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NONLINEAR SCHRÖDINGER EQUATION WITH LANDAU DAMPING ON A HALF–LINE

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Abstract. We consider the initial-boundary value problem for the modified Schrödinger equation, posed on positive half-line $x > 0$:

$$\begin{cases} u_t + \mathbb{K}u + i|u|^2 u = 0, \ t \geqslant 0, \ x \geqslant 0; \\ u(x,0) = u_0(x), \ \ x > 0 \\ u(0,t) = h(t), \ t > 0. \end{cases}$$

where the operator $\mathbb{K}$ is defined as

$$\mathbb{K}(u) = \alpha u_{xx} + \lambda |\partial_x|^\gamma u$$

with $\alpha \in \mathbb{C}$, $\lambda > 0$ and $|\partial_x|^\gamma$ is the module-fractional derivative operator defined by

$$|\partial_x|^\gamma u = R^\gamma \partial_x u.$$

Here R^γ is the modified Riesz Potential

$$R^\gamma u = \frac{1}{2\sqrt{\pi}\sin\left(\frac{\pi}{4}\right)} \int_0^\infty \frac{sign\,(x-y)}{\sqrt{|x-y|}} u(y)dy.$$

We study the local and global existence in time of solutions to the initial-boundary value problem.

1. Introduction

We consider the initial-boundary value problemfor a modified Schrödinger equation with Landau damping on a half-line

$$\begin{cases} u_t + \mathbb{K}u + i|u|^2 u = 0, \ t \geqslant 0, \ x \geqslant 0; \\ u(x,0) = u_0(x), \ \ x > 0 \\ u(0,t) = h(t), \ t > 0. \end{cases} \tag{1.1}$$

where the operator $\mathbb{K}$ is defined as

$$\mathbb{K} = \alpha u_{xx} + \lambda |\partial_x|^\gamma u \tag{1.2}$$

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with $\alpha, \lambda \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $|\partial_x|^\gamma$ is the module-fractional derivative operator giben by $|\partial_x|^\gamma u = R^\gamma \partial_x u$. Here R^γ is the modified Riesz Potential

$$R^\gamma u = \frac{1}{2\Gamma(\gamma)\sin\left(\frac{\pi}{2}\gamma\right)} \int_0^\infty \frac{sign\,(x-y)}{|x-y|^{1-\gamma}} u(y)dy.$$

This paper is the first attempt to give a rigorous analysis of the initial-boundary value problem (IBV problem) for the nonlinear Schrödinger equation with Landau damping possed on a half-line. It combines the quantum state equation with the fractional derivative term that produce a wave damping.

Nowadays, both the theory as the applications of the partial differential equations with fractional derivative are being widely studied. Fractional Schrödinger equation with $\alpha = 0$ and $\lambda = i$ was discover by Laskin [15], he applied Feynman path integral approach to the Levy-like quantum mechanical paths. Feynman path integral approach to quantum mechanics is in fact integration over Brownian-like quantum mechanical paths.

The Brownian motion is a special case of the Levy γ-stable random process, when $\gamma = 2$ the Levy γ-stable distribution is transformed to the well-known Gaussian probability distribution or in other words, the Levy motion is transformed to the Brownian motion. The fractional Schrödinger equation includes the space derivative of order γ instead of the second order space derivative in the standar Schrödinger equation. Thus, the fractional Quantum mechanical includes the standar quantum mechanial as a particular Gaussian case at $\gamma = 2$. Quantum mechanical paths integral over the Levy paths at $\gamma = 2$ becomes the well known Feynmas path integral. Some physical applications of the fractional Schrödinger equation are the energy spectrum for a hydrogen-like atom-fractional “Bohr atom” and the energy spectrum of fractional oscilator in the semiclassical approximation (see [15],[8]).

There exist many works in which has been researched the local and global existence in time of solutions to the Cauchy problem for nonlinear Scrödinger equation (NLS), which are the most related to our problem. In the book [3] can us find a study of several problems of local nature and global nature for the initial value problem for NLS. In paper [12] showed the asymptotic behavior of small solutions to NLS with cubic nonlinearity. In the case of the fractional NLS there few works about the well-posedness of Cauchy problem. For example in [9], [10], [16], [18] the authors used methods such as Strichartz estimates, Fourier analysis among others and the solutions appear in the Sobolev spaces, where s is related to the nonlinearity studied in each case.

In spite of the importance to describe several physical problems and, in general, in application to natural sciences, the study of IBV to linear and nonlinear PDE's is less extensive that Cauchy problem. The IBVP have serious analytical difficulties such as the presence of unknown boundary values in the relevant equations. As far as we knowthere are not results about the IBV problem for the fractional NLS. Also if we consider IBV problem for the nonlinear Schrödinger equation the information concerning to this problem is much less than those relating to Cauchy problem. In the paper [4] the authors considered both linear and nonlinear integrable cases, and initial-boundary value problems associated with the Schrödinger equation. They present a method of solution, which is based on the elimination of the unknown boundary values by proper restrictions of the functional space and of the spectral variable complex domain. On the other hand, Fokas [5], assuming that the solution of the nonlinear Schrödinger equation on the half-line exists, showed that solution can be represented in terms of the solution of a matrix Riemann Hilbert, and in [6] the authors prove that given appropriate initial and boundary conditions, the solution of the nonlinear Schrödinger equation exists globally. In paper [14] was showed the existence of global in time solutions as well as the asymptotic behavior of this solutions for a IBVP for NLS with boundary data of Dirichlet type.

In this paper we study IBV-problem for nonlinear Schrödinger equation with Landau damping. In this case the symbol of the pseudodifferential operator $\mathbb{K}$ given by (1.2) is

$$K(p) = \alpha p^2 + \beta |p|^\gamma. \tag{1.3}$$

For a general theory of the initial-boundary value problems for evolution equations with pseudodifferential operators on a half-line you can see the book [11]. Also, we can find a few number of publications have dealt with asymptotic representation of solution to the boundary- initial value problem of nonlinear equation on a half-line, for example [1], [2] and [13] where the authors have considered homogeneous boundary value problem.

Since the symbol K given by (1.3) is nonanalytic in the right half plane, we can not use method of the [11] directly. We adopt the analytic continuation method proposed in the paper [13] to derive an integral representation for the solution of the linear problem associated with (1.1), reducing this linear problem with a corresponding Riemann boundary value problem. We will show that only one boundary data is necessary to put in the problem (1.1) for its solubility.

To state precisely the results of the present paper we give some notations. Direct Laplace transformation is

$$\widehat{u}(p) = \mathscr{L}u = \int_0^\infty e^{-px} u(x)\, dx,$$

and the inverse Laplace transformation $\mathscr{L}^{-1}u$ is defined by

$$u(x) = \mathscr{L}^{-1}u = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{px} \widehat{u}(p)\, dp.$$

Remembering the Weighted spaces:

$$\mathbf{L}^{s,\mu}(\mathbb{R}^+) = \{\phi : \|\phi\|_{\mathbf{L}^{s,\mu}} < \infty\},\ \|\phi\|_{\mathbf{L}^{s,\mu}} = \|\langle\cdot\rangle^\mu \phi\|_s.$$

We now introduce the space

$$\mathbf{Z} := \mathbf{L}^{1,\mu}(\mathbb{R}^+) \cap \mathbf{L}^\infty(\mathbb{R}^+),$$

where $\mu \in \left(0, \frac{1}{4}\right)$ with the norm

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{L}^2} + \|\phi\|_{\mathbf{L}^{1,\mu}} + \|\phi\|_{\mathbf{L}^\infty},$$

and

$$\mathbf{Y}_\beta := \left\{ \phi \in \mathbf{L}^1 : \|\phi\|_{\mathbf{Y}} < \infty \right\},$$

with $\beta > 1$ and the norm

$$\|\phi\|_{\mathbf{Y}_\beta} = \left\|\widehat{\phi}\right\|_{\mathbf{L}^{1,1}} + \left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty} + \|\phi\|_{\mathbf{L}^1} + \left\|\langle t\rangle^\beta \phi\right\|_{\mathbf{L}^\infty}.$$

Different positive constants we denote by the same letter C. For simplicity we put $\alpha = i, \lambda = 1, \gamma = \frac{1}{2}$ in (1.2). Now we state the main results.

THEOREM 1. *Suppose that* $u \in \mathbf{Z}$ *and* $h \in \mathbf{Y}_\beta$ *with* $\|u_0\|_{\mathbf{Z}} + \|h\|_{\mathbf{Y}_\beta} \leqslant \varepsilon$, *where* $\varepsilon > 0$ *is sufficiently small and* $\beta > 1$. *Then there exist a unique global solution*

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\left(\mathbb{R}^+\right)\right) \cap \mathbf{C}\left((0,\infty);\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}\left(\mathbb{R}^+\right) \cap \mathbf{L}^\infty\left(\mathbb{R}^+\right)\right),$$

with $\mu \in \left(0, \frac{1}{2}\right)$ *to the initial-boundary value problem* (1.1).

2. Preliminaries

We consider the following initial-boundary value problem

$$\begin{cases} u_t + \mathbb{K}(u) = 0, \;\; x \geqslant 0, \;\; t \geqslant 0. \\ u(x,0) = u_0(x), \;\; x > 0, \\ u(0,x) = h(t), \; t > 0 \end{cases} \tag{2.1}$$

where $\mathbb{K}$ was given in (1.2).

