Differential Equations & Applications
Volume 7, Number 2 (2015), 245–262 doi:10.7153/dea-07-14

EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS TO A COUPLED SYSTEM OF NONLINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH ANTI PERIODIC BOUNDARY CONDITIONS

KAMAL SHAH AND RAHMAT ALI KHAN

(*Communicated by Yuji Liu*)

Abstract. In this article, we study sufficient conditions for existence and uniqueness of positive solutions to the following coupled system of fractional order differential equations with anti-periodic boundary conditions

$$\begin{cases} {}^cD^{\alpha}u(t)+f(t,v(t),{}^cD^{\alpha-1}v(t))=0,\ {}^cD^{\beta}v(t)+g(t,u(t),{}^cD^{\beta-1}u(t))=0, \quad 0<t<1,\\ u(0)=-u(1), v(0)=-v(1), D^pu(0)=-D^pu(1),\ D^qv(0)=-D^qv(1), \end{cases}$$

where $1<\alpha,\beta\leqslant 2, \alpha-p\geqslant 1, \beta-q\geqslant 1$ and $0<p,q<1, f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and D stands for Caputo derivative. We use Banach and Schauder fixed point theorems to develop sufficient conditions for existence and uniqueness of positive solutions. We also study sufficient conditions for existence of multiple positive solutions and conditions for non existence of solutions. We provide several examples to show the applicability of our results. We also link our analysis for the problem to equivalent integral equations.

1. Introduction

The study of fractional differential equations has attracted the attention of many researchers, because of their applications in various fields of science and engineering. Fractional differential equations arise in the field of physics, electrochemistry, viscoelasticity, Control theory, image and signal processing.

Recently, the study of existence and uniqueness of solutions to boundary value problems for fractional order differential equations has attracted the attentions of many scientists and a number of research articles are available in the literature, we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9] and the reference therein for some of the recent development in the theory. The study of fractional differential equations with periodic boundary conditions has also attracted some attention, we refer to [8, 10].

In [10], the authors have studied existence and uniqueness of solutions for the following anti-periodic boundary value problem

$$\begin{aligned} &{}^cD^{\alpha}y(t)=f(t,y(t),{}^cD^{\alpha-1}y(t)) \ \ for \ \ t\in J=[0,b] \\ &y(0)=-y(b), y'(0)=-y'(b) \end{aligned}$$

Mathematics subject classification (2010): 47J05, 92D25, 34A08, 34A34, 34K15.
Keywords and phrases: Coupled system, Fractional differential equations, Anti-periodic boundary conditions, Fixed point theorems.

Paper DEA-07-14

where ${}^cD^{\alpha}, 1 < \alpha \leqslant 2$ is the Caputo fractional derivative and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function. Many problems in applied sciences can be modeled as coupled system of differential equations with different type of boundary conditions. Boundary values problems for coupled systems with ordinary derivatives are well studied, however, coupled systems with fractional derivatives have attracted the attention quite recently. For example Su [11] developed sufficient conditions for existence of solutions for the coupled system with two point boundary conditions of the form

$$D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), D^{\beta}v(t) = g(t, u(t), D^{\nu}v(t)), 0 < t < 1$$
$$u(0) = u(1) = v(0) = v(1) = 0,$$

where $1 < \alpha, \beta < 2,\ \mu, \nu > 0,\ \alpha - \nu \geqslant 1,\ \beta - \mu \geqslant 1,\ f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and D is the standard Riemann-Liouville fractional derivative. B.Ahmad and Neito [12] extended the results of [11] to a three-point boundary value problem for the following coupled system of fractional differential equations

$$D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)),\ \ D^{\beta}v(t) = g(t, u(t), D^{\nu}v(t)),\ \ 0 < t < 1$$
$$u(0) = 0,\ u(1) = \gamma u(\eta),\ v(0) = 0,\ v(1) = \gamma v(\eta),$$

where $1 < \alpha, \beta < 2,\ \mu, \nu, \gamma > 0,\ 0 < \eta < 1, \alpha - \nu \geqslant 1,\ \beta - \mu \geqslant 1,\ \gamma\eta^{\alpha-1} < 1,\ \gamma\eta^{\beta-1} < 1, f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and D is standard Riemann-Liouville fractional derivative. Wang et al. [13] studied existence and uniqueness of positive solutions to a three-point boundary value problems for the coupled system

$$D^{\alpha}u(t) = f(t, v(t)),\ D^{\beta}v(t) = g(t, u(t)), 0 < t < 1$$
$$u(0) = 0 = v(0),\ u(1) = au(\xi),\ v(1) = bv(\xi),$$

where $1 < \alpha,\ \beta < 2,\ 0 \leqslant a,\ b \leqslant 1, 0 < \xi < 1,\ f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and D is the standard Riemann-Liouville fractional derivative.

Motivated by the above work, we study existence as well as non-existence of positive solutions to boundary value problems for the coupled system with anti-periodic boundary conditions of the type

$$\begin{aligned} &{}^cD^{\alpha}u(t) + f(t, v(t), {}^cD^{\alpha-1}v(t)) = 0, \\ &{}^cD^{\beta}v(t) + g(t, u(t), {}^cD^{\beta-1}u(t)) = 0, 0 < t < 1 \\ &u(0) = -u(1), v(0) = -v(1), D^p u(0) = -D^p u(1), D^q v(0) = -D^q v(1), \end{aligned} \tag{1.1}$$

where $1 < \alpha,\ \beta \leqslant 2,\ 0 < p, q < 1$, and $f, g \in C([0,1] \times [0,\infty), [0,\infty))$ are continuous and ${}^cD$ is the Caputo fractional derivative. We apply Banach fixed point theorem, Leray-Schauder fixed point theorem and fixed point theorems of cone expansion, to obtain sufficient conditions for existence and non-existence of positive solutions of (1.1). We also provide some examples to illustrate our main results.

2. Preliminaries

We recall some basic definitions and lemmas from fractional calculus [14, 15, 17].

DEFINITION 1. The fractional integral of order $q > 0$ of a function $u : (0,\infty) \to R$ is defined by

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{u(s)}{(t-s)^{1-q}} ds,$$

provided the integral converges.

DEFINITION 2. The Caputo fractional derivative of order $q > 0$ of a function $u \in C^n[0,1]$ is defined by

$${}^cD^q_{0+} u(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{u^n(s)}{(t-s)^{q-n+1}} ds, \text{ where } n = \lceil q \rceil,$$

provided that the right side is point wise defined on $(0,\infty)$.

The following results needed in the sequel.

