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EXISTENCE OF SOLUTIONS AND SEMI–DISCRETIZATION FOR PDE WITH INFINITE DELAY

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Abstract. In this paper, we obtain an existence theorem for a Semi-Linear PDE with infinite delay employing a phase space in which discretizations can naturally be performed. Further, for linear PDEs with infinite delay we show that the solutions of the ODE with infinite delay obtained by the semi-discretization converge to the original solution. Our results cover various types of PDEs under the assumption that semi-discretization of the PDEs without the delay terms can be performed. The method of our proof is applicable for the case of finite delays too.

1. Introduction

The study of delay differential equations is motivated by the fact to model some evolution phenomena arising in physics, biology, engineering etc, some hereditary characteristics such as after effect, time lag and time delay can appear in variables. Complex phenomena in biological, chemical and physical systems can sometimes be modelled by delay partial differential equations (DPDEs). Such equations are naturally more difficult than ordinary differential equations with delay since these are infinite dimensional both in time and space variables. Since 1970's such equations have been widely studied and several important properties such as existence and stability of the solution are nowadays fairly well understood.

However, the exact solution is not available in general, so one has to resort to numerical methods when solving such equations. The numerical analysis of computational methods for delay PDEs has not received too much attention yet as notices in the literature surveyed.

It is a well known fact that there are numerous technical difficulties in dealing with partial differential equations with infinite delay due to the unboundedness of the delay involved. Most numerical schemes for PDEs without delay can be adopted to the solution delay PDEs, when they are combined with an appropriate interpolation procedure for the evaluation of the delay argument.

An evolutionary (time dependent) PDE can be reduced to a system of ODEs by replacing the spatial derivatives with finite difference approximations. The resulting approximations is called semi-discrete since the time variable is left continuous. The procedure of reducing a PDE to an ODE system is often called the method of lines

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since a solution of the ODE system gives an approximation to the PDE solution along x equals constant lines (x,t) space. In the second part of this paper, the process of semi-discretization is applied to PDE with infinite delay which can be approximated by a sequence of ODE with infinite delay.

The focus in this research paper is on showing the existence of non-linear PDE with infinite delay and convergence of the sequence of solutions of linear ODE with infinite delay to the solution of linear PDE with infinite delay. Two new theorems are presented in this paper to prove the results and the proposed results cover a broad spectrum of PDEs with infinite delay.

Unlike in the case of infinite delay, the choice of the phase space for the study of infinite delay equation is a difficult one. This has motivated the introduction of an axiomatic approach. In 1983, Pazy proved the existence results of the equation of the type

$$\begin{aligned} u'(t) &= Au(t) + F(t,u_t), \ \ t \geqslant 0, \\ u(\theta) &= \phi(\theta), \ \ \theta \in (-\infty, 0], \end{aligned}$$

where A is not necessarily densely defined. Semi-discretization is a well known technique used in the finite element analysis as well as finite difference schemes in computational fluid mechanics.

While Rey and Mackey (1993) applied Galerkin finite element method, Houwen et al. (1986), Higham and Sardar (1995), Zubic-Kowal and Vandewalle (1999) used finite difference schemes to semi-discretize PDE with delays. Homotopy analysis is used to approximate solutions of initial-boundary value problem for delay parabolic equations by Agirseven (2012), Roales et al. (2012) applied stable difference methods for delay partial differential equations. The phase space

$$\left\{ \phi \in \mathbf{C}(-\infty, 0] : \sum_{k=1}^{\infty} |\beta_k| \sup_{\theta \in [-k\tau_1, 0]} |\phi(\theta)| < \infty \right\}$$

has been introduced to study discretization of ODE with infinite delay by Sengadir (2006). In this paper, appropriate modification has been made to the phase space to cover PDEs and a broader class of delay equations. Inspired by above mentioned work, in this paper the research work has taken the direction to study the existence of solution to the non-linear abstract infinite delay equation as well as approximation of solutions of sequence of delay differential equations to the proposed partial differential equation.

In this paper, the following non-linear abstract infinite delay equation is considered.

$$\begin{aligned} u'(t) &= Au(t) + F(u_t), \ \ t \geqslant 0, \\ u(\theta) &= \phi(\theta), \ \ \theta \in (-\infty, 0], \end{aligned} \tag{1.1}$$

in a phase space which makes the analysis of numerical approximations convenient. Also the convergence of the sequence of solutions of the linear delay equation

$$x'(t) = A_n x(t) + L_n(x_t), \ \ t \geqslant 0,$$

$$x(\theta) = \phi_n(\theta), \;\; \theta \in (-\infty, 0], \tag{1.2}$$

to the solution of the abstract linear infinite delay equation

$$\begin{aligned} u'(t) &= Au(t) + Lu_t, \;\; t \geqslant 0, \\ u(\theta) &= \phi(\theta), \;\; \theta \in (-\infty, 0], \end{aligned} \tag{1.3}$$

in an appropriate sense is proven in this paper. Here $A : \mathbf{D}(A) \to X$ is the generator of a C_0 semigroup and A_n are bounded linear maps on finite dimensional spaces X_n which approximate A in an appropriate sense. This paper is structured as follows: In section 2, the attention is fixed on preliminary results and in sections 3 and 4, the discussions are focused on non-linear as well as linear PDE with delay respectively.

2. Preliminaries

This section is devoted to some preliminary definitions and facts which are used throughout this paper.

DEFINITION 1. Let X and Y be normed linear spaces. A function $f : X \to Y$ is said to be Lipschitz, if there exists a constant $L > 0$ (called as a Lipschitz constant of f) such that for all $x, y \in X$, $||f(x) - f(y)|| \leqslant L||x - y||$.

PROPOSITION 1. *Let X be a Banach space with the norm $\| \; \|_X$ and $\{\beta_k\}$ be a sequence of strictly positive reals such that $\sum_{k=1}^{\infty} \beta_k < \infty$. Define the vector space $\mathbf{C}_\sigma((-\infty, 0]; X)$ as*

$$\left\{ \phi \in \mathbf{C}((-\infty, 0]; X) : \sum_{k=1}^{\infty} |\beta_k| \sup_{\theta \in [-k, -k+1]} \|\phi(\theta)\|_X < \infty \right\}.$$

Then $\mathbf{C}_\sigma((-\infty, 0]; X)$ is a Banach space with the norm $\| \; \|_{\sigma, X}$ defined as

$$\|\phi\|_{\sigma, X} = \sum_{k=1}^{\infty} |\beta_k| \sup_{\theta \in [-k, -k+1]} \|\phi(\theta)\|_X.$$

Further, a sequence $\phi_n \in \mathbf{C}_\sigma((-\infty, 0]; X)$ converges to $\phi \in \mathbf{C}_\sigma((-\infty, 0]; X)$ if for every fixed $k \in \mathbb{N}$,

$$\lim_{n \to \infty} \sup_{\theta \in [-k, -k+1]} \|\phi_n(\theta) - \phi(\theta)\|_X = 0$$

and there exists a sequence $\alpha_k \geqslant 0$ such that $\sum_{k=1}^{\infty} b_k \alpha_k < \infty$ with

$$\sup_{\theta \in [-k, -k+1]} \|\phi_k(\theta)\|_X \leqslant \alpha_k \text{ for all } k \in \mathbb{N}.$$

THEOREM 1. (**Trotter-Kato Approximation Theorem**) *Let* $(X_n, \| \quad \|_n)$, $n = 1,2,3,...$ *and* $(X, \| \ \|)$ *be the Banach spaces. Further, let there be bounded linear maps* $P_n : X \longrightarrow X_n$ *and* $E_n : X_n \longrightarrow X$ *such that*

(i) $\|P_n\| \leqslant C_1$, $\|E_n\| \leqslant C_2$, *with* C_1 *and* C_2 *are constants independent of* n.