We define the operator $\mathbb{P}$ as

$$\mathbb{P}\phi(z) = -\frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{1}{q-z}\phi(q)\, dq, \quad \operatorname{Re} z \neq 0,$$

for a function ϕ of the complex variable q, which obeys the Hölder condition for Re $q \neq 0$ and tends to 0 as $q \to \pm i\infty$. We can note that $\mathbb{P}\phi$ is a analytic function for Re $z \neq 0$. The boundary values of this function are given by

$$(\mathbb{P}\phi)^{\pm}(p) = \lim_{z \to p,\, \pm \operatorname{Re} z > 0} \mathbb{P}\phi(z).$$

We will show two important lemmas about this boundary values. (see [7])

LEMMA 1. *Let* ϕ *be a function of complex variable* z*, which satisfies the Hölder condition on* $i\mathbb{R}$. *Then the boundary values of the function* $\mathbb{P}\phi$ *are such that*

$$\mathbb{P}^+\phi(p) - \mathbb{P}^-\phi(p) = \phi(p), \qquad Re\; p = 0. \tag{2.2}$$

The equality (2.2) *is known as Sokhotski Plemelj formula.*

LEMMA 2. ***Index Zero*** *Let ϕ a function of complex variable z, which satisfies the Hölder condition on $i\mathbb{R}$. If index of ϕ is equal to zero,*

$$Ind\,\phi = \int_{i\mathbb{R}} d\ln\,\phi(q) = 0,$$

then, there exist a analytic function X on $Re\,z \neq 0$ such that

$$\phi(q) = \frac{X^{+}(p)}{X^{-}(p)},\ \ Re\,q = 0.$$

Moreover, this function is given, up to an arbitrary constant, by formula

$$X(z) = \exp\left[\frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{\ln\phi(q)}{q-z}\,dq\right],\quad Re\,z \neq 0.$$

Proof of these Lemmas can be find in [7].

Setting

$$K(p) = ip^2 + \sqrt{|p|},\quad K_1(p) = ip^2 + \sqrt{p}. \tag{2.3}$$

Here and below, we denote the inverse functions $\varphi(\xi) = K_1^{-1}(\xi)$. We define the sectionally analytic function $Y(z,\xi)$ by the formula

$$Y(z,\xi) = e^{\Gamma(z,\xi)},\quad \mathrm{Re}\,z \neq 0, \tag{2.4}$$

where

$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{1}{q-z}\ln\left(\frac{K(p)+\xi}{K_1(q)+\xi}\right)dq,\ \ \mathrm{Re}\,\xi > 0. \tag{2.5}$$

We introduce the Green operator

$$\mathscr{G}(t)\phi = \frac{1}{2\pi i}\int_0^{\infty} G(x,y,t)\,\phi(y)\,dy, \tag{2.6}$$

where the function $G(x,y,t)$ is given by formula

$$G(x,y,t) = \mathscr{L}_{tx}^{-2}\left(\frac{Y^{+}(p,\xi)}{K(p)+\xi}(\varphi(\xi)-p)\,\mathbb{I}_{p,\xi}^{-}\left(e^{-(\cdot)y}\right)\right). \tag{2.7}$$

where

$$\mathbb{I}_{p,\xi}(\phi) = \frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{\phi(q)}{(\varphi(\xi)-q)(q-p)}\frac{1}{Y^{+}(q,\xi)}dq. \tag{2.8}$$

The boundary operator $\mathscr{H}(t)$ is defined as following

$$\mathscr{H}(t)\phi = \mathscr{L}_{tx}^{-2}\left(\frac{Y^{+}(p,\xi)}{K(p)+\xi}i[\varphi(\xi)-p]\left(\mathbb{I}_{p,\xi}\left(q^{-1}\sqrt{|q|}\right)-1\right)\widehat{\phi}\right). \tag{2.9}$$

In the next proposition, we analyze the linear initial-boundary value problem (2.1).

PROPOSITION 1. *Let the initial data* $u_0 \in \mathbf{L}^1(\mathbb{R}^+)$ *and boundary data* $h \in \mathbf{L}^1(\mathbb{R}^+)$. *Then the solution* $u(x,t)$ *of the initial-boundary value problem* (2.1) *has the following integral representation*

$$u(x,t) = \mathscr{G}(t)u_0 + \mathscr{H}(t)h, \tag{2.10}$$

where the operators $\mathscr{G}(t)$ *and* $\mathscr{H}(t)$ *are defined by* (2.6) *and* (2.9) *respectively.*

Proof. To derive an integral representation for the solution of (2.1) problem we adopt the analytic continuation method proposed in the paper [13]. We suppose that exists a solution $u(x,t)$ for the problem (2.1), such that

$$u(x,t) = 0 \ \text{ for all } x < 0.$$

Let $\Psi(p,t)$ be a some complex function such that $\mathbb{P}^-\{\Psi(\cdot,\xi)\} = 0$ and $|\Psi(p,\xi)| < C\langle p\rangle^{-\delta}$, $\delta > 0$. Applying the Laplace transforms with respect to space and time variables we obtain for the solution of (2.1)

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p)+\xi}\left[\widehat{u}_0(p) + \frac{K(p)}{p}\widehat{u}(0,\xi) + i\widehat{u}_x(0,\xi) + \widehat{\Psi}(p,\xi)\right], \tag{2.11}$$

where $\widehat{\widehat{u}}(p,\xi)$, $\widehat{\Psi}(p,\xi)$, $\widehat{u}(0,\xi)$ and $\widehat{u}_x(0,\xi)$ are Laplace transforms of $u(x,t)$, $\Psi(p,t)$, $u(0,t)$ and $u_x(0,t)$ respectively. To find the unknown function Ψ we need to solve the nonhomogeneous Riemann boundary value problem (see paper [13])

$$\Omega^+(p,\xi) = \frac{K(p)+\xi}{\xi}\Omega^-(p,\xi) - K(z)\Delta^-(z,\xi), \quad \text{for } \operatorname{Re} p = 0,\ \operatorname{Re}\xi > 0. \tag{2.12}$$

with

$$\Omega(z,\xi) = \mathbb{P}\left(\frac{K(\cdot)}{K(\cdot)+\xi}\Psi(\cdot,\xi)\right)(z), \tag{2.13}$$

$$\Delta(z,\xi) = \mathbb{P}\left(\frac{1}{K(\cdot)+\xi}\left(\widehat{u}_0(\cdot) + \frac{K(\cdot)}{(\cdot)}\widehat{u}(0,\xi) + i\widehat{u}_x(0,\xi)\right)\right). \tag{2.14}$$

We note that

$$\operatorname{Ind}\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right) = \frac{1}{2\pi i}\int_{i\mathbb{R}} d\ln\left(\frac{K(q)+\xi}{K_1(q)+\xi}\right) = 0.$$

Therefore, from Lemma 2 we infer

$$\frac{K(p)+\xi}{K_1(p)+\xi} = \frac{Y^+(p,\xi)}{Y^-(p,\xi)}. \tag{2.15}$$

So we can rewrite the Riemann problem (2.12) as

$$\frac{\Omega^+(p,\xi)}{Y^+(p,\xi)} = \frac{K_1(p)+\xi}{\xi}\frac{\Omega^-(p,\xi)}{Y^-(p,\xi)} - \frac{K(z)\Delta^-(z,\xi)}{Y^+(p,\xi)}, \tag{2.16}$$

Using the definition of $\Delta(p,\xi)$ in (2.14) and via Sokhotski-Plemelj formula (2.2), we rewrite $K(z)\Delta^-(z,\xi)$ as

$$K(z)\Delta^-(z,\xi) = (K(p)+\xi)\Delta^-(p,\xi) + \psi(p,\xi) - \xi\Delta^+(p,\xi),$$

where

$$\psi(p,\xi) = \widehat{u}_0(p) + \frac{K(p)}{p}\widehat{u}(0,\xi) - i\widehat{u}_x(0,\xi).$$

Replacing the last formula in equation (2.16) we reduce the nonhomogeneous Riemann problem in the form

$$F^+(p,\xi) + U^+(p,\xi) = F^-(p,\xi) + U^-(p,\xi), \text{ for } \operatorname{Re} p = 0,\ \operatorname{Re} \xi > 0, \tag{2.17}$$

where the sectionally analytic function U and the functions F^+, F^- are defined as

$$U(z,\xi) = \mathbb{P}\left(\frac{1}{Y^+(\cdot,\xi)}\left(\widehat{u}_0(\cdot) + \frac{|\cdot|^{\frac{1}{2}}}{(\cdot)}\widehat{u}(0,\xi)\right)\right), \tag{2.18}$$

$$F^+(z,\xi) = \frac{\Omega(z,\xi) - \xi\Delta(z,\xi) + iz\widehat{u}(0,\xi) + i\widehat{u}_x(0,\xi)}{Y(z,\xi)}, \quad \operatorname{Re} z < 0,$$

$$F^-(z,\xi) = \frac{K_1(z)+\xi}{\xi}\frac{\Omega(z,\xi) - \xi\Delta(z,\xi)}{Y(z,\xi)}, \quad \operatorname{Re} z > 0.$$

The relation (2.17) indicates that the functions $F^+ + U^+$, $F^- + U^-$ are branches of a unique analytic function in the complex plane. Moreover, this function has a pole of order one in infinity. So by the Liouville theorem this function is a polynomial of one degree $A(\xi)p + B(\xi)$. Hence

$$\begin{aligned}\Omega^+(p,\xi) = & Y^+(p,\xi)\left(A(\xi)p + B(\xi) - U^+(p,\xi)\right) \\ & + \xi\Delta^+(p,\xi) - ip\widehat{u}(0,\xi) - i\widehat{u}_x(0,\xi), \\ \Omega^-(p,\xi) = & \frac{\xi}{K(p)+\xi}Y^-(p,\xi)\left(A(\xi)p + B(\xi) - U^-(p,\xi)\right) + \xi\Delta^-(p,\xi).\end{aligned}$$

