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GLOBAL EXISTENCE AND BLOW–UP FOR NONAUTONOMOUS SYSTEMS WITH NON–LOCAL SYMMETRIC GENERATORS AND DIRICHLET CONDITIONS

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Abstract. We study a semilinear system of the form

$$\frac{\partial u_i(t,x)}{\partial t} = k_i(t)\mathscr{A}_i u_i(t,x) + u_{i'}^{\beta_i}(t,x), \quad t>0, \; x\in D,$$
$$u_i(0,x) = f_i(x), \quad x\in D, \quad u_i|_{D^c}\equiv 0,$$

where $D\subset\mathbb{R}^d$ is a bounded open domain, $k_i:[0,\infty)\to[0,\infty)$ is continuous, $\mathscr{A}_i$ is the infinitesimal generator of a symmetric jump-type process $Z_i\equiv\{Z_i(t)\}_{t\geqslant 0}$, $\beta_i>1$, $i\in\{1,2\}$ and $i'=3-i$. Under some assumptions on the infinitesimal generator $\mathscr{A}_i^D$ of the subprocess Z_i killed upon leaving D, $i=1,2$, we give sufficient conditions for global existence or finite-time blow-up of the positive mild solutions of our system. This paper can be considered as a continuation of the article [16].

1. Introduction

After the pioneering work of Fujita [7], many authors have studied global existence *vs.* blow-up in finite time of positive solutions for semilinear problems of the form

$$\frac{\partial u}{\partial t} = k(t)\mathscr{A}u + h(t)u^{\beta},$$
$$u(0) = f,$$

where $k,h:[0,\infty)\to[0,\infty)$ are not identically zero, the initial condition f is nonnegative, β is a positive constant and $\mathscr{A}$ is the infinitesimal generator of a Lévy process. The articles [5, 14, 15, 18, 20], are only a few examples for the autonomous ($k\equiv 1$) and nonautonomous (k not identically zero) cases with initial value problems or initial-boundary value problems for the above prototype when $\mathscr{A}$ is the Laplacian Δ, the fractional Laplacian $\Delta_\alpha\equiv-(-\Delta)^{\alpha/2}$, $0<\alpha\leqslant 2$ or the fractional Laplacian with gradient perturbation $\Delta_\alpha+b\cdot\nabla$, where the symbol ∇ is the gradient in $x\in\mathbb{R}^d$ and $b(x):=\big(b_1(x),\dots,b_d(x)\big)$ is an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$.

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The dichotomy global existence vs blow up in finite time for weakly coupled systems of the form

$$\frac{\partial u_i}{\partial t} = k_i(t)\mathscr{A}_i u_i + h_i(t) u_{i'}^{\beta_i}$$
$$u_i(0) = f_i \geqslant 0,$$

$i \in \{1,2\}$ and $i' = 3-i$, was initially considered by Escobedo and Herrero [6] for the case when $k_i, h_i \equiv 1$, and $\mathscr{A}_i = \Delta$, $i = 1,2$, is the Laplacian operator. More general cases for initial value ploblems involving $\mathscr{A}_i = \Delta_{\alpha_i}$, $0 < \alpha_i \leqslant 2$ and even systems with fractional derivatives in space and time, and cases for the initial-boundary value problems for Laplacians, can be found for instance in [8, 9, 11, 12, 13, 19, 23] and references therein. In this paper we have considered the nonautonomous semilinear system with Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial u_i(t,x)}{\partial t} &= k_i(t)\mathscr{A}_i u_i(t,x) + u_{i'}^{\beta_i}(t,x), \quad t>0, \ x \in D, \\ u_i(0,x) &= f_i(x), \quad x \in D, \quad u_i|_{D^c} \equiv 0, \end{aligned} \tag{1.1}$$

$i \in \{1,2\}$ and $i' = 3-i$, where D is a $C^{1,1}$ bounded open domain of $\mathbb{R}^d$ with $d \geqslant 1$, $k_i : [0,\infty) \to [0,\infty)$ is continuos and not identically zero, $\beta_i > 1$ is constant, the initial condition f_i is a nonnegative function in the space $C_0(D)$ of continuous functions on D vanishing on D^c, $i = 1,2$ and $\mathscr{A}_i$ is the infinitesimal generator of a Lévy process Z_i whose corresponding subprocesses Z_D^i killed upon leaving D satisfies some conditions given below and that are satisfied by many Lévy processes with theoretical and applied importance, such as, for example, symmetric α-stable processes, mixed symmetric stable processes and relativistic α-stable processes. The infinitesimal generator of a relativistic α-stable process is of the form

$$m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}, \ 0 < \alpha \leqslant 2 \ \text{ and } \ m > 0;$$

when $\alpha = 1$, this operator corresponds to the kinetic energy of a relativistic particle with mass m and when $m = 1$ it is just the Bessel potential kernel. Of course, when $m \downarrow 0$, $m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}$ converges (in the distributional sense) to the fractional Laplacian $\Delta_\alpha \equiv \left(-\Delta\right)^{\alpha/2}$. Moreover the results of this paper are also worth for mixed symmetric stable processes with infinitesimal generator $\Delta_\alpha + a\Delta_\beta$ with $0 < \beta < \alpha < 2$ and $0 \leqslant a \leqslant 1$. Mixed symmetric stable processes are the independent sum of a symmetric α-stable process and a symmetric β-stable process with weight a; a such Lévy process runs on two different scales: on the small spatial scale, the α component dominates, while on the large one the β component takes over. Since both components play essential roles, these type of processes can not be considered as a perturbation of the α-stable process or the β-stable process (see [2, 4]).

