GLOBAL EXISTENCE AND BLOW–UP FOR NONAUTONOMOUS SYSTEMS WITH NON–LOCAL SYMMETRIC GENERATORS AND DIRICHLET CONDITIONS

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Abstract. We study a semilinear system of the form
\[
\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t} &= k_i(t) \mathcal{A}_i(t,x) + u_i^{\beta_i}(t,x), \quad t > 0, \ x \in D, \\
u_i(0,x) &= f_i(x), \ x \in D, \quad u_i|_{D^c} \equiv 0,
\end{aligned}
\]
where \( D \subset \mathbb{R}^d \) is a bounded open domain, \( k_i : [0,\infty) \to [0,\infty) \) is continuous, \( \mathcal{A}_i \) is the infinitesimal generator of a symmetric jump-type process \( Z_i \equiv \{Z_i(t)\}_{t \geq 0} \), \( \beta_i > 1, \ i \in \{1,2\} \) and \( \iota' = 3 - i \). Under some assumptions on the infinitesimal generator \( \mathcal{A}_i^D \) of the subprocess \( Z_i \) killed upon leaving \( D \), \( i = 1,2 \), we give sufficient conditions for global existence or finite-time blow-up of the positive mild solutions of our system. This paper can be considered as a continuation of the article [16].

1. Introduction

After the pioneering work of Fujita [7], many authors have studied global existence vs. blow-up in finite time of positive solutions for semilinear problems of the form
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= k(t) \mathcal{A} u + h(t) u^\beta, \\
u(0) &= f,
\end{aligned}
\]
where \( k, h : [0,\infty) \to [0,\infty) \) are not identically zero, the initial condition \( f \) is nonnegative, \( \beta \) is a positive constant and \( \mathcal{A} \) is the infinitesimal generator of a Lévy process. The articles [5, 14, 15, 18, 20], are only a few examples for the autonomous \( (k \equiv 1) \) and nonautonomous \( (k \) not identically zero) cases with initial value problems or initial-boundary value problems for the above prototype when \( \mathcal{A} \) is the Laplacian \( \Delta \), the fractional Laplacian \( \Delta_\alpha \equiv (-\Delta)^{\alpha/2} \), \( 0 < \alpha \leq 2 \) or the fractional Laplacian with gradient perturbation \( \Delta_\alpha + b \cdot \nabla \), where the symbol \( \nabla \) is the gradient in \( x \in \mathbb{R}^d \) and \( b(x) := \left(b_1(x), \ldots, b_d(x)\right) \) is an \( \mathbb{R}^d \)-valued function on \( \mathbb{R}^d \).


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The dichotomy global existence vs blow up in finite time for weakly coupled systems of the form

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= k_i(t)\mathcal{A}_i u_i + h_i(t)u_i^{\beta_i}, \\
u_i(0) &= f_i \geq 0,
\end{align*}
\]

\(i \in \{1, 2\}\) and \(i' = 3 - i\), was initially considered by Escobedo and Herrero [6] for the case when \(k_i, h_i \equiv 1\), and \(\mathcal{A}_i = \Delta\), \(i = 1, 2\), is the Laplacian operator. More general cases for initial value problems involving \(\mathcal{A}_i = \Delta_{\alpha_i}\), \(0 < \alpha_i \leq 2\) and even systems with fractional derivatives in space and time, and cases for the initial-boundary value problems for Laplacians, can be found for instance in [8, 9, 11, 12, 13, 19, 23] and references therein. In this paper we have considered the nonautonomous semilinear system with Dirichlet boundary conditions

\[
\begin{align*}
\frac{\partial u_i(t, x)}{\partial t} &= k_i(t)\mathcal{A}_i u_i(t, x) + u_i^{\beta_i}(t, x), \quad t > 0, \quad x \in D, \\
u_i(0, x) &= f_i(x), \quad x \in D, \quad u_i|_{D^c} \equiv 0,
\end{align*}
\]

\(i \in \{1, 2\}\) and \(i' = 3 - i\), where \(D\) is a \(C^{1,1}\) bounded open domain of \(\mathbb{R}^d\) with \(d \geq 1\), \(k_i : [0, \infty) \to [0, \infty)\) is continuous and not identically zero, \(\beta_i > 1\) is constant, the initial condition \(f_i\) is a nonnegative function in the space \(C_0(D)\) of continuous functions on \(D\) vanishing on \(D^c\), \(i = 1, 2\) and \(\mathcal{A}_i\) is the infinitesimal generator of a Lévy process \(Z_i\) whose corresponding subprocesses \(Z^D_{\beta_i}\) killed upon leaving \(D\) satisfies some conditions given below and that are satisfied by many Lévy processes with theoretical and applied importance, such as, for example, symmetric \(\alpha\)-stable processes, mixed symmetric stable processes and relativistic \(\alpha\)-stable processes. The infinitesimal generator of a relativistic \(\alpha\)-stable process is of the form

\[
m - (m^{2/\alpha} - \Delta)^{\alpha/2}, \quad 0 < \alpha \leq 2 \quad \text{and} \quad m > 0;
\]

when \(\alpha = 1\), this operator corresponds to the kinetic energy of a relativistic particle with mass \(m\) and when \(m = 1\) it is just the Bessel potential kernel. Of course, when \(m \downarrow 0\), \(m - (m^{2/\alpha} - \Delta)^{\alpha/2}\) converges (in the distributional sense) to the fractional Laplacian \(\Delta_{\alpha} \equiv (-\Delta)^{\alpha/2}\). Moreover the results of this paper are also worth for mixed symmetric stable processes with infinitesimal generator \(\Delta_{\alpha} + a\Delta_{\beta}\) with \(0 < \beta < \alpha < 2\) and \(0 \leq a \leq 1\). Mixed symmetric stable processes are the independent sum of a symmetric \(\alpha\)-stable process and a symmetric \(\beta\)-stable process with weight \(a\); a such Lévy process runs on two different scales: on the small spatial scale, the \(\alpha\) component dominates, while on the large one the \(\beta\) component takes over. Since both components play essential roles, these type of processes can not be considered as a perturbation of the \(\alpha\)-stable process or the \(\beta\)-stable process (see [2, 4]).