(ii) $\|E_nP_nu - u\| \to 0$ *as* $n \to \infty$ *for every* $u \in X$.

(iii) $P_nE_n = I_n$, *where* I_n *is the identity operator on* X_n.
Further, assume that

(iv) $A : \mathbf{D}(A) \longrightarrow X$ *is a closed and densely defined operator in class* $G(M,w,X)$ *and* $A_n : X_n \longrightarrow X_n$ *are bounded linear maps in* $G(M,w,X_n)$ *respectively generating the semigroups* T_t *and* T_t^n.

Then the following are equivalent:

(a) *For all* $u \in \mathbf{D}(A)$ *there exists a sequence* $\overline{u}_n \in X_n$ *such that* $\lim_{n\to\infty} E_n\overline{u}_n = u$ *and* $\lim_{n\to\infty} E_nA_n\overline{u}_n = Au$.

(b) $\lim_{n\to\infty} \|E_nT_t^nP_nx - T_tx\| \to 0$ *as* $n \to \infty$ *for all* $x \in X$.

THEOREM 2. *Let* $(X_n, \| \ \|_n)$, $n = 1,2,3,...$, $(X, \| \ \|)$ *be Banach spaces and assume that* **(i)**, **(ii)** *and* **(iii)** *of Theorem 1 are true. Define*
$\overline{P}_n : \mathbf{C}_\sigma((-\infty,0];X) \longrightarrow \mathbf{C}_\sigma((-\infty,0];X_n)$ *and*
$\overline{E}_n : \mathbf{C}_\sigma((-\infty,0];X_n) \longrightarrow \mathbf{C}_\sigma((-\infty,0];X)$ *as* $\overline{P}_nf(\theta) = P_n[f(\theta)]$ *and*
$\overline{E}_ng(\theta) = E_n[g(\theta)]$. *Then the operators* $\overline{E}_n$ *and* $\overline{P}_n$ *satisfy conditions analogous to that of* **(i)**, **(ii)** *and* **(iii)** *of Theorem 1.*

Proof. Since **(i)** and **(ii)** are obvious, to prove **(iii)**, let $f \in \mathbf{C}_\sigma((-\infty,0];X)$. For a fixed $\theta \in (-\infty,0]$, $E_nP_n(f(\theta))$ converges to $f(\theta)$ and the sequence of functions $\overline{E}_n\overline{P}_nf$ is equi-continuous on any fixed interval $[-k,-k+1]$. Hence the sequence $\sup_{\theta\in[-k,-k+1]} \|E_nP_n(f(\theta)) - f(\theta)\|_X$ converges to 0. Next letting

$$\alpha_k = C_1C_2 \sup_{\theta\in[-k,-k+1]} \|f(\theta)\|_X,$$

the hypotheses of Proposition 1 are satisfied for $\phi_n = E_nP_nf$.
Hence **(iii)** follows.

REMARK 1. It is clear that Theorem 2 is valid if the spaces $\mathbf{C}_\sigma((-\infty,0];X)$ and $\mathbf{C}_\sigma((-\infty,0];X_n)$ are replaced by $\mathbf{C}([-\tau,0];X)$ and $\mathbf{C}([-\tau,0];X_n)$ respectively.

3. Non-linear PDE with infinite delay

This section is concerned with the existence of solution to non-linear partial differential equation with infinite delay. The following theorem facilitates the existence result for equation (1.1) by a semigroup of Lipchitz maps. In the sequel, $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

THEOREM 3. *Let X be a Banach space and let $A : \mathbf{D}(A) \longrightarrow X$ be a closed and densely defined linear operator in class $G(M,w,X)$ generating the semigroup S_t. Let β_k be a sequence of positive reals such that $\sum_{k=1}^{\infty} \beta_k < \infty$ and for every fixed $t \geqslant 0$,*

$$\sup_{k \geqslant \lfloor t \rfloor + 2} \max\left(\frac{\beta_k}{\beta_{k-\lfloor t \rfloor}}, \frac{\beta_k}{\beta_{-\lfloor t \rfloor + k - 1}}\right) < \infty. \tag{3.1}$$

Assume that $F : \mathbf{C}_\sigma((-\infty,0];X) \longrightarrow X$ is a Lipchitz continuous map with Lipchitz constant $\alpha \geqslant 0$. Then there is a unique mild solution to (1.1) which is given by a semigroup $\overline{S}_t$ of Lipchitz maps.

Proof. The quantity $\|y\|_T = \sup_{t \in [0,T]} \|y(t)\|_X$ which is the standard norm in the space $\mathbf{C}([0,T];X)$. Define

$$H : [0,T] \times \{(y,\phi) \in \mathbf{C}([0,T];X) \times \mathbf{C}_\sigma((-\infty,0];X) : y(0) = \phi(0)\} \to \mathbf{C}_\sigma((-\infty,0];X)$$

as

$$\begin{aligned} H(s,y,\phi)(\theta) &= y(s+\theta),\ s+\theta \geqslant 0, \\ &= \phi(s+\theta),\ s+\theta < 0, \end{aligned}$$

$$\Sigma : [0,T] \times \mathbf{C}_\sigma((-\infty,0];X) \longrightarrow \mathbf{C}_\sigma((-\infty,0];X)$$

as

$$\begin{aligned} \Sigma(s,\phi)(\theta) &= 0,\ s+\theta \geqslant 0, \\ &= \phi(s+\theta) - \phi(0),\ s+\theta < 0 \end{aligned}$$

and

$$K : [0,T] \times \mathbf{C}([0,T];X) \longrightarrow \mathbf{C}_\sigma((-\infty,0];X)$$

as

$$\begin{aligned} K(s,y)(\theta) &= y(s+\theta) - y(0),\ s+\theta \geqslant 0, \\ &= 0,\ s+\theta < 0. \end{aligned}$$

Now clearly, $H(s,y,\phi) = K(s,y) + \Sigma(s,\phi) + c_{y(0)}$, where $c_{y(0)}$ is the constant function assuming the value $y(0) \in X$ for all $\theta \in (-\infty, 0]$.

It can be easily shown that $\|K(s,y)\|_\sigma = 2\sum_{k=1}^{\infty} |\beta_k| \|y\|_T$.

$\|\Sigma(t,\phi)\|_\sigma$ can be estimated as follows.
Let t be fixed. Then $\lfloor t \rfloor \leqslant t < \lfloor t \rfloor + 1$ and hence $-\lfloor t \rfloor - 1 < -t \leqslant -\lfloor t \rfloor$.