This work can be considered as a continuation of the article [16]. In [16] system (1.1) was studied with $k_1 = k_2 \equiv k$ and $\mathscr{A}_1 = \mathscr{A}_2 \equiv \mathscr{A}$ being the infinitesimal generator of a symmetric Lévy processes Z; it is shown that the positive mild solution of (1.1) blows up in finite time for any initial conditions f_1, f_2 in the space $C_0(D)$ of continuous functions on D vanishing on D^c provided that

$$\min_{i\in\{1,2\}} \int_D f_i(x)\varphi_0(x)dx$$
$$> \max_{i\in\{1,2\}} \left[\left(\frac{\beta_1\beta_2-1}{\beta_i+1}\right)\left(\frac{\beta_i+1}{\beta_{i'}+1}\right)^{\frac{\beta_i}{\beta_i+1}} \int_0^\infty \min_{i\in\{1,2\}} \left(\frac{e^{-\lambda_0 K(r,0)}}{\|\varphi_0\|_1}\right)^{\beta_i-1} dr\right]^{\frac{\beta_i+1}{1-\beta_1\beta_2}},$$

where φ_0 is the eigenfunction corresponding to the first eigenvalue of $\mathscr{A}$ on D, and

$$K(t,s) = \int_s^t k(r)dr, \quad 0 \leqslant s \leqslant t.$$

On the other hand, in [16] was also proved that the positive mild solution for this case is global if

$$(\beta_i - 1)\int_0^\infty \|S_D\big(K(t,0)\big)g\|_\infty^{\beta_i-1} dt < 1, \ \ i = 1,2,$$

where $\{S_D(t)\}_{t\geqslant 0}$ is the semigroup with generator $\mathscr{A}_D$. The results that we are going to present in this paper extend and are consistent with the above results.

System (1.1) can be used as a model to describe heat and burning in a two-component media with temporary-inhomogeneous thermal conductivity, where u_1 and u_2 represent the temperatures of the two reactant components. Unlike article [16], in our case, the thermal conductivity may be different for each substance.

2. Killed additive process and assumptions

Throughout this paper we assume that D is a $C^{1,1}$ bounded open domain of $\mathbb{R}^d$ with $d \geqslant 1$, $k_i : [0,\infty) \to [0,\infty)$ is continuous and not identically zero and $\mathscr{A}_i$ is the infinitesimal generator of a symmetric jump process Z_i in $\mathbb{R}^d$, $i = 1,2$. Letting

$$K_i(t,s) = \int_s^t k_i(r)dr, \quad 0 \leqslant s \leqslant t, \tag{2.1}$$

it is known (see [16], p.3 and 4) that the time-inhomogeneous Markov process $W_i \equiv \{W_i(t)\}_{t\geqslant 0}$, where $W_i(t) \stackrel{D}{=} Z_i\big(K_i(t,0)\big)$ (here $\stackrel{D}{=}$ means equality in distribution) has the transition probability

$$\begin{aligned} P_i(s,x,t,B) &= \Pr\Big[Z_i\big(K_i(t,s)\big) \in B - x\Big] \\ &= \Big(S_i\big(K_i(t,s)\big)1_B\Big)(x), \end{aligned}$$

where $\{S_i(t)\}_{t\geqslant 0}$ denotes the semigroup with generator $\mathscr{A}_i$, $i = 1,2$ and 1_B is the indicator function of B. Moreover, the function $(t,x) \mapsto S_i\big(K_i(t,s)\big)f(x)$, $(t,x) \in [s,\infty)\times\mathbb{R}^d$, is the unique solution of

$$\frac{\partial w(t,x)}{\partial t} = k_i(t)\mathscr{A}_i w(t,x), \quad t > s, \ \ x \in \mathbb{R}^d,$$

$$w(s,x) = f(x), \quad f \in C_0(\mathbb{R}^d),$$

$i \in \{1,2\}$. For this reason we call $\{W_i(t)\}_{t\geqslant 0}$ the time-inhomogeneous Markov process corresponding to the family of generators $\{k_i(t)\mathscr{A}_i\}_{t\geqslant 0}$. Letting

$$p_i(s,x,t,y) = p_i\big(K_i(t,s),x,y\big), \ \ 0 \leqslant s \leqslant t, \quad x,y \in \mathbb{R}^d,$$

we see that $p_i(s,x,t,y)$ is a transition density function for the process $\{W_i(t)\}_{t\geqslant 0}$. We define

$$\tau_D^i = \inf\{t > 0 : W_i(t) \notin D\} \quad \text{and} \quad \widehat{\tau}_D^i = \inf\{t > 0 : Z_i(t) \notin D\}.$$

Using that $W_i(t) \stackrel{D}{=} Z_i\big(K_i(t,0)\big)$ we get

$$\widehat{\tau}_D^i = K_i\big(\tau_D^i, 0\big). \tag{2.2}$$

Let us consider the Z_D^i process killed on leaving D, which is given by

$$Z_D^i(t) = \begin{cases} Z_i(t) & \text{on} \quad \{t < \widehat{\tau}_D^i\}, \\ \partial & \text{on} \quad \{t \geqslant \widehat{\tau}_D^i\}, \end{cases}$$

where ∂ is a cemetery state. The state space of $\{Z_D^i(t)\}_{t\geqslant 0}$ is the set $D_\partial = D \cup \{\partial\}$ and its transition probability is

$$P_D^i(t,x,\Gamma) = P_x^i\big[Z_i(t) \in \Gamma; t < \widehat{\tau}_D^i\big], \ \ 0 < t, \ \ x \in D, \ \ \Gamma \in \mathscr{B}(D),$$

where $\mathscr{B}(D)$ denotes the Borel σ-field on D. Here and in the sequel P_x^i and E_x^i denote, respectively, the distribution and expectation with respect to the process $\{x + Z_i(t)\}_{t\geqslant 0}$ starting in $x \in \mathbb{R}^d$, but we use the same symbol $\{Z_i(t)\}_{t\geqslant 0}$ for the resulting process.

