}{(\alpha-1)(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_1+1)}\right]
\end{aligned}$$

and

$$\begin{aligned}
\Big|D_{0^+}^{q_2}T_1v(t)\Big| &= \Bigg|D_{0^+}^{q_2}I_{0^+}^{\alpha}f_1\Big(t,v(t),D_{0^+}^{p_1}v(t),D_{0^+}^{p_2}v(t)\Big)\\
&\quad - \frac{1}{(1-\xi\eta^{\alpha-2})}\Big[I_{0^+}^{\alpha}f_1\Big(1,v(1),D_{0^+}^{p_1}v(1),D_{0^+}^{p_2}v(1)\Big)\\
&\quad - \xi I_{0^+}^{\alpha}f_1\Big(\eta,v(\eta),D_{0^+}^{p_1}v(\eta),D_{0^+}^{p_2}v(\eta)\Big)D_{0^+}^{q_2}t^{\alpha-1}\Big]\Bigg|\\
&= \Big|I_{0^+}^{\alpha-q_2}f_1\Big(t,v(t),D_{0^+}^{p_1}v(t),D_{0^+}^{p-2}v(t)\Big)\\
&\quad - \frac{1}{(1-\xi\eta^{\alpha-2})}\Big[I_{0^+}^{\alpha}f_1\Big(1,v(1),D_{0^+}^{p_1}v(1),D_{0^+}^{p_2}v(1)\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\xi I_{0^+}^{\alpha} f_1\Big(\eta, v(\eta), D_{0^+}^{p_1} v(\eta), D_{0^+}^{p_2} v(\eta)\Big)\Big] \frac{\Gamma\alpha}{\Gamma(\alpha-q_2)} t^{\alpha-q_2-1}\Big| \\
\leqslant{}& \frac{1}{\Gamma(\alpha-q_2)}\Bigg[\int_0^t (t-s)^{\alpha-q_2-1} l(s)ds + \Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big) \\
&\times \int_0^t (t-s)^{\alpha-q_2-1} ds + \frac{1}{(1-\xi\eta^{\alpha-2})}\Bigg\{\int_0^1 (1-s)^{\alpha-2} l(s)ds \\
&+\Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big)\int_0^1 (1-s)^{\alpha-2} ds \\
&+\xi\int_0^\eta (\eta-s)^{\alpha-2} l(s)ds + \xi\Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big) \\
&\times\int_0^1 (1-s)^{\alpha-2} ds + \xi\int_0^\eta (\eta-s)^{\alpha-2} l(s)ds \\
&+\xi\Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big)\int_0^\eta (\eta-s)^{\alpha-2} ds\Bigg\}\Bigg] \\
\leqslant{}& \frac{1}{\Gamma(\alpha-q_2)(1-\xi\eta^{\alpha-2})}\Bigg[(2-\xi\eta^{\alpha-2})\int_0^1 (1-s)^{\alpha-q_2-1} l(s)ds \\
&+\xi\int_0^\eta (\eta-s)^{\alpha-2} l(s)ds + \Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big) \\
&\times\Bigg\{(2-\xi\eta^{\alpha-2})\int_0^1 (1-s)^{\alpha-q_2-1} ds + \xi\int_0^\eta (\eta-s)^{\alpha-2} ds\Bigg\}\Bigg] \\
\leqslant{}& \frac{1}{\Gamma(\alpha-q_2)(1-\xi\eta^{\alpha-2})}\Bigg[(2-\xi\eta^{\alpha-2})\int_0^1 (1-s)^{\alpha-q_2-1} l(s)ds \\
&+\xi\int_0^\eta (\eta-s)^{\alpha-2} l(s)ds\Bigg] + \Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big) \\
&\times\left[\frac{(\alpha-1)(2-\xi\eta^{\alpha-2})+\xi\eta^{\alpha-1}(\alpha-q_2)}{(\alpha-1)(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_2+1)}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
||T_1 v(t)||_X ={}& \max_{t\in[0,1]} |T_1 v(t)| + \max_{t\in[0,1]} \left|D_{0^+}^{q_1} T_1 v(t)\right| + \max_{t\in[0,1]} \left|D_{0^+}^{q_2} T_2 v(t)\right| \\
\leqslant{}& \int_0^1 \Big|G_1(t,s) l(s)ds\Big| + \Big(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\Big) \\
&\times\left[\frac{1-\xi\eta^{\alpha-2}+\alpha\xi(\eta^{\alpha-2}-\eta^{\alpha-1})}{(\alpha-1)\Gamma(\alpha+1)(1-\xi\eta^{\alpha-2})}\right] \\
&+\frac{1}{\Gamma(\alpha-q_1)(1-\xi\eta^{\alpha-2})}\Bigg[(2-\xi\eta^{\alpha-2})\int_0^1 (1-s)^{\alpha-q_1-1} l(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\xi\int_0^\eta(\eta-s)^{\alpha-2}l(s)ds\Bigg]+\left(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\right)\\
&\quad\times\left[\frac{(\alpha-1)(2-\xi\eta^{\alpha-2})+\xi\eta^{\alpha-1}(\alpha-q_1)}{(\alpha-1)(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_1+1)}\right]\\
&+\frac{1}{\Gamma(\alpha-q_2)(1-\xi\eta^{\alpha-2})}\Bigg[(2-\xi\eta^{\alpha-2})\int_0^1(1-s)^{\alpha-q_2-1}l(s)ds\\
&\quad+\xi\int_0^\eta(\eta-s)^{\alpha-2}l(s)ds\Bigg]+\left(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\right)\\
&\quad\quad\times\left[\frac{(\alpha-1)(2-\xi\eta^{\alpha-2})+\xi\eta^{\alpha-1}(\alpha-q_2)}{(\alpha-1)(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_2+1)}\right]\\
\leqslant{}&\gamma+\left(\varepsilon_1|R|^{\theta_1}+\varepsilon_2|R|^{\theta_2}+\varepsilon_3|R|^{\theta_3}\right)\Lambda_1\\
\leqslant{}&\frac{R}{4}+\frac{R}{4}+\frac{R}{4}+\frac{R}{4}\\
={}&R.
\end{aligned}
$$