LEMMA 1. [10], *let* $\alpha > 0$, *then* $I^\alpha\, {}^cD^\alpha h(t) = h(t) + C_0 + C_1 t + \ldots + C_{n-1} t^{n-1}$ *for arbitrary* $C_i \in R$, $i = 0,1,2,\ldots,n-1$, $n = [\alpha]+1$

LEMMA 2. [14], *Let* E *be a Banach space with* $C \subseteq E$ *closed and convex. Let* U *be a relatively open subset of* C *with* $0 \in U$ *and* $T : U \to U$ *be a continuous and compact mapping. Then either*

1. *The mapping* T *has a fixed point in* U *or*
2. *There exist* $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $U = \lambda T u$

LEMMA 3. [16], *Let* P *be a cone of real Banach space* E *and let* Ω_1 *and* Ω_2 *be two bounded open sets in* E *such that* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. *Let* $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *be completely continuous operator. Suppose that one of the two conditions holds:*

1. $\|Au\| \leqslant \|u\|$ *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \geqslant \|u\|$, *for all* $u \in P \cap \partial\Omega_2$
2. $\|Au\| \geqslant \|u\|$ *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \leqslant \|u\|$, *for all* $u \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

LEMMA 4. *For* $y \in C(0,1]$, *the boundary value problem*

$$\begin{aligned} &{}^cD^\alpha u(t) + y(t) = 0,\ 1 < \alpha \leqslant 2,\ 0 < t < 1, \\ &u(0) = -u(1),\ D^p u(0) = -D^p u(1),\ \ 0 < p < 1, \end{aligned} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G_\alpha(t,s) y(s) ds$$

where

$$G_\alpha(t,s)=\begin{cases}\dfrac{1}{2\Gamma\alpha}(1-s)^{\alpha-1}+\dfrac{1}{2\Gamma(\alpha-p)}(2t-1)(1-s)^{\alpha-p-1} \\ \qquad\qquad -\dfrac{1}{\Gamma\alpha}(t-s)^{\alpha-1} & ,0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{1}{2\Gamma\alpha}(1-s)^{\alpha-1}+\dfrac{1}{2\Gamma(\alpha-p)}(2t-1)(1-s)^{\alpha-p-1} & ,0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

Proof. In view of Lemma (1), we obtain $I^\alpha\, {}^cD^\alpha u(t)=-I^\alpha y(t)$, that is,

$$u(t)=C_0+C_1t-I^\alpha y(t).$$

Using the boundary conditions $u(0)=-u(1)$ and $D^pu(0)=-D^pu(1)$, we obtain $C_1=I^{\alpha-p}y(1)$ and $C_0=\frac{I^\alpha y(1)-I^{\alpha-p}y(1)}{2}$. Thus, we have

$$u(t)=\frac{I^\alpha}{2}+\frac{2t-1}{2}I^{\alpha-p}y(1)-I^\alpha y(t)=\int_0^1 G_\alpha(t,s)y(s)ds, \tag{2.2}$$

where

$$G_\alpha(t,s)=\begin{cases}\dfrac{1}{2\Gamma\alpha}(1-s)^{\alpha-1}+\dfrac{1}{2\Gamma(\alpha-p)}(2t-1)(1-s)^{\alpha-p-1} \\ \qquad\qquad -\dfrac{1}{\Gamma\alpha}(t-s)^{\alpha-1} & ,0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{1}{2\Gamma\alpha}(1-s)^{\alpha-1}+\dfrac{1}{2\Gamma(\alpha-p)}(2t-1)(1-s)^{\alpha-p-1} & ,0\leqslant t\leqslant s\leqslant 1.\end{cases} \tag{2.3}$$

Now, define $G(t,s)=(G_\alpha(t,s),\ G_\beta(t,s))$, then $G(t,s)$ is the Green's function corresponding to the system of BVP (1.1). It is easy to verify the following lemma:

LEMMA 5. *The Green's function $G(t,s)$ has the following properties*

- (P_1) $G(t,s)\geqslant 0$ *for all* $t,s\in[0,1]$ *and* $G(t,s)>0$ *for all* $t,s\in(0,1)$;
- (P_2) $\max\limits_{0\leqslant t\leqslant 1} G(t,s)=G(s,s), s\in[0,1]$;
- (P_3) $\min\limits_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4}} G(t,s)\geqslant\gamma(s)G(s,s)$ *for* $s\in[0,1]$.

3. Existence of positive solutions

In this section we investigate the existence of positive solution for boundary value problem (1.1). Let us define $X=\{u(t):u(t)\in C[0,1]\}$ and $Y=\{v(t):v(t)\in C[0,1]\}$ endowed with the norm

$$\|u\|=\max_{t\in[0,1]}|u(t)|+\max_{t\in[0,1]}|{}^cD^{\alpha-1}u(t)| \text{ and }, \|v\|=\max_{t\in[0,1]}|v(t)|+\max_{t\in[0,1]}|{}^cD^{\beta-1}v(t)|.$$

The product space $X\times Y$ is a Banach space under the norm $\|(u,v)\|=\max\{\|u\|,\|v\|\}$. Define a cone $P=\{(u,\ v)\in X\times Y: u(t)\geqslant 0,\ v(t)\geqslant 0\}$, Take $J=[\frac{1}{4},\frac{3}{4}]$ and subsets

$$B=\{(u,v)\in P,\ \min_{t\in J}u(t)\geqslant \gamma_\alpha\|u\|,\ \min_{t\in J}v(t)\geqslant \gamma_\beta\|v\|\},$$
$$B_r=\{(u,v)\in B:\|(u,v)\|\leqslant r\},\ \partial B_r=\{(u,v)\in B:\|(u,v)\|=r\}.$$

In view of Lemma (4), we can write the system of BVP (1.1) as an equivalent system of integral equations

$$\begin{cases} u(t)=\int_0^1 G_\alpha(t,s)f(s,v(s),{}^cD^{\alpha-1}v(s))ds, & 0\leqslant t\leqslant 1,\\ v(t)=\int_0^1 G_\beta(t,s)g(s,u(s),{}^cD^{\beta-1}u(s))ds, & 0\leqslant t\leqslant 1,\end{cases} \tag{3.1}$$

where f and g are continuous, then $(u,v)\in X\times Y$ is a solution of BVP (1.1) if and only if $(u,v)\in X\times Y$ is a solution of the above integral equations. Define $T:X\times Y\to X\times Y$ as follows

$$\begin{aligned} &T(u,v)(t)\\ &=\left(\int_0^1(G_\alpha(t,s)f(s,v(s),\ {}^cD^{\alpha-1}v(s))ds,\int_0^1(G_\beta(t,s)g(s,u(s),{}^cD^{\beta-1}u(s))ds\right)\\ &=(T_1v(t),T_2u(t)). \end{aligned} \tag{3.2}$$

Then the fixed point of operator T coincide with the solution of system BVP (1.1).

