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EXISTENCE RESULTS FOR A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS ON THE HALF–LINE

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Abstract. By means of nonlinear alternative theorem of Leray-Schauder, some new results on the existence of positive solutions for a nonlinear fractional differential equations with integral boundary conditions on unbounded domain are established. The paper concludes with an illustrative example.

1. Introduction

The study of fractional differential equations has become a very important and useful area of mathematics over the last few decades due to its numerous applications in various areas of physics, chemistry and engineering such as viscoelasticity.

Recently, the theory on existence and uniqueness of solutions (or positive solution) of nonlinear fractional differential equations with finite domain by the use of techniques of nonlinear analysis has attracted the attention of many authors, (see, for example [2, 3, 4, 6, 15, 16, 17]) and references therein.

Some recent results on fractional differential equations with infinite domain, for instance, can be found in papers [5, 8, 11, 13, 18, 19, 20] and references therein. In this paper, we study the existence of positive solutions for a boundary value problem of nonlinear fractional differential equations with integral boundary conditions on an infinite interval. Precisely, we consider the following problem (FBVP for short):

$$D_{0^+}^{\alpha}u(t)+f(t,u(t))=0,\ t\in J=[0,+\infty), \tag{1.1}$$

$$u(0)=u'(0)=0,\ D_{0^+}^{\alpha-1}u(\infty)=\int_0^{\infty}\varphi(t)u(t)dt, \tag{1.2}$$

where $2<\alpha\leqslant 3$, $f\in C(J\times\mathbb{R}^+,\mathbb{R}^+)$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\alpha-1}$ are the standard Riemann-Liouville fractional derivatives of order α and $\alpha-1$, respectively and $D_{0^+}^{\alpha-1}u(\infty)=\lim_{t\to+\infty}D_{0^+}^{\alpha-1}u(t)$.

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Here by a positive solution we mean a function $u(t)$ which is positive on $(0,+\infty)$ and satisfies (1.1)-(1.2).

Throughout this paper, we assume that the following conditions hold:

(H1) for any $t \in (0,+\infty)$ assume that

$$\int_0^\infty t^{\alpha-1}\varphi(t)dt = \mu < +\infty, \ \mu \in (0,\Gamma(\alpha));$$

(H2) there exist nonnegative functions $a(t)$ and $b(t)$ defined on $[0,+\infty)$ and a constant $\rho > 0$ such that

$$|f(t,u(t))| \leqslant a(t) + b(t)\,|u(t)|^\rho$$

and

$$\int_0^{+\infty} a(t)dt = a < +\infty, \quad \int_0^{+\infty} \left(1+t^{\alpha-1}\right)^\rho b(t)dt = b < +\infty.$$

The rest of this paper is organized as follows: In section 2, we present some preliminaries and lemmas that will be used to prove our main results. Section 3 is devoted to prove the existence of positive solutions for FBVP (1.1)-(1.2). In section 4 an example is worked out to demonstrate our main result.

2. Background and Preliminary lemmas

In this section, we present some notations, definitions and preliminary lemmas which are used throughout the paper. We also state in this section the nonlinear alternative theorem of Leray-Schauder.

For the definitions of fractional integral, fractional derivative, related proprieties and existence theorems of fractional differential equations we refer the reader to [9, 14].

DEFINITION 1. ([9, 14]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

where $\Gamma(\alpha)$ is the gamma function, provided that the integral exists.

DEFINITION 2. ([9, 14]) The Riemann-Liouville fractional derivative of order $\alpha > 0$, of a continuous function $u : (0,+\infty) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α, provided that the right side is pointwise defined on $(0,\infty)$.

LEMMA 1. *([9]) Let* $u \in C(0,+\infty) \cap L^1(0,+\infty)$, $\beta \geqslant \alpha > 0$, *then*

$$D_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\beta-\alpha} u(t).$$

LEMMA 2. *([9]) Let* $\alpha > 0$, *then*

(i) *If* $\mu > -1$, $\mu \neq \alpha - i$ *with* $i = 1,2,...,[\alpha]+1$, $t > 0$, *then*

$$D_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha} \quad , \ \alpha > 0.$$

(ii) *For* $i = 1,2,...,[\alpha]+1$, *we have*

$$D_{0^+}^{\alpha} t^{\alpha-i} = 0.$$

(iii) *For every* $t \in (0,+\infty)$, $u \in L^1(0,+\infty)$

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t).$$

(iv) *For every* $t \in (0,+\infty)$, $u \in L^1(0,+\infty)$

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-i}, \ c_i \in R, n = [\alpha]+1.$$

(v) $D_{0^+}^{\alpha} u(t) = 0$ *if and only if* $u(t) = \sum_{i=1}^{n} c_i t^{\alpha-i}$, $c_i \in \mathbb{R}$, $n = [\alpha]+1$.