Let $\{S_D^i(t)\}_{t\geqslant 0}$ be the semigroup associated to the process $\{Z_i(t)\}_{t\geqslant 0}$ killed on exiting D, and let $p_D^i(t,x,y)$ be the transition density function of $\{S_D^i(t)\}_{t\geqslant 0}$, i.e.

$$S_D^i(t)f(x) = E_x^i\big[f\big(Z_i(t)\big); t < \widehat{\tau}_D^i\big] = \int_D f(y) p_D^i(t,x,y)dy, \ x \in D, \ t > 0, \ f \in \mathbb{B}^+(\mathbb{R}^d),$$

where $\mathbb{B}^+\big(\mathbb{R}^d\big)$ is the space of nonnegative bounded measurable functions on $\mathbb{R}^d$.

Throughout the remainder of this paper, we assume that $p_D^i(t,x,y)$ is a strictly positive and continuous function on $(0,\infty) \times D \times D$ such that $p_D^i(t,x,y) = p_D^i(t,y,x)$ for all $t > 0$ and $x,y \in D$. We also assume that $\{S_D^i(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup of contractions on the space $L^2(D)$ such that $S_D^i(t) \in C_0(D)$ for any $f \in C_0(D)$, i.e, we suppose that $\{S_D^i(t)\}_{t\geqslant 0}$ is a Feller semigroup. We suppose also that the linear operators $S_D^i(t)$, $t \geqslant 0$ are compact, and that there exists an orthonormal basis of eigenfunctions $\{\varphi_n^i\}_{n=0}^\infty$ of the operator $S_D^i(t)$ with corresponding eigenvalues $\{e^{-\lambda_n^i t}\}_{n=0}^\infty$ satisfying $0 < \lambda_0^i < \lambda_1^i \leqslant \lambda_2^i \ldots$, and $\lim_{n\to\infty} \lambda_n^i = \infty$, such that all eigenfunctions φ_n^i are continuous and real-valued and that the eigenfunction φ_0^i is strictly positive on D. Moreover, we assume that $\{S_D^i(t)\}_{t\geqslant 0}$ is an intrinsically ultracontractive semigroup. The infinitesimal generators $\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, $\Delta_\alpha + a\Delta_\beta$, $0 < \beta < \alpha < 2$,

$0 \leqslant a \leqslant 1$ and $m-\left(m^{2/\alpha}-\Delta\right)^{\alpha/2}$, $0<\alpha<2$, $m>0$ are examples of generators such that their semigroups generated by their corresponding processes killed on exiting D satisfies the above properties (see [1, 2, 3, 10, 21]).

Finally, we assume that there exist constants $c_1>0$ and $c_2>0$ such that

$$c_1\varphi_0^1(x) \leqslant \varphi_0^2(x) \leqslant c_2\varphi_0^1(x), \quad x\in D. \tag{2.3}$$

This property holds, for example, when

$$\mathscr{A}_i = m_i-\left(m_i^{2/\alpha}-\Delta\right)^{\alpha/2},\ m_i>0,\ i=1,2$$

or

$$\mathscr{A}_i = \Delta_\alpha + a\Delta_{\beta_i},\ 0\leqslant a\leqslant 1 \text{ and } 0<\beta_i<\alpha,\ i=1,2$$

or even, if $\mathscr{A}_1 = m-\left(m^{2/\alpha}-\Delta\right)^{\alpha/2}$, $m>0$ and $\mathscr{A}_2=\Delta_\alpha+a\Delta_\beta$, $0\leqslant a\leqslant 1$ and $0<\beta<\alpha<2$. In fact, when $\mathscr{A}_1$ and $\mathscr{A}_2$ are the infinitesimal generators of relativistic α-stable processes with mass $m_1>0$ and $m_2>0$, respectively, are obtained from [1], Theorem 1.1(ii), constants $b_1>0$ and $b_2>0$ such that for all $x\in D$,

$$b_1 p_D^1(1,x,x) \leqslant p_D^2(1,x,x) \leqslant b_2 p_D^1(1,x,x). \tag{2.4}$$

Now, by the intrinsic ultracontractive property of $S_D^i(t)$, $i=1,2$ (see [21], p. 11) there exist constants $B_1^i>0$ and $B_2^i>0$ such that for all $x\in D$,

$$B_1^i\left(\varphi_0^i(x)\right)^2 \leqslant p_D^i(1,x,x) \leqslant B_2^i\left(\varphi_0^i(x)\right)^2, \quad i=1,2. \tag{2.5}$$

Finally, (2.3) it follows easily from (2.4), (2.5) and the fact that φ_0^i is strictly positive on D, $i=1,2$. The proof of (2.3) for $\mathscr{A}_i=\Delta_\alpha+a_i\Delta_{\beta_i}$, $0<\beta_i<\alpha<2$, $0\leqslant a_i\leqslant 1$, $i=1,2$ follows in a similar form using Theorem 1.1(ii) of [2] and the validity for the last case it follows from Theorem 1.1(ii) of [1] and Theorem 1.1(ii) of [2].

Let us consider now the additive process $\{W_i(t)\}_{t\geqslant 0}$ killed on exiting D, namely

$$W_D^i(t) = \begin{cases} W_i(t) & \text{on} \quad \{t<\tau_D^i\}, \\ \partial & \text{on} \quad \{t\geqslant \tau_D^i\}. \end{cases}$$

The transition function of $\{W_D^i(t)\}_{t\geqslant 0}$ is given by

$$P_D^i(s,x,t,\Gamma) = P_x^i\left[Z_i\left(K_i(t,s)\right)\in\Gamma; K_i(t,s)<\widehat{\tau}_D^i\right],\ \ 0\leqslant s<t,\ x\in D,\ \Gamma\in\mathscr{B}(D).$$

Hence the transition density function of $\{W_D^i(t)\}_{t\geqslant 0}$ is given by $p_D^i(s,x,t,y) = p_D^i\left(K_i(t,s),x,y\right)$ and thus, for every $f\in L^2(D)$,

$$U_D^i(t,s)f(x) \equiv \int_D f(y)p_D^i(s,x,t,y)dy = S_D^i\left(K_i(t,s)\right)f(x), \quad 0\leqslant s<t, \ \ x\in D. \tag{2.6}$$

PROPOSITION 1. *The function* $p_D^i(s,x,t,y)$ *is a density of* $P_D^i(s,x,t,\Gamma)$, *which is strictly positive and continuous for* $0<s<t<\infty$ *and* $x,y\in D$.