THEOREM 1. *Let $f,g:[0,1]\times[0,\infty)\to[0,\infty)$ be continuous, then $T:P\to P$ and $T:B\to B$ defined by (3.2) are completely continuous.*

Proof. From the continuity of f, g, it follows that T is continuous. Let $D\subseteq P$ be bounded set, that is, there exist a positive constant $l>0$ such that $\|(u,v)\|\leqslant \ell$ for all $(u,v)\in D$. Let

$$M=\max\{|f(t,v(t),{}^cD^{\alpha-1}v(t))|+1:\ 0\leqslant t\leqslant 1,\ 0\leqslant v\leqslant h\},$$
$$N=\max\{|g(t,u(t),{}^cD^{\beta-1}u(t))|+1:\ 0\leqslant t\leqslant 1,\ 0\leqslant u\leqslant h\}.$$

Then we have

$$\begin{aligned}|T_1v(t)|&=|\int_0^1 G_\alpha(t,s)f(s,v(s),{}^cD^{\alpha-1}v(s))ds|\\ &\leqslant\int_0^1 G_\alpha(t,s)|f(s,v(s)),{}^cD^{\alpha-1}v(s)|ds\leqslant M\int_0^1 G_\alpha(s,s)ds\end{aligned}$$

and $|T_2u(t)| \leqslant N\int\limits_0^1 G_\beta(s,s)ds$. Hence, it follows that

$$\|T(u,v)\| = \|(T_1v, T_2u)\| \leqslant \max\{(M\int_0^1 G_\alpha(s,s)ds, N\int_0^1 G_\beta(s,s)ds)\} = A,$$

that is, $T(D)$ is bounded so is uniformly bounded. From the uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$ it follows that for a fixed $s\in[0,1]$ and $\varepsilon>0$, there exist a constant $\delta>0$ such that for $|t_2-t_1|<\delta$ we have

$$|G_\alpha(t_2,s)-G_\alpha(t_1,s)| < \frac{\varepsilon}{M} \text{ and } |G_\beta(t_2,s)-G_\beta(t_1,s)| < \frac{\varepsilon}{N}.$$

It follows that

$$|T_1v(t_2)-T_1v(t_1)| \leqslant M\int_0^1 |G_\alpha(t_2,s)-G_\alpha(t_1,s)|ds < M\frac{\varepsilon}{M} = \varepsilon.$$

Similarly, we have

$$|T_2u(t_2)-T_2u(t_1)| < \varepsilon.$$

For Euclidean distance d on R^2, if $0\leqslant t_1,t_2\leqslant 1$ are such that $|t_2-t_1|<\delta$, then

$$\begin{aligned} d(T(u,v)t_2, T(u,v)t_1) &= \sqrt{|T_1v(t_2)-T_1v(t_1)|^2+|T_2u(t_2)-T_2u(t_1)|^2)} \\ &< \sqrt{2\varepsilon^2} = \sqrt{2}\varepsilon, \end{aligned}$$

which implies that $T(P)$ is equi-continuous. By Arzela Ascoli's theorem, $T: P\to P$ is completely continuous.

THEOREM 2. *Let there exist two positive constants K_α and K_β such that the following hold for $t\in[0,1]$, $u_1,u_2,v_1,v_2\in[0,\infty)$:*

(A_1) $|f(t,u_2,v_2)-f(t,u_1,v_1)| \leqslant K_\alpha(|u_2-u_1|+|v_2-v_1|)$,

(A_2) $|g(t,u_2,v_2)-g(t,u_1,v_1)| \leqslant K_\beta(|u_2-u_1|+|v_2-v_1|$.

Then the system (1.1) *has a unique positive solution if*

$$\max\{2G^*_\alpha K_\alpha, 2K_\alpha(\frac{1+\Gamma(3-\alpha)\Gamma(\alpha-p+1)}{\Gamma(3-\alpha)\Gamma(\alpha-p+1)})\} < 1,$$
$$\max\{2G^*_\beta K_\beta, 2K_\beta(\frac{1+\Gamma(3-\beta)\Gamma(\beta-q+1)}{\Gamma(3-\beta)\Gamma(\beta-q+1)})\} < 1.$$

Proof. Before proving the above result, since f,g are continuous and $Tu(t), Tv(t)$ and ${}^cD^{\alpha-1}Tu(t)$, ${}^cD^{\beta-1}v(t)$ are continuous on $[0,1]\times[0,\infty)\to[0,\infty)$.
For v, $\bar{v}\in Y$ and each $t\in[0,1]$, we have

$$|T_1v(t)-T_1\bar{v}(t)| \leqslant \max_{t\in[0,1]}\int_0^1 |G_\alpha(t,s)|[|f(s,v(s),{}^cD^{\alpha-1}v(s))-f(s,\bar{v}(s),{}^cD^{\alpha-1}\bar{v}(s))|]ds$$

which implies that

$$\begin{aligned}|T_1v(t)-T_1\bar{v}(t)| &\leqslant G^*_\alpha K_\alpha\{|v-\bar{v}|+|^cD^{\alpha-1}v-{}^cD^{\alpha-1}\bar{v}|\}\\ &\leqslant G^*_\alpha K_\alpha \max\{\|v-\bar{v}\|_\infty,\ \|D^{\alpha-1}v-D^{\alpha-1}\bar{v}\|_\infty\}\\ &=2G^*_\alpha K_\alpha\|v-\bar{v}\|, \end{aligned}\tag{3.3}$$

where $G^*_\alpha=\max_{t\in[0,1]}\int_0^1|G_\alpha(t,s)|ds$. Further, using the following relation

$$\begin{aligned}&{}^cD^{\alpha-1}T_1v(t)\\ &=\frac{t^{2-\alpha}}{\Gamma(3-\alpha)\Gamma(\alpha-p+1)}\int_0^1(1-s)^{\alpha-p+1}f(s,v(s),{}^cD^{\alpha-1}v(s))ds\\ &\qquad-\int_0^t f(s,v(s),{}^cD^{\alpha-1}v(s))ds,\end{aligned}$$

we obtain

$$|D^{\alpha-1}T_1v(t)-D^{\alpha-1}T_1\bar{v}(t)|\leqslant 2K_\alpha\big(\frac{1+\Gamma(3-\alpha)\Gamma(\alpha-p+1)}{\Gamma(3-\alpha)\Gamma(\alpha-p+1)}\big)||v-\bar{v}||\tag{3.4}$$

Choose

$$\rho_1=\max\Big\{2G^*_\alpha K_\alpha, 2K_\alpha\big(\frac{1+\Gamma(3-\alpha)\Gamma(\alpha-p+1)}{\Gamma(3-\alpha)\Gamma(\alpha-p+1)}\big)\Big\},$$

then from (3.3) and (3.4), it follows that

$$|T_1v(t)-T_1\bar{v}(t)|\leqslant\rho_1||v-\bar{v}||,\tag{3.5}$$

which implies that T_1 is a contraction as $\rho_1<1$. Similarly we can show that

$$|T_2u(t)-T_2\overline{u}(t)|\leqslant\rho_2\|u-\bar{u}\|\tag{3.6}$$

where

$$\rho_2=\max\Big\{2G^*_\beta K_\beta, 2K_\beta 2K_\beta\big(\frac{1+\Gamma(3-\beta)\Gamma(\beta-q+1)}{\Gamma(3-\beta)\Gamma(\beta-q+1)}\big)\Big\}<1$$

and

$$G^*_\beta=\max_{t\in[0,1]}\int_0^1|G_\beta(t,s)|ds,$$

that is, T_2 is a contraction. From (3.5) and (3.6), we have

$$\|T(u_2,v_2)-T(u_1,v_1)\|\leqslant\max(\rho_1,\rho_2)\|(u_2,v_2)-(u_1,v_1)\|\leqslant L\|(u_2,v_2)-(u_1,v_1)\|$$

where $L=\max(\rho_1,\rho_2)<1$. By Banach contraction theorem T has a unique fixed point which is unique positive solution of system of BVP (1.1).