The following lemma is crucial in finding an integral representation of the boundary value problem (1.1)-(1.2).

LEMMA 3. *Let* $h \in C([0,+\infty))$ *such that*

$$0 < \int_0^{\infty} h(s)ds < +\infty.$$

For $\Gamma(\alpha) \neq \mu$, *the fractional boundary-value problem*

$$D_{0^+}^{\alpha} u(t) + h(t) = 0, \ t \in (0,+\infty), \ 2 < \alpha \leqslant 3, \tag{2.1}$$

$$u(0) = u'(0) = 0, \ D_{0^+}^{\alpha-1} u(\infty) = \int_0^{\infty} \varphi(t) u(t) dt, \tag{2.2}$$

has a unique solution given by

$$u(t) = \int_0^{\infty} G(t,s) h(s) ds, \tag{2.3}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.4}$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant +\infty \\ t^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant +\infty \end{cases} \tag{2.5}$$

and

$$G_2(t,s) = \frac{t^{\alpha-1}}{(\Gamma(\alpha)-\mu)} \int_0^{\infty} \varphi(t) G_1(t,s) dt. \tag{2.6}$$

The function $G(t,s)$ *is called Green's function of boundary-value problem (2.1)-(2.2).*

Proof. By Lemmas 1 and 2, we can reduce the equation (2.1) to an equivalent integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3}, \tag{2.7}$$

for some $c_1, c_2, c_3 \in \mathbb{R}$.

From $u(0) = u'(0) = 0$, we know that $c_2 = c_3 = 0$. Thus,

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1t^{\alpha-1}. \tag{2.8}$$

On the other hand

$$D_{0^+}^{\alpha-1}u(t) = -\int_0^t h(s)ds + c_1\Gamma(\alpha),$$

combining with

$$D_{0^+}^{\alpha-1}u(\infty) = \int_0^\infty \varphi(s)u(s)ds,$$

we have

$$c_1 = \frac{1}{\Gamma(\alpha)}\left(\int_0^\infty \varphi(s)u(s)ds + \int_0^\infty h(s)ds\right).$$

Therefore, the unique solution of (2.1)-(2.2) is

$$\begin{aligned}
u(t) &= -\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^\infty \varphi(s)u(s)ds + \int_0^\infty h(s)ds\right) \\
&= -\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}h(s)ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\varphi(s)u(s)ds \\
&= \int_0^\infty G_1(t,s)h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\varphi(s)u(s)ds
\end{aligned}$$

where $G_1(t,s)$ is defined by (2.5).

From

$$u(t) = \int_0^\infty G_1(t,s)h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\varphi(s)u(s)ds \tag{2.9}$$

we have

$$\begin{aligned}
\int_0^\infty \varphi(t)u(t)dt &= \int_0^\infty \varphi(t)\left(\int_0^\infty G_1(t,s)h(s)ds + \tfrac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\varphi(s)u(s)ds\right)dt \\
&= \int_0^\infty \varphi(t)\left(\int_0^\infty G_1(t,s)h(s)ds\right)dt \\
&\quad + \tfrac{1}{\Gamma(\alpha)}\int_0^\infty \varphi(t)\left(\int_0^\infty t^{\alpha-1}\varphi(s)u(s)ds\right)dt
\end{aligned}$$

$$= \int_0^\infty \varphi(t)\int_0^\infty G_1(t,s)h(s)dsdt$$
$$+ \tfrac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\varphi(t)dt\int_0^\infty \varphi(t)u(t)dt$$

it follows that

$$\int_0^\infty \varphi(t)u(t)dt = \frac{1}{\left(1-\tfrac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\varphi(t)dt\right)}\int_0^\infty \varphi(t)\int_0^\infty G_1(t,s)h(s)dsdt,$$

then

$$\int_0^\infty \varphi(t)u(t)dt = \frac{\Gamma(\alpha)}{(\Gamma(\alpha)-\mu)}\int_0^\infty \varphi(t)\int_0^\infty G_1(t,s)h(s)dsdt. \tag{2.10}$$

Substituting (2.10) into (2.9), we obtain

$$\begin{aligned} u(t) &= \int_0^\infty G_1(t,s)h(s)ds + \frac{t^{\alpha-1}}{(\Gamma(\alpha)-\mu)}\int_0^\infty \varphi(t)\int_0^\infty G_1(t,s)h(s)dsdt \\ &= \int_0^\infty G_1(t,s)h(s)ds + \frac{t^{\alpha-1}}{(\Gamma(\alpha)-\mu)}\int_0^\infty \left(\int_0^\infty \varphi(t)G_1(t,s)dt\right)h(s)ds \\ &= \int_0^\infty G_1(t,s)h(s)ds + \int_0^\infty G_2(t,s)h(s)ds \\ &= \int_0^\infty G(t,s)h(s)ds \end{aligned}$$

where $G(t,s), G_1(t,s)$, and $G_2(t,s)$ are defined by (2.4), (2.5) and (2.6) respectively. The proof is complete.