Proof. This follows easily from the fact that $p_D^i(t,x,y)$ is a density of $P_D^i(t,x,\Gamma)$, which is strictly positive and continuous on $(0,\infty)\times D\times D$. □

Using (2.6) and the fact that $\{S_D^i(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup of contractions on $L^2(D)$, we obtain that $\{U_D^i(t,s)\}_{t\geqslant s\geqslant 0}$ is an evolution family of contractions on $L^2(D)$.

3. Local existence of a mild solution

A solution of the integral system

$$u_i(t,x) = U_D^i(t,0)f_i(x) + \int_0^t U_D^i(t,r)u_{i'}^{\beta_i}(r,x)dr, \quad t\geqslant 0, \ \ x\in D, \tag{3.1}$$

is called a mild solution of (1.1); here and in the sequel, $i\in\{1,2\}$ and $i' = 3-i$.

We are going to assume that f_1 and f_2 are nonnegative functions in $L^\infty(D)$, where $L^\infty(D)$ is the space of real-valued essentially bounded functions defined on D.

Since the evolution system $\{U_D^i(t,s)\}_{t\geqslant s\geqslant 0}$ preserves positivity (due to (2.6)) we have that

$$u_{i,0}(t,x)\equiv U_D^i(t,0)f_i(x)\geqslant 0, \quad t\geqslant 0, \ \ x\in D. \tag{3.2}$$

Define

$$u_{i,n+1}(t,x) = \mathscr{F}_i u_{i,n}(t,x), \quad t\geqslant 0, \ \ x\in D, \quad n=0,1,\dots, \tag{3.3}$$

where $\mathscr{F}_i$ is given by

$$\mathscr{F}_i v_i(t,x) = U_D^i(t,0)f_i(x) + \int_0^t U_D^i(t,r)v_{i'}^{\beta_i}(r,x)dr \tag{3.4}$$

for any nonnegative $v_i\in L^\infty(D)$, $i=1,2$. Therefore $u_{i,0}(t,x)\leqslant u_{i,1}(t,x)$ for all $t\geqslant 0$, $x\in D$. Using again that $\{U_D^i(t,s)\}_{t\geqslant s\geqslant 0}$ preserves positivity it follows by induction that $u_{i,n}(t,x)\leqslant u_{i,n+1}(t,x)$ for $n=0,1,\dots$, $i=1,2$. Hence the limit

$$u_i(t,x)\equiv \limsup_{n\to\infty} u_{i,n}(t,x), \quad i=1,2, \tag{3.5}$$

exists for all $t\geqslant 0$ and $x\in D$, $i=1,2$. From the monotone convergence theorem we conclude that $u_i(t,x)$ satisfies (3.1). This shows that if the integral system (3.1) has a solution, such solution is given by the increasing limit (3.5).

Our proof of the existence of local solutions follows closely the proof of Theorem 2 in [16], see also [22]. For any constant $\tau>0$ let

$$E_\tau \equiv \{(u_1,u_2):[0,\tau]\to L^\infty(D)\times L^\infty(D), |||(u_1,u_2)|||<\infty\},$$

where

$$|||(u_1,u_2)||| \equiv \sup_{0\leqslant t\leqslant\tau}\{\|u_1(t,\cdot)\|_\infty + \|u_2(t,\cdot)\|_\infty\}.$$

The couple $\left(E_\tau, |||\cdot|||\right)$ is a Banach space and $P_\tau\equiv\{(u_1,u_2)\in E_\tau : u_1\geqslant 0, u_2\geqslant 0\}$ and $C_R\equiv\{(u_1,u_2)\in E_\tau : |||(u_1,u_2)|||\leqslant R\}$, $R>0$, are closed subsets of E_τ.

THEOREM 1. *Let $f_i : D \to [0,\infty)$ be in $L^\infty(D)$, $i = 1,2$. There exists a constant $\tau = \tau(\|f_1\|_\infty, \|f_2\|_\infty) > 0$ such that the integral system (3.1) possesses a unique non-negative local solution in $L^\infty([0,\tau]\times D)\times L^\infty([0,\tau]\times D)\cap C_R$.*

Proof. Let us define the operator Ψ on $C_R \cap P_\tau$ by

$$\Psi\big(u_1,u_2\big)(t,x) = \big(U_D^1(t,0)f_1(x), U_D^2(t,0)f_2(x)\big) \\ + \Big(\int_0^t U_D^1(t,r)u_2^{\beta_1}(r,x)dr, \int_0^t U_D^2(t,r)u_1^{\beta_2}(r,x)dr\Big).$$

We are going to show that Ψ is a contraction on $C_R \cap P_\tau$ for suitably chosen $R > 0$ and $\tau > 0$. In fact, if $(u_1,u_2),(\tilde{u}_1,\tilde{u}_2) \in C_R \cap P_\tau$, then

$$|||\Psi\big(u_1,u_2\big) - \Psi\big(\tilde{u}_1,\tilde{u}_2\big)||| \\ \leqslant \sup_{0\leqslant t\leqslant\tau}\int_0^t \|u_2^{\beta_1}(r,\cdot) - \tilde{u}_2^{\beta_1}(r,\cdot)\|_\infty dr + \sup_{0\leqslant t\leqslant\tau}\int_0^t \|u_1^{\beta_2}(r,\cdot) - \tilde{u}_1^{\beta_2}(r,\cdot)\|_\infty dr,$$

and using the elementary inequality $|a^p - b^p| \leqslant p(a\vee b)^{p-1}|a-b|$, which holds for all $a,b > 0$ and $p \geqslant 1$, we get

$$|||\Psi\big(u_1,u_2\big) - \Psi\big(\tilde{u}_1,\tilde{u}_2\big)||| \leqslant \beta_1 R^{\beta_1-1}\int_0^\tau \|u_2(r,\cdot) - \tilde{u}_2(r,\cdot)\|_\infty dr \\ + \beta_2 R^{\beta_2-1}\int_0^\tau \|u_1(r,\cdot) - \tilde{u}_1(r,\cdot)\|_\infty dr \\ \leqslant (\beta_1 R^{\beta_1-1} \vee \beta_2 R^{\beta_2-1})|||(u_1,u_2) - (\tilde{u}_1,\tilde{u}_2)|||\tau. \tag{3.6}$$