THEOREM 3. *Assume that f and g are continuous on $[0,1]\times[0,\infty)\to[0,\infty)$ and satisfying*

(A_3) $|f(t,v(t),\ {}^cD^{\alpha-1}v(t))| \leqslant c_1(t)+c_2(t)(|v(t)|+|{}^cD^{\alpha-1}v(t)|)$

(A_4) $|g(t,u(t),\ {}^cD^{\beta-1}u(t))| \leqslant d_1(t)+d_2(t)(|u(t)|+|{}^cD^{\beta-1}u(t)|)$

(A_5) $C_1=\int_0^1 G_\alpha(s,s)c_1(s)ds<1,\ D_1=\int_0^1 G_\alpha(s,s)d_1(s)ds<\infty$

(A_6) $C_2=\int_0^1 G_\beta(s,s)d_2(s)ds<1,\ D_2=\int_0^1 G_\beta(s,s)d_1(s)ds<\infty.$

Then the system (1.1) *has at least one positive solution (u,v) in*

$$S=\left\{(u,v)\in P\,\middle|\,\|(u,v)\|<\min\left(\frac{D_1}{1-C_1},\frac{D_2}{1-C_2}\right)\right\}.$$

Proof. Choose $r=\min\left(\dfrac{D_1}{1-C_1},\dfrac{D_2}{1-C_2}\right)$ and define

$$S=\{(u,v)\in P|\|(u,v)\|<r\}.$$

For $(u,v)\in S$, we have consider

$$\begin{aligned}\|T_1v\| &= \max_{t\in[0,1]}\left|\int_0^1 G_\alpha(t,s)f(s),v(s),\ {}^cD^{\alpha-1}v(s)ds\right| \\ &\leqslant \int_0^1 G_\alpha(s,s)(c_1t)+c_2(t)(|v(t)|+|D^{\alpha-1}v(t)|)ds\leqslant D_1+C_1\|v\|\leqslant r.\end{aligned}$$

Similarly, we obtain $\|T_2u\|\leqslant r$. Hence, $\|T(u,v)\|\leqslant r$ which implies that $T(u,v)\in\overline{S}$. Further, $T:\overline{S}\to\overline{S}$ is completely continuous. Let $(u,v)\in S$ such that

$$(u,v)=\lambda T(u,v),0<\lambda<1, \tag{3.7}$$

then, we have

$$\begin{aligned}\|u\|=\|\lambda T_1v\| &= \lambda\max_{t\in[0,1]}\left|\int_0^1 G_\alpha(t,s)f(s,v(s),\ {}^cD^{\alpha-1}v(s))ds\right| \\ &< \int_0^1 G_\alpha(s,s)(c_1(t))+c_2(t)(|v(t)|+|{}^cD^{\alpha-1}v(t)|)ds\leqslant D+C_1\|v\|\leqslant r,\end{aligned}$$

which implies that $\|u\|<r$. Similarly, $\|v\|<r$. Hence, $\|(u,v)\|<r$ which implies that $(u,v)\notin\partial S$. Thus T has a fixed point in $\overline{S}$. □

Now, we introduce the following assumptions and notations.

(B_1) $f,g:[0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty)$ and $f(t,0,0)=g(t,0,0)=0$ are uniformly continuous with respect to t on $[0,1]$.

(B_2) Let $f_0=\lim\limits_{u\to 0}\dfrac{f(u)}{u}$, $g_0=\lim\limits_{u\to 0}\dfrac{g(u)}{u}$, $f_\infty=\lim\limits_{u\to\infty}\dfrac{f(u)}{u}$ and $g_\infty=\lim\limits_{u\to\infty}\dfrac{g(u)}{u}$.

THEOREM 4. *In addition to the assumptions (B_1) and (B_2), suppose that one of the following conditions holds.*

(C_1) $f_0^{-1}<\gamma_\alpha^2\int_{\frac14}^{\frac34}G_\alpha(s,s)ds$, $(f^\infty)^{-1}>G_\alpha^*$, $g_0^{-1}<\gamma_\beta^2\int_{\frac14}^{\frac34}G_\beta(s,s)ds$ *and* $(g^\infty)^{-1}>G_\beta^*$;

(C_2) *there exist four constants a,A,b,B with $0<a\leqslant A$ and $0<b\leqslant B$ such that $f(t,v(t),{}^cD^{\alpha-1}v(t))$ and $g(t,u(t),{}^cD^{\alpha-1}u(t))$ are non decreasing on $[0,1]\times[0,A]\times[0,B]$, $f(t,A,B)\leqslant(A+B)/G_\alpha^*$, $g(t,A,B)\leqslant(A+B)/G_\beta^*$, for all $t\in[0,1]$ and*

$$f^{-1}(t,\gamma_\alpha a,\gamma_\alpha b)<\gamma_\alpha^2\int_{\frac14}^{\frac34}G_\alpha(s,s)ds,\ g^{-1}(t,\gamma_\beta a,\gamma_\beta b)<\gamma_\beta^2\int_{\frac14}^{\frac34}G_\beta(s,s)ds.$$

Then the boundary value problem (1.1) has at least one positive solution.