This work can be considered as a continuation of the article [16]. In [16] system (1.1) was studied with \(k_1 = k_2 \equiv k\) and \(\mathcal{A}_1 = \mathcal{A}_2 \equiv \mathcal{A}\) being the infinitesimal generator of a symmetric Lévy processes \(Z\); it is shown that the positive mild solution of (1.1)
blows up in finite time for any initial conditions \( f_1, f_2 \) in the space \( C_0(D) \) of continuous functions on \( D \) vanishing on \( D^c \) provided that

\[
\min_{i \in \{1,2\}} \int_D f_i(x) \varphi_0(x) dx > \max_{i \in \{1,2\}} \left[ \left( \frac{\beta_1 \beta_2 - 1}{\beta_1 + 1} \right) \left( \frac{\beta_2 + 1}{\beta_2 + 1} \right) \beta_i \int_0^\infty \min_{i \in \{1,2\}} \left( \frac{e^{-\lambda_0(K(t,0))}}{\| \varphi_0 \|_1} \right) \beta_i - 1 \right] \frac{\beta_i + 1}{\beta_1 \beta_2},
\]

where \( \varphi_0 \) is the eigenfunction corresponding to the first eigenvalue of \( \mathcal{A} \) on \( D \), and

\[ K(t,s) = \int_s^t k(r) dr, \quad 0 \leq s \leq t. \]

On the other hand, in [16] was also proved that the positive mild solution for this case is global if

\[ (\beta_i - 1) \int_0^\infty \| S_D(K(t,0)) g \|^{\beta_i - 1}_\infty dt < 1, \quad i = 1, 2, \]

where \( \{S_D(t)\}_{t \geq 0} \) is the semigroup with generator \( \mathcal{A}_D \). The results that we are going to present in this paper extend and are consistent with the above results.

System (1.1) can be used as a model to describe heat and burning in a two-component medium with temporary-inhomogeneous thermal conductivity, where \( u_1 \) and \( u_2 \) represent the temperatures of the two reactant components. Unlike article [16], in our case, the thermal conductivity may be different for each substance.

2. Killed additive process and assumptions

Throughout this paper we assume that \( D \) is a \( C^{1,1} \) bounded open domain of \( \mathbb{R}^d \) with \( d \geq 1 \), \( k_i : [0, \infty) \to [0, \infty) \) is continuous and not identically zero and \( \mathcal{A}_i \) is the infinitesimal generator of a symmetric jump process \( Z_i \) in \( \mathbb{R}^d \), \( i = 1, 2 \). Letting

\[ K_i(t,s) = \int_s^t k_i(r) dr, \quad 0 \leq s \leq t, \tag{2.1} \]

it is known (see [16], p.3 and 4) that the time-inhomogeneous Markov process \( W_i \equiv \{W_i(t)\}_{t \geq 0} \), where \( W_i(t) \overset{D}{=} Z_i(K_i(t,0)) \) (here \( \overset{D}{=} \) means equality in distribution) has the transition probability

\[ P_i(s,x,t,B) = \Pr \left[ Z_i(K_i(t,s)) \in B - x \right] = \left( S_i(K_i(t,s)) 1_B \right)(x), \]

where \( \{S_i(t)\}_{t \geq 0} \) denotes the semigroup with generator \( \mathcal{A}_i \), \( i = 1, 2 \) and \( 1_B \) is the indicator function of \( B \). Moreover, the function \( (t,x) \mapsto S_i(K_i(t,s)) f(x), \quad (t,x) \in [s, \infty) \times \mathbb{R}^d \), is the unique solution of

\[ \frac{\partial w(t,x)}{\partial t} = k_i(t) \mathcal{A}_i w(t,x), \quad t > s, \quad x \in \mathbb{R}^d, \]
\[ w(s,x) = f(x), \quad f \in C_0(\mathbb{R}^d), \]

\( i \in \{1, 2\}. \) For this reason we call \( \{W_t(t)\}_{t \geq 0} \) the time-inhomogeneous Markov process corresponding to the family of generators \( \{k_i(t) \} \).

Let us consider the \( \phi_i \) operators

\[ \left\{ D \phi_i \right\} \]

functions \( \phi_i \in \mathbb{R}^d \),

where \( s \) satisfying \( 0 < \lambda_1 < \lambda_2 \), and \( \lim \lambda_n = \infty \), such that all eigenfunctions \( \phi_n \) are continuous and real-valued and that the eigenfunction \( \phi_0 \) is strictly positive on \( D \).

Moreover, we assume that \( \Delta \alpha \equiv (-\Delta)^{\alpha/2} \), \( 0 < \alpha < 2 \), \( \Delta_\alpha + a\Delta_\beta \), \( 0 < \beta < \alpha < 2 \),

\[ \tau_D^i = \inf\{t > 0 : W_t(t) \notin D\} \quad \text{and} \quad \tilde{\tau}_D^i = \inf\{t > 0 : Z_i(t) \notin D\}. \]

Using that \( W_t(t) \overset{D}{=} Z_i(K_i(t), 0) \) we get

\[ \tilde{\tau}_D^i = K_i(t_D^i, 0). \quad (2.2) \]