Noticing that

$$|||\Psi\big(u_1,u_2\big)||| \leqslant \|f_1\|_\infty + \|f_2\|_\infty + \tau\big(R^{\beta_1} + R^{\beta_2}\big),$$

and taking $R > 0$ big enough and $\tau > 0$ sufficiently small we get from (3.6) that Ψ is a contraction mapping on $C_R \cap P_\tau$. Thus, the Banach fixed-point theorem implies that (3.1) posesses a unique solution (u_1,u_2) such that $u_i \geqslant 0$, $i = 1,2$. □

4. Global existence of the mild solution

Here we suppose again that $f_i \in L^\infty(D)$, $i = 1,2$. Our proof of the next theorem follows closely the proof of Theorem 3 in [16], see also Theorem 2.2 in [17].

THEOREM 2. *Let f_1, f_2 be nonnegative. If*

$$(\beta_i - 1)\int_0^\infty \|U_D^i(t,0)f_i\|_\infty^{\beta_i-1}dt < 1, \quad i = 1,2,$$

then the solution of the integral system (3.1) is global.

Proof. If f_1, f_2 are identically zero, the nonnegative solution of (3.1) is clearly $(u_1, u_2) \equiv (0,0)$, which is global. Now, if f_1, f_2 are not identically zero, putting

$$B_i(t) = \left[1 - (\beta_i - 1)\int_0^t \|U_D^i(r,0)f_i\|_\infty^{\beta_i - 1} dr\right]^{-\frac{1}{\beta_i - 1}}$$

we get $B_i(0) = 1$ and

$$\begin{aligned}
&\frac{d}{dt}B_i(t) \\
&= -\frac{1}{\beta_i - 1}\left[1 - (\beta_i - 1)\int_0^t \|U_D^i(r,0)f_i\|_\infty^{\beta_i - 1} dr\right]^{-\frac{1}{\beta_i - 1} - 1}\left[-(\beta_i - 1)\|U_D^i(t,0)f_i\|_\infty^{\beta_i - 1}\right] \\
&= \|U_D^i(t,0)f_i\|_\infty^{\beta_i - 1} B_i^{\beta_i}(t),
\end{aligned}$$

which gives

$$B_i(t) = 1 + \int_0^t \|U_D^i(r,0)g\|_\infty^{\beta_i - 1} B_i^{\beta_i}(r)dr, \quad i = 1,2. \tag{4.1}$$

Since the evolution system $\{U_D^i(t,s)\}_{t \geqslant s \geqslant 0}$ is positivity-preserving we can choose two continuous functions $v_i : [0,\infty) \times D \longrightarrow [0,\infty)$, $i = 1,2$, such that $v_i(t,\cdot) \in C_b(D)$ for all $t \geqslant 0$ and

$$0 \leqslant v_i(t,x) \leqslant \min\{B_1(t)U_D^1(t,0)f_1(x), B_2(t)U_D^2(t,0)f_2(x)\}, \quad t \geqslant 0, \ \ i = 1,2.$$

Let us define $\mathscr{F}_i$ on the space of nonnegative functions in $L^\infty(D)$ as in (3.4). Then

$$\begin{aligned}
0 \leqslant \mathscr{F}_i v_i(t,x) &\leqslant U_D^i(t,0)f_i(x) + \int_0^t B_i^{\beta_i}(r)U_D^i(t,r)\big(U_D^i(r,0)f_i(x)\big)^{\beta_i} dr \\
&\leqslant U_D^i(t,0)f_i(x) + \int_0^t B_i^{\beta_i}(r)U_D^i(t,r)U_D^i(r,0)f_i(x)\|U_D^i(r,0)f_i\|_\infty^{\beta_i - 1} dr \\
&= U_D^i(t,0)f_i(x)\left[1 + \int_0^t \|U_D^i(r,0)f_i\|_\infty^{\beta_i - 1} B_i^{\beta_i}(r)dr\right] \\
&= B_i(t)U_D^i(t,0)f_i(x),
\end{aligned}$$

where we used (4.1) in the last equality. Therefore,

$$0 \leqslant \mathscr{F}_i v_i(t,x) \leqslant \max\{B_1(t)U_D^1(t,0)f_1(x), B_2(t)U_D^2(t,0)f_2(x)\}, \quad t \geqslant 0, \ \ x \in D.$$

Defining now the sequence $\{u_{i,n}(t,x)\}_{n=0}^\infty$ as in (3.2) and (3.3) it is follows as in Section 3 that $u_{i,n}(t,x) \leqslant u_{i,n+1}(t,x)$, $n \geqslant 0$. Hence

$$u_i(t,x) \equiv \limsup_{n \to \infty} u_{i,n}(t,x) \leqslant \max\{B_1(t)U_D^1(t,0)f_1(x), B_2(t)U_D^2(t,0)f_2(x)\} < \infty$$

for all $t \geqslant 0$ and $x \in D$. Therefore (u_1, u_2) is a global mild solution of (1.1). □