Similarly, it can be shown that

$$\|T_2u(t)\|_Y\leqslant\nu+\left(\delta_1|R|^{\psi_1}+\delta_2|R|^{\psi_2}+\delta_3|R|^{\psi_3}\right)\Lambda_2\leqslant R.$$

Hence, we conclude that

$$||T(u,v)||_{X\times Y}\leqslant R.$$

Since $T_1v(t),\ T_2u(t),\ D_{0^+}^{q_1}T_1v(t),\ \ D_{0^+}^{p_1}T_2u(t),\ D_{0^+}^{q_2}T_1v(t)$ and $D_{0^+}^{p_2}T_2u(t)$ are continuous on $[0,1]$. Therefore, $T:B\to B$.

Now, we show that T is a completely continuous operator. For this, we let

$$
\begin{aligned}
M&=\max_{t\in[0,1]}\left|f_1(t,v(t),D_{0^+}^{p_1}v(t),D_{0^+}^{p_2}v(t))\right|,\\
N&=\max_{t\in[0,1]}\left|f_2(t,u(t),D_{0^+}^{q_1}u(t),D_{0^+}^{q_2}u(t))\right|.
\end{aligned}
$$

For $(u,v)\in B,\ t,\tau\in[0,1],\ t<\tau$, we have

$$
\begin{aligned}
|T_1v(t)-T_1v(\tau)|=&\left|\int_0^1\Big(G_1(t,s)-G_1(\tau,s)\Big)f_1\Big(s,v(s),D_{0^+}^{p_1}v(s),D_{0^+}^{p_2}v(s)\Big)\right|\\
\leqslant{}&\frac{M}{\Gamma\alpha(1-\xi\eta^{\alpha-2})}\Bigg[\int_0^t[(\tau-s)^{\alpha-1}-(t-s)^{\alpha-1}](1-\xi\eta^{\alpha-2})\\
&+(\tau^{\alpha-1}-t^{\alpha-1})(1-s)^{\alpha-2}+\xi(\tau^{\alpha-1}-t^{\alpha-1})(\eta-s)^{\alpha-2}\Bigg]ds\\
&+\int_t^\tau(\tau^{\alpha-1}-t^{\alpha-1})\Big[(1-s)^{\alpha-2}+\xi(\eta-s)^{\alpha-2}\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\eta}^{1}(\tau^{\alpha-1}-t^{\alpha-1})(1-s)^{\alpha-2}ds\Bigg]\\
&\leqslant M\left[\frac{\tau^{\alpha}-t^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\tau^{\alpha-1}-t^{\alpha-1})(1-\xi\eta^{\alpha-1})}{(\alpha-1)\Gamma(\alpha)(1-\xi\eta^{\alpha-2})}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|D_{0^+}^{q_1}T_1v(t)-D_{0^+}^{q_1}T_1v(\tau)\right|\\
&\quad=\Bigg|I_{0^+}^{\alpha-q_1}f_1\Big(t,v(t),D_{0^+}^{p_1}v(t),D_{0^+}^{p_2}v(t)\Big)\\