However, by definition of the sectionally analytic function Ω given in (2.13), Ω must satisfy the Hölder condition in consequence this function vanishes at infinity, i.e

$$\operatorname{Res}_{p=\infty}\left\{p\left(Y^+(p,\xi)A(\xi) - i\widehat{u}(0,\xi)\right) + \left(B(\xi) - i\widehat{u}_x(0,\xi)\right)\right\} = 0, \tag{2.19}$$

as $Y(\cdot,\xi)$ is a Hölder continuous function, we have

$$\lim_{|p|\to\infty} Y^{\pm}(p,\xi) = 1. \tag{2.20}$$

From (2.19) and (2.20) we conclude

$$A(\xi) = i\widehat{u}(0,\xi) \quad \text{y} \quad B(\xi) = \widehat{u}_x(0,\xi).$$

So we obtain

$$\Omega^{+}(p,\xi) = i\left(p\widehat{u}(0,\xi)+\widehat{u}_x(0,\xi)\right)\left(Y^{+}(p,\xi)-1\right) \\ -Y^{+}(p,\xi)U^{+}(p,\xi)+\xi\Delta^{+}(p,\xi),$$

$$\Omega^{-}(p,\xi) = \frac{\xi}{K(p)+\xi}Y^{-}(p,\xi)\left(ip\widehat{u}(0,\xi)+i\widehat{u}_x(0,\xi)-U^{-}(p,\xi)\right)+\xi\Delta^{-}(p,\xi).$$

Via Sokhotski Plemelj formula (2.2) and (2.13) we have

$$\begin{aligned}\Psi(p,\xi) &= \frac{K(p)+\xi}{\xi}\left(\Omega^{+}(p,\xi)-\Omega^{-}(p,\xi)\right) \\ &= \frac{K(p)}{\xi}Y^{+}(p,\xi)\left[ip\widehat{u}(0,\xi)+i\widehat{u}_x(0,\xi)-\psi(p,\xi)-U^{-}(p,\xi)\right].\end{aligned}$$

Substituting the previous relation in (2.11) we get

$$\widehat{\widehat{u}}(p,\xi) = \frac{Y^{-}(p,\xi)}{K_1(p)+\xi}\left[ip\widehat{u}(o,\xi)+i\widehat{u}_x(0,\xi)-U^{-}(p,\xi)\right].$$

Note that the equality $K_1(p)+\xi=0$ have only one root $\varphi(\xi)$ in $\operatorname{Re}\xi>0$ such that $\operatorname{Re}\varphi(\xi)>0$. So, for the analyticity of the function $\widehat{\widehat{u}}$ in the right half-plane we must to put the following condition

$$i\varphi(\xi)\widehat{u}(0,\xi)-i\widehat{u}_x(0,\xi)-U^{-}(\varphi(\xi),\xi)=0. \tag{2.21}$$

With this assumption we only need to put one boundary data in the initial boundary value problem (2.1). Thus, for example if we consider Dirichlet boundary condition, $u(0,\xi)=h(t)$ another unknown boundary data $u_x(0,t)$ is completely determined by

$$\widehat{u}_x(0,\xi) = h(\xi)\varphi(\xi)-iU^{-}(\varphi(\xi),\xi).$$

Finally, we obtain for solution of (2.1)

$$\widehat{\widehat{u}}(p,\xi) = \frac{Y^{-}(p,\xi)}{K_1(p)+\xi}\left[i\left(p-\varphi(\xi)\right)\widehat{h}(\xi)+\left(U^{-}(\varphi(\xi),\xi)-U^{-}(p,\xi)\right)\right],$$

where $K_1(p)$, $Y^{+}(p,\xi)$ and $U(p,\xi)$ are defined by (2.3), (2.4) and (2.18) respectively. Taking inverse Laplace transform with respect to time and space variables, we obtain (2.10), so Proposition 1 has been proved.

3. Estimation for the Green operator and Boundary operator

The following inequalities will be important to prove some estimates for the Green operator $\mathscr{G}(t)$ and the boundary operator $\mathscr{H}(t)$:

- Let $a,b\in\mathbb{C}$ with $a\neq b$ and $\gamma\in[0,1]$ then

$$\frac{1}{|a-b|} \leqslant \frac{1}{|a|^{1-\gamma}}\frac{1}{|b|^{\gamma}}. \tag{3.1}$$

- For Re $z>0$ and $\eta\in(0,1)$ we have

$$\left|e^{-z}\right| \leqslant C\,|z|^{-\eta}. \tag{3.2}$$

- For $z\in\mathbb{C}$ with Re $z>0$ and $\gamma\in(0,1)$ we have

$$\left|e^{-z}-1\right| \leqslant |z|^{\gamma}. \tag{3.3}$$

- Let be $z\in\mathbb{C}$, Re $z\neq 0$ and $\gamma\in(0,1)$. Then

$$\int_{i\mathbb{R}} \frac{1}{|q|^{\gamma}|q-z|}dq \leqslant \frac{1}{|z|^{\gamma}}. \tag{3.4}$$

We defined the contours $\mathscr{C}_j$, $j=1,2$, as

$$\begin{aligned}\mathscr{C}_1 &= \left\{\xi\in\left(\infty e^{-i\left(\frac{\pi}{2}+\varepsilon_1\right)},0\right)\cup\left(0,\infty e^{i\left(\frac{\pi}{2}+\varepsilon_1\right)}\right)\right\},\\ \mathscr{C}_2 &= \left\{p\in\left(\infty e^{-i\left(\frac{\pi}{2}+\varepsilon_2\right)},0\right)\cup\left(0,\infty e^{i\left(\frac{\pi}{2}+\varepsilon_2\right)}\right)\right\}.\end{aligned} \tag{3.5}$$

We take the ε_1, ε_2 sufficiently small such that the functions $\varphi(\xi)$ and $K(p)$ are analytic for $\xi\in\mathscr{C}_1$ and $p\in\mathscr{C}_2$.

In the next lemma we obtain some estimates of the Green operator $\mathscr{G}(t)$ defined by (2.6) in the spaces $\mathbf{L}^2$ and $\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}$.

LEMMA 3. *The estimates are valid for* $\mu\in\left(0,\frac{1}{2}\right)$, $t>0$

1. $\|\mathscr{G}(t)\phi\|_{\mathbf{L}^2} \leqslant C\left(\|\phi\|_{\mathbf{L}^1}+\|\phi\|_{\mathbf{L}^\infty}\right)$.
2. $\|\mathscr{G}_2(t)\phi\|_{\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} \leqslant C\langle t\rangle^{-\left(\frac{1}{2}-\mu\right)}\left(\|\phi\|_{\mathbf{L}^1}+\|\phi\|_{\mathbf{L}^{1,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}}\right)$,
   provident that the right-hand sides are finite.

Proof. Applying the Sokhotzki-Plemelj formula and changing the contour of the integration via Cauchy Theorem, we can rewrite the Green operator $\mathscr{G}(t)$ given in (2.6) as $\mathscr{G}(t)=\mathscr{G}_1(t)+\mathscr{G}_2(t)$, where the operators $\mathscr{G}_j(t)$, $j=1,2$ are defined as following

$$\mathscr{G}_1(t) = \int_0^\infty \mathrm{G}_1(x,y,t)\,\phi(y)\,dy,$$

$$\mathscr{G}_2(t) = \int_0^\infty \mathrm{G}_2(x-y,t)\,\phi(y)\,dy.$$

with

$$\mathrm{G}_1(x,y,t) = \left(\frac{1}{2\pi i}\right)^2\int_{\mathscr{C}_1} e^{\xi t}\int_{\mathscr{C}_2} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}\left(p-\varphi(\xi)\right)\mathbb{I}_{p,\xi}\left(e^{-qy}\right)dp\,d\xi, \tag{3.6}$$

$$\mathrm{G}_2(r,t) = \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{pr-K(p)t}\, dp.\ r\in\mathbb{R} \tag{3.7}$$

Here $\mathscr{C}_1$, $\mathscr{C}_2$ were given by (3.5).