5. Blow up in finite time of the positive mild solution

Recall that φ_0^i is the eigenfunction corresponding to the first eigenvalue λ_0^i of the infinitesimal generator of the semigroup $\{S_D^i(t)\}_{t\geqslant 0}$, $i=1,2$. Arguing as in the case of Brownian motion in a bounded domain (see [18], p. 287), it can be shown that $\left(\varphi_0^i\right)^2(x)dx$ is the unique invariant measure of the semigroup $\{Q_i(t)\}_{t\geqslant 0}$ given by

$$Q_i(t)g(x)=\frac{e^{\lambda_0^i t}}{\varphi_0^i(x)}S_D^i(t)\left(g\varphi_0^i\right)(x),\ \ x\in D,\ g\in C_b\left(D\right),\ t\geqslant 0.$$

Thus, defining

$$E_i[h]:=\int h(x)\left(\varphi_0^i\right)^2(x)dx,\quad h\in C_b(D),$$

and

$$T_i(t,s)g(x)=\frac{e^{\lambda_0^i K_i(t,s)}}{\varphi_0^i(x)}S_D^i\left(K_i(t,s)\right)\left(g\varphi_0^i\right)(x),\quad x\in D,\ \ g\in C_b\left(D\right),\ \ t\geqslant s\geqslant 0,$$

we have that for any $t\geqslant s\geqslant 0$ and $g\in C_b(D)$,

$$E_i[Q_i(t)g]=E_i[g]\quad\text{and}\quad T_i(t,s)g=Q_i(K_i(t,s))g. \tag{5.1}$$

LEMMA 1. *For any $t\geqslant s\geqslant 0$ and $g_i\in C_b(D)$, $i=1,2$,*

$$E_i[T_i(t,s)g_i]=E_i[g_i],\quad i=1,2.$$

Proof. This is a direct consequence of (5.1). □

THEOREM 3. *Let $f_i=g_i\varphi_0^i$, where $g_i\in C_b(D)$ is nonnegative and not identically zero, $i=1,2$. If*

$$\min_{i\in\{1,2\}}\left\langle f_i,\varphi_0^i\right\rangle>\max_{i\in\{1,2\}}\left[\left(\frac{\beta_1\beta_2-1}{\beta_i+1}\right)\left(\frac{\beta_i+1}{\beta_{i'}+1}\right)^{\frac{\beta_i}{\beta_i+1}}\times\int_0^\infty\min_{i\in\{1,2\}}\left\{C_i\frac{e^{-\lambda_0^{i'}K_{i'}(r,0)\beta_i}e^{\lambda_0^iK_i(r,0)}}{\|\varphi_0^{i'}\|_i^{\beta_i-1}}\right\}dr\right]^{\frac{\beta_i+1}{1-\beta_1\beta_2}}, \tag{5.2}$$

where $C_1=\left(1/c_2\right)\left(c_1/c_2\right)^{\beta_1}$ and $C_2=c_1\left(c_1/c_2\right)^{\beta_2}$, then the mild solution of (1.1) blows up in finite time.

Proof. Notice that $\left\langle f_i,\varphi_0^i\right\rangle=E_i[g_i]>0$, $i=1,2$. We define

$$w_i(t,x)=\frac{e^{\lambda_0^iK_i(t,0)}u_i(t,x)}{\varphi_0^i(x)},\quad i=1,2,$$

where (u_1,u_2) is the mild solution of (1.1), i.e., (u_1,u_2) solves the integral system (3.1). Multiplying both sides of (3.1) by $\varphi_0^1(x)^{-1}\exp\left(\lambda_0^1 K_1(t,0)\right)$ we get

$$
\begin{aligned}
& w_1(t,x) \\
& = T_1(t,0)g_1(x) + \int_0^t \frac{e^{\lambda_0^1 K_1(t,0)}}{\varphi_0^1(x)} U_D^1(t,r) u_2^{\beta_1}(r,x)dr \\
& = T_1(t,0)g_1(x) + \int_0^t \frac{e^{\lambda_0^1 K_1(t,0)}}{\varphi_0^1(x)} U_D^1(t,r)\left(\frac{u_2^{\beta_1}(r,x)}{\left(\varphi_0^1\right)^{\beta_1-1}(x)}\left(\varphi_0^1\right)^{\beta_1-1}(x)\right)dr \\
& = T_1(t,0)g_1(x) + \int_0^t e^{\lambda_0^1 K_1(r,0)} \frac{e^{\lambda_0^1 K_1(t,r)}}{\varphi_0^1(x)} U_D^1(t,r)\left(\frac{u_2^{\beta_1}(r,x)}{\left(\varphi_0^1\right)^{\beta_1-1}(x)}\left(\varphi_0^1\right)^{\beta_1-1}(x)\right)dr \\
& = T_1(t,0)g_1(x) + \int_0^t e^{\lambda_0^1 K_1(r,0)} T_1(t,r)\left(\frac{u_2^{\beta_1}(r,x)}{\left(\varphi_0^1\right)^{\beta_1}(x)}\left(\varphi_0^1\right)^{\beta_1-1}(x)\right)dr \\
& = T_1(t,0)g_1(x) + \int_0^t T_1(t,r)\left(\frac{e^{\lambda_0^2 K_2(r,0)\beta_1} u_2^{\beta_1}(r,x)}{\left(\varphi_0^1\right)^{\beta_1}(x)}\right) e^{-\lambda_0^2 K_2(r,0)\beta_1} \\
& \cdot e^{\lambda_0^1 K_1(r,0)}\left(\varphi_0^1\right)^{\beta_1-1}(x)dr.
\end{aligned}
$$

The last equality renders

$$
\begin{aligned}
E_1[w_1(t,\cdot)] &= E_1[T_1(t,0)g_1] \\
&\quad + \int_0^t E_1\left[T_1(t,r)\left(\frac{e^{\lambda_0^2 K_2(r,0)\beta_1} u_2^{\beta_1}(r,\cdot)}{\left(\varphi_0^1\right)^{\beta_1}(\cdot)}\right) e^{-\lambda_0^2 K_2(r,0)\beta_1} e^{\lambda_0^1 K_1(r,0)}\left(\varphi_0^1\right)^{\beta_1-1}(\cdot)\right]dr,
\end{aligned}
$$

and due to Lemma 1,

$$
\begin{aligned}
E_1[w_1(t,\cdot)] &= E_1[g_1] \\
&\quad + \int_0^t E_1\left[\left(\frac{e^{\lambda_0^2 K_2(r,0)\beta_1} u_2^{\beta_1}(r,\cdot)}{\left(\varphi_0^1\right)^{\beta_1}(\cdot)}\right) e^{-\lambda_0^2 K_2(r,0)\beta_1} e^{\lambda_0^1 K_1(r,0)}\left(\varphi_0^1\right)^{\beta_1-1}(\cdot)\right]dr.
\end{aligned}
$$