For $k < \lfloor t \rfloor + 1$ and $\theta \in [-k, -k+1]$, one can have $\theta \geqslant -k \geqslant -\lfloor t \rfloor - 1 > -t$ and hence $\theta + t > 0$. So, $|\Sigma(t,\phi)(\theta)| = 0$.

For $k = \lfloor t \rfloor + 1$ and $\theta \in [-k, -k+1] = [-\lfloor t \rfloor - 1, -\lfloor t \rfloor]$, then $\theta \geqslant -t$. Hence $|\Sigma(t,\phi)(\theta)| = 0$.

Let $\theta \in [-\lfloor t \rfloor - 1, -\lfloor t \rfloor]$. Then $-1 \leqslant t - \lfloor t \rfloor - 1 \leqslant \theta + t \leqslant 0$.
So,

$$\begin{aligned}
\beta_{\lfloor t \rfloor + 1} \sup_{\theta \in [-\lfloor t \rfloor - 1, -\lfloor t \rfloor]} \|\Sigma(t,\phi)\|_\sigma &\leqslant \sup_{\theta \in [-1,0]} |\phi(\theta)| + |\phi(0)| \\
&\leqslant 2\beta_{\lfloor t \rfloor + 1} \sup_{\theta \in [-1,0]} |\phi(\theta)| \\
&\leqslant 2\frac{\beta_{\lfloor t \rfloor + 1}}{\beta_1} \|\phi\|_\sigma.
\end{aligned}$$

For $k > \lfloor t \rfloor + 1$, if $\theta \in [-k, -k+1]$, then $t + \theta \leqslant 0$. Further,

$$\lfloor t \rfloor - k \leqslant t - k \leqslant \theta + t \leqslant -k + 1 + \lfloor t \rfloor + 1 = -k + \lfloor t \rfloor + 2.$$

So,

$$\begin{aligned}
&\beta_k \sup_{\theta \in [-k, -k+1]} \|\Sigma(t,\phi)(\theta)\| \\
&\quad \leqslant \beta_k \max\left(\sup_{\theta \in [\lfloor t \rfloor - k, \lfloor t \rfloor - k + 1]} \|\phi(\theta)\|, \sup_{\theta \in [\lfloor t \rfloor - k + 1, \lfloor t \rfloor - k + 2]} \|\phi(\theta)\| \right) \\
&\quad \leqslant \max\left(\beta_k \sup_{\theta \in [\lfloor t \rfloor - k, \lfloor t \rfloor - k + 1]} \|\phi(\theta)\|, \beta_k \sup_{\theta \in [\lfloor t \rfloor - k + 1, \lfloor t \rfloor - k + 2]} \|\phi(\theta)\| \right) \\
&\quad \leqslant \max\left(\beta_{k - \lfloor t \rfloor} \frac{\beta_k}{\beta_{k - \lfloor t \rfloor}} \sup_{\theta \in [\lfloor t \rfloor - k, \lfloor t \rfloor - k + 1]} \|\phi(\theta)\|, \right. \\
&\qquad\qquad \left. \beta_{-\lfloor t \rfloor + k - 1} \frac{\beta_k}{\beta_{-\lfloor t \rfloor + k - 1}} \sup_{\theta \in [\lfloor t \rfloor - k + 1, \lfloor t \rfloor - k + 2]} \|\phi(\theta)\| \right) \\
&\quad \leqslant \max\left(\frac{\beta_k}{\beta_{k - \lfloor t \rfloor}}, \frac{\beta_k}{\beta_{-\lfloor t \rfloor + k - 1}} \right) \times \\
&\qquad\qquad \left(\beta_{k - \lfloor t \rfloor} \sup_{\theta \in [\lfloor t \rfloor - k, \lfloor t \rfloor - k + 1]} \|\phi(\theta)\| + \right. \\
&\qquad\qquad \left. \beta_{-\lfloor t \rfloor + k + 1} \sup_{\theta \in [\lfloor t \rfloor - k + 1, \lfloor t \rfloor - k + 2]} \|\phi(\theta)\| \right).
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\|\Sigma(t,\phi)\|_\sigma \\
&\quad \leqslant 2\frac{\beta_{\lfloor t\rfloor+1}}{\beta_1}\|\phi\|_\sigma + \sum_{k=\lfloor t\rfloor+2}^{\infty} \max\left(\frac{\beta_k}{\beta_{k-\lfloor t\rfloor}}, \frac{\beta_k}{\beta_{-\lfloor t\rfloor+k-1}}\right) \times \\
&\qquad \left(\beta_{k-\lfloor t\rfloor} \sup_{\theta\in[\lfloor t\rfloor-k,\lfloor t\rfloor-k+1]} |\phi(\theta)| + \beta_{-\lfloor t\rfloor+k+1} \sup_{\theta\in[\lfloor t\rfloor-k+1,\lfloor t\rfloor-k+2]} |\phi(\theta)|\right) \\
&\quad \leqslant 2\frac{\beta_{\lfloor t\rfloor+1}}{\beta_1}\|\phi\|_\sigma + \max\left\{k \geqslant \lfloor t\rfloor+2 : \max\left(\frac{\beta_k}{\beta_{k-\lfloor t\rfloor}}, \frac{\beta_k}{\beta_{-\lfloor t\rfloor+k-1}}\right)\right\} 2\|\phi\|_\sigma \\
&\quad \leqslant 2\max\left[\left\{\frac{\beta_{\lfloor t\rfloor+1}}{\beta_1}\right\} \cup \left\{k \geqslant \lfloor t\rfloor+2 : \max\left(\frac{\beta_k}{\beta_{k-\lfloor t\rfloor}}, \frac{\beta_k}{\beta_{-\lfloor t\rfloor+k-1}}\right)\right\}\right] \|\phi\|_\sigma.
\end{aligned}
$$

Denote the quantity

$$
2\max\left[\left\{\frac{\beta_{\lfloor t\rfloor+1}}{\beta_1}\right\} \cup \left\{k \geqslant \lfloor t\rfloor+2 : \max\left(\frac{\beta_k}{\beta_{k-\lfloor t\rfloor}}, \frac{\beta_k}{\beta_{-\lfloor t\rfloor+k-1}}\right)\right\}\right]
$$

by $\gamma(t)$ and $\sup_{t\in[0,T]}\gamma(t) = \gamma_T$. Then

$$
\|\Sigma(t,\phi)\|_\sigma \leqslant \gamma(t)\|\phi\|_\sigma.
$$

Define $G_\phi : \mathbf{C}([0,T];X) \longrightarrow \mathbf{C}([0,T];X)$ as

$$
G_\phi y(t) = S_t\phi(0) + \int_0^t S_{t-s}F[H(s,y,\phi)]ds.
$$

It is obtained that

$$
\begin{aligned}
&\|[H(s,y,\phi_1)] - [H(s,z,\phi_2)]\|_\sigma \\
&\leqslant \|K(s,y) - K(s,z) + \Sigma(s,\phi_1) - \Sigma(s,\phi_2) + c_{\phi_1(0)-\phi_2(0)}\|_\sigma \\
&\leqslant \|K(s,y-z)\|_\sigma + \|\Sigma(s,\phi_1-\phi_2)\|_\sigma + \sum_{i=1}^{\infty}|\beta_i|\|\phi_1(0)-\phi_2(0)\| \\
&\leqslant 2\sum_{k=1}^{\infty}|\beta_k|\|y-z\|_T + [2\gamma(s) + \frac{1}{|\beta_1|}\sum_{i=1}^{\infty}|\beta_i|]\|\phi_1-\phi_2\|_\sigma.
\end{aligned}
$$