&\qquad-\frac{\Gamma\alpha}{\Gamma(\alpha-q_1)(1-\xi\eta^{\alpha-2})}\Big\{I_{0^+}^{\alpha}f_1\Big(1,v(1),D_{0^+}^{p_1}v(1),D_{0^+}^{p_2}v(1)\Big)\\
&\qquad\quad-\xi I_{0^+}^{\alpha}f_1\Big(\eta,v(\eta),D_{0^+}^{p_1}v(\eta),D_{0^+}^{p_2}v(\eta)\Big)\Big\}t^{\alpha-q_1-1}\\
&\qquad-I_{0^+}^{\alpha-q_1}f_1\Big(\tau,v(\tau),D_{0^+}^{p_1}v(\tau),D_{0^+}^{p_2}v(\tau)\Big)\\
&\qquad+\frac{\Gamma\alpha}{\Gamma(\alpha-q_1)(1-\xi\eta^{\alpha-2})}\Big\{I_{0^+}^{\alpha}f_1\Big(1,v(1),D_{0^+}^{p_1}v(1),D_{0^+}^{p_2}v(1)\Big)\\
&\qquad\quad-\xi I_{0^+}^{\alpha}f_1\Big(\eta,v(\eta),D_{0^+}^{p_1}v(\eta),D_{0^+}^{p_2}v(\eta)\Big)\Big\}\tau^{\alpha-q_1-1}\Bigg|\\
&\quad\leqslant\frac{1}{\Gamma(\alpha-q_1)}\Bigg[\Bigg|\int_0^t(t-s)^{\alpha-q_1-1}f_1\Big(s,v(s),D_{0^+}^{p_1}v(s),D_{0^+}^{p_2}v(s)\Big)ds\\
&\qquad\quad-\int_0^{\tau}(\tau-s)^{\alpha-q_1-1}f_1\Big(s,v(s),D_{0^+}^{p_1}v(s),D_{0^+}^{p_2}v(s)\Big)ds\Bigg|\Bigg]\\
&\qquad+\frac{M(\tau^{\alpha-q_1-1}-t^{\alpha-q_1-1})}{\Gamma(\alpha-q_1)(1-\xi\eta^{\alpha-2})}\left|\left\{\int_0^1(1-s)^{\alpha-1}ds-\xi\int_0^{\eta}(\eta-s)^{\alpha-1}ds\right\}\right|\\
&\quad\leqslant\left[\frac{M(\tau^{\alpha-q_1}-t^{\alpha-q_1})}{\Gamma(\alpha-q_1+1)}+\frac{M(1-\xi\eta^{\alpha})(\tau^{\alpha-q_1-1}-t^{\alpha-q_1-1})}{(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_1)\alpha}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|D_{0^+}^{q_2}T_1v(t)-D_{0^+}^{q_2}T_1v(\tau)\right|\\
&\quad=\Bigg|I_{0^+}^{\alpha-q_2}f_1\Big(t,v(t),D_{0^+}^{p_1}v(t),D_{0^+}^{p_2}v(t)\Big)\\
&\qquad-\frac{\Gamma\alpha}{\Gamma(\alpha-q_2)(1-\xi\eta^{\alpha-2})}\Big\{I_{0^+}^{\alpha}f_1\Big(1,v(1),D_{0^+}^{p_1}v(1),D_{0^+}^{p_2}v(1)\Big)\\
&\qquad-\xi I_{0^+}^{\alpha}f_1\Big(\eta,v(\eta),D_{0^+}^{p_1}v(\eta),D_{0^+}^{p_2}v(\eta)\Big)\Big\}t^{\alpha-q_2-1}
\end{aligned}
$$