We need some properties of functions $G_1(t,s)$, $G_2(t,s)$ and $G(t,s)$ in order to discuss the existence of positive solutions.

LEMMA 4. *([11]) The function $G_1(t,s)$ defined by (2.5) satisfies*
(i) *$G_1(t,s)$ is a continuous function and $G_1(t,s) \geqslant 0$ for $(t,s) \in [0,+\infty)\times[0,+\infty)$;*
(ii) *$G_1(t,s)$ is strictly increasing in the first variable;*
(iii) *$G_1(t,s)$ is concave in the first variable for $0 < s < t < +\infty$;*
(iv) *$G_1(t,s)/(1+t^{\alpha-1}) \leqslant 1/\Gamma(\alpha)$, for all $(t,s) \in [0,+\infty)\times[0,+\infty)$ and,*

$$\int_0^\infty \varphi(t)G_1(t,s)dt < \mu/\Gamma(\alpha) \ \textit{for all } s \in [0,+\infty).$$

PROPOSITION 1. If $\mu \in (0,\Gamma(\alpha))$, the function $G_2(t,s)$ defined by (2.6) satisfies:
(i) $G_2(t,s)$ is a continuous function and $G_2(t,s) \geqslant 0$ for all $(t,s) \in [0,+\infty)\times[0,+\infty)$;
(ii) $G_2(t,s) \leqslant \mu t^{\alpha-1}/\Gamma(\alpha)(\Gamma(\alpha)-\mu)$, for all $(t,s) \in [0,+\infty)\times[0,+\infty)$;
(iii) $G_2(t,s)/(1+t^{\alpha-1}) \leqslant \mu/\Gamma(\alpha)(\Gamma(\alpha)-\mu)$, for all $(t,s) \in [0,+\infty)\times[0,+\infty)$.

Proof. Using the properties of $G_1(t,s)$, definition of $G_2(t,s)$, it can easily be shown that (i), (ii) and (iii) hold.

LEMMA 5. *If* $\mu \in (0,\Gamma(\alpha))$, *the function* $G(t,s)$ *defined by (2.4) satisfies:*
(i) $G(t,s)$ *is a continuous function and* $G(t,s) \geqslant 0$ *for all* $(t,s) \in [0,+\infty) \times [0,+\infty)$;
(ii) $G(t,s)/(1+t^{\alpha-1}) \leqslant 1/(\Gamma(\alpha)-\mu)$, *for all* $(t,s) \in [0,+\infty) \times [0,+\infty)$, *where*

$$L = \frac{1}{(\Gamma(\alpha)-\mu)}.$$

Proof. It follows from Lemme 1 and Lemme 2 that $G(t,s)$ is a continuous function and $G(t,s) \geqslant 0$ for all $(t,s) \in [0,+\infty) \times [0,+\infty)$;
(ii) If $0 \leqslant s,t < +\infty$, we have

$$\begin{aligned}
\frac{G(t,s)}{1+t^{\alpha-1}} &= \frac{G_1(t,s)}{1+t^{\alpha-1}} + \frac{G_2(t,s)}{1+t^{\alpha-1}} \\
&\leqslant \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} + \frac{t^{\alpha-1}}{1+t^{\alpha-1}} \frac{1}{(\Gamma(\alpha)-\mu)} \int_0^\infty \varphi(t) G_1(t,s)dt \\
&\leqslant \frac{1}{\Gamma(\alpha)} + \frac{1}{(\Gamma(\alpha)-\mu)} \int_0^\infty \varphi(t) G_1(t,s)dt \\
&\leqslant \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)(\Gamma(\alpha)-\mu)} \int_0^\infty t^{\alpha-1}\varphi(t)dt \\
&= \frac{1}{\Gamma(\alpha)} + \frac{\mu}{\Gamma(\alpha)(\Gamma(\alpha)-\mu)} \\
&= \frac{1}{\Gamma(\alpha)} \left(1 + \frac{\mu}{\Gamma(\alpha)-\mu}\right) \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha)-\mu}\right) \\
&= \frac{1}{\Gamma(\alpha)-\mu} = L.
\end{aligned}$$

This complete the proof of Lemma 5.