Estimation of $\mathscr{G}_1(t)$: Since $\operatorname{Re} p < 0$ by Cauchy Theorem and the equality (2.15), we rewrite the function $\mathbb{I}_{p,\xi}(e^{-qy})$ given by (2.8) in the next way

$$\begin{aligned}(p-\varphi(\xi))\mathbb{I}_{p,\xi}\left(e^{-qy}\right) &= \frac{1}{2\pi i}\int_{i\mathbb{R}}\frac{e^{-qy}}{q-p}\left[\frac{1}{Y^+(q,\xi)}-\frac{1}{Y^-(q,\xi)}\right]dq\\ &\quad+\frac{1}{2\pi i}\int_{i\mathbb{R}}\frac{e^{-qy}}{q-\varphi(\xi)}\frac{1}{Y^+(q,\xi)}dq\\ &= \frac{1}{2\pi i}\int_{i\mathbb{R}}\frac{e^{-qy}}{q-p}\frac{1}{Y^+(q,\xi)}\frac{\sqrt{q}-\sqrt{|q|}}{K_1(q)+\xi}\,dq\\ &\quad+\frac{1}{2\pi i}\int_{i\mathbb{R}}\frac{e^{-qy}}{q-\varphi(\xi)}\frac{1}{Y^+(q,\xi)}dq\end{aligned}$$

Therefore, we infer

$$\mathrm{G}_1(x,y,t) = \mathrm{G}_{11}(x,y,t)+\mathrm{G}_{12}(x,y,t), \tag{3.8}$$

where

$$\begin{aligned}\mathrm{G}_{11}(x,y,t) &= C_\pi^3\int_{\mathscr{C}_1}e^{\xi t}\int_{\mathscr{C}_2}e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}\int_{i\mathbb{R}}\frac{e^{-qy}}{q-p}\frac{1}{Y^+(q,\xi)}\frac{\sqrt{q}-\sqrt{|q|}}{K_1(q)+\xi}\,dq\,dp\,d\xi,\\ \mathrm{G}_{12}(x,y,t) &= C_\pi^3\int_{\mathscr{C}_1}e^{\xi t}\int_{\mathscr{C}_2}e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}\int_{i\mathbb{R}}\frac{e^{-qy}}{q-\varphi(\xi)}\frac{1}{Y^+(q,\xi)}dq\,dp\,d\xi.\end{aligned}$$

Since $K_1(q)+\xi\neq 0$ for $\xi\in\mathscr{C}_1$ via (3.1) we have

$$\left|\int_{i\mathbb{R}}\frac{e^{-qy}}{(q-p)}\frac{1}{Y^+(q,\xi)}\frac{\sqrt{q}-\sqrt{|q|}}{K_1(q)+\xi}\,dq\right| \leqslant C\frac{1}{\langle|\xi|\rangle^\gamma}\frac{1}{|p|^\alpha},\ \text{ with } \gamma,\alpha\in(0,1), \tag{3.9}$$

such that $\frac{3}{2}-\alpha-2\gamma>0$. Applying (3.9) and via estimate

$$\left\|e^{-p(\cdot)}\right\|_{\mathbf{L}^2}\leqslant C\frac{1}{|p|^{\frac{1}{2}}},\ \ \operatorname{Re} p<0,$$

the $\mathbf{L}^2(\mathbb{R}^+)$ norm of $G_{11}(\cdot,y,t)$ satisfies

$$\|\mathrm{G}_{11}(\cdot,y,t)\|_{\mathbf{L}^2}\leqslant C\int_{\mathscr{C}_1}\frac{1}{\langle|\xi|\rangle^\gamma}\int_{\mathscr{C}_2}\frac{1}{|p|^{\frac{1}{2}+\alpha}}\frac{1}{|K(p)+\xi|}\,d|p|\,d|\xi|,$$

since all integrals in the before expression converge, we get

$$\|\mathrm{G}_{11}(\cdot,y,t)\|_{\mathbf{L}^2}\leqslant C. \tag{3.10}$$

To calculate the $\mathbf{L}^2$-norm of $G_{12}(\cdot,y,t)$ we have

$$\left|\int_{i\mathbb{R}} \frac{e^{-qy}}{q-\varphi(\xi)} \frac{1}{Y^+(q,\xi)} dq\right| \leqslant Cy^{-\eta} \frac{1}{|\xi|^{\frac{\eta}{2}}},\ \eta \in \left(\frac{1}{2},1\right).$$

Using that $K(p)+\xi \neq 0$ we use (3.1) with $\eta \in \left(\frac{1}{2},1\right)$ and the above inequality to obtain

$$\|\mathrm{G}_{12}(\cdot,y,t)\|_{\mathbf{L}^2} \leqslant Cy^{-\eta} \int_{\mathscr{C}_1} \frac{1}{|\xi|^{\frac{\eta}{2}}} \int_{\mathscr{C}_2} \frac{1}{|p|^{\frac{1}{2}}} \frac{1}{|K(p)+\xi|} d|p|\, d|\xi|\,,$$

in consequence

$$\|\mathrm{G}_{12}(\cdot,y,t)\|_{\mathbf{L}^2} \leqslant Cy^{-\eta}. \tag{3.11}$$

Hence by (3.8), (3.10) and (3.11) we have

$$\|\mathrm{G}_1(\cdot,y,t)\|_{\mathbf{L}^2} \leqslant Cy^{-\eta}.$$

Therefore

$$\|\mathscr{G}_1(t)\phi\|_{\mathbf{L}^2} \leqslant \int_0^\infty \|\mathrm{G}_1(\cdot,y,t)\|_{\mathbf{L}^2} |\phi(y)|\, dy \leqslant C(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^\infty}). \tag{3.12}$$

Estimation of $\mathscr{G}_2(t)$: Applying the Plancherel theorem we have

$$\|\mathscr{G}_2(t)\phi\|_{\mathbf{L}^2} = \left\|e^{-py}e^{-K(p)t}\widehat{\phi}\right\|_{\mathbf{L}^2} \leqslant \left\|\widehat{\phi}\right\|_{\mathbf{L}^2} \leqslant \|\phi\|_{\mathbf{L}^2}\,,$$

and using the interpolation inequality

$$\|\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{L}^\infty}^{\frac{1}{2}} \|\phi\|_{\mathbf{L}^1}^{\frac{1}{2}} \leqslant C(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^\infty}),$$

we get

$$\|\mathscr{G}_2(t)\phi\|_{\mathbf{L}^2} \leqslant C(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^\infty}). \tag{3.13}$$

From the estimates for $\mathscr{G}_1$ and $\mathscr{G}_2$ showing in (3.12) and (3.13) respectively one has that

$$\|\mathscr{G}(t)\phi\|_{\mathbf{L}^2} \leqslant C(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^\infty}).$$

So the first estimates of Lemma 3 has been proved.

Now we prove the second estimation. Let us denote $\gamma = \frac{1}{2}\left(\frac{1}{2}+\mu\right)$, $\mu \in \left(0,\frac{1}{2}\right)$. Firstly we consider the case $t<1$. By a similar procedure to made in the calculation of $\mathbf{L}^2$ norm of $\mathscr{G}_1(t)$ we get

$$\|\mathscr{G}_1(t)\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{L}^1}. \tag{3.14}$$

On the other hand, we can rewrite the function $G_2(r,t)$ in the form

$$\mathrm{G}_2(r,t) = \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{pr-ip^2t} e^{-\sqrt{|p|}t}\, dp = \frac{e^{-\left(\frac{r}{2t}\right)^2}}{2\pi i}\int_{i\mathbb{R}} e^{-it\left(p+\frac{ir}{2t}\right)^2} e^{-\sqrt{|p|}t}\, dp$$

$$= \frac{e^{-\left(\frac{r}{2t}\right)^2}}{2\pi i}\int_{i\mathbb{R}} e^{-itp^2} e^{-t\sqrt{|p-\frac{ir}{2t}|}}\, dp.$$

We can change the contour of the integration via the Cauchy theorem:

$$\mathrm{G}_2(r,t) = \frac{e^{-\left(\frac{r}{2t}\right)^2}}{2\pi i}\int_{\mathscr{C}} e^{-itp^2} e^{-t\sqrt{|p-\frac{ir}{2t}|}}\, dp = \frac{e^{-\left(\frac{r}{2t}\right)^2}}{2\pi i}\int_{\mathscr{C}} e^{-it\left(p+\frac{ir}{2t}\right)^2} e^{-\sqrt{|p|t}}\, dp, \quad (3.15)$$

where the contour $\mathscr{C}$ is taken such that $\operatorname{Re} ip^2 > 0$ for all $p \in \mathscr{C}$. So we infer

$$\|\mathrm{G}(\cdot,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant C\left\| e^{-\left(\frac{r}{2t}\right)^2}\right\|_{\mathbf{L}^{2,\gamma}} \int_{\mathscr{C}} e^{-|p|^2 t}\, d|p|.$$

Since

$$\left\| e^{-\left(\frac{r}{2t}\right)^2}\right\|_{\mathbf{L}^{2,\gamma}} \leqslant Ct^{\frac{1}{2}+\gamma} \text{ for } \gamma \in (0,\frac{1}{2}), \quad (3.16)$$

and by the substitution $w = p\sqrt{t}$ we have

$$\|\mathrm{G}(\cdot,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant C\sqrt{t}\int_0^\infty e^{-|p|^2 t}\, d|p| \leqslant C\int_0^\infty e^{-|w|^2} d|w|,$$

in consequence

$$\|\mathrm{G}(\cdot,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant C, \text{ for } \gamma \in (0,\frac{1}{2}),\ t < 1. \quad (3.17)$$

From the estimation

$$|x|^\gamma \leqslant C|x-y|^\gamma + C|y|^\gamma,$$

and the Young inequality we conclude

$$\|\mathscr{G}_2(t)\phi\|_{\mathbf{L}^{2,\gamma}} \leqslant \|\mathrm{G}(\cdot,t)\|_{\mathbf{L}^{2,\gamma}}\|\phi\|_{\mathbf{L}^1} + \|\mathrm{G}(\cdot,t)\|_{\mathbf{L}^2}\|\phi\|_{\mathbf{L}^{1,\gamma}}. \quad (3.18)$$

By (3.17) and (3.18) we get

$$\|\mathscr{G}_2(t)\phi\|_{\mathbf{L}^{2,\gamma}} \leqslant C\left(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,\gamma}}\right). \quad (3.19)$$

Replacing $\gamma = \frac{1}{2}\left(\frac{1}{2}+\mu\right)$ in (3.19) we get

$$\|\mathscr{G}_2(t)\phi\|_{\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} \leqslant C\left(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}}\right). \quad (3.20)$$