Proof. **Case 1:** Assume that (C_1) holds. Since,

$$f_0^{-1}<\gamma_\alpha^2\int_{\frac14}^{\frac34}G_\alpha(s,s)ds,$$

it follows that there exists $a_1>0$ such that $f(t,v,\ {}^cD^{\alpha-1}v(t)\geqslant(f_0-\varepsilon_1-\varepsilon_2)v$ for all $t\in[0,1]$, $v\in[0,a_1]$ where $\varepsilon_1,\varepsilon_2>0$ are such that

$$(f_0-\varepsilon_1-\varepsilon_2)\gamma_\alpha^2\int_{\frac14}^{\frac34}G_\alpha(s,s)ds\geqslant 1.$$

Hence, for $(u,v)\in\partial S_r$, we have

$$\begin{aligned}T_1v(t)&=\int_0^1 G_\alpha(t,s)f(s,v(s),\ {}^cD^{\alpha-1}v(s))ds\\&\geqslant\gamma_\alpha\int_0^1 G_\alpha(s,s)f(s,v(s),\ D^{\alpha-1}v(s))ds\\&\geqslant\gamma_\alpha\int_0^1 G_\alpha(s,s)(f_0-\varepsilon_1-\varepsilon_2)v(s)ds\end{aligned}$$

$$\geqslant (f_0-\varepsilon_1-\varepsilon_2)\gamma_\alpha^2\int_0^1 G_\alpha(s,s)\|v\|ds \geqslant \|v\|.$$

Similarly, we obtain $T_2u(t)\geqslant \|u\|$. Hence,

$$\|T(u,v)\|\geqslant \|(u,v)\| \text{ for } (u,v)\in\partial S_r. \tag{3.8}$$

On the other hand for $(f^\infty)^{-1}>G_\alpha^*$, there exist $\overline{A}>0$ such that $f(t,v,{}^cD^{\alpha-1}v(t))\leqslant (f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})v$, for $t\in[0,1], v\in(\overline{A},\infty)$, where $\varepsilon_1^{'},\varepsilon_2^{'}>0$ satisfies $G_\alpha^*(f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})^{-1}\leqslant 1,$. Put

$$Z=\max_{t\in[0,1],v\in[0,\overline{A}]} f(t,v(t),{}^cD^{\alpha-1}v(t),$$

then $f(t,v(t),{}^cD^{\alpha-1})\leqslant Z+(f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})v$ choosing

$$A>\max\{a,\overline{A},ZG_\alpha^*(1-G_\alpha^*)(f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})^{-1}\}.$$

Then for $t\in[0,1],(u,v)\in\partial S_A$, we get

$$\begin{aligned}
T_1v(t) &= \int_0^1 G_\alpha(t,s)f(s,v(s),\ {}^cD^{\alpha-1}v(s)ds)\\
&\leqslant \int_0^1 G_\alpha(s,s)f(s,v(s),\ {}^cD^{\alpha-1}v(s))ds\\
&\leqslant \int_0^1 G_\alpha(s,s)(Z+(f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})v(s))ds\\
&\leqslant Z\int_0^1 G_\alpha(s,s)ds+(f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})\int_0^1(s,s)\|v\|ds\\
&< A-G_\alpha^*(f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})A+(f^\infty+\varepsilon_1^{'}+\varepsilon_2^{'})G_\alpha^*\|v\|\leqslant A.
\end{aligned}$$

Similarly we have $T_2u(t)<A$, that is

$$(u,v)\in\partial S_A \Rightarrow \|T(u,v)\|<\|(u,v)\|. \tag{3.9}$$

Case 2: In view of (C_2) for $(u,v)\in S$ and from the definitions of S, we obtain that

$$\min_{t\in[\frac14,\frac34]} u(t)\geqslant \gamma_\alpha\|u\|\text{and } \min_{t\in[\frac14,\frac34]} v(t)\geqslant \beta_\alpha\|u\|.$$

Therefore for $(u,v)\in\partial S_b$, we have $\|(u,v)\|=b$, for $t\in[\frac14,\frac34]$. From (C_2) we have

$$T_1v(t)=\int_0^1 G_\alpha(t,s)f(s,v(s),\ {}^cD^{\alpha-1}v(s))ds$$

$$\geqslant \gamma_\alpha \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s) f(s, v(s),\ {}^cD^{\alpha-1}v(s))ds$$

$$\geqslant \gamma_\alpha \frac{b}{\gamma_\alpha \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds} \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds = b$$

Similarly, we obtain $T_2u(t) \geqslant b$, which implies that

$$\|T(u,v)\| \geqslant \|(u,v)\| \text{ for } (u,v) \in \partial S_b. \tag{3.10}$$

On the other hand for $(u,v) \in \partial K_B$, we have that $\|(u,v)\| = B$ and using (C_2), we get

$$T_1v(t) = \int_0^1 G_\alpha(t,s) f(s, v(s),\ {}^cD^{\alpha-1}v(t))ds$$

$$\leqslant \int_0^1 G_\alpha(s,s) f(s, v(s),\ {}^cD^{\alpha-1}v(s))ds$$

$$\leqslant \frac{B}{G_\alpha^*} \int_0^1 G_\alpha(s,s)ds = B.$$

Similarly, $T_2u(t) \leqslant B$, hence,

$$\|T(u,v)\| \leqslant \|(u,v)\| \text{ for } (u,v) \in \partial S_B. \tag{3.11}$$

Thus (1.1) has at least one positive solution.

THEOREM 5. *Suppose that (B_1)-(B_2) hold and let the following conditions are satisfied:*

(C_3) $(f^0)^{-1} > G_\alpha^*$ *and* $f_\infty^{-1} < \gamma_\alpha^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds$, *specially* $f^0 = 0$ *and* $f_\infty = \infty$;

(C_4) $(g^0)^{-1} > G_\beta^*$ *and* $g_\infty^{-1} < \gamma_\beta^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\beta(s,s)ds$, *specially* $g^0 = 0$ *and* $g_\infty = \infty$.

Then the boundary value problem (1.1) *has at least one positive solution.*

Proof. The proof is similar to that of Theorem 4 and we omit it.

THEOREM 6. *Suppose (C_1)-(C_3) holds and also let the following two conditions are satisfied:*

(C_5) *f_∞^{-1} and $f_0^{-1} < \gamma_\alpha^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds$, particularly $f_0 = f_\infty = \infty$ and*

$$g_\infty^{-1} \text{ and } g_0^{-1} < \gamma_\beta^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds, \text{ particularly } g_0 = g_\infty = \infty.$$

(C_6) *There exist $d > 0$ such that*

$$\max_{0 \leqslant t \leqslant 1, (u,v) \in \partial S_d} f(t,v,D^{\alpha-1}v(t)) < \frac{d}{G_\alpha^*}, \text{ and } \max_{0 \leqslant t \leqslant 1, (u,v) \in \partial S_d} g(t,u, {}^cD^{\alpha-1}u(t)) < \frac{d}{G_\beta^*}.$$

Then boundary value problem (1.1) has at least two positive solution (u_1, v_1) and (u_2, v_2) which obey

$$0 < \|(u_1, v_1)\| < b < \|(u_2, v_2)\|. \tag{3.12}$$

Proof. **Case 1:** Consider (C_4), selecting $0 < a < d < A$, if

$$f_0^{-1} < \gamma_\alpha^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds \text{ and } g_0^{-1} < \gamma_\beta^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\beta(s,s)ds.$$

Then we have

$$\|T(u,v\|) \geqslant \|(u,v)\|, \text{ for } (u,v) \in \partial S_r. \tag{3.13}$$