It follows that for any $\varepsilon > 0$,

$$
\begin{aligned}
& E_1[w_1(t+\varepsilon,\cdot)] - E_1[w_1(t,\cdot)] \\
& \quad = \int_t^{t+\varepsilon} E_1\left[\left(\frac{e^{\lambda_0^2 K_2(r,0)\beta_1} u_2^{\beta_1}(r,\cdot)}{\left(\varphi_0^1\right)^{\beta_1}(\cdot)}\right) e^{-\lambda_0^2 K_2(r,0)\beta_1} e^{\lambda_0^1 K_1(r,0)}\left(\varphi_0^1\right)^{\beta_1-1}(\cdot)\right]dr. \qquad (5.3)
\end{aligned}
$$

Using (2.3) we obtain

$$
E_1\left[\left(\frac{e^{\lambda_0^2 K_2(r,0)\beta_1} u_2^{\beta_1}(r,\cdot)}{\left(\varphi_0^1\right)^{\beta_1}(\cdot)}\right) e^{-\lambda_0^2 K_2(r,0)\beta_1} e^{\lambda_0^1 K_1(r,0)}\left(\varphi_0^1\right)^{\beta_1-1}(\cdot)\right]
$$

$$
\begin{aligned}
&\geqslant E_1\left[\left(\frac{e^{\lambda_0^2K_2(r,0)\beta_1}u_2^{\beta_1}(r,\cdot)}{\left(\frac{1}{c_1}\varphi_0^2(\cdot)\right)^{\beta_1}}\right)e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\left(\frac{1}{c_2}\varphi_0^2(\cdot)\right)^{\beta_1-1}\right]\\
&\geqslant c_1\left(\frac{c_1}{c_2}\right)^{\beta_1-1}E_1\left[\left(\frac{e^{\lambda_0^2K_2(r,0)\beta_1}u_2^{\beta_1}(r,\cdot)}{\left(\varphi_0^2(\cdot)\right)^{\beta_1}}\right)e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\left(\varphi_0^2(\cdot)\right)^{\beta_1-1}\right]\\
&=c_1\left(\frac{c_1}{c_2}\right)^{\beta_1-1}E_1\left[w_2(s,\cdot)^{\beta_1}e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\left(\varphi_0^2(\cdot)\right)^{\beta_1-1}\right]\\
&=c_1\left(\frac{c_1}{c_2}\right)^{\beta_1-1}e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}E_1\left[w_2(s,\cdot)^{\beta_1}\left(\varphi_0^2(\cdot)\right)^{\beta_1-1}\right]\\
&=c_1\left(\frac{c_1}{c_2}\right)^{\beta_1-1}e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\int w_2(s,x)^{\beta_1}\left(\varphi_0^2(x)\right)^{\beta_1-1}\left(\varphi_0^1(x)\right)^2dx.
\end{aligned}
$$

From here, using again (2.3) we get

$$
\begin{aligned}
&E_1\left[\left(\frac{e^{\lambda_0^2K_2(r,0)\beta_1}u_2^{\beta_1}(r,\cdot)}{\left(\varphi_0^1\right)^{\beta_1}(\cdot)}\right)e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\left(\varphi_0^1\right)^{\beta_1-1}(\cdot)\right]\\
&\geqslant c_1\left(\frac{c_1}{c_2}\right)^{\beta_1-1}e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\int w_2(r,x)^{\beta_1}\left(\varphi_0^2(x)\right)^{\beta_1-1}\left(\frac{1}{c_2}\varphi_0^2(x)\right)^2dx\\
&=\frac{1}{c_2}\left(\frac{c_1}{c_2}\right)^{\beta_1}e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\|\varphi_0^2\|_1\int\left[w_2(r,x)\varphi_0^2(x)\right]^{\beta_1}\frac{\varphi_0^2(x)}{\|\varphi_0^2\|_1}dx\\
&\geqslant\frac{1}{c_2}\left(\frac{c_1}{c_2}\right)^{\beta_1}e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}\|\varphi_0^2\|_1\left(\int w_2(r,x)\frac{\left(\varphi_0^2\right)^2(x)}{\|\varphi_0^2\|_1}dx\right)^{\beta_1}\\
&=\frac{1}{c_2}\left(\frac{c_1}{c_2}\right)^{\beta_1}\frac{e^{-\lambda_0^2K_2(r,0)\beta_1}e^{\lambda_0^1K_1(r,0)}}{\|\varphi_0^2\|_1^{\beta_1-1}}E_2\left[w_2(r,\cdot)\right]^{\beta_1},
\end{aligned}
\tag{5.4}
$$

where we have used Jensen's inequality with respect to the probability measure $\|\varphi_0^2\|_1^{-1}\varphi_0^2(x)dx$. Let $h_i(t):=E_i[w_i(t,\cdot)]$, $i=1,2$. Plugging (5.4) into (5.3), and afterward multiplying the resulting inequality by ε^{-1} with $\varepsilon\to 0$, we obtain that

$$
h_1'(t)\geqslant C_1\frac{e^{-\lambda_0^2K_2(t,0)\beta_1}e^{\lambda_0^1K_1(t,0)}}{\|\varphi_0^2\|_1^{\beta_1-1}}h_2^{\beta_1}(t),\quad h_1(0)=\langle f_1,\varphi_0^1\rangle. \tag{5.5}
$$