Now, we consider the case $t > 1$. We express the function G_1 as:

$$\mathrm{G}_1(x,y,t) = J_1(x,y,t) + J_2(x,y,t),$$

where

$$J_1(x,y,t) = \left(\frac{1}{2\pi i}\right)^3 \int_{\mathscr{C}_1} e^{\xi t}\int_{\mathscr{C}_2} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}\int_{i\mathbb{R}} \frac{e^{-qy}}{q-p}\frac{1}{Y^+(q,\xi)}\, dq\ dp\, d\xi,$$

$$J_2(x,y,t) = \left(\frac{1}{2\pi i}\right)^3 \int_{\mathscr{C}_1} e^{\xi t} \int_{\mathscr{C}_2} e^{px} \frac{Y^+(p,\xi)}{K(p)+\xi} \int_{i\mathbb{R}} \frac{e^{-qy}}{q-\varphi(\xi)} \frac{1}{Y^+(q,\xi)}\, dq\ dp\, d\xi.$$

To calculate the $\mathbf{L}^{2,\gamma}$ norm of $J_1(\cdot,y,t)$ we use (3.2) and (3.4) with $\eta \in (0,1)$ to obtain:

$$\begin{aligned}\|J_1(\cdot,y,t)\|_{\mathbf{L}^{2,\gamma}} &\leqslant Cy^{-\eta} \int_{\mathscr{C}_1} e^{-C|\xi|t} \int_{\mathscr{C}_2} \frac{1}{|p|^{\frac{1}{2}+\gamma}} \frac{1}{|K(p)+\xi|} \\ &\qquad \times \int_{i\mathbb{R}} \frac{1}{|q|^{\eta}\,|q-p|}\, d|q|\, d|p|\, d|\xi| \\ &\leqslant Cy^{-\eta} \int_{\mathscr{C}_1} e^{-C|\xi|t} \int_{\mathscr{C}_2} \frac{1}{|p|^{\frac{1}{2}+\gamma+\eta}} \frac{1}{\left||p|^{\frac{1}{2}}+\xi\right|}\, d|p|\, d|\xi|,\end{aligned}$$

via the sustitution $z = \xi t,\ z = pt^2$ we get

$$\|J_1(\cdot,y,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant Cy^{-\eta} t^{-(1-2\gamma-2\eta)}.$$

In a similar form to preceding we get

$$\|J_2(\cdot,y,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant Cy^{-\eta} t^{-(1-2\gamma-2\eta)}.$$

So, we obtain

$$\|\mathrm{G}_1(\cdot,y,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant Cy^{-\eta} t^{-(1-2\gamma-2\eta)}.$$

Therefore for $\gamma = \frac{1}{2}\left(\frac{1}{2}+\mu\right)$ and $\eta \in (0,1)$ we have

$$\|\mathscr{G}_1(t)\,\phi\|_{\mathbf{L}^{\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} \leqslant Ct^{-\left(\frac{1}{2}-\mu\right)} \left(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}}\right). \tag{3.21}$$

Moreover, from the integral representation of G_2 in (3.15) and the inequality (3.16) we get

$$\|\mathrm{G}_2(\cdot,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant Ct^{\frac{1}{2}+\gamma} \int_{\mathscr{C}} e^{-|p|^{\frac{1}{2}}t}\ d|p|.$$

Changing the variable: $z = pt^2$ we conclude

$$\|\mathrm{G}_2(\cdot,t)\|_{\mathbf{L}^{2,\gamma}} \leqslant Ct^{-(1-2\gamma)}, \tag{3.22}$$

As consequence from the estimations (3.18) and (3.22) for $\gamma = \frac{1}{2}\left(\frac{1}{2}+\mu\right)$ we conclude

$$\|\mathscr{G}_2(t)\,\phi\|_{\mathbf{L}^{\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} \leqslant t^{-\left(\frac{1}{2}-\mu\right)} \left(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}}\right). \tag{3.23}$$

We deduce the second estimation from the inequalities (3.14), (3.20), (3.21) and (3.23). So the Lemma 3 has been proved.

In the next lemma we estimate the Green operator $\mathscr{G}(t)$ in the Lebesgue space $\mathbf{L}^{\infty}$.

LEMMA 4. *The following estimates are true, provided that the right-hand sides are finite:*

$$\|\mathscr{G}(t)\phi\|_{\mathbf{L}^\infty} \leqslant C\{t\}^{-\frac{1}{2}}\langle t\rangle^{-2}\left(\|\phi\|_{\mathbf{L}^1}+\|\phi\|_{\mathbf{L}^{1,\mu}}\right),$$

for $\mu \in (0,1)$.

Proof. We consider $t>1$ and will denote $\gamma = \frac{1}{2}+\mu$, with $\mu \in \left(0,\frac{1}{2}\right)$. By the Cauchy theorem and the equations (3.6) and (3.7) we rewrite the Green function in the next way

$$G(x,y,t) = G_1(x,y,t) + G_2(x,y,t), \tag{3.24}$$

with

$$G_1(x,y,t) = C_\pi^2 \int_{\mathscr{C}_1} e^{\xi t} \int_{\mathscr{C}_2} e^{px} Y^+(p,\xi) \frac{p-\varphi(\xi)}{K(p)+\xi} \mathbb{I}_{p,\xi}\left(e^{-qy}-1\right) dp\, d\xi, \tag{3.25}$$

$$G_2(x,y,t) = C_\pi \int_{i\mathbb{R}} e^{px-K(p)t}\left(e^{-py}-1\right) dp. \tag{3.26}$$

were $\mathscr{C}_1$ and $\mathscr{C}_2$ given by (3.5) and $C_\pi = \frac{1}{2\pi i}$.

Estimation of $G_1(x,y,t)$ **:** Using the equality

$$\frac{p}{(q-p)(q-\varphi(\xi))} = \frac{p}{q(q-p)} + \frac{\varphi(\xi)}{(q-p)(q-\varphi(\xi))} - \frac{\varphi(\xi)}{q(q-\varphi(\xi))},$$

we rewrite $G_1(x,y,t)$ as

$$G_1(x,y,t) = J_1(x,y,t) + J_2(x,y,t),$$

where

$$J_1(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{\xi t} \int_{\mathscr{C}_2} e^{px} p \frac{Y^+(p,\xi)}{K(p)+\xi} \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,\xi)}\, dq\, dp\, d\xi.$$

$$J_2(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{\xi t}\varphi(\xi) \int_{\mathscr{C}_2} e^{px} \frac{Y^+(p,\xi)}{K(p)+\xi} \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-\varphi(\xi))} \frac{1}{Y^+(q,\xi)}\, dq\, dp\, d\xi.$$

Estimation of $J_1(x,y,t)$: Making the change of variables $\xi = -K_1(z)$ we get

$$J_1(x,y,t) = C_\pi^3 \int_{\widetilde{C}} e^{-K_1(z)t} K_1'(z) \int_{\mathscr{C}_2} e^{px} p\, \frac{Y^+(p,-K_1(z))}{K(p)-K_1(z)} \times \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,-K_1(z))}\, dq\, dp\, dz,$$

where $\widetilde{C} = \{z \in \mathbb{C} : \operatorname{Re} K_1(z) = 0\}$. Now, using Cauchy Theorem we obtain

$$J_1(x,y,t) = J_{11}(x,y,t) + J_{12}(x,y,t),$$

with

$$J_{11}(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{-K_1(z)t} K_1'(z) \int_{\mathscr{C}_2} e^{px} p \frac{Y^+(p,-K_1(z))}{K(p)-K_1(z)} \times \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,-K_1(z))}\, dq\, dp\, dz.$$

$$J_{12}(x,y,t) = C_\pi^2 \int_{\mathscr{C}_2} e^{px-K(p)t}\; pY^+(p,-K(p)) \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,-K(z))}\, dq\, dp.$$

We can note

$$J_{11}(x,y,t) = \sum_{k=1}^{5} J_{11k}(x,y,t),$$

where

$$J_{111}(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{-\sqrt{z}t} \frac{1}{\sqrt{z}} \int_{\mathscr{C}_2} e^{px} pY^+(p,0) \left[\frac{1}{K(p)-K_1(z)} - \frac{1}{ip^2+\sqrt{|p|}} \right] \times \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,0)}\, dq\, dp\, dz.$$

$$J_{112}(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{-\sqrt{z}t} \frac{1}{\sqrt{z}} \int_{\mathscr{C}_2} e^{px} p \frac{Y^+(p,-K_1(z)) - Y^+(p,0)}{K(p)-K_1(z)} \times \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,0)}\, dq\, dp\, dz.$$

$$J_{113}(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{-\sqrt{z}t} \frac{1}{\sqrt{z}} \int_{\mathscr{C}_2} e^{px} p \frac{Y^+(p,-K_1(z))}{K(p)-K_1(z)} \times \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \left[\frac{1}{Y^+(p,-K_1(z))} - \frac{1}{Y^+(q,0)} \right] dq\, dp\, dz.$$

$$J_{114}(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{-\sqrt{z}t} (e^{iz^2 t}-1) \frac{1}{\sqrt{z}} \int_{\mathscr{C}_2} e^{px} p \frac{Y^+(p,-K_1(z))}{K(p)-K_1(z)} \times \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,-K_1(z))}\, dq\, dp\, dz.$$

$$J_{115}(x,y,t) = C_\pi^3 \int_{i\mathbb{R}} e^{-K_1(z)t} z \int_{\mathscr{C}_2} e^{px} p \frac{Y^+(p,-K_1(z))}{K(p)-K_1(z)} \times \int_{i\mathbb{R}} \frac{e^{-qy}-1}{q(q-p)} \frac{1}{Y^+(q,-K_1(z))}\, dq\, dp\, dz.$$