If

$$f_\infty^{-1} < \gamma_\alpha^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds \text{ and } g_\infty^{-1} < \gamma_\beta^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\beta(s,s)ds,$$

then we have

$$\|T(u,v\|) \geqslant \|(u,v)\|, \text{ for } (u,v) \in \partial S_A. \tag{3.14}$$

Case 2: From (C_6) and $(u,v) \in \partial S_b$ we have

$$\begin{aligned} T_1v(t) &= \int_0^1 G_\alpha(t,s) f(s,v(s), {}^cD^{\alpha-1}v(s))ds \\ &\leqslant \int_0^1 G_\alpha(s,s) f(s,v(s), {}^cD^{\alpha-1}v(s))ds \\ &< \frac{b}{G_\alpha^*} \int_0^1 G_\alpha(s,s)ds = b, \end{aligned}$$

that implies $T_1v(t) < b$. Similarly $T_2u(t) < b$. It implies that $(u,v) \in \partial K_b$ which gives

$$\|T(u,v)\| < \|(u,v)\|. \tag{3.15}$$

REMARK 1. By applying lemma (3) to (3.8) and (3.9) or (3.10) and (3.11) which yields that T has a fixed point $(\overline{u},\overline{v}) \in \overline{S}_{a,A}$ or $(\overline{u},\overline{v})$ in $\overline{S}_{a_i,A_i}$ $(i=1,2)$ with $\overline{u}(t) \geqslant \gamma_\alpha \|\overline{u}\| > 0$ and $\overline{v}(t) \geqslant \gamma_\beta \|\overline{v}\| > 0$, $t \in [0,1]$.

Thus BVP (1.1) has a positive solution $(\overline{u},\overline{v})$ which complete the proof of Theorem 4.

REMARK 2. By applying lemma (3) to (3.8) and (3.12) or (3.13) and (3.15) yields that T has a fixed point $(u_1,v_1) \in \partial\overline{S}_{a,b}$ and a fixed point $(u_2,v_2) \in \partial\overline{S}_{b,A}$.

Thus BVP (1.1) has at least two positive solutions (u_1,v_1) and (u_2,v_2). By assuming $\|(u_1,v_1)\| \neq b$ and $\|(u_2,v_2)\| \neq b$ holds, and thus proof of Theorem 6 is completed. □

Going on above fashion we have the following result as in [16].

THEOREM 7. *Let (B_1) and (B_2) holds and suppose that the following condition hold.*

(C_7) $(f^0)^{-1}$ *and* $(g^\infty)^{-1} < G^*_\alpha$, $(g^0)^{-1}$ *and* $(g^\infty)^{-1} < G^*_\beta$;

(C_8) *There exist $E > 0$ constant such that:*

$$\max_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4},\ (u,v)\in\partial S_E} f(t,v(t)^c D^{\alpha-1}v(t)) > B\left(\gamma_\alpha \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds\right)^{-1},$$

$$\max_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4},(u,v)\in\partial S_E} g(t,u,^c D^{\beta-1}u(t)) > B\left(\gamma_\beta \int_{\frac{1}{4}}^{\frac{3}{4}} G_\beta(s,s)ds\right)^{-1}.$$

Then the BVP(1.1) has at least two positive solutions (u_1,u_2) and (u_2,v_2) which obey

$$0 < \|(u_1,v_1)\| < E < \|(u_2,v_2)\|. \tag{3.16}$$

THEOREM 8. *Let (B_1)-(B_3) hold. If there exist $2n$ positive numbers ℓ_k, L_k, $k = 1,2\ldots n$, with*

$$\ell_1 < \gamma_\alpha L_1 < L_1 < \ell_2 < \gamma_\alpha L_2 < L_2 \ldots d_n < \gamma_\alpha D_n < D_n$$
$$\ell_1 < \gamma_\beta L_1 < L_1 < \ell_2 < \gamma_\beta L_2 < L_2 \ldots d_n < \gamma_\beta D_n < D_n,$$

such that

(C_9) $f \geqslant d_k(\gamma_\alpha \int_0^1 G_\alpha(s,s)ds)^{-1}$, *for* $(t,v) \in [0,1]\times[\gamma_\alpha \ell_k, \ell_k]$,

$f \leqslant (G^*_\alpha)^{-1} D_k$, *for* $(t,v) \in [0,1]\times[\gamma_\alpha L_k, L_k], k = 1,2\ldots n$,

$g \geqslant d_k(\gamma_\beta \int_0^1 G_\beta(s,s)ds)^{-1}$, *for* $(t,u) \in [0,1]\times[\gamma_\beta \ell_k, \ell_k]$,

$g \leqslant (G^*_\beta)^{-1} D_k$ *for* $(t,u) \in [0,1]\times[\gamma_\beta L_k, L_k]$, $k = 1,2\ldots n$.

Then the BVP (1.1) has at least n positive solutions (u_k, v_k) *obeying* $\ell_k \leqslant \|(u_k, v_k)\| \leqslant L_k,\ k = 1,2\ldots n.$

THEOREM 9. *Suppose that* $(B_1)-(B_3)$ *holds. If there exist* $2n$ *positive numbers* $\ell_k, L_k,\ k = 1,2\ldots n$ *with* $\ell_1 < L_1 < \ell_2 < L_2 \ldots < \ell_n < L_n$, *such that*

(C_{10}) f *and* g *are non-decreasing on* $[0, L_n]$, *for all* $t \in [0,1]$,

(C_{11}) $f(t,\cdot) \geqslant \ell_k(\gamma_\alpha \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds)^{-1},\ f(t,\cdot) \leqslant \frac{D_k}{G_\alpha^*},\ k = 1,2\ldots n,$

and $g(t,\cdot) \geqslant \ell_k(\gamma_\beta \int_{\frac{1}{4}}^{\frac{3}{4}} G_\beta(s,s)ds)^{-1},\ g(t,\cdot) \leqslant \frac{D_k}{G_\beta^*},\ k = 1,2\ldots n.$

Then the BVP (1.1) has at least n positive solutions (u_k, v_k) *obeying* $\ell_k \leqslant \|(u_k, v_k)\| \leqslant L_k,\ v = 1,2\ldots n.$