$$
\begin{aligned}
&- I_{0^+}^{\alpha-q_2} f_1\Big(\tau, v(\tau), D_{0^+}^{p_1} v(\tau), D_{0^+}^{p_2} v(\tau)\Big) \\
&+ \frac{\Gamma\alpha}{\Gamma(\alpha-q_2)(1-\xi\eta^{\alpha-2})}\Big\{ I_{0^+}^{\alpha} f_1\Big(1, v(1), D_{0^+}^{p_1} v(1), D_{0^+}^{p_2} v(1)\Big) \\
&\qquad - \xi I_{0^+}^{\alpha} f_1\Big(\eta, v(\eta), D_{0^+}^{p_1} v(\eta), D_{0^+}^{p_2} v(\eta)\Big)\Big\}\tau^{\alpha-q_2-1}\Bigg| \\
\leqslant {} & \frac{1}{\Gamma(\alpha-q_2)}\Bigg[\Bigg|\int_0^t (t-s)^{\alpha-q_2-1} f_1\Big(s, v(s), D_{0^+}^{p_1} v(s), D_{0^+}^{p_2} v(s)\Big)ds \\
&\qquad - \int_0^\tau (\tau-s)^{\alpha-q_2-1} f_1\Big(s, v(s), D_{0^+}^{p_1} v(s), D_{0^+}^{p_2} v(s)\Big)ds\Bigg|\Bigg] \\
&+ \frac{M}{\Gamma(\alpha-q_2)(1-\xi\eta^{\alpha-2})}\Bigg|\Big(\tau^{\alpha-q_2-1} - t^{\alpha-q_2-1}\Big) \\
&\qquad \times\Big\{\int_0^1 (1-s)^{\alpha-1} ds - \xi\int_0^\eta (\eta-s)^{\alpha-1} ds\Big\}\Bigg| \\
\leqslant {} & \frac{1}{\Gamma(\alpha-q_2)}\Bigg[\Bigg|\int_0^t (t-s)^{\alpha-q_2-1} f_1\Big(s, v(s), D_{0^+}^{p_1} v(s), D_{0^+}^{p_2} v(s)\Big)ds \\
&- \int_0^\tau (\tau-s)^{\alpha-q_2-1} f_1\Big(s, v(s), D_{0^+}^{p_1} v(s), D_{0^+}^{p_2} v(s)\Big)ds\Bigg|\Bigg] \\
&+ \frac{M(1-\xi\eta^{\alpha})}{\Gamma(\alpha-q_2)(1-\xi\eta^{\alpha-2})\alpha}\Big(\tau^{\alpha-q_2-1} - t^{\alpha-q_2-1}\Big) \\
\leqslant {} & \frac{M}{\Gamma(\alpha-q_2)}\Bigg[\int_0^\tau (t-s)^{\alpha-q_2-1} - (\tau-s)^{\alpha-q_2-1}) ds \\
&+ \int_t^\tau (\tau-s)^{\alpha-q_2-1} ds\Bigg] + \frac{M(1-\xi\eta^{\alpha})(\tau^{\alpha-q_2-1} - t^{\alpha-q_2-1})}{(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_2)\alpha} \\
\leqslant {} & \Bigg[\frac{M(\tau^{\alpha-q_2} - t^{\alpha-q_2})}{\Gamma(\alpha-q_2+1)} + \frac{M(1-\xi\eta^{\alpha})(\tau^{\alpha-q_2-1} - t^{\alpha-q_2-1})}{(1-\xi\eta^{\alpha-2})\Gamma(\alpha-q_2)\alpha}\Bigg].
\end{aligned}
$$