To establish the existence of solutions for FBVP (1.1)-(1.2), we need the following known Leray-Schauder nonlinear alternative.

THEOREM 1. *([1] Leray-Schauder nonlinear alternative). Let* C *be a convex subset of a Banach space, and let* U *be an open subset of* C *with* $0 \in U$. *Then every completely continuous map* $T : \overline{U} \longrightarrow C$ *has at least one of the following two properties:*
(i) T *has a fixed point in* $\overline{U}$;
(ii) *There is an* $x \in \partial\ U$ *and* $\lambda \in (0,1)$ *with* $x = \lambda Tx$.

3. Existence of solutions

In this section, we will apply the Leray-Schauder nonlinear alternative theorem 1 to the problem (1.1)-(1.2). For our constructions, we shall consider the space

$$X = \left\{ u \in C(J,\mathbb{R}) : \lim_{t\to\infty} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty \right\}, \tag{3.1}$$

equipped with the norm

$$\|u\|_X = \lim_{t\to\infty} \frac{|u(t)|}{1+t^{\alpha-1}}.$$

LEMMA 6. *([11]) $(X, \|.\|_X)$ is a Banach space.*

The Arzela-Ascoli theorem fails to work in the Banach space X due to the fact that the infinite interval $J = [0,+\infty)$ is noncompact. The following compctness criterion will help us to resolve this problem.

THEOREM 2. *([12]) Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following conditions hold:*
(i) *For any $u(t) \in U$, $u(t)/(1+t^{\alpha-1})$ is equicontinuous on any compact interval of J.*
(ii) *For any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that*

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \varepsilon$$

for any $t_1, t_2 \geqslant T$ and $u(t) \in U$.

Define the operator $T : X \to X$ by

$$Tu(t) = \int_0^\infty G(t,s) f(s,u(s))ds \ , 0 \leqslant t < +\infty \tag{3.2}$$

where $G(t,s)$ defined by (2.4),(2.5) and (2.6).

Observe that the problem (1.1)-(1.2) has a solution if and only if the operator T defined by (3.2) has a fixed point.

In order to use Theorem 1, we must show that $T: X \to X$ is completely continuous.

LEMMA 7. *Assume that (H_1)-(H_2) hold. Then the operator $T: X \to X$ is completely continuous.*

Proof. We divide the proof into several steps.

(a) The operator $T : X \to X$ is uniformly bounded. Let Ω be any bounded subset of X; then there exists a constant $R > 0$ such that $\|u\|_X \leqslant R$. By (H_2), we have

$$\begin{aligned}
\|Tu\|_X &= \sup_{t\in J} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} |f(s,u(s))|\, ds \\
&\leqslant \frac{1}{(\Gamma(\alpha)-\mu)} \int_0^{+\infty} \left[a(s) + b(s) \left(1+s^{\alpha-1}\right)^\rho \frac{|u(s)|^\rho}{(1+s^{\alpha-1})^\rho} \right] ds \\
&\leqslant \frac{a + b\,\|u\|_X^\rho}{(\Gamma(\alpha)-\mu)}
\end{aligned}$$

$$\leqslant \frac{a+bR^{\sigma}}{(\Gamma(\alpha)-\mu)} < +\infty.$$

This shows that $T\Omega$ is uniformly bounded.

(b) The operator $T : X \to X$ is continuous. Take u_n, $u \in X$ such that $\|u_n\|_X < +\infty$, $\|u\|_X < +\infty$ and $u_n \to u$ as $n \to \infty$. Then by (H_2), we have

$$\begin{aligned}
&\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s,u_n(s))ds \\
&\leqslant \frac{1}{(\Gamma(\alpha)-\mu)} \int_0^\infty \left[a(s)+b(s)\left(1+s^{\alpha-1}\right)^{\rho} \frac{|u_n(s)|^{\rho}}{(1+s^{\alpha-1})^{\rho}}\right] ds \\
&\leqslant \frac{a+b\,\|u_n\|_X^{\rho}}{(\Gamma(\alpha)-\mu)} \\
&< +\infty.
\end{aligned}$$

By the Lebesgue dominated convergence theorem and continuity of f, we obtain

$$\lim_{n\to\infty} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s,u_n(s))ds = \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s,u(s))ds.$$

Taking the limit $n \to \infty$, we get

$$\|Tu_n - Tu\|_X = \sup_{t\in J} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} \left|f(s,u_n(s)) - f(s,u(s))\right| ds \to 0.$$

Therefore T is continuous.

(c) The operator $T : X \to X$ is equicontinuous. We consider tow cases.