Using (3.3) for $\mu \in (0,1)$ we get

$$\|J_{111}(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu \int_{i\mathbb{R}} e^{-C\sqrt{|z|}t} \frac{|K_1(z)|}{\sqrt{|z|}} \int_{\mathscr{C}_2} \frac{|p|}{|K(p)-K_1(z)| \left|p^2+\sqrt{|p|}\right|} \times \int_{i\mathbb{R}} \frac{1}{|q|^{1-\mu}|q-p|}\, d|q|\, d|p|\, d|z|,$$

therefore

$$\|J_{111}(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu \int_{i\mathbb{R}} e^{-C\sqrt{|z|}t}(|z|^{\frac{3}{2}}+1)\int_{\mathscr{C}_2} \frac{|p|}{\left|p^2+\sqrt{|p|}\right|^2} \times \int_{i\mathbb{R}} \frac{1}{|q|^{1-\mu}|q-p|}\, d|q|\, d|p|\, d|z|,$$

taking $w = zt^2$ we obtain

$$\|J_{111}(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu t^{-2}. \tag{3.27}$$

By Mean Value Theorem

$$\|J_{112}(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu \int_{i\mathbb{R}} e^{-\sqrt{|z|}t}\sqrt{|z|}\int_{\mathscr{C}_2} \frac{|p|}{\left|p^2+\sqrt{|p|}\right|} \times \int_{i\mathbb{R}} \frac{1}{|q|^{1-\mu}|q-p|}\, d|q|\, d|p|\, d|z| \leqslant Cy^\mu t^{-2}. \tag{3.28}$$

In a similar way we have

$$\|J_{113}(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu t^{-2}. \tag{3.29}$$

On the other hand, we note

$$\|J_{114}(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu t \int_{i\mathbb{R}} e^{-\sqrt{|z|}t}|z|^{\frac{3}{2}}\int_{\mathscr{C}_2} \frac{|p|}{\left|p^2+\sqrt{|p|}\right|} \times \int_{i\mathbb{R}} \frac{1}{|q|^{1-\mu}|q-p|}\, d|q|\, d|p|\, d|z| \leqslant Cy^\mu t^{-2}, \tag{3.30}$$

Changing of variables $w = zt^2$ we have

$$\|J_{115}(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu t^{-2}. \tag{3.31}$$

Moreover by analogy to (3.28) it easy to prove

$$\begin{aligned}\|J_{12}(\cdot,y,t)\|_{\mathbb{L}^\infty} &\leqslant Cy^\mu \int_{\mathscr{C}_2} e^{-\sqrt{|p|}t}\,|p| \int_{i\mathbb{R}} \frac{1}{|q|^{1-\mu}|q-p|}\, d|q|\, d|p| \\ &\leqslant Cy^\mu \int_{\mathscr{C}_2} e^{-\sqrt{|p|}t}\,|p|^\mu\, d|p| \\ &\leqslant Cy^\mu t^{-2(1+\mu)}.\end{aligned} \tag{3.32}$$

Thus from (3.27), (3.28), (3.29), (3.30), (3.31) and (3.32) we get

$$\|J_1(\cdot,y,t)\|_{\mathbb{L}^\infty} \leqslant Cy^\mu t^{-2}. \tag{3.33}$$

The calculation of $\mathbf{L}^{\infty}$ norm of $J_2(\cdot,y,t)$ can be done in a similar way

$$\|J_2(\cdot,y,t)\|_{\mathbb{L}^{\infty}} \leqslant Cy^{\mu}t^{-2} \tag{3.34}$$

Via (3.33) and (3.34) we infer

$$\|G_1(\cdot,y,t)\|_{\mathbb{L}^{\infty}} \leqslant Cy^{\mu}t^{-2}. \tag{3.35}$$

Estimation of $G_2(x,y,t)$: Via the definition of G_2 given by (3.26) and the inequality (3.3) for $\gamma < 1$, we conclude

$$\|G_2(\cdot,y,t)\|_{\mathbf{L}^{\infty}} \leqslant Cy^{\gamma}\int_{i\mathbb{R}} e^{-|p|^{\frac{1}{2}}t}p^{\gamma}d|p|.$$

Changing of variable $w = pt^2$ we have

$$\|G_2(\cdot,y,t)\|_{\mathbf{L}^{\infty}} \leqslant Cy^{\gamma}t^{-2(1+\gamma)}. \tag{3.36}$$

So, for $t > 1$ from (3.35) and (3.36) we have

$$\|\mathscr{G}(t)\phi\|_{\mathbf{L}^{\infty}} \leqslant Ct^{-2}\|\phi\|_{\mathbf{L}^{1,\mu}}, \text{ for } \mu \in (0,1). \tag{3.37}$$

Now, for the case $t < 1$ we remember the representation of the Green function G as

$$G(x,y,t) = \mathrm{G}_1(x,y,t) + \mathrm{G}_2(x-y,t),$$

where the functions $\mathrm{G}_1, \mathrm{G}_2$ was defined (3.6) and (3.7) respectively. By a similar procedure to exhibited in the previous Lemma, we get that the function $\mathrm{G}_1(x,y,t)$ satisfied in this case that

$$\|\mathrm{G}_1(\cdot,y,t)\|_{\mathbf{L}^{\infty}} \leqslant C.$$

Moreover, reminding the definition of the function G_2 :

$$\mathrm{G}_2(r,t) = \frac{1}{2\pi i}\int_{\mathscr{C}} e^{pr-K(p)t}\,dp,$$

where the contour $\mathscr{C}$ is taken such that $\operatorname{Re} pr < 0$ and $\operatorname{Re} p^2 < 0$, making use of the inequality

$$\|e^{pr}\|_{\mathbf{L}^{\infty}} \leqslant C \text{ for } p \in \mathscr{C},$$

and by the sustitution $z = pt^{\frac{1}{2}}$ we get

$$\|\mathrm{G}_2(r,t)\|_{\mathbf{L}^{\infty}} \leqslant C\int_{\mathscr{C}} e^{-C|p|^2 t}\,d|p| \leqslant Ct^{-\frac{1}{2}}.$$

In consequence for $t < 1$ we obtain

$$\|\mathscr{G}(t)\phi\|_{\mathbf{L}^{\infty}} \leqslant Ct^{-\frac{1}{2}}\|\phi\|_{\mathbf{L}^{1}}. \tag{3.38}$$

From (3.38) and (3.37) we deduce the estimation of the Green operator in $\mathbf{L}^{\infty}$ space.

LEMMA 5. *Let us be* $h \in \mathbf{Y}$, *then the following estimates are true*

1. $\|\mathscr{H}(t)h\|_{\mathbf{L}^2} \leqslant C \ \|h\|_{\mathbf{Y}}$.
2. $\|\mathscr{H}(t)h\|_{\mathbf{L}_x^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} \leqslant C\|h\|_{\mathbf{Y}}$
3. $\|\mathscr{H}(t)h\|_{\mathbf{L}^\infty} \leqslant C\langle t\rangle^{-1}\|h\|_{\mathbf{Y}}$
   where $0 \leqslant \mu \leqslant \frac{1}{2}$ *and* $t > 0$.

Proof. Remembering that the boundary operator $\mathscr{H}(t)$ is defined as

$$\mathscr{H}(t)h = \mathscr{L}_{tx}^{-2}\left(\frac{Y^+(p,\xi)}{K(p)+\xi}i(\varphi(\xi)-p)\left(I_{p,\xi}^-\left(q^{-1}\sqrt{|q|}\right)-1\right)\widehat{h}(\xi)\right).$$

We note that

$$\mathscr{H}(t) = \sum_{k=1}^{4}\mathscr{H}_k(t),$$

where

$$\mathscr{H}_1(t)h = \left(\frac{1}{2\pi i}\right)^2\int_{i\mathbb{R}} e^{\xi t}\widehat{h}(\xi)\varphi(\xi)\int_{\mathscr{C}_2} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}\,dp\,d\xi,$$

$$\mathscr{H}_2(t)h = \left(\frac{1}{2\pi i}\right)^2\int_{i\mathbb{R}} e^{\xi t}\widehat{h}(\xi)\int_{\mathscr{C}_2} e^{px}p\frac{Y^+(p,\xi)}{K(p)+\xi}\,dp\,d\xi,$$

$$\mathscr{H}_3(t)h = \left(\frac{1}{2\pi i}\right)^2\int_{i\mathbb{R}} e^{\xi t}\widehat{h}(\xi)\varphi(\xi)\int_{\mathscr{C}_2} e\frac{Y^+(p,\xi)}{K(p)+\xi} \times\int_{i\mathbb{R}}\frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))}dq\,dp\,d\xi,$$

$$\mathscr{H}_4(t)h = \left(\frac{1}{2\pi i}\right)^2\int_{i\mathbb{R}} e^{\xi t}\widehat{h}(\xi)\int_{\mathscr{C}_2} e^{px}p\frac{Y^+(p,\xi)}{K(p)+\xi} \times\int_{i\mathbb{R}}\frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))}dq\,dp\,d\xi.$$

In this proof we only show the estimates for $\mathscr{H}_1(t)$. We can estimate the other operators $\mathscr{H}_k(t)$ with $k = 2,3,4$ by a similar proceeding.