Let us consider the \( Z_D^i \) process killed on leaving \( D \), which is given by

\[ Z_D^i(t) = \begin{cases} Z_i(t) & \text{on} \quad \{t < \tilde{\tau}_D^i\} , \\ \partial & \text{on} \quad \{t \geq \tilde{\tau}_D^i\}, \end{cases} \]

where \( \partial \) is a cemetery state. The state space of \( \{Z_D^i(t)\}_{t \geq 0} \) is the set \( D = D \cup \{\partial\} \)

and its transition probability is

\[ P_D^i(t, x, \Gamma) = P_t^i(\{Z_i(t) \in \Gamma ; t < \tilde{\tau}_D^i\}, 0 < t, x \in D, \ \Gamma \in \mathcal{B}(D), \]

where \( \mathcal{B}(D) \) denotes the Borel \( \sigma \)-field on \( D \). Here and in the sequel \( P_t^i \) and \( E_t^i \) denote, respectively, the distribution and expectation with respect to the process \( \{x + Z_i(t)\}_{t \geq 0} \) starting in \( x \in \mathbb{R}^d \), but we use the same symbol \( \{Z_i(t)\}_{t \geq 0} \) for the resulting process.

Let \( \{S_D^i(t)\}_{t \geq 0} \) be the semigroup associated to the process \( \{Z_i(t)\}_{t \geq 0} \) killed on exiting \( D \), and let \( S_D^i(t, x, y) \) be the transition density function of \( \{S_D^i(t)\}_{t \geq 0} \), i.e.

\[ S_D^i(t, x, y, \Gamma) = P_t^i(\{Z_i(t) \in \Gamma ; t < \tilde{\tau}_D^i\}, 0 < t, x \in D, \ \Gamma \in \mathcal{B}(D). \]

Throughout the remainder of this paper, we assume that \( p_D^i(t, x, y) \) is a strictly positive and continuous function on \( (0, \infty) \times D \times D \) such that \( p_D^i(t, x, y) = p_D^i(t, y, x) \) for all \( t > 0 \) and \( x, y \in D \). We also assume that \( \{S_D^i(t)\}_{t \geq 0} \) is a strongly continuous semigroup of contractions on the space \( L^2(D) \) such that \( S_D^i(t) \in C_0(D) \) for any \( f \in C_0(D) \), i.e., we suppose that \( \{S_D^i(t)\}_{t \geq 0} \) is a Feller semigroup. We suppose also that the linear operators \( S_D^i(t), t \geq 0 \) are compact, and that there exists an orthonormal basis of eigenfunctions \( \{\phi_n^i\}_{n=0}^\infty \) of the operator \( S_D^i(t) \) with corresponding eigenvalues \( \{e^{-\lambda_n^i}t\}_{n=0}^\infty \) satisfying \( 0 < \lambda_0^i < \lambda_1^i < \lambda_2^i \),... , and \( \lim_{n \to \infty} \lambda_n^i = \infty \), such that all eigenfunctions \( \phi_n^i \) are continuous and real-valued and that the eigenfunction \( \phi_0^i \) is strictly positive on \( D \).

The infinitesimal generators \( \Delta_\alpha \equiv (-\Delta)^{\alpha/2} \), \( 0 < \alpha < 2 \), \( \Delta_\alpha + a\Delta_\beta \), \( 0 < \beta < \alpha < 2 \),
\[ 0 \leq a \leq 1 \text{ and } m - \left( m^2/\alpha - \Delta \right)^{\alpha/2}, \quad 0 < \alpha < 2, \quad m > 0 \text{ are examples of generators such that their semigroups generated by their corresponding processes killed on exiting } D \text{ satisfies the above properties (see [1, 2, 3, 10, 21]).} \]

Finally, we assume that there exist constants \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[ c_1 \phi_0^i(x) \leq \phi_0^i(x) \leq c_2 \phi_0^i(x), \quad x \in D. \tag{2.3} \]
This property holds, for example, when
\[ \mathcal{A}_i = m_i - \left( m_i^2/\alpha - \Delta \right)^{\alpha/2}, \quad m_i > 0, \quad i = 1, 2 \]
or
\[ \mathcal{A}_i = \Delta \alpha + a \Delta \beta_i, \quad 0 \leq a \leq 1 \text{ and } 0 < \beta_i < \alpha, \quad i = 1, 2 \]
or even, if \( \mathcal{A}_1 = m - \left( m^2/\alpha - \Delta \right)^{\alpha/2}, \quad m > 0 \) and \( \mathcal{A}_2 = \Delta \alpha + a \Delta \beta, \quad 0 \leq a \leq 1 \text{ and } 0 < \beta < \alpha < 2 \). In fact, when \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are the infinitesimal generators of relativistic \( \alpha \)-stable processes with mass \( m_1 > 0 \) and \( m_2 > 0 \), respectively, are obtained from [1], Theorem 1.1(ii), constants \( b_1 > 0 \) and \( b_2 > 0 \) such that for all \( x \in D \),
\[ b_1 p_D^i(1) \phi_0^i(x) \leq p_D^i(1) \phi_0^i(x) \leq b_2 p_D^i(1) \phi_0^i(x). \tag{2.4} \]
Now, by the intrinsic ultracontractive property of \( S_D^i(t) \), \( i = 1, 2 \) (see [21], p. 11) there exist constants \( B_1^i > 0 \) and \( B_2^i > 0 \) such that for all \( x \in D \),
\[ B_1^i \phi_0^i(x) \leq p_D^i(1) \phi_0^i(x) \leq B_2^i \phi_0^i(x), \quad i = 1, 2. \tag{2.5} \]
Finally, (2.3) it follows easily from (2.4), (2.5) and the fact that \( \phi_0^i \) is strictly positive on \( D, \quad i = 1, 2 \). The proof of (2.3) for \( \mathcal{A}_i = \Delta \alpha + a_i \Delta \beta_i, \quad 0 < \beta_i < \alpha < 2, \quad 0 \leq a_i \leq 1, \quad i = 1, 2 \) follows in a similar form using Theorem 1.1(ii) of [2] and the validity for the last case it follows from Theorem 1.1(ii) of [1] and Theorem 1.1(ii) of [2].