Analogously, it can be proved that

$$
\Big|T_2u(t) - T_2u(\tau)\Big| \leqslant N\Bigg[\frac{\tau^{\beta} - t^{\beta}}{\Gamma(\beta+1)} + \frac{(\tau^{\beta-1} - t^{\beta-1})(1-\xi\eta^{\beta-1})}{(\beta-1)\Gamma(\beta)(1-\xi\eta^{\beta-2})}\Bigg],
$$

$$
\begin{aligned}
&\Big|D_{0^+}^{p_1} T_2u(t) - D_{0^+}^{p_1} T_2u(\tau)\Big| \\
&\qquad\qquad \leqslant N\Bigg[\frac{(\tau^{\beta-p_1} - t^{\beta-p_1})}{\Gamma(\beta-p_1+1)} + \frac{(1-\xi\eta^{\beta})(\tau^{\beta-p_1-1} - t^{\beta-p_1-1})}{(1-\xi\eta^{\beta-2})\Gamma(\beta-p_1)\beta}\Bigg]
\end{aligned}
$$

and

$$\left|D_{0^+}^{p_2} T_2 u(t) - D_{0^+}^{p_2} T_2 u(\tau)\right| \leqslant N\left[\frac{(\tau^{\beta-p_2} - t^{\beta-p_2})}{\Gamma(\beta-p_2+1)} + \frac{(1-\xi\eta^{\beta})(\tau^{\beta-p_2-1} - t^{\beta-p_2-1})}{(1-\xi\eta^{\beta-2})\Gamma(\beta-p_2)\beta}\right].$$

Therefore, it follows from the above estimates that T_B is an equicontinuous. Also it is uniformly bounded as $T_B \subset B$. Thus, we conclude that T is a completely continuous operator. Hence by the Schauder fixed point theorem, there exists a solution to the coupled system of fractional order boundary value problem (1)-(4). This completes the proof.

4. An example

In this section, as an application, we demonstrate our results with an example. Consider the system of fractional order differential equations,

$$\left. D_{0^+}^{\frac{23}{9}} u(t) = \frac{1}{\sqrt{t(1-t^2)}}\left[\left(D_{0^+}^{\frac{3}{2}} v(t)\right)^{\theta_3} + \left(D_{0^+}^{\frac{1}{5}} v(t)\right)^{\theta_2} + \left(v(t)\right)^{\theta_1}\right] + \frac{99}{100}, \quad t \in (0,1), \right\} \tag{9}$$

$$\left. D_{0^+}^{\frac{16}{7}} v(t) = \frac{1}{\sqrt{t(1-t^2)}}\left[\left(D_{0^+}^{\frac{4}{3}} u(t)\right)^{\psi_3} + \left(D_{0^+}^{\frac{1}{7}} u(t)\right)^{\psi_2} + \left(u(t)\right)^{\psi_1}\right] + \frac{1}{2}, \quad t \in (0,1), \right\} \tag{10}$$

satisfying the three-point boundary conditions,

$$u(0) = u'(0) = 0,\ u'(1) - \frac{7}{9}u'\left(\frac{5}{7}\right) = 0, \tag{11}$$

$$v(0) = v'(0) = 0,\ v'(1) - \frac{7}{9}v'\left(\frac{5}{7}\right) = 0, \tag{12}$$

where $0 < \theta_i,\ \psi_i < 1,\ i = 1,2,3$. By direct calculations, one can determine

$$\Lambda_1 = 29.285308, \Lambda_2 = 15.356694, \gamma = 0.475942, \nu = 0.811456$$

and

$$f_1(t,x,y,z) = \frac{1}{\sqrt{t(1-t^2)}}\left[\left(x(t)\right)^{\theta_1} + \left(y(t)\right)^{\theta_2} + \left(z(t)\right)^{\theta_3}\right] + \frac{99}{100},$$

$$f_2(t,x,y,z) = \frac{1}{\sqrt{t(1-t^2)}}\left[\left(x(t)\right)^{\psi_1} + \left(y(t)\right)^{\psi_2} + \left(z(t)\right)^{\psi_3}\right] + \frac{1}{2},$$

for $t \in (0,1)$, we have

$$|f_1(t,x,y,z)| \leqslant l + \varepsilon_1 |x|^{\theta_1} + \varepsilon_2 |y|^{\theta_2} + \varepsilon_3 |z|^{\theta_3},$$
$$|f_2(t,x,y,z)| \leqslant l^* + \delta_1 |x|^{\psi_1} + \delta_2 |y|^{\psi_2} + \delta_3 |z|^{\psi_3},$$

where l, l^* are constants, $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $\delta_1, \delta_2, \delta_3 > 0$. Applying the Theorem 1, the fractional order boundary value problem (9)-(12) has a solution $(u,v) \in B$, where B is determined from (8).

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