Applying Plancherel theorem we have

$$\|\mathscr{H}_1(t)h\|_{\mathbf{L}_x^2} = C\left\|\int_{i\mathbb{R}} e^{\xi t}\widehat{h}(\xi)\frac{Y^+(p,\xi)}{K(p)+\xi}\varphi(\xi)\ d\xi\right\|_{\mathbf{L}_p^2},$$

consequently for $\gamma \in \left(\frac{1}{2},1\right)$ we get

$$\|\mathscr{H}_1(t)h\|_{\mathbf{L}_x^2} \leqslant C\int_{i\mathbb{R}}\sqrt{|\xi|}\left|\widehat{h}(\xi)\right|\left\|\frac{Y^+(p,\xi)}{K(p)+\xi}\right\|_{\mathbf{L}_p^2}\ d|\xi|$$

$$\leqslant C \int_{i\mathbb{R}} |\xi|^{\frac{1}{2}-\gamma} \left| \widehat{h}(\xi) \right| d|\xi|,$$

from which we conclude that

$$\|\mathscr{H}_1(t)h\|_{\mathbf{L}^2_x} \leqslant C\left(\left\|\widehat{h}\right\|_{\mathbf{L}^\infty} + \left\|\widehat{h}\right\|_{\mathbf{L}^1}\right) \leqslant C\|h\|_{\mathbf{Y}}. \tag{3.39}$$

To prove the second estimation via a similar procedure to preceding we can obtain

$$\|\mathscr{H}_1(t)h\|_{\mathbf{L}_x^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} \leqslant C\|h\|_{\mathbf{Y}}.$$

Now, we will study calculate the $\mathbf{L}_x^\infty$ norm of the operator $\mathscr{H}$. First, we suppose that $t > 1$. We note that

$$\mathscr{H}(t)h = \int_0^t H(x,t-\tau)h(\tau)\,d\tau, \tag{3.40}$$

where

$$H(x,t) = H_1(x,t) + H_2(x,t), \;\; x > 0, \;\; t \in \mathbb{R},$$

with

$$\begin{aligned} H_1(x,t) = \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}} e^{\xi t} \int_{\mathscr{C}_2} e^{px} \frac{Y^+(p,\xi)}{K(p)+\xi}(p-\varphi(\xi)) \\ \times \int_{i\mathbb{R}} \frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y^+(q,\xi)}\,dq\,dp\,d\xi. \\ H_2(x,t) = \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}} e^{\xi t} \int_{\mathscr{C}_2} e^{px} \frac{Y^+(p,\xi)}{K(p)+\xi}(p-\varphi(\xi))\,dp\,d\xi. \end{aligned}$$

We note that

$$\begin{aligned} \int_{i\mathbb{R}} \frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y^+(q,\xi)} dq = \int_{i\mathbb{R}} \frac{K(q)+\xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y^+(q,\xi)} dq \\ - \int_{i\mathbb{R}} \frac{iq^2+\xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y^+(q,\xi)}. \end{aligned}$$

Remembering that $\operatorname{Re} p > 0$, $\operatorname{Re} \varphi(\xi) > 0$ and using (2.15)

$$\frac{K(p)+\xi}{Y^+(p,\xi)} = \frac{K_1(p)+\xi}{Y^-(p,\xi)},$$

via the Cauchy theorem we have

$$\begin{aligned} \int_{i\mathbb{R}} \frac{K(q)+\xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y^+(q,\xi)} dq = \frac{K(p)+\xi}{p(p-\varphi(\xi))} \frac{1}{Y^+(p,\xi)} \\ + \frac{1}{2} \frac{\xi}{p\varphi(\xi)} \frac{1}{Y^+(0,\xi)} - \frac{1}{2}, \\ \int_{i\mathbb{R}} \frac{iq^2+\xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y^+(q,\xi)} dq = \frac{1}{2} - \frac{1}{2} \frac{\xi}{p\varphi(\xi)} \frac{1}{Y^+(0,\xi)}. \end{aligned}$$

As consequence we obtain

$$\int_{i\mathbb{R}} \frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y^{+}(q,\xi)} dq = \frac{K(p)+\xi}{p(p-\varphi(\xi))} \frac{1}{Y^{+}(p,\xi)} + \frac{\xi}{p\varphi(\xi)} \frac{1}{Y^{+}(0,\xi)} - 1.$$

Thus making use of Cauchy Theorem we get

$$H(x,t) = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{\xi t}\, d\xi + \left(\frac{1}{2\pi i}\right)^2 \int_{\mathscr{C}_1} e^{\xi t} \frac{\xi}{\varphi(\xi)} \int_{\mathscr{C}_2} e^{px} \frac{Y^{+}(p,\xi)}{Y^{-}(0,\xi)} \frac{p-\varphi(\xi)}{p(K(p)+\xi)}\, dp\, d\xi. \quad (3.41)$$

where $\mathscr{C}_1$, $\mathscr{C}_2$ are defined in (3.5). Combining the equalities (3.40), (3.41) and the Cauchy Theorem we conclude

$$\mathscr{H}(t)h = h(t) + \widetilde{H}(t)h, \quad (3.42)$$

with

$$\widetilde{H}(t)h = \int_0^t \widetilde{H}(x,t-\tau)h(\tau)d\tau,$$

Here

$$\widetilde{H}(x,t) = \widetilde{H}_1(x,t) + \widetilde{H}_2(x,t) + \widetilde{H}_3(x,t),$$

where by the Cauchy theorem

$$\begin{aligned} \widetilde{H}_1(x,t) &= \frac{1}{2\pi i} \int_{\mathscr{C}_2} e^{px-K(p)t} \frac{K(p)}{p}\, dp. \\ \widetilde{H}_2(x,t) &= \left(\frac{1}{2\pi i}\right)^2 \int_{\mathscr{C}_1} e^{\xi t} \frac{\xi}{\varphi(\xi)} \int_{\mathscr{C}_2} e^{px} \frac{Y^{+}(p,\xi)}{Y^{-}(0,\xi)} \frac{1}{K(p)+\xi}\, dp\, d\xi. \\ \widetilde{H}_3(x,t) &= \left(\frac{1}{2\pi i}\right)^2 \int_{\mathscr{C}_1} e^{\xi t} \xi \int_{\mathscr{C}_2} e^{px} \frac{Y^{+}(p,\xi) - Y^{-}(0,\xi)}{Y^{-}(0,\xi)} \frac{1}{p(K(p)+\xi)}\, dp d\xi. \end{aligned} \quad (3.43)$$

By directly calculation we obtain

$$\left\|\widetilde{H}_1(\cdot,t)\right\|_{\mathbf{L}^\infty} \leqslant C \int_{\mathscr{C}_2} e^{-\sqrt{|p|}t} \left(|p| + \frac{1}{\sqrt{|p|}}\right) dp \leqslant Ct^{-1}. \quad (3.44)$$

Now, using that $\varphi(\xi) = \sqrt{\langle\xi\rangle}$ and via the inequality (3.1) we have

$$\left\|\widetilde{H}_2(\cdot,t)\right\|_{\mathbf{L}^\infty} \leqslant C \int_{\mathscr{C}_1} e^{-C|\xi|t} |\xi| \int_{\mathscr{C}_2} \frac{1}{\left|ip^2 + |p|^{\frac{1}{2}}\right|} d|p|\ d|\xi| \leqslant Ct^{-2}. \quad (3.45)$$

Since by Mean Value Theorem

$$\left|Y^{+}(p,\xi)-Y^{-}(0,\xi)\right| \leqslant C|p|,$$

and via the inequality (3.2) for $\delta \in \left(0,\frac{1}{2}\right)$ and mediantly the substitution $w=\xi t$ we infer

$$\left\|\widetilde{H}_3(\cdot,t)\right\|_{\mathbf{L}^\infty} \leqslant C\int_{\mathscr{C}_1} e^{-C|\xi|t}|\xi|\int_{\mathscr{C}_2}\frac{1}{\left|ip^2+|p|^{\frac{1}{2}}\right|}\,d|p|\;\,d|\xi| \leqslant Ct^{-2}. \tag{3.46}$$

From (3.44), (3.45) and (3.46) we conclude

$$\left\|\widetilde{H}(\cdot,t)\right\|_{\mathbf{L}^\infty} \leqslant Ct^{-1}.$$

The above inequality, the equation (3.42) and the definition of $\mathbf{Y}_\beta$ norm imply that

$$\|\mathscr{H}(t)h\|_{\mathbf{L}^\infty} \leqslant \|h(t)\|_{\mathbf{L}^\infty} + \left\|\widetilde{H}(t)h\right\|_{\mathbf{L}^\infty} \leqslant Ct^{-1}\|h\|_{\mathbf{Y}_\beta}, \quad \text{for } t>1. \tag{3.47}$$

Moreover for $0<t<1$ by analogy to (3.39) it easy to prove

$$\|\mathscr{H}(t)h\|_{\mathbf{L}^\infty} \leqslant C\|h\|_{\mathbf{Y}_\beta}. \tag{3.48}$$

From (3.47) and (3.48) it follows the Lemma 5.

In based to book [11] and the results presented in the previous Lemmas we can infer the next result.

THEOREM 2. *Let the initial data* $u_0 \in \mathbf{Z}$ *and the boundary data* $h \in Y_\beta$, *where* $\|u_0\|_{\mathbf{Z}} + \|h\|_{Y_\beta} = \varepsilon$, $\varepsilon > 0$ *is sufficiently small and* $\beta > 1$. *Then for some* $T > 0$ *there exists a unique solution*

$$u \in C\left([0,T];\,\mathbf{L}^2(\mathbb{R}^+)\right) \cap C\left((0,T];\mathbf{L}^{2,\mu}(\mathbb{R}^+) \cap \mathbf{L}^\infty\left(\mathbb{R}^+\right)\right),$$

for the initial boundary-value problem (1.1).