Now, multiplying both sides of (3.1) (with $i=2$) by $\varphi_0^2(x)^{-1}\exp\left(\lambda_0^2K_2(t,0)\right)$, it can be obtained, similarly as above, that

$$
h_2'(t)\geqslant C_2\frac{e^{-\lambda_0^1K_1(t,0)\beta_2}e^{\lambda_0^2K_2(t,0)}}{\|\varphi_0^1\|_1^{\beta_2-1}}h_1^{\beta_2}(t),\quad h_2(0)=\langle f_2,\varphi_0^2\rangle. \tag{5.6}
$$

Let

$$
c(t)=\min_{i\in\{1,2\}}\left\{C_i\frac{e^{-\lambda_0^{i'}K_{i'}(t,0)\beta_i}e^{\lambda_0^iK_i(t,0)}}{\|\varphi_0^{i'}\|_1^{\beta_i-1}}\right\},\quad N=\min_{i\in\{1,2\}}\left\{\langle f_i,\varphi_0^i\rangle\right\}>0,
$$

and consider the ordinary differential system

$$p_1'(t) = c(t)p_2^{\beta_1}(t), \quad p_2'(t) = c(t)p_1^{\beta_2}(t), \quad p_i(0) = N, \; i = 1,2. \tag{5.7}$$

It follows that $\int_0^t p_1^{\beta_2}(r)p_1'(r)\,dr = \int_0^t p_2^{\beta_1}(r)p_2'(r)dr$, and

$$\frac{1}{\beta_2+1}\left[p_1^{\beta_2+1}(t) - N^{\beta_2+1}\right] = \frac{1}{\beta_1+1}\left[p_2^{\beta_1+1}(t) - N^{\beta_1+1}\right].$$

Notice that if $N \leqslant \Big((\beta_2+1)/(\beta_1+1)\Big)^{1/(\beta_2+1)} N^{(\beta_1+1)/(\beta_2+1)}$, then

$$\frac{1}{\beta_2+1} p_1^{\beta_2+1}(t) \leqslant \frac{1}{\beta_1+1} p_2^{\beta_1+1}(t) \tag{5.8}$$

and, if $N \geqslant \Big((\beta_2+1)/(\beta_1+1)\Big)^{1/(\beta_2+1)} N^{(\beta_1+1)/(\beta_2+1)}$, then

$$\frac{1}{\beta_2+1} p_1^{\beta_2+1}(t) \geqslant \frac{1}{\beta_1+1} p_2^{\beta_1+1}(t). \tag{5.9}$$

If (5.8) holds, then

$$p_2(t) \geqslant \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{1}{\beta_1+1}} p_1^{\frac{\beta_2+1}{\beta_1+1}}(t).$$

Substituting this into the first equation of (5.7), we get

$$p_1'(t) \geqslant c(t)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} p_1^{\frac{\beta_1(\beta_2+1)}{\beta_1+1}}(t),$$

which is the same as

$$p_1^{-\frac{\beta_1(\beta_2+1)}{\beta_1+1}}(t)p_1'(t) \geqslant c(t)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}}.$$

Integrating both sides of the above inequality from 0 to t yields

$$\frac{\beta_1+1}{1-\beta_1\beta_2}\left[p_1^{\frac{1-\beta_1\beta_2}{\beta_1+1}}(t) - N^{\frac{1-\beta_1\beta_2}{\beta_1+1}}\right] \geqslant \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^t c(r)dr.$$

Thus, in view of $\beta_1, \beta_2 > 1$,

$$p_1(t) \geqslant \left[N^{\frac{1-\beta_1\beta_2}{\beta_1+1}} - \left(\frac{\beta_1\beta_2-1}{\beta_1+1}\right)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^t c(r)dr\right]^{\frac{\beta_1+1}{1-\beta_1\beta_2}}. \tag{5.10}$$

Similarly, if (5.9) holds, we can show that

$$p_2(t) \geqslant \left[N^{\frac{1-\beta_1\beta_2}{\beta_2+1}} - \left(\frac{\beta_1\beta_2-1}{\beta_2+1}\right)\left(\frac{\beta_2+1}{\beta_1+1}\right)^{\frac{\beta_2}{\beta_2+1}} \int_0^t c(r)dr\right]^{\frac{\beta_2+1}{1-\beta_1\beta_2}}. \tag{5.11}$$

Since the function $\int_0^t c(r)dr$ is continuous and increases to $\int_0^\infty c(r)dr$, (5.10) and (5.11) imply that for some $0 < t_0 < \infty$,

$$\lim_{t\uparrow t_0} \|w_i(t,\cdot)\|_\infty \geqslant \lim_{t\uparrow t_0} p_i(t) = \infty \quad \text{for } i=1 \text{ o } i=2,$$

whenever

$$\min_{i\in\{1,2\}} \langle f_i, \varphi_0^i \rangle >$$

$$\max_{i\in\{1,2\}} \left[\left(\frac{\beta_1\beta_2 - 1}{\beta_i + 1}\right)\left(\frac{\beta_i + 1}{\beta_{i'} + 1}\right)^{\frac{\beta_i}{\beta_i+1}} \int_0^\infty \min_{i\in\{1,2\}} \left\{ C_i \frac{e^{-\lambda_0^{i'} K_{i'}(r,0)\beta_i} e^{\lambda_0^i K_i(r,0)}}{\|\varphi_0^{i'}\|_i^{\beta_i - 1}} \right\} dr \right]^{\frac{\beta_i+1}{1-\beta_1\beta_2}}.$$

□

Using the intrinsic ultracontractivity of the semigroups $\{S_D^i(t)\}_{t\geqslant 0}$, $i = 1,2$, the proof of the following theorem follows in exactly the same form as in Theorem 6 of [16] with appropiate changes in the notations.

THEOREM 4. *Let $f_1, f_2 \in C_0(D)$ be two nonnegative functions which are not identically zero. If Condition (5.2) holds, then the mild solution of (1.1) blows up in finite time.*

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