Now, let $y = G_{\phi_1}(y)$ and $z = G_{\phi_2}(z)$. Then

$$
\begin{aligned}
\|G_{\phi_1}y - G_{\phi_2}z\|_T \leqslant& \sup_{t\in[0,T]} \Big(\|S_t\phi_1(0) - S_t\phi_2(0)\| + \\
&\left\|\int_0^t S_{t-s}\left(F[H(s,y,\phi_1)] - F[H(s,z,\phi_2)]\right)ds\right)\right\| \\
\leqslant& \sup_{t\in[0,T]}\left(e^{\omega t}\right)\frac{M}{\beta_1}\|\phi_1-\phi_2\|_\sigma +
\end{aligned}
$$

$$\sup_{t\in[0,T]}\int_0^t \|S_{t-s}\left(F[H(s,y,\phi)]-F[H(s,z,\phi)]\right)\|\,ds$$
$$\leqslant \max\left(1,e^{\omega T}\right)\frac{M}{\beta_1}\|\phi_1-\phi_2\|_\sigma +$$
$$\sup_{t\in[0,T]}\int_0^t \|S_{t-s}\left(F[H(s,y,\phi)]-F[H(s,z,\phi)]\right)\|\,ds.$$

Now,

$$\sup_{t\in[0,T]}\int_0^t \|S_{t-s}\left(F[H(s,y,\phi_1)]-F[H(s,z,\phi_2)]\right)\|\,ds$$
$$\leqslant \sup_{t\in[0,T]}\int_0^t Me^{\omega(t-s)}\|F(H(s,y,\phi_1))-F(H(s,z,\phi_2))\|ds$$
$$\leqslant \sup_{t\in[0,T]}\int_0^t Me^{\omega(t-s)}\alpha\|H(s,y,\phi_1)-(H(s,z,\phi_2))\|_\sigma ds$$
$$\leqslant M\sup_{t\in[0,T]}\left(\frac{e^{\omega t}-1}{\omega}\right)\times$$
$$\alpha\left(2\sum_{k=1}^{\infty}|\beta_k|\|y-z\|_T+[2\gamma_T+\frac{1}{|\beta_1|}\sum_{i=1}^{\infty}|\beta_i|]\|\phi_1-\phi_2\|_\sigma\right)$$
$$\leqslant M\left(\frac{e^{\omega T}-1}{\omega}\right)\times$$
$$\alpha\left(2\sum_{k=1}^{\infty}|\beta_k|\|y-z\|_T+[2\gamma_T+\frac{1}{|\beta_1|}\sum_{i=1}^{\infty}|\beta_i|]\|\phi_1-\phi_2\|_\sigma\right).$$

So,

$$\|G_{\phi_1}y-G_{\phi_2}z\|_T\leqslant \sup_{t\in[0,T]}\left(e^{\omega t}\right)\frac{M}{\beta_1}\|\phi_1-\phi_2\|_\sigma$$
$$+2\alpha M\left(\frac{e^{\omega T}-1}{\omega}\right)\left(\sum_{k=1}^{\infty}|\beta_k|\right)\|y-z\|_T$$
$$+\left(2\alpha M\frac{e^{\omega T}-1}{\omega}\left(\gamma_T+\frac{1}{2|\beta_1|}\sum_{i=1}^{\infty}|\beta_i|\right)\|\phi_1-\phi_2\|_\sigma\right).$$

Now, it follows that

$$\|G_\phi y-G_\phi z\|_T\leqslant 2M\frac{e^{\omega T}-1}{\omega}\left(\alpha\sum_{k=1}^{\infty}|\beta_k|\right)\|y-z\|_T.$$

Next, choose $0<T$ sufficiently small such that

$$\kappa\doteq\left(\frac{e^{\omega T}-1}{\omega}\right)\left(2\alpha M\sum_{k=1}^{\infty}|\beta_k|\right)<1.$$

Thus, G_ϕ is a contraction and hence has a unique fixed point. Denote the unique fixed point of G_ϕ by $N\phi$. If $N\phi_1 = y$ and $N\phi_2 = z$, then

$$\begin{aligned}\|N\phi_1 - N\phi_2\|_T &= \|y-z\|_T = \|G_{\phi_1}y - G_{\phi_2}z\|_T \\ &\leqslant \max\left(1, e^{\omega T}\right)\frac{M}{\beta_1}\|\phi_1-\phi_2\|_\sigma + \kappa\|y-z\|_T \\ &+ \left(2\alpha M\left(\frac{e^{\omega T}-1}{\omega}\right)\right)\left(\gamma_T + \frac{1}{2|\beta_1|}\sum_{k=1}^{\infty}|\beta_k|\right)\|\phi_1-\phi_2\|_\sigma.\end{aligned}$$

Denoting $\sum_{k=1}^{\infty}|\beta_k|$ by b, one can write

$$\begin{aligned}&(1-\kappa)\|y-z\|_T \leqslant \\ &\left(\left(\max\left(1,e^{\omega T}\right)\right)\frac{M}{\beta_1} + \left(2\alpha M\left(\frac{e^{\omega T}-1}{\omega}\right)\right)\left(\gamma_T + \frac{b}{2|\beta_1|}\right)\right)\|\phi_1-\phi_2\|_\sigma.\end{aligned}$$

So,

$$\begin{aligned}&\|N\phi_1 - N\phi_2\|_T \leqslant \\ &\frac{\left(\max\left(1,e^{\omega T}\right)\right)\frac{M}{\beta_1} + \left(2\alpha M\left(\frac{e^{\omega T}-1}{\omega}\right)\right)\left(\gamma_T + \frac{b}{2|\beta_1|}\right)}{1 - \left(\left(\frac{e^{\omega T}-1}{\omega}\right)(2\alpha M b)\right)}\|\phi_1-\phi_2\|_\sigma.\end{aligned}$$

Therefore, a Lipschitz map is obtained,

$$N : \{\phi \in \mathbf{C}_\sigma(-\infty,0] : \phi(0) = x_0\} \longrightarrow \{y \in \mathbf{C}[0,T] : y(0) = x_0\}.$$