Let us consider now the additive process \( \{ W_i(t) \}_{t \geq 0} \) killed on exiting \( D \), namely
\[ W_D^i(t) = \begin{cases} W_i(t) & \text{on } \{ t < \tau_D^i \}, \\ \partial & \text{on } \{ t \geq \tau_D^i \}. \end{cases} \]
The transition function of \( \{ W_D^i(t) \}_{t \geq 0} \) is given by
\[ P_D^i(s,x,t,\Gamma) = P_x^i \left[ Z_i \left( K_i(t,s) \right) \right] \in \Gamma; K_i(t,s) < \tau_D^i \], \quad 0 \leq s < t, \quad x \in D, \quad \Gamma \in \mathcal{B}(D). \]

Hence the transition density function of \( \{ W_D^i(t) \}_{t \geq 0} \) is given by \( p_D^i(s,x,t,y) = P_D^i \left( K_i(t,s),x,y \right) \) and thus, for every \( f \in L^2(D) \),
\[ U_D^i(t,s)f(x) = \int_D f(y)p_D^i(s,x,t,y)dy = S_D^i \left( K_i(t,s) \right) f(x), \quad 0 \leq s < t, \quad x \in D. \tag{2.6} \]

**PROPOSITION 1.** The function \( p_D^i(s,x,t,y) \) is a density of \( P_D^i(s,x,t,\Gamma) \), which is strictly positive and continuous for \( 0 < s < t < \infty \) and \( x, y \in D \).
Proof. This follows easily from the fact that $p^i_D(t,x,y)$ is a density of $P^i_D(t,x,\Gamma)$, which is strictly positive and continuous on $(0,\infty) \times D \times D$. \hfill \Box

Using (2.6) and the fact that $\{S^i_D(t)\}_{t \geq 0}$ is a strongly continuous semigroup of contractions on $L^2(D)$, we obtain that $\{U^i_D(t,s)\}_{t \geq s \geq 0}$ is an evolution family of contractions on $L^2(D)$.

3. Local existence of a mild solution

A solution of the integral system

$$u_i(t,x) = U^i_D(t,0) f_i(x) + \int_0^t U^i_D(t,r) u_{i'}(r,x) dr, \quad t \geq 0, \ x \in D, \quad (3.1)$$

is called a mild solution of (1.1); here and in the sequel, $i \in \{1,2\}$ and $i' = 3 - i$.

We are going to assume that $f_1$ and $f_2$ are nonnegative functions in $L^\infty(D)$, where $L^\infty(D)$ is the space of real-valued essentially bounded functions defined on $D$.

Since the evolution system $\{U^i_D(t,s)\}_{t \geq s \geq 0}$ preserves positivity (due to (2.6)) we have that

$$u_{i,0}(t,x) = U^i_D(t,0) f_i(x) \geq 0, \quad t \geq 0, \ x \in D. \quad (3.2)$$

Define

$$u_{i,n+1}(t,x) = F_i u_{i,n}(t,x), \quad t \geq 0, \ x \in D, \quad n = 0,1,\ldots, \quad (3.3)$$

where $F_i$ is given by

$$F_i v_i(t,x) = U^i_D(t,0) f_i(x) + \int_0^t U^i_D(t,r) v_{i'}(r,x) dr \quad (3.4)$$

for any nonnegative $v_i \in L^\infty(D)$, $i = 1,2$. Therefore $u_{i,0}(t,x) \leq u_{i,1}(t,x)$ for all $t \geq 0$, $x \in D$. Using again that $\{U^i_D(t,s)\}_{t \geq s \geq 0}$ preserves positivity it follows by induction that $u_{i,n}(t,x) \leq u_{i,n+1}(t,x)$ for $n = 0,1,\ldots$, $i = 1,2$. Hence the limit

$$u_i(t,x) = \lim_{n \to \infty} \sup_{n \geq 0} u_{i,n}(t,x), \quad i = 1,2, \quad (3.5)$$

exists for all $t \geq 0$ and $x \in D$, $i = 1,2$. From the monotone convergence theorem we conclude that $u_i(t,x)$ satisfies (3.1). This shows that if the integral system (3.1) has a solution, such solution is given by the increasing limit (3.5).

Our proof of the existence of local solutions follows closely the proof of Theorem 2 in [16], see also [22]. For any constant $\tau > 0$ let

$$E_\tau \equiv \{(u_1,u_2) : [0,\tau] \to L^\infty(D) \times L^\infty(D), \|\| (u_1,u_2) \|\| < \infty\},$$

where

$$\|\| (u_1,u_2) \|\| \equiv \sup_{0 \leq t \leq \tau} \{ \|u_1(t,\cdot)\|_\infty + \|u_2(t,\cdot)\|_\infty \}.$$

The couple $(E_\tau,\|\| \cdot \|\|)$ is a Banach space and $P_\tau \equiv \{(u_1,u_2) \in E_\tau : u_1 \geq 0, u_2 \geq 0\}$ and $C_R \equiv \{(u_1,u_2) \in E_\tau : \|\| (u_1,u_2) \|\| \leq R\}, \ R > 0$, are closed subsets of $E_\tau$. 

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THEOREM 1. Let \( f_i : D \to [0, \infty) \) be in \( L^\infty(D) \), \( i = 1, 2 \). There exists a constant \( \tau = \tau(\|f_1\|_\infty, \|f_2\|_\infty) > 0 \) such that the integral system (3.1) possesses a unique non-negative local solution in \( L^\infty([0, \tau] \times D) \times L^\infty([0, \tau] \times D) \cap C_R \).