4. Some examples

EXAMPLE 1. Consider the coupled system as follow

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) = \frac{-1}{24e^t+8}\left(\frac{1}{1+|v(t)|+|{}^cD^{\frac{1}{2}}v(t)|}\right),\ t \in [0,1], \\ {}^cD^{\frac{5}{2}}v(t) = \frac{-1}{24e^t+8}\left(\frac{1}{1+|u(t)|+|{}^cD^{\frac{3}{2}}u(t)|}\right),\ t \in [0,1], \\ u(0) = -u(1),\ D^{\frac{1}{2}}u(0) = -D^{\frac{1}{2}}u(1), \\ v(0) = -v(1),\ D^{\frac{1}{2}}v(0) = -D^{\frac{1}{2}}v(1), \end{cases}$$

where

$$f(t, v(t), {}^cD^{\frac{1}{2}}v(t)) = \frac{-1}{(24e^t+8)(1+|v(t)|+{}^c|D^{\frac{1}{2}}v(t)|)},$$
$$g(t, u(t), {}^cD^{\frac{3}{2}}u(t)) = \frac{-1}{(24e^t+8)(1+|u(t)|+{}^c|D^{\frac{3}{2}}u(t)|)}.$$

Then

$$|f(t, v_2, {}^cD^{\frac{1}{2}}v_2) - f(t_1, v_1, {}^cD^{\frac{1}{2}}v_1)| \leqslant \frac{2}{32}|v_2 - v_1| = \frac{1}{16}|v_2 - v_1|,$$

and

$$|g(t, u_2, {}^cD^{\frac{3}{2}}u_2) - g(t_1, u_1, {}^cD^{\frac{1}{2}}u_2)| \leqslant \frac{1}{16}|u_2 - u_1|,$$

where

$$G_\alpha^* \leqslant \frac{1}{2\Gamma\frac{3}{2}}\int_0^1 (1-s)^{\frac{1}{2}}ds + \frac{1}{2\Gamma(\frac{3}{2}-\frac{1}{2})}\int_0^1 (1-s)^{\frac{3}{2}-\frac{1}{2}-1}ds$$

which implies that $G_\alpha^* \leqslant 0.876126$, where $K_\alpha = \frac{1}{16}$. Similarly $G_\beta^* \leqslant 0.475675$, where $K_\beta = \frac{1}{16}$.

Now let

$$\max\{2G_\alpha^* K_\alpha, 2K_\alpha\Big(\frac{1+\Gamma(3-\alpha)\Gamma(\alpha-p+1)}{\Gamma(3-\alpha)\Gamma(\alpha-p+1)}\Big)\},$$

where: $\alpha=\frac{3}{2}, p=\frac{1}{2}$, $\rho_1=\max\{.10951575, .26611\}=.26611<1$ and

$$\max\{2G_\beta^* K_\beta, 2K_\beta\Big(\frac{1+\Gamma(3-\beta)\Gamma(\beta-p+1)}{\Gamma(3-\beta)\Gamma(\beta-q+1)}\Big)\}=\rho_2.$$

Putting $\beta=\frac{5}{2}, q=\frac{1}{2}$, we get

$$\rho_2=\max\{.05945, .160270\}=.160270<1.$$

Since $\rho_1<1$ and $\rho_2<1$. Thus by the use of Theorem 2, BVP (1) has a unique positive solution.

EXAMPLE 2. Consider the system of non-linear fractional differential equations.

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t)+\sqrt{v(t)}=0,\ {}^cD^{\frac{5}{2}}v(t)+\sqrt{u(t)}=0,\ 0<t<1,\\ u(0)=-u(1) \text{ and } v(0)=-v(1),\\ D^{\frac{1}{2}}u(0)=-D^{\frac{1}{2}}u(1) \text{ and } D^{\frac{1}{2}}v(0)=-D^{\frac{1}{2}}v(1). \end{cases}$$

Let $f=\sqrt{v}$ and $g=\sqrt{u}$. Now $f^0=\lim\limits_{v\to 0}\frac{f(v)}{v}=\infty$, $g^0=\infty$, $f^\infty=0=g^\infty$, thus clearly we can see that

$$0<\int_0^1 G_{\frac{3}{2}}(s,s)ds<\infty \text{ and } 0<\int_0^1 G_{\frac{5}{2}}(s,s)ds<\infty.$$

Thus by (C_1) $f^0=\infty$ and $f^\infty=0$ and $g^0=\infty$ and $g^\infty=0$. Hence by Theorem 5, BVP (2) has a positive solution.

EXAMPLE 3. Consider the following boundary value problem as

$$\begin{cases} {}^cD^{\frac{5}{2}}u(t)+[v(t)]^{100}=0,\ {}^cD^{\frac{7}{2}}v(t)+[u(t)]^{1000}=0,\ 0<t<1\\ u(0)=-u(1),\ {}^cD^{\frac{1}{2}}u(0)=-{}^cD^{\frac{1}{2}}u(1),\\ v(0)=-v(1),\ {}^cD^{\frac{1}{2}}v(0)=-{}^cD^{\frac{1}{2}}v(1). \end{cases}$$

Clearly (B_1) and (B_2) holds as $f^0=g^0=0$ and $f^\infty=g^\infty=\infty$. Thus by (C_3) and using Theorem 6, the BVP (3) has a positive solution.

5. Non-existence of positive solution

In this section we discuss the non-existence criteria for positive solution and present an example for the illustration of our result.

THEOREM 10. *Suppose* (B_1) *and* (B_2) *hold true and* $f < \frac{v}{G_\alpha^*}$ *and* $g < \frac{u}{G_\beta^*}$, *for all* $t \in [0,1], u > 0, v > 0$, *then the BVP* (1.1) *has no positive solution.*

Proof. On contrary let (u,v) be the positive solution of (1.1). Then $(u,v) \in B, u(t)$ and $v(t)$ both are positive for $0 < t < 1$ and

$$\|u\| = \max_{t\in[0,1]}|u(t)| + \max_{t\in[0,1]}|^cD^{\alpha-1}u(t)| = \max_{t\in[0,1]}|\int_0^1 G_\alpha(t,s)f(s,v(s),^cD^{\alpha-1}v(s)|ds$$

implies

$$\|u\| \leqslant \int_0^1 G_\alpha(s,s)f(s,v(s),^cD^{\alpha-1}v(s))ds < \int_0^1 G_\alpha(s,s)\frac{\|v\|}{G_\alpha^*}ds = \|v\|.$$

Similarly $\|v\| < \|u\|$ which is acontradiction. So BVP (1.1) has no positive solution. Hence proof is completed. □

THEOREM 11. *Let* (B_1) *and* (B_2) *holds and*

$$f(t,v(t),^cD^{\alpha-1}v(t)) > v\left(\gamma_\alpha^2\int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds\right)^{-1}$$

$$g(t,u(t),^cD^{\beta-1}u(t)) > u\left(\gamma_\beta^2\int_{\frac{1}{4}}^{\frac{3}{4}} G_\beta(s,s)ds\right)^{-1}$$

for all $t \in [0,1], u > 0$ *and* $v > 0$, *then BVP* (1.1) *has no positive solution.*

Proof. Proof is just like Theorem 11 so we omit it.