Define $\overline{S}_t : (\mathbf{C}_\sigma(-\infty,0];X) \longrightarrow (\mathbf{C}_\sigma(-\infty,0];X)$ as

$$\begin{aligned}\overline{S}_t\phi(\theta) &= N\phi(t+\theta), \ \ \theta + t \geqslant 0, \\ &= \phi(t+\theta), \ \ \theta+t < 0.\end{aligned}$$

$$\begin{aligned}\|\overline{S}_t\phi_1 - \overline{S}_t\phi_2)\| &= \|H(t,N(\phi_1),\phi_1) - H(t,N(\phi_2),\phi_2)\| \\ &\leqslant 2b\|N(\phi_1)-N(\phi_2)\|_T + [2\gamma(t) + \frac{1}{|\beta_1|}b]\|\phi_1-\phi_2\|_\sigma \\ &\leqslant \left(\frac{\left(\max\left(1,e^{\omega T}\right)\right)\frac{M}{\beta_1} + \left(2M\left(\frac{e^{\omega T}-1}{\omega}\right)\right)\left(\gamma_T + \frac{1}{2|\beta_1|}b\right)}{1-\left(\left(\frac{e^{\omega T}-1}{\omega}\right)(2\alpha M b)\right)}\right. \\ &+ \left.[2\gamma_T + \frac{1}{|\beta_1|}b]\right)\|\phi_1-\phi_2\|_\sigma.\end{aligned}$$

Also, it is standard to show that for $t,s \in [0,T]$ with $t+s \leqslant T$, $\overline{S}_t\overline{S}_s\phi = \overline{S}_{t+s}\phi$. For arbitrary $t > T$, it is obtained that for $n \in \mathbb{N}$ such that $t = nT + \delta$, where $0 < \delta < T$. Defining

$$\overline{S}_t = (\overline{S}_T)^n\overline{S}_\delta$$

the proof is complete.

EXAMPLE 1. Let $\beta > 1$. The sequence $\beta_k = \beta^{-k}$ satisfy (3.1). Further, if $\log_e \beta > \gamma > 0$, then the standard phase space

$$\mathbf{C}_\gamma = \left\{ \phi \in \mathbf{C}((-\infty,0];X) : \sup_{\theta \in (-\infty,0]} \|\phi(\theta)\| e^{-\gamma\theta} < \infty \right\}$$

is contained in $\mathbf{C}_\sigma((-\infty,0];X)$.

EXAMPLE 2. Let $p > 1$. The sequence $\beta_k = \frac{1}{k^p}$ satisfies (3.1).

EXAMPLE 3. Let $K : (-\infty,0] \longrightarrow I\!R$ be measurable function and let there exist a sequence β_k such that

$$\int_{-k}^{-k+1} |K(\theta)| d\theta \leqslant \beta_k$$

for all $k \in I\!N$ and let β_k satisfy (3.1). Then $L : \mathbf{C}_\sigma((-\infty,0];X) \longrightarrow X$ defined as

$$L\phi = \int_{-\infty}^{0} K(\theta)\phi(\theta)d\theta \tag{3.2}$$

is a bounded linear map from $\mathbf{C}_\sigma((-\infty,0];X)$ into X.

Let $\gamma, p, q > 0$ and it is not difficult to check that $K_1, K_2 : (-\infty,0] \longrightarrow I\!R$ defined as

$$K_1(\theta) = |\theta|^p e^{-\gamma\theta} \quad \text{and} \quad K_2(\theta) = \frac{1}{(|\theta| + q)^{p+1}}$$

satisfy the above conditions.

EXAMPLE 4. Let $\eta : (-\infty,0] \longrightarrow I\!R$ be a function of bounded variation on each of the interval $[-k,-k+1]$. Let there exist a sequence β_k such that

$$Var_\eta(-k,-k+1) \leqslant \beta_k \text{ for all } k \in I\!N \text{ and let } \beta_k \text{ satisfy (3.1).}$$

Then $L : \mathbf{C}_\sigma((-\infty,0];X)) \longrightarrow X$ defined as

$$L\phi = \int_{-\infty}^{0} d\eta(\theta)\phi(\theta)d\theta \tag{3.3}$$

is a bounded linear map from $\mathbf{C}_\sigma((-\infty,0];X)$ into X.

4. Linear PDE with infinite delay

In this section, the sequence of solutions to (1.2) converge to the solution of (1.3) is shown. The following theorem proves this result.

THEOREM 4. *Let* X , $A:\mathbf{D}(A)\longrightarrow X$,X_n, E_n, P_n A_n *satisfy* **(i)**, **(ii)**, **(iii)** *and* **(iv)** *of Theorem 1. The maps* $\overline{E}_n$ *and* $\overline{P}_n$ *are defined as in Theorem 2. In addition, assume that for every* $u\in\mathbf{D}(A)$, *there exists a sequence* $u_n\in X_n$ *such that* $E_n\overline{u}_n\to u$ *and* $E_nA_n\overline{u}_n\to Au$. *Further, let* $L:\mathbf{C}_\sigma((-\infty,0];X)\longrightarrow X$ *be a bounded linear map. Define* $\mathbf{D}(\tilde{A})$ *as*

$$\mathbf{D}(\tilde{A}) = \big\{\phi\in\mathbf{C}_\sigma((-\infty,0];X):\phi'\in\mathbf{C}_\sigma((-\infty,0];X),\\ \phi(0)\in\mathbf{D}(A),\phi'(0)=A[\phi(0)]+L(\phi)\big\},$$

$\tilde{A}$ *as* $\tilde{A}\phi=\phi'$, $L_n:\mathbf{C}_\sigma((-\infty,0];X_n)\longrightarrow X_n$ *as* $L_n\varphi=P_n[L\overline{E}_n(\varphi)]$,
$\mathbf{D}(\tilde{A_n})$ *as*

$$\mathbf{D}(\tilde{A_n}) = \big\{\varphi\in\mathbf{C}_\sigma((-\infty,0];X_n):\varphi'\in\mathbf{C}_\sigma((-\infty,0];X_n),\\ \varphi(0)\in\mathbf{D}(A_n),\quad \varphi'(0)=A_n[\varphi(0)]+P_nL(\overline{E}_n\varphi)\big\}$$

and $\tilde{A_n}$ *as* $\tilde{A_n}\varphi=\varphi'$. *Then*

(A) $\tilde{A}_n$ *generates the semigroup* $\overline{T}_t^{(n)}$ *such that* $\overline{T}_t^{(n)}\varphi$ *is the mild solution to the delay equation (1.2)*

(B) $\tilde{A}$ *generates the semigroup* $\overline{T}_t$ *such that* $\overline{T}_t\phi$ *is the mild solution to the delay equation (1.3)*

(C) *For every* $\phi\in\mathbf{C}_\sigma((-\infty,0];X)$, $\overline{E}_n\overline{T}_t^{(n)}\overline{P}_n\phi$ *converges to* $\overline{T}_t\phi$.

Proof. If the operator norm $\|L\|$ is denoted by α, then the operator norm $\|L_n\|$ satisfies $\|L_n\|\leqslant\alpha C_1C_2$. Choose $T>0$ such that

$$\left(\frac{e^{\omega T}-1}{\omega}\right)\left(2M\sum_{k=1}^{\infty}|\beta_k|\right)\max(\alpha,\alpha C_1C_2)<1.$$

Thus, using Theorem 3 one obtains the existence of linear semigroups $\overline{T}_t$ and $\overline{T}_t^{(n)}$ which satisfy

$$\sup_{t\in[0,T]}\max(\|T_t\|,\|T_t^{(n)}\|)\\ \leqslant\left(\frac{\left(\max\left(1,e^{\omega T}\right)\right)\frac{M}{\beta_1}+\left(2M\left(\frac{e^{\omega T}-1}{\omega}\right)\right)\left(\gamma_T+\frac{b}{2|\beta_1|}\right)}{1-\left(\left(\frac{e^{\omega T}-1}{\omega}\right)(2\max(\alpha,\alpha C_1C_2)Mb)\right)}[2\gamma_T+\frac{b}{|\beta_1|}]\right).$$

Denoting the quantity on the right hand side of the above inequality by μ, it is standard to show that the above semigroups satisfy

$$\max(\|T_t\|,\|T_t^{(n)}\|)\leqslant\mu e^{\frac{\ln\mu}{T}t}$$

for all $t\in[0,\infty)$.