**Proof.** Let us define the operator \( \Psi \) on \( C_R \cap P_\tau \) by

\[
\Psi(u_1, u_2)(t, x) = (U^1_D(t, 0)f_1(x), U^2_D(t, 0)f_2(x)) + \left( \int_0^t U^1_D(t, r)u^\beta_1_2 (r, x)dr, \int_0^t U^2_D(t, r)u^\beta_2_1 (r, x)dr \right).
\]

We are going to show that \( \Psi \) is a contraction on \( C_R \cap P_\tau \) for suitably chosen \( R > 0 \) and \( \tau > 0 \). In fact, if \( (u_1, u_2), (\bar{u}_1, \bar{u}_2) \in C_R \cap P_\tau \), then

\[
|||\Psi(u_1, u_2) - \Psi(\bar{u}_1, \bar{u}_2)||| \\
\leq \sup_{0 \leq t \leq \tau} \int_0^t ||u^\beta_2_1 (r, \cdot) - \bar{u}^\beta_2_1 (r, \cdot)||_\infty dr + \sup_{0 \leq t \leq \tau} \int_0^t ||u^\beta_1_2 (r, \cdot) - \bar{u}^\beta_1_2 (r, \cdot)||_\infty dr,
\]

and using the elementary inequality \(|a^p - b^p| \leq p(a \vee b)^{p-1}|a - b|\), which holds for all \( a, b > 0 \) and \( p \geq 1 \), we get

\[
|||\Psi(u_1, u_2) - \Psi(\bar{u}_1, \bar{u}_2)||| \\
\leq \beta_1 R^{\beta_1 - 1} \int_0^\tau ||u_2(r, \cdot) - \bar{u}_2(r, \cdot)||_\infty dr \\
+ \beta_2 R^{\beta_2 - 1} \int_0^\tau ||u_1(r, \cdot) - \bar{u}_1(r, \cdot)||_\infty dr \\
\leq (\beta_1 R^{\beta_1 - 1} \vee \beta_2 R^{\beta_2 - 1})|||(u_1, u_2) - (\bar{u}_1, \bar{u}_2)|||\tau \quad (3.6)
\]

Noticing that

\[
|||\Psi(u_1, u_2)||| \leq ||f_1||_\infty + ||f_2||_\infty + \tau(R^{\beta_1} + R^{\beta_2}),
\]

and taking \( R > 0 \) big enough and \( \tau > 0 \) sufficiently small we get from (3.6) that \( \Psi \) is a contraction mapping on \( C_R \cap P_\tau \). Thus, the Banach fixed-point theorem implies that (3.1) possesses a unique solution \((u_1, u_2)\) such that \( u_i \geq 0, \ i = 1, 2 \). \( \square \)

4. Global existence of the mild solution

Here we suppose again that \( f_i \in L^\infty(D), \ i = 1, 2 \). Our proof of the next theorem follows closely the proof of Theorem 3 in [16], see also Theorem 2.2 in [17].

THEOREM 2. Let \( f_1, f_2 \) be nonnegative. If

\[
(\beta_i - 1) \int_0^\infty ||U^i_D(t, 0)f_i||_\infty^{\beta_i - 1} dt < 1, \ i = 1, 2,
\]

then the solution of the integral system (3.1) is global.
Proof. If $f_1, f_2$ are identically zero, the nonnegative solution of (3.1) is clearly $(u_1, u_2) \equiv (0, 0)$, which is global. Now, if $f_1, f_2$ are not identically zero, putting

$$B_i(t) = \left[1 - \left(\beta_i - 1\right) \int_0^t \|U_D^j(r, 0)f_i\|_{\infty}^{\beta_i-1} dr \right]^{-\frac{1}{\beta_i-1}}$$

we get $B_i(0) = 1$ and

$$\frac{d}{dt}B_i(t) = -\frac{1}{\beta_i-1} \left[1 - \left(\beta_i - 1\right) \int_0^t \|U_D^j(r, 0)f_i\|_{\infty}^{\beta_i-1} dr \right]^{-\frac{1}{\beta_i-1}} \left[- \left(\beta_i - 1\right) \|U_D^j(t, 0)f_i\|_{\infty}^{\beta_i-1}\right] = \|U_D^j(t, 0)f_i\|_{\infty}^{\beta_i-1} \left\|\frac{\partial_t}{B_i}(t)\right\|_{\infty} \tag{4.1}$$

which gives

$$B_i(t) = 1 + \int_0^t \|U_D^j(r, 0)f_i\|_{\infty}^{\beta_i-1} \left\|\frac{\partial_t}{B_i}(r)\right\|_{\infty} dr, \quad i = 1, 2.$$

Since the evolution system $\{U_D^j(t, s)\}_{t \geq s \geq 0}$ is positivity-preserving we can choose two continuous functions $v_i : [0, \infty) \times D \to [0, \infty), \quad i = 1, 2,$ such that $v_i(t, \cdot) \in C_b(D)$ for all $t \geq 0$ and

$$0 \leq v_i(t, x) \leq \min\{B_1(t)U_D^j(t, 0)f_1(x), B_2(t)U_D^2(t, 0)f_2(x)\}, \quad t \geq 0, \quad i = 1, 2.$$

Let us define $\mathcal{F}_i$ on the space of nonnegative functions in $L^\infty(D)$ as in (3.4). Then

$$0 \leq \mathcal{F}_i v_i(t, x) \leq U_D^j(t, 0)f_i(x) + \int_0^t B_i(t)(U_D^j(r, 0)f_i(x))^{\beta_i} dr$$

$$\leq U_D^j(t, 0)f_i(x) + \int_0^t B_i(t)U_D^j(r, 0)f_i(x)U_D^j(r, 0)f_i(x)\|U_D^j(r, 0)f_i\|_{\infty}^{\beta_i-1} dr$$

$$= U_D^j(t, 0)f_i(x)\left[1 + \int_0^t \|U_D^j(r, 0)f_i\|_{\infty}^{\beta_i-1} B_i(r) dr \right]$$

where we used (4.1) in the last equality. Therefore,

$$0 \leq \mathcal{F}_i v_i(t, x) \leq \max\{B_1(t)U_D^j(t, 0)f_1(x), B_2(t)U_D^2(t, 0)f_2(x)\}, \quad t \geq 0, \quad x \in D.$$