EXAMPLE 4. Consider the system of non linear fractional differential equations:

$$\begin{cases} ^cD^{\frac{5}{2}}u(t) + \dfrac{(2v^2(t)+v(t))(20+\cos v(t))}{v(t)+1} = 0,\ t \in [0,1], \\ ^cD^{\frac{5}{2}}v(t) + \dfrac{(2u^2(t)+u(t))(20+\cos u(t))}{u(t)+1} = 0,\ t \in [0,1], \\ u(0) = -u(1) \text{ and } ^cD^{\frac{1}{2}}u(0) = -^cD^{\frac{1}{2}}u(1), \\ v(0) = -v(1) \text{ and } ^cD^{\frac{1}{2}}v(0) = -^cD^{\frac{1}{2}}v(1). \end{cases}$$

Since (B_1) and (B_2) holds and also $f^0 = g^0 = 20$, $f^\infty = g^\infty = 43$, and

$$20v < f(t,v(t),{}^cD^{\frac{1}{2}}v(t)) < 43v$$
$$20u < g(t,u(t),{}^cD^{\frac{1}{2}}u(t)) < 43u$$
$$f(t,v(t),{}^cD^{\frac{1}{2}}u(t)) < 43v < \frac{v}{G^*_\alpha},$$

where $G^*_\alpha \approx 0.876126$ and $G^*_\beta \approx .475675$. Now

$$f(t,v(t),D^{\frac{1}{2}}v(t)) < \frac{v}{G^*_\alpha} \approx 1.1413v$$

implies

$$\Rightarrow f(t,\cdot) < 43v \approx 1.1413v \text{ and } g(t,\cdot) < 43u \approx 2.1022u.$$

Thus by Theorem 10, BVP (4) has no positive solution.
(ii) Also

$$f(t,\cdot) > 20v > v(\gamma_\alpha^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\alpha(s,s)ds)^{-1} \approx 15.8688v$$

and

$$g(t,\cdot) > 2u > u(\gamma_\beta^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G_\beta(s,s)ds)^{-1} \approx 12.678u.$$

Then by Theorem 11, BVP (4) has no solution.

REFERENCES

[1] V. Lakshmikantham and A. Vatsala, *General uniqueness and monotone iterative technique for fractional differential equation*, Applied Mathematics Letter., **21**, (2008), 828–834.

[2] V. Lakshmikantham and S. Leel, *Naguma-type uniqueness result for fractional differential equations*, Nonlinear Analysis., **71**, (2009), 2886–2889.

[3] M. Feng and W. Ge, *Existence result for a class of nth order m-point boundary value problem in Banach space*, Applied Mathematics Letter., **22**, (2009), 1303–1308.

[4] C. Goodrich, *Existence of a positive solution to a class of Fractional differential equations*, Computer Mathematics with Appllication., **59**, (2010), 3889-3999.

[5] Z. Bai, *On positive solution of a non local fractional boundary value problem*, Nonlinear Analysis, **72**, (2010), 916–924.

[6] B. Ahmad and J. Nieto, *Existence of solutions for non local boundary value problems of higher order non linear fractional differential equations*, Abstract and Applied Analysis, (2009), Article ID 494720.

[7] B. Ahmad and J.J. Nieto, *Existence and approximation of solution for a class of non-linear impulsive fractional differential equation with anti-periodic boundary conditions*, Nonlinear Analysis, **69**, (2008, 3291–3298.

[8] M. Benchohra, N. Hamidi and J. Henderson, *Existence and uniqueness of solution for fractional differential equations with Anti-periodic boundary condition*, Numerical Functional Analysis and Optimization, **34**,4(2013), 404–414.

[9] M. BENCHOHRA, J. R. GRAEF AND S. HAMANI, *Existence result for boundary value problem with non-linear fractional differential equations*, Applied Analysis, **87**,(2008), 851–863.
[10] B. AHMAD AND V. OTERO ESPINAR, *Existence of solution for fractional differential inclusions with anti-periodic boundary conditions* Boundary valaue problems, (2009), Art. 6259347,pp 11.
[11] X. SU, *Boundary value problem for a coupled system of non linear fractional differential equations*, Applied Mathematical Letter, **22**, (2009), 64–69.
[12] B. AHMAD AND J. J. NIETO, *Existence result for a coupled system of non linear fractional differential equations with three point boundary conditions*, Computer Mathematics with Appllication, **58**, (2009), 1838–1843.
[13] J. WANG, M. XIANG AND Z. LIU, *Positive solution to a non zero boundary value problem for a coupled system of non linear fractional differential equations*, International Journal of Differential Equations, **12**,(2010), Article ID 186928.
[14] I. PODLUBNY, *Fractional differential equations, Academic press*, New York 1993.
[15] A.A. KILBAS, H. SRIVASTAVA AND J. TRUJILLO, *Theory and application of fractional differential equations*, North Holland Mathematical Studies., **204**, Elseveir, Amsterdam, 2006.
[16] M. FENG, X. ZHANG AND W.GE, *New existence results for higher-order nonlinear fractional differential equation with intigral boundary conditions* , Boundary Value Problems, **20**,(2011), ArticleID720702.
[17] E. ZEIDLER, *Non linear fractional analysis and its application-I Fixed-point Theorem*, Springer, New York, USA 1986.
[18] M. REHMAN AND R. A. KHAN, *A note on boundary value problems for a coupled system of fractional differential equations*, computer Mathematics with Appllication, **61**, (2011), 2630-2637.
[19] B. AHMAD, AND S. K. NTOUYAS, *A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations*, Fractional Calculus and Applied Analysis, **17**, 2(2014), 348–360.
[20] Y. LIU AND B. AHMAD, *A Study of Impulsive Multiterm Fractional Differential Equations with Single and Multiple Base Points and Applications*, Scientific World Journal, (2014), 1–28.
[21] Y. LIU, *Solvability of multi-point boundary value problems for multiple term Riemann-Liouville fractional differential equations*, Computet Mathematics with Appllications, **64**, (2012), 413–431.
[22] Y. LIU, *Positive solutions for singular FDES*, U.P.B.Sci. Series A,. **73**,(2011), 89–100.
[23] Y. LIU AND X. LIU, *Existence of solutions of resonant boundary value problems for fractional differential equations*, Differential Equations and Control Process, **3**,(2012), 70–90.
[24] Y. LIU, T. HE AND H. SHI, *Three positive solutions of Sturm-Liouville boundary value problems for fractional differential equations*, Differ. Equ. Appl., **5**, (2013), 127–152, doi:10.7153/dea-05-09.

(Received August 8, 2014)
(Revised May 18, 2015)

Kamal Shah
Department of Mathematics
University of Malakand Chakdara
Dir(Lower), Khyber Pakhtunkhwa
Pakistan
e-mail: kamalshah408@gmail.com

Rahmat Ali Khan
Department of Mathematics
University of Malakand Chakdara
Dir(Lower), Khyber Pakhtunkhwa
Pakistan
e-mail: alipk@yahoo.com