Let $\phi \in \mathbf{D}(\tilde{A})$. Then $\phi'(0) = A[\phi(0)] + L\phi$.

Define $\lambda_n : (-\infty, 0] \longrightarrow I\!R$ as

$$\begin{aligned}\lambda_n(\theta) &= \theta\left(\theta + \frac{1}{n}\right)\left(\theta + \frac{1}{2n}\right)[12n^4\theta^2 + 26n^3\theta + 10n^2] \\ &\quad -4n\theta\left(\theta + \frac{1}{n}\right), \ \ \theta \in \left[\frac{-1}{n}, 0\right] \\ &= 0, \ \ \theta < \frac{-1}{n}.\end{aligned}$$

Note that $\lambda_n(0) = \lambda_n(-1/n) = \lambda_n'(-1/n) = \lambda_n'(-1/2n) = 0$ and $\lambda_n'(0) = 1$.

For $\theta \in \left[-\frac{1}{n}, 0\right]$,

$$\begin{aligned}|\lambda_n(\theta)| &\leqslant \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{2n}\right)\left(\frac{12n^4}{n^2} + 26\frac{n^3}{n} + 10n^2\right) + 4n\left(\frac{1}{n}\right)\left(\frac{2}{n}\right) \\ &\leqslant \frac{144}{n} + \frac{8}{n} \\ &\leqslant \frac{152}{n}\end{aligned}$$

and hence as $n \to \infty$, λ_n converges to zero uniformly on $(-\infty, 0]$. Further, it is obtained that there exists $C_3 > 0$ such that

$$|\lambda_n'(\theta)| \leqslant C_3 \ \text{ for all } \ \theta \in (-\infty, 0]. \tag{4.1}$$

Fixing a $v \in X_n$, denote the X_n valued function defined on $(-\infty, 0]$ which takes the value $\lambda_n(\theta)v$ at every θ by $\lambda_n v$. It is clear that $\lambda_n v \in \mathbf{C}_\sigma((-\infty, 0]; X_n)$. Denoting the norm on $\mathbf{C}_\sigma((-\infty, 0]; X_n)$ by $\| \ \|_{\sigma,n}$, it is obtained that

$$\|\lambda_n v\|_{\sigma,n} \leqslant \beta_1 \left(\sup_{\theta \in [-1/n, 0]} |\lambda_n(\theta)|\right) \|v\|_n \leqslant \beta_1 \frac{152}{n} \|v\|_n.$$

Recalling that $\phi \in \mathbf{D}(\tilde{A})$, $\phi(0) \in \mathbf{D}(A)$ and there exists $\overline{u}_n \in X_n$ such that $E_n\overline{u}_n \to \phi(0)$ and $E_nA_n\overline{u}_n \to A[\phi(0)]$.

Define ψ_n as

$$\psi_n(\theta) = P_n[\phi(\theta)] + \lambda_n(\theta)v_n - P_n[\phi(0)] + \overline{u}_n,$$

where $v_n \in X_n$ is the unique fixed point of $K_n : X_n \longrightarrow X_n$ defined as

$$\begin{aligned}K_n v = {} & A_n\overline{u}_n - P_nA(\phi(0) \\ & +P_n[L(\overline{E}_n\overline{P}_n\phi) - (L\phi)] \\ & +P_n[-L(\overline{E}_n\,\overline{P}_n\phi(0) + L\overline{E}_n\overline{u}_n)] \\ & +P_n[L(\overline{E}_n\,\overline{P}_n\lambda_n v)].\end{aligned}$$

Here, it is observed that K_n is an affine map such that

$$\|K_n v - K_n w\|_{\sigma,n} \leqslant C_1 C_2 \|L\| \; \|\lambda_n (v-w)\|_{\sigma,n}.$$

Thus, for large enough n, K_n is a contraction.

Now, one can claim that $\psi_n \in \mathbf{D}(\tilde{A_n})$.

In the following computations, the constant function $c_y : (-\infty, 0] \longrightarrow Y$ defined as $c_y(\theta) = y$, where Y is a Banach space and $y \in Y$ is denoted by y itself.

Now, $\psi_n(0) = P_n[\phi(0)] + \lambda(0)v_n - P_n[\phi(0)] + \overline{u}_n = \overline{u}_n$,

$$\psi_n'(\theta) = P_n[\phi'(\theta)] + \lambda_n'(\theta)v_n \tag{4.2}$$

and thus,

$$\begin{aligned}
\psi_n'(0) &= P_n[\varphi'(0)] + \lambda'(0)v_n \\
&= P_n[\varphi'(0)] + v_n \\
&= P_n[A(\phi(0))] + P_n(L\phi) + v_n \\
&= P_n[A(\phi(0))] + P_n L(\phi) \\
&\quad + A_n\overline{u}_n - P_n A(\phi(0) \\
&\quad + P_n[L(\overline{E}_n \overline{P}_n \phi) - (L\phi)] \\
&\quad + P_n[-L(\overline{E}_n \overline{P}_n \phi(0) + L\overline{E}_n \overline{u}_n)] \\
&\quad + P_n[L(\overline{E}_n \overline{P}_n \lambda_n v)] \\
&= A_n(\overline{u}_n) + P_n L\overline{E}_n \overline{P}_n(\phi + \lambda_n v) - P_n L E_n P_n[\phi(0)] + P_n L E_n \overline{u}_n.
\end{aligned}$$

Then

$$A_n[\psi_n(0)] + P_n L(\overline{E}_n \psi_n) = A_n\overline{u}_n + P_n L\overline{E}_n \overline{P}_n(\phi + \lambda_n v_n) - P_n L E_n P_n[\varphi(0)] + P_n L E_n \overline{u}_n$$

and hence

$$\psi_n'(0) = A_n[\psi_n(0)] + P_n L(\psi_n).$$

Thus, $\psi_n \in \mathbf{D}(\tilde{A_n})$. Further, $\tilde{A_n}\psi_n = \psi_n'$. Next,

$$\begin{aligned}
\|E_n v_n\| = \|E_n K_n v_n\| \leqslant\; & \|E_n A_n \overline{u}_n - E_n P_n A(\phi(0)\| + \|E_n P_n L(\overline{E}_n \overline{P}_n \phi) - E_n P_n(L\phi)]\| \\
& + \|E_n P_n L\overline{E}_n \overline{u}_n - E_n P_n L\overline{E}_n P_n[\phi(0))]\| + \|E_n P_n[L(\overline{E}\lambda_n v)]\|.
\end{aligned} \tag{4.3}$$