Defining now the sequence $\{u_{i,n}(t, x)\}_{n=0}^\infty$ as in (3.2) and (3.3) it is follows as in Section 3 that $u_{i,n}(t, x) \leq u_{i,n+1}(t, x), \quad n \geq 0.$ Hence

$$u_i(t, x) \equiv \limsup_{n \to \infty} u_{i,n}(t, x) \leq \max\{B_1(t)U_D^j(t, 0)f_1(x), B_2(t)U_D^2(t, 0)f_2(x)\} < \infty$$

for all $t \geq 0$ and $x \in D$. Therefore $(u_1, u_2)$ is a global mild solution of (1.1). $\Box$
5. Blow up in finite time of the positive mild solution

Recall that \( \varphi_0^i \) is the eigenfunction corresponding to the first eigenvalue \( \lambda_0^i \) of the infinitesimal generator of the semigroup \( \{S_D^t(t)\}_{t \geq 0}, \ i = 1, 2 \). Arguing as in the case of Brownian motion in a bounded domain (see [18], p. 287), it can be shown that \( (\varphi_0^i)^2(x)dx \) is the unique invariant measure of the semigroup \( \{Q_i^t(t)\}_{t \geq 0} \) given by

\[
Q_i^t(g(x)) = \frac{e^{\lambda_0^i t}}{\varphi_0^i(x)} S_D^t(t)(g\varphi_0^i)(x), \quad x \in D, \ g \in C_b(D), \ t \geq 0.
\]

Thus, defining

\[
E_i[h] := \int h(x)(\varphi_0^i)^2(x)dx, \quad h \in C_b(D),
\]

and

\[
T_i(t,s)g(x) = \frac{e^{\lambda_0^i K_i(t,s)}}{\varphi_0^i(x)} S_D^t(K_i(t,s))(g\varphi_0^i)(x), \quad x \in D, \ g \in C_b(D), \ t \geq s \geq 0,
\]

we have that for any \( t \geq s \geq 0 \) and \( g \in C_b(D) \),

\[
E_i[Q_i^t(t)g] = E_i[g] \quad \text{and} \quad T_i(t,s)g = Q_i(K_i(t,s))g. \tag{5.1}
\]

**Lemma 1.** For any \( t \geq s \geq 0 \) and \( g_i \in C_b(D) \), \( i = 1, 2 \),

\[
E_i[T_i(t,s)g_i] = E_i[g_i], \quad i = 1, 2.
\]

**Proof.** This is a direct consequence of (5.1). \( \square \)

**Theorem 3.** Let \( f_i = g_i \varphi_0^i \), where \( g_i \in C_b(D) \) is nonnegative and not identically zero, \( i = 1, 2 \). If

\[
\min_{i \in \{1,2\}} \langle f_i, \varphi_0^i \rangle > \max_{i \in \{1,2\}} \left[ \left( \frac{\beta_1 \beta_2 - 1}{\beta_1 + 1} \right) \left( \frac{\beta_1 + 1}{\beta_2 + 1} \right)^{\frac{\beta_2}{\beta_1 + 1}} \times \int_0^{\infty} \min_{i \in \{1,2\}} \left\{ C_i e^{-\lambda_0^i K_i(r,0)} \beta_i e^{\lambda_0^i K_i(r,0)} \right\} \frac{\beta_2^{\beta_2 + 1}}{\beta_1^{\beta_1 + 1}} dr \right]
\]

where \( C_1 = (1/c_2) (c_1/c_2)^{\beta_1} \) and \( C_2 = c_1 (c_1/c_2)^{\beta_2} \), then the mild solution of (1.1) blows up in finite time.

**Proof.** Notice that \( \langle f_i, \varphi_0^i \rangle = E_i[g_i] > 0, \ i = 1, 2 \). We define

\[
w_i(t,x) = \frac{\varphi_0^i(x)}{e^{\lambda_0^i K_i(t,x)}} , \quad i = 1, 2,
\]
where \((u_1, u_2)\) is the mild solution of (1.1), i.e., \((u_1, u_2)\) solves the integral system (3.1). Multiplying both sides of (3.1) by \(\varphi_0^1(x)^{-1}\exp(\lambda_0^1 K_1(t, 0))\) we get

\[
\begin{align*}
w_1(t, x) &= T_1(t, 0)g_1(x) + \int_0^t \frac{e^{\lambda_0^1 K_1(t, 0)}}{\varphi_0^1(x)} U_D^1(t, r)u_2^1(r, x)dr \\
&= T_1(t, 0)g_1(x) + \int_0^t \frac{e^{\lambda_0^1 K_1(t, 0)}}{\varphi_0^1(x)} U_D^1(t, r)\left(\frac{u_2^1(r, x)}{\varphi_0^1}\right)^{\beta_1^{-1}}(x)dr \\
&= T_1(t, 0)g_1(x) + \int_0^t \frac{e^{\lambda_0^1 K_1(t, 0)}}{\varphi_0^1(x)} U_D^1(t, r)\left(\frac{u_2^1(r, x)}{\varphi_0^1}\right)^{\beta_1^{-1}}(x)dr \\
&= T_1(t, 0)g_1(x) + \int_0^t \frac{e^{\lambda_0^1 K_1(r, 0)}}{\varphi_0^1(x)} T_1(t, r)\left(\frac{u_2^1(r, x)}{\varphi_0^1}\right)^{\beta_1^{-1}}(x)dr \\
&= T_1(t, 0)g_1(x) + \int_0^t T_1(t, r)\left(\frac{e^{\lambda_0^1 K_2(r, 0)}u_2^1(r, x)}{\varphi_0^1}\right)e^{-\lambda_0^1 K_2(r, 0)\beta_1} \cdot e^{\lambda_0^1 K_1(r, 0)}(\varphi_0^1)^{\beta_1^{-1}}(x)dr.
\end{align*}
\]