4. Proof of the Main Theorem

We introduce the spaces

$$\mathbf{Z} := \mathbf{L}^2(\mathbb{R}^+) \cap \mathbf{L}^{1,\frac{1}{4}+\mu}(\mathbb{R}^+) \cap \mathbf{L}^\infty(\mathbb{R}^+),$$

where $\mu \in \left(0,\frac{1}{4}\right)$ with the norm

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{L}^2} + \|\phi\|_{\mathbf{L}^{1,\frac{1}{4}+\mu}} + \|\phi\|_{\mathbf{L}^\infty},$$

and

$$\mathbf{Y}_\beta := \left\{\phi \in \mathbf{L}^1 : \|\phi\|_{\mathbf{Y}_\beta} < \infty\right\},$$

with

$$\|\phi\|_{\mathbf{Y}_\beta} := \left\|\widehat{\phi}\right\|_{\mathbf{L}^{1,1}} + \left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty} + \left\|\langle t\rangle^\beta \phi\right\|_{\mathbf{L}^\infty}.$$

Let us define the functional space

$$\mathbf{X} := \left\{\phi \in C\left([0,\infty);\mathbf{L}^2(\mathbb{R}^+)\right) \cap C\left((0,\infty);\mathbf{L}^{2,\frac{1}{2}(1-\mu)}(\mathbb{R}^+) \cap \mathbf{L}^\infty(\mathbb{R}^+)\right) : \|\phi\|_{\mathbf{X}} < \infty\right\},$$

where

$$\|\phi\|_{\mathbf{X}} := \sup_{t\geqslant 0}\|\phi(t)\|_{\mathbf{L}^2} + \sup_{t>0}\{t\}^{\frac{1}{4}\left(\frac{1}{2}-\mu\right)}\langle t\rangle^{\frac{1}{2}-\mu}\|\phi(t)\|_{\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} + \{t\}^{\frac{1}{2}}\langle t\rangle^2\|\phi\|_{\mathbf{L}^\infty}.$$

By Proposition 1 where $u_0 \in \mathbf{Z}$, $h \in \mathbf{Y}$ are such that $\|u_0\|_{\mathbf{Z}} + \|h\|_{\mathbf{Y}} = \varepsilon$ is sufficiently small. Let us define the operator

$$\mathscr{A}u := \mathscr{G}(t)u_0 - \int_0^t \mathscr{G}(t-\tau)\mathscr{N}(u)(\tau)\,d\tau + \mathscr{H}(t)h. \tag{4.1}$$

We prove that $\mathscr{A}$ is a contraction mapping on a ball $\mathbf{X}_\rho = \{v \in \mathbf{X} : \|v\|_{\mathbf{X}} \leqslant \rho\}$ where $\rho = 2C(\|u_0\|_{\mathbf{Z}} + \|h\|_{\mathbf{Y}})$. Firstly we need to prove that

$$\left\|\int_0^t \mathscr{G}(t-\tau)(|v|^2 v)d\tau\right\|_{\mathbf{X}} \leqslant C\|v\|_{\mathbf{X}}^3. \tag{4.2}$$

Since for $v \in \mathbf{X}$

$$\left\||v|^2 v\right\|_{\mathbf{L}^1} \leqslant \|v\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^2}^2 \leqslant C\{\tau\}^{-\frac{1}{2}}\langle\tau\rangle^{-2}\|v\|_{\mathbf{X}}^3,$$

$$\left\||v|^2 v\right\|_{\mathbf{L}^\infty} \leqslant \|v\|_{\mathbf{L}^\infty}^3 \leqslant C\{\tau\}^{-\frac{3}{2}}\langle\tau\rangle^{-6}\|v\|_{\mathbf{X}}^3,$$

$$\left\||v|^2 v\right\|_{\mathbf{L}^{1,\frac{1}{2}+\mu}} \leqslant C\|v\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}}^2 \leqslant C\{\tau\}^{-\frac{1}{2}\left(\frac{3}{2}-\mu\right)}\langle\tau\rangle^{-(3-2\mu)}\|v\|_{\mathbf{X}}^3,$$

$$\begin{aligned}\left\||v|^2 v\right\|_{\mathbf{L}^{1,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} &\leqslant C\left(\|v\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^2}^2 + \|v\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}}^2\right)\\ &\leqslant C\left(\{\tau\}^{-\frac{1}{2}}\langle\tau\rangle^{-1} + \{\tau\}^{-\frac{1}{2}\left(\frac{3}{2}-\mu\right)}\langle\tau\rangle^{-(3-2\mu)}\right)\|v\|_{\mathbf{X}}^3.\end{aligned}$$

Via Lemmas 3 and Lemma 4 we get

$$\begin{aligned}\left\|\int_0^t \mathscr{G}(t-\tau)(|v|^2 v)d\tau\right\|_{\mathbf{L}^2} &\leqslant \int_0^t \left(\left\||v|^2 v\right\|_{\mathbf{L}^1} + \left\||v|^2 v\right\|_{\mathbf{L}^\infty}\right)d\tau\\ &\leqslant C\|v\|_{\mathbf{X}}^3\int_0^\infty\left(\{\tau\}^{-\frac{1}{2}}\langle\tau\rangle^{-1} + \{\tau\}^{-\frac{3}{2}}\langle\tau\rangle^{-6}\right)d\tau\\ &\leqslant C\|v\|_{\mathbf{X}}^3.\end{aligned}$$

$$\begin{aligned}\left\| \int_0^t \mathscr{G}(t-\tau)(|v|^2 v)d\tau \right\|_{\mathbf{L}^{2,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} &\leqslant \int_0^t \{t-\tau\}^{-\frac{1}{4}\left(\frac{1}{2}-\mu\right)} \langle t-\tau\rangle^{-\left(\frac{1}{2}-\mu\right)} \\ &\quad \times \left(\left\| |v|^2 v \right\|_{\mathbf{L}^1} + \left\| |v|^2 v \right\|_{\mathbf{L}^{1,\frac{1}{2}\left(\frac{1}{2}+\mu\right)}} \right) d\tau \\ &\leqslant C\{t\}^{-\frac{1}{4}\left(\frac{1}{2}-\mu\right)} \langle t\rangle^{-\left(\frac{1}{2}-\mu\right)} \|v\|_{\mathbf{X}}^3 .\end{aligned}$$

$$\begin{aligned}&\left\| \int_0^t \mathscr{G}(t-\tau)(|v|^2 v)d\tau \right\|_{\mathbf{L}^\infty} \\ &\qquad \leqslant C \int_0^t \{t-\tau\}^{-\frac{1}{2}} \langle t-\tau\rangle^{-2} \left(\left\| |v|^2 v \right\|_{\mathbf{L}^1} + \left\| |v|^2 v \right\|_{\mathbf{L}^{1,\frac{1}{2}+\mu}} \right) d\tau \\ &\qquad \leqslant C\langle t\rangle^{-1} \|v\|_{\mathbf{X}}^3 .\end{aligned}$$

Thus (4.2) is proved.

By the same way, we can prove that for all $v,\ w \in \mathbf{X}$ is true

$$\left\| \int_0^t \mathscr{G}(t-\tau)(|v|^2 v - |w|^2 w)d\tau \right\|_{\mathbf{X}} \leqslant C\|v-w\|_{\mathbf{X}} (\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}})^2 .$$

By Lemma 3 and Lemma 4 we have $\|\mathscr{G}(t)u_0\|_{\mathbf{X}} \leqslant C\|u_0\|_{\mathbf{Z}}$, and for the Lemma 5 we have $\|\mathscr{H}(t)h\|_{\mathbf{X}} \leqslant C\|h\|_{\mathbf{Y}}$. In consequence for $v \in \mathbf{X}_\rho$ we get

$$\begin{aligned}\|\mathscr{A}(v)\|_{\mathbf{X}} &\leqslant \|\mathscr{G}(t)u_0\|_{\mathbf{X}} + \left\| \int_0^t \mathscr{G}(t-\tau)\mathbb{N}(v)(\tau)d\tau \right\|_{\mathbf{X}} + \|\mathscr{H}(t)h\|_{\mathbf{X}} \\ &\leqslant C\left(\|u_0\|_{\mathbf{Z}} + \|v\|_{\mathbf{X}}^3 + \|h\|_{\mathbf{Y}} \right) \leqslant \frac{\rho}{2} + \rho^3 \\ &\leqslant \rho .\end{aligned}$$

Therefore, the operator $\mathscr{A}$ transforms the ball $\mathbf{X}_\rho$ into itself. In the same way, we estimate the difference of two functions $v,\ w \in \mathbf{X}_\rho$

$$\begin{aligned}\|\mathscr{A}(v) - \mathscr{A}(w)\|_{\mathbf{X}} &\leqslant \left\| \int_0^t \mathscr{G}(t-\tau)(\mathbb{N}(v)(\tau) - \mathbb{N}(w)(\tau))\, d\tau \right\|_{\mathbf{X}} \\ &\leqslant C\|v-w\|_{\mathbf{X}} (\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}})^2 \\ &\leqslant C\rho^2 \|v-w\|_{\mathbf{X}} \leqslant \frac{1}{2}\|v-w\|_{\mathbf{X}} ,\end{aligned}$$

where $\rho > 0$ is sufficient small. Therefore $\mathscr{A}$ is a contraction mapping in $\mathbf{X}_\rho$ and thus by fixed point theorem there exists a unique solution $u \in \mathbf{X}$ to the initial boundary value problem (1.1). Hence the Theorem (1) is proved.

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