Since

$$\begin{aligned}
\lim_{n\to\infty} \big(& \|E_n A_n \overline{u}_n - E_n P_n A(\phi(0))\| + \|E_n P_n L(\overline{E}_n \overline{P}_n \phi) - E_n P_n(L\phi)\| \\
& + \|E_n P_n L\overline{E}_n \overline{u}_n - E_n P_n L\overline{E}_n P_n(\phi(0))\|\big) = 0
\end{aligned}$$

there exists a C such that

$$\|E_n v_n\| \leqslant C + \|E_n P_n[L(\overline{E}_n \lambda_n v_n)]\|.$$

For large enough n, one obtains $k \in (0,1)$ such that $\|L(\overline{E}_n\lambda_n v_n)\|_n \leqslant \frac{k}{C_1}\|v_n\|_n$. Hence

$$\begin{aligned}\|P_nE_nv_n\|_n &\leqslant C_1C + k\|v_n\|_n\\ (1-k)\|v_n\|_n &\leqslant C_1C\\ \|v_n\|_n &\leqslant C_1C/(1-k).\end{aligned} \tag{4.4}$$

From (4.4) and the estimation,

$$\|E_n(L\overline{E}_n\lambda_n v_n)\| \leqslant C_2^2\|L\|\frac{152}{n}\|v_n\|_n$$

$$\lim_{n\to\infty}\|E_n(L\overline{E}_n\lambda_n v_n)\| = 0. \tag{4.5}$$

Using (4.3) and (4.5),

$$\lim_{n\to\infty}\|v_n\|_n = \lim_{n\to\infty}\|P_nE_nK_nv_n\|_n = 0. \tag{4.6}$$

With the help of (4.1), (4.2) and (4.6),

$$\begin{aligned}\lim_{n\to\infty}\overline{E}_n\tilde{A}_n\psi_n &= \lim_{n\to\infty}\overline{E}_n\overline{P}_n\varphi\prime + \lim_{n\to\infty}\overline{E}_n\lambda_n' v_n\\ &= \varphi\prime = \tilde{A}\varphi.\end{aligned}$$

Thus **(a)** of Theorem 1 is satisfied and hence **(b)** of that same holds.

REMARK 2. If $\lim_{n\to\infty} E_nA_nP_nu = Au$ for all $u \in \mathbf{D}(A)$, then it can be taken $\overline{u}_n = P_nu$. There are many interesting cases one can have $E_nA_nP_nu \to Au$, where it can be considered $\overline{u}_n$ to be just P_nu. For example, refer to Theorem 4.4 of Kulkarni and Ramesh (2008). But the general case is useful in examples like Example 5 and Example 8 of next section.

REMARK 3. The proposed method of proof works without any major modifications for the finite delay case also.

5. Examples

In this section, based on this research work few examples have presented here.

EXAMPLE 5. (First Order Hyperbolic Equation With Delay in $L^1[0,1]$)

$$\begin{aligned}u_t(x,t) + u_x(x,t) &= \int_{-\infty}^{0} K(\theta)u(x,t+\theta)d\theta,\ \ x\in[0,1],\ t\geqslant 0,\\ u(x,\theta) &= \phi(x,\theta),\ \ \theta\in(-\infty,0],\\ u(1,\theta) &= 0,\ \ \theta\in(-\infty,0].\end{aligned} \tag{5.1}$$

Let X, A, X_n, E_n, P_n and A_n be as in Case 1 in Example 3.1 of Ito and Kappel (1998) and let K be as in Example 3. Define $L : \mathbf{C}_\sigma((-\infty,0];L^1[0,1]) \longrightarrow L^1[0,1]$ as in (3.3). The solutions of the system of DDE

$$v'(t)_i = \frac{v(t)_{i-1} - v(t)_i}{h} + \int_{-\infty}^{0} K(\theta)v_i(t+\theta)d\theta,\ i = 1,2,\ldots,n$$

approximate the solution to (5.1).

EXAMPLE 6. (First Order Hyperbolic Equation With Delay in $\mathbf{C}[0, 1]$) This example is related to the equation studied in Dyson et al. (2003).

Let $a,b,c : [0,1] \longrightarrow I\!R$ be continuous functions such that

$$a(x) > 0 \text{ for all } x \in [0,1),\ a(1) = 0 \text{ and } \int_0^1 \frac{d\xi}{a(\xi)} = \infty.$$

Take $0 < \alpha < 1$.

Consider the delayed PDE

$$\begin{aligned} u_t(x,t) &= a(x)u_x(x,t) + b(x)u(x,t) + c(x)u(\alpha x, t-\tau),\ \ x \in [0,l], t \geqslant 0,\\ u(x,\theta) &= \phi(x,\theta),\ \ \theta \in [-\tau,0],\\ u(1,s) &= 0,\ \ \theta \in [-\tau,0]. \end{aligned} \tag{5.2}$$

Let $X = \{u \in C[0,1] : u(1) = 0\}$ and $\beta(x) = \int_0^x \frac{d\xi}{a(\xi)}$.
Now, $S_t^B f(x) = f[\beta^{-1}(t+\beta(x))]$ defines a contraction semigroup on $\mathbf{C}[0,1]$ whose generator B is given by

$$\mathbf{D}(B) = \left\{ g \in \mathbf{C}[0,1] : g' \in \mathbf{C}[0,1) \ \text{ and } \ \lim_{x\to 1} a(x)g'(x) = 0 \right\}$$

and

$$\begin{aligned} Bg(x) &= a(x)g'(x),\ \ x \in [0,1),\\ Bg(1) &= 0. \end{aligned}$$

With $\mathbf{D}(A) = \mathbf{D}(B)$ and $Ag(x) = Bg(x) + b(x)g(x)$, A generates a semigroup which gives the mild solution to the following PDE by Theorem 1.1 from Chapter 3 of Pazy (1983),

$$\begin{aligned} u_t(x,t) &= a(x)u_x(x,t) + b(x)u(x,t),\ \ x \in [0,1], t \geqslant 0,\\ u(x,0) &= u_0(x),\ \ x \in [0,1] \end{aligned} \tag{5.3}$$

where $u_0(1) = 0$. Define $L : \mathbf{C}([-\tau,0];\mathbf{C}[0,1]) \longrightarrow \mathbf{C}[0,1]$ is defined as $L\phi(x) = \phi(-\tau)(\alpha x)$.

Let $X_n = I\!R^{n+1}$. For $v \in I\!R^{n+1}$, choose the notation $v = (v_0, v_1, \ldots, v_n)$. Both spaces X and X_n are normed with the usual supremum norm.

Now define $P_n : X \to X_n$ as $P_n f = (f(0), f(1/n), f(2/n), \ldots, f(1))$.
$E_n : X_n \to X$ is defined as follows:

for $v \in I\!R^{n+1}$, $E_n v$ the function f which is equal to v_i at the point $x = i/n$, $i = 0, 1, \ldots, n$ and is linear between any two consecutive points i/n and $(i+1)/n$. It is easy to see that $\|E_n\| \leqslant 1$ and $\|P_n\| \leqslant 1$, $P_n E_n V = V$ for all $V \in I\!R^{n+1}$ and $\lim\limits_{n\to\infty} E_n P_n f = f$ for every $f \in X$.