The last equality renders

\[
E_1[w_1(t, \cdot)] = E_1[T_1(t, 0)g_1] + \int_0^t E_1\left[T_1(t, r)\left(\frac{e^{\lambda_0^1 K_2(r, 0)}u_2^1(r, \cdot)}{\varphi_0^1}\right)e^{-\lambda_0^1 K_2(r, 0)\beta_1} e^{\lambda_0^1 K_1(r, 0)}(\varphi_0^1)^{\beta_1^{-1}}(\cdot)\right]dr,
\]

and due to Lemma 1,

\[
E_1[w_1(t, \cdot)] = E_1[g_1] + \int_0^t E_1\left[\left(\frac{e^{\lambda_0^1 K_2(r, 0)}u_2^1(r, \cdot)}{\varphi_0^1}\right)e^{-\lambda_0^1 K_2(r, 0)\beta_1} e^{\lambda_0^1 K_1(r, 0)}(\varphi_0^1)^{\beta_1^{-1}}(\cdot)\right]dr.
\]

It follows that for any \(\varepsilon > 0\),

\[
E_1[w_1(t + \varepsilon, \cdot)] - E_1[w_1(t, \cdot)] = \int_t^{t+\varepsilon} E_1\left[\left(\frac{e^{\lambda_0^1 K_2(r, 0)}u_2^1(r, \cdot)}{\varphi_0^1}\right)e^{-\lambda_0^1 K_2(r, 0)\beta_1} e^{\lambda_0^1 K_1(r, 0)}(\varphi_0^1)^{\beta_1^{-1}}(\cdot)\right]dr. \tag{5.3}
\]

Using (2.3) we obtain

\[
E_1\left[\left(\frac{e^{\lambda_0^1 K_2(r, 0)}u_2^1(r, \cdot)}{\varphi_0^1}\right)e^{-\lambda_0^1 K_2(r, 0)\beta_1} e^{\lambda_0^1 K_1(r, 0)}(\varphi_0^1)^{\beta_1^{-1}}(\cdot)\right]
\]
\[
\geq E_1 \left[ \left( \frac{e^{\lambda_0^2 K_2(r,0)} u_2^2}{\varphi_0^2(x)} \right)^{\beta_1} \right] e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \left( \frac{1}{c_2} \varphi_0^2(x) \right)^{\beta_1 - 1}
\]

\[
\geq c_1 \left( \frac{c_1}{c_2} \right)^{\beta_1 - 1} E_1 \left[ \left( \frac{e^{\lambda_0^2 K_2(r,0)} u_2^2}{\varphi_0^2(x)} \right)^{\beta_1} \right] e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \left( \varphi_0^2(x) \right)^{\beta_1 - 1}
\]

\[
= c_1 \left( \frac{c_1}{c_2} \right)^{\beta_1 - 1} \int w_2(s, \cdot) \beta_1 e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \left( \varphi_0^2(x) \right)^{\beta_1 - 1} dx
\]

From here, using again (2.3) we get

\[
E_1 \left[ \left( \frac{e^{\lambda_0^2 K_2(r,0)} u_2^2}{\varphi_0^2(x)} \right)^{\beta_1} \right] e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \left( \varphi_0^2(x) \right)^{\beta_1 - 1}
\]

\[
\geq c_1 \left( \frac{c_1}{c_2} \right)^{\beta_1 - 1} e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \int w_2(r, x) \beta_1 \left( \varphi_0^2(x) \right)^{\beta_1 - 1} \left( \frac{1}{c_2} \varphi_0^2(x) \right)^2 dx
\]

\[
= \frac{1}{c_2} \left( \frac{c_1}{c_2} \right)^{\beta_1} e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \left\| \varphi_0^2(x) \right\|_1 \int \left[ w_2(r, x) \varphi_0^2(x) \right] \beta_1 \varphi_0^2(x) dx
\]

\[
\geq \frac{1}{c_2} \left( \frac{c_1}{c_2} \right)^{\beta_1} e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \left\| \varphi_0^2(x) \right\|_1 \int \left[ w_2(r, x) \frac{\varphi_0^2(x)}{\left\| \varphi_0^2(x) \right\|_1} \right] \beta_1 dx
\]

\[
= \frac{1}{c_2} \left( \frac{c_1}{c_2} \right)^{\beta_1} e^{-\lambda_0^2 K_2(r,0)} \beta_1 e^{\lambda_1^1 K_1(r,0)} \frac{\left\| \varphi_0^2(x) \right\|_1^{\beta_1 - 1}}{\left\| \varphi_0^2(x) \right\|_1} E_2 \left[ w_2(r, \cdot) \right] \beta_1 , \tag{5.4}
\]

where we have used Jensen’s inequality with respect to the probability measure \( \left\| \varphi_0^2(x) \right\|_1^{-1} \varphi_0^2(x) dx \). Let \( h_i(t) := E_1[w_i(t, \cdot)] \), \( i = 1, 2 \). Plugging (5.4) into (5.3), and afterward multiplying the resulting inequality by \( e^{-1} \) with \( \varepsilon \to 0 \), we obtain that

\[
h_1'(t) \geq C_1 e^{-\lambda_0^2 K_2(t,0)} \beta_1 e^{\lambda_1^1 K_1(t,0)} \frac{1}{\left\| \varphi_0^2(x) \right\|_1^{\beta_1 - 1}} h_2(t), \quad h_1(0) = \langle f_1, \varphi_0^1 \rangle . \tag{5.5}
\]