Define $A_n : I\!R^n \longrightarrow I\!R^n$ as

$$(A_n v)_i = na(i/n)[v_{i+1} - v_i] + b(i/n)v_i, \;\; i = 0, 1, 2, \ldots, n-1$$
$$(A_n v)_n = b(1)v_{n-1}.$$

Using arguments analogous to those in Case 3 of Example 4.1 in Ito and Kappel (1998) and the fact that product of bounded uniformly convergent sequences converges uniformly, it can be shown that $E_n A_n P_n u \longrightarrow Au$ and hence by taking $\overline{u}_n = P_n u$, all the hypothesis of Theorem 4 are true.

Now, the explicit form for $P_n L \overline{E}_n \varphi$ can be presented here. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. For $\varphi \in \mathbf{C}_\sigma \left((-\infty, 0]; I\!R^{n+1}\right)$,

$$[P_n L \overline{E}_n \varphi]_i = \varphi(-\tau)_{\lfloor \alpha i \rfloor} + \frac{\varphi(-\tau)_{\lfloor \alpha i \rfloor + 1} - \varphi(-\tau)_{\lfloor \alpha i \rfloor}}{h} \left(\alpha i n - \frac{\lfloor \alpha i \rfloor}{n}\right).$$

Thus, the system of differential equations approximating equation 5.2 is given by

$$\begin{aligned}
[v'(t)]_i = {} & na(i/n)[v(t)_{i+1} - v(t)_i] + b(i/n)v(t)_i + v(t-\tau)_{\lfloor \alpha i \rfloor} \\
& + \frac{v(t-\tau)_{\lfloor \alpha i \rfloor + 1} - v(t-\tau)_{\lfloor \alpha i \rfloor}}{h} \left(\alpha i n - \frac{\lfloor \alpha i \rfloor}{n}\right), \;\; i = 0, 1, 2, \ldots, n-1, \\
(v'(t))_n = {} & b(1)v(t)_n + v_n(t-\tau)_{\lfloor \alpha n \rfloor} \\
& + \frac{v(t-\tau)_{\lfloor \alpha n \rfloor + 1} - v(t-\tau)_{\lfloor \alpha n \rfloor}}{h} \left(\alpha - \frac{\lfloor \alpha n \rfloor}{n}\right).
\end{aligned}$$

EXAMPLE 7. (Parabolic Equation With Delay) Let Ω be a bounded open set of class $\mathbf{C}^\infty$ with boundary Γ. Consider

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \triangle u(x,t) + \int_{-\infty}^{0} d\eta(\theta) u(x, \theta + t), \\
u(x,\theta) &= \phi(x,\theta), \;\; \theta \in (-\infty, 0], \;\; x \in \Omega, \\
u(x,t) &= 0 \;\text{ on } \Gamma \times [0, \infty).
\end{aligned}$$

Take $X = L^2(\Omega)$, $\mathbf{D}(A) = H^2(\Omega) \cup H_0^1(\Omega)$, $A : \mathbf{D}(A) \longrightarrow X$ as $Au = \triangle u$, $X_n = I\!R^n$ with the norm $\|v\|_n = \|E_n P_n v\|_{L^2}$, where $E_n : X_n \longrightarrow X$ and $P_n : X \longrightarrow X_n$ are defined as below:

Let $\{g_n : n \in I\!N\}$ be an ortho-normal basis of $L^2(\Omega)$ consisting of eigen functions of $\triangle$ and $\{\lambda_n : n \in I\!N\}$ be the corresponding sequence of negative eigen values with $\lambda_n \to -\infty$ as $n \to \infty$.

$$(P_n f)_i = \langle f, g_i \rangle, \;\; E_n v = \sum_{i=1}^{n} v_i g_i.$$

Since $E_nP_n = \Pi_n f$, where Π_n is the orthogonal projection given by

$$\Pi_n f = \sum_{i=1}^{n} \langle f, g_i \rangle g_i,$$

one can verify **(i), (ii)** and **(iii)** of Theorem 1. Let $A_n : X_n \longrightarrow X_n$ be defined as $[A_n v] = P_n A E_n v$. For $u \in \mathbf{D}(A)$,

$$E_n A_n P_n u = E_n P_n A E_n P_n u = \Pi_n A \Pi_n u.$$

Now, using Theorem 4.4 of Kulkarni and Ramesh (2008), $\|Au - A\Pi_n u\| \to 0$. Since

$$\begin{aligned} \|Au - E_n A_n P_n u\| &= \|Au - \Pi_n Au + \Pi_n Au - \Pi_n A \Pi_n u\| \\ &\leqslant \|Au - \Pi Au\| + \|\Pi_n\| \|Au - A\Pi_n u\|, \end{aligned}$$

(a) of Theorem 1 is true for $\overline{u} = P_n u$. Letting L be as in Example 4, note that $P_n L \overline{E}_n$ is given by an identical expression to (3.3). Since $(A_n v_n)_i = \lambda_i v_i$, one can obtain the approximating DDE as

$$\begin{aligned} v'(t)_i &= \lambda_i v(t)_i + \int_{-\infty}^{0} d\eta(\theta) v_i(t+\theta),\ i = 1, 2, \ldots, n, \\ v(\theta) &= \varphi(\theta), \theta \in (-\infty, 0]. \end{aligned}$$

EXAMPLE 8. (Wave Equation with Delay)
For equations such as

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= a^2 \frac{\partial^2 u}{\partial x^2}(x,t) + bu(x, t-\tau) + c\frac{\partial u}{\partial t}(x, t-\tau), \\ u(x,\theta) &= \phi(x,\theta), \quad \theta \in [-\tau, 0], \quad x \in [0, l], \\ \frac{\partial u}{\partial t}(x,\theta) &= \phi_1(x,\theta), \quad \theta \in [-\tau, 0], \quad x \in [0, 1], \\ u(0,t) &= 0, \\ ku(1,t) + \frac{\partial}{\partial x} u(1,t) &= 0, \quad t \in [0, \infty) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= a^2 \frac{\partial^2 u}{\partial x^2}(x,t) + \int_{-\infty}^{0} d\eta(\theta) u(x, t+\theta) + \int_{-\infty}^{0} d\eta_1(\theta) \frac{\partial u}{\partial t}(x, t+\theta), \\ u(x,\theta) &= \phi(x,\theta), \quad \theta \in [-\tau, 0], \quad x \in [0, l], \\ \frac{\partial u}{\partial t}(x,\theta) &= \phi_1(x,\theta), \quad \theta \in [-\tau, 0], \quad x \in [0, l], \\ u(0,t) &= 0, \\ ku(1,t) + \frac{\partial}{\partial x} u(1,t) &= 0, \quad t \in [0, \infty). \end{aligned} \tag{5.5}$$

One can find approximating DDEs with infinite delay by choosing X, A, X_n, P_n, E_n and A_n as in Example 4.2 of Ito and Kappel (1998).

6. Conclusion

In this paper, the existence of solution to non-linear partial differential equation with infinite delay by semigroup of Lipchitz maps is proved and examples are given to support the result.

Also the convergence of solutions of linear infinite delay equation to the solution of linear partial differential equation with infinite delay is proved.

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