Now, multiplying both sides of (3.1) (with \( i = 2 \)) by \( \varphi_0^2(x)^{-1} \exp\left( \lambda_0^2 K_2(t,0) \right) \), it can be obtained, similarly as above, that

\[
h_2'(t) \geq C_2 e^{-\lambda_1^1 K_1(t,0)} \beta_2 e^{\lambda_0^2 K_2(t,0)} \frac{1}{\left\| \varphi_0^2(x) \right\|_1^{\beta_2 - 1}} h_1(t), \quad h_2(0) = \langle f_2, \varphi_0^2 \rangle . \tag{5.6}
\]

Let

\[
c(t) = \min_{i \in \{1, 2\}} \left\{ C_i e^{-\lambda_0^2 K_2(t,0)} \varphi_0^2(x)^{\beta_1 - 1} \right\}, \quad N = \min_{i \in \{1, 2\}} \left\{ \langle f_i, \varphi_0^i \rangle \right\} > 0,
\]

\[
\]
and consider the ordinary differential system

\[ p_1'(t) = c(t)p_2^{\beta_1}(t), \quad p_2'(t) = c(t)p_1^{\beta_2}(t), \quad p_i(0) = N, \ i = 1, 2. \quad (5.7) \]

It follows that

\[ \frac{1}{\beta_2 + 1} \left[ p_1^{\beta_2+1}(t) - N^{\beta_2+1} \right] = \frac{1}{\beta_1 + 1} \left[ p_2^{\beta_1+1}(t) - N^{\beta_1+1} \right]. \]

Notice that if \( N \leq \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{1/\beta_2+1} N^{(\beta_1+1)/\beta_2+1}, \) then

\[ \frac{1}{\beta_2 + 1} p_1^{\beta_2+1}(t) \leq \frac{1}{\beta_1 + 1} p_2^{\beta_1+1}(t) \quad (5.8) \]

and, if \( N \geq \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{1/\beta_2+1} N^{(\beta_1+1)/\beta_2+1}, \) then

\[ \frac{1}{\beta_2 + 1} p_1^{\beta_2+1}(t) \geq \frac{1}{\beta_1 + 1} p_2^{\beta_1+1}(t). \quad (5.9) \]

If (5.8) holds, then

\[ p_2(t) \geq \left( \frac{\beta_1 + 1}{\beta_2 + 1} \right)^{\frac{1}{\beta_1+1}} \frac{\beta_2+1}{p_1^{\beta_1+1}}. \]

Substituting this into the first equation of (5.7), we get

\[ p_1'(t) \geq c(t) \left( \frac{\beta_1 + 1}{\beta_2 + 1} \right)^{\frac{\beta_1}{p_1^{\beta_1+1}}} \left( \frac{\beta_1 + 1}{\beta_2 + 1} \right)^{\frac{\beta_2+1}{p_1^{\beta_1+1}}} (t), \]

which is the same as

\[ p_1 \left( \frac{\beta_1^{(\beta_2+1)}}{p_1^{\beta_1+1}} \right) p_1'(t) \geq c(t) \left( \frac{\beta_1 + 1}{\beta_2 + 1} \right)^{\frac{\beta_1}{p_1^{\beta_1+1}}}. \]

Integrating both sides of the above inequality from 0 to \( t \) yields

\[ \frac{\beta_1 + 1}{1 - \beta_1 \beta_2} \left[ p_1^{\frac{1-\beta_1 \beta_2}{p_1^{\beta_1+1}}} (t) - N^{1-\beta_1 \beta_2} \right] \geq \left( \frac{\beta_1 + 1}{\beta_2 + 1} \right)^{\frac{\beta_1}{p_1^{\beta_1+1}}} \int_0^t c(r) dr. \]

Thus, in view of \( \beta_1, \beta_2 > 1, \)

\[ p_1(t) \geq \left[ N^{\frac{1-\beta_1 \beta_2}{p_1^{\beta_1+1}}} - \left( \frac{\beta_1 \beta_2 - 1}{\beta_1 + 1} \right) \left( \frac{\beta_1 + 1}{\beta_2 + 1} \right)^{\frac{\beta_1}{p_1^{\beta_1+1}}} \int_0^t c(r) dr \right] \frac{\beta_1+1}{\beta_1 - \beta_1 \beta_2}. \quad (5.10) \]

Similarly, if (5.9) holds, we can show that

\[ p_2(t) \geq \left[ N^{\frac{1-\beta_1 \beta_2}{p_2^{\beta_1+1}}} - \left( \frac{\beta_1 \beta_2 - 1}{\beta_2 + 1} \right) \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{\frac{\beta_2+1}{p_2^{\beta_1+1}}} \int_0^t c(r) dr \right] \frac{\beta_2+1}{\beta_2 - \beta_1 \beta_2}. \quad (5.11) \]
Since the function $\int_0^t c(r)dr$ is continuous and increases to $\int_0^\infty c(r)dr$, (5.10) and (5.11) imply that for some $0 < t_0 < \infty$,

$$\lim_{t \uparrow t_0} \|w_i(t, \cdot)\|_\infty \geq \lim_{t \uparrow t_0} p_i(t) = \infty \quad \text{for } i = 1 \text{ or } i = 2,$$

whenever

$$\min_{i \in \{1, 2\}} \langle f_i, \varphi_0^i \rangle > 0.$$ 

Using the intrinsic ultracontractivity of the semigroups $\{S_D^i(t)\}_{t \geq 0}$, $i = 1, 2$, the proof of the following theorem follows in exactly the same form as in Theorem 6 of [16] with appropriate changes in the notations.

**Theorem 4.** Let $f_1, f_2 \in C_0(D)$ be two nonnegative functions which are not identically zero. If Condition (5.2) holds, then the mild solution of (1.1) blows up in finite time.

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**References**


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