(c_1) Let $I \subset J$ be any compact interval and $t_1, t_2 \in I$ are such that $t_1 < t_2$. Let Ω be any bounded subset of X, then for any $u \in \Omega$, we have

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| \\
&\quad = \left|\int_0^\infty \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right) f(s,u(s))ds\right| \\
&\quad \leqslant \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| \left(a(s)+b(s)\left(1+s^{\alpha-1}\right)^{\rho} \frac{|u(s)|^{\rho}}{(1+s^{\alpha-1})^{\rho}}\right) ds \\
&\quad \leqslant \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| \left(a(s)+b(s)\left(1+s^{\alpha-1}\right)^{\rho} \|u\|_X^{\rho}\right) ds.
\end{aligned}$$

So,

$$\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|$$
$$\leqslant \int_0^{\infty}\left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right|\left(a(s)+b(s)\left(1+s^{\alpha-1}\right)^{\rho}\|u\|_X^{\rho}\right)ds. \quad (3.3)$$

Since $G(t,s)$ is continuous on $J\times J$, we have that $G(t,s)/(1+t^{\alpha-1})$ is a uniformly continuous function on the compact set $I\times I$. Moreover, for $s\geqslant t$, we have that this function only depends on t, in consequence it is uniformly continuous on $I\times(J\setminus I)$. So we have that for all $s\in J$ and $t_1,t_2\in I$ the following property holds:

For all $\varepsilon>0$ there is $\delta(\varepsilon)>0$ such that if $|t_2-t_1|<\delta$ then,

$$\left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right|<\varepsilon.$$

By this, together with (3.3), and the fact that

$$\int_0^{\infty}\left(a(s)+b(s)\left(1+s^{\alpha-1}\right)^{\rho}R^{\rho}\right)ds<\infty, \quad (3.4)$$

we can get that $T\Omega$ is equicontinuous on I.

(c_2) In fact, when $t\to\infty$, we have

$$\lim_{t\to\infty}\frac{G(t,s)}{1+t^{\alpha-1}}=0. \quad (3.5)$$

From this, it is not difficult to verify that for any given $\varepsilon>0$, there is a constant $T'=T'(\varepsilon)>0$ such that

$$\left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right|<\varepsilon.$$

for any $t_1,t_2\geqslant T'$ and $s\in J$. Hence, T is equiconvergent at ∞.

Thus the conclusion of Theorem 2 applies that hence T is relatively compact on J. So, $T:X\to X$ is completely continuous. This completes the proof.

We are now in a position to state and prove our existence result for the FBVP (1.1)-(1.2).

THEOREM 3. *Assume that (H_1) and (H_2) with $\sigma=1$ hold. If there exists $r>0$ such that*

$$r\left(1-\frac{b}{(\Gamma(\alpha)-\mu)}\right)>\frac{a}{(\Gamma(\alpha)-\mu)}. \quad (3.6)$$

Then problem (1.1)-(1.2) has a solution $u(t)$ satisfying

$$0\leqslant u(t)\leqslant\left(1+t^{\alpha-1}\right)r\ ,\ for\ t\in J.$$

Proof. Let $U = \{u \in X,\ \|u\|_X < r\}$. For $u \in \partial U$, if there exist $\lambda \in (0,1)$ such that $u = \lambda Tu$, then we have

$$\begin{aligned}
\|u\|_X &= \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}} \\
&= \sup_{t\in J} \frac{|\lambda\,(Tu)\,(t)|}{1+t^{\alpha-1}} \\
&\leqslant \sup_{t\in J} \int_0^{\infty} \frac{G(t,s)}{1+t^{\alpha-1}} \,|f(s,u(s))|\, ds \\
&\leqslant \frac{1}{(\Gamma(\alpha)-\mu)} \int_0^{\infty} |a(s)+b(s)u(s)|\, ds \\
&\leqslant \frac{a+b\,\|u\|_X}{(\Gamma(\alpha)-\mu)}.
\end{aligned}$$

This implies that

$$r\left(1-\frac{b}{(\Gamma(\alpha)-\mu)}\right) \leqslant \frac{a}{(\Gamma(\alpha)-\mu)},$$

which contradicts (3.6). By Lemma 7 and Theorem 1, we conclude that problem (1.1)-(1.2) has a solution $u(t)$ satisfying

$$0 \leqslant u(t) \leqslant \left(1+t^{\alpha-1}\right) r \ \text{, for } t \in J.$$

This completes the proof.

THEOREM 4. *Assume that (H_1) and (H_2) with $0 < \sigma < 1$ hold. If there exists $r > 0$ such that*

$$r > \max\left\{\frac{2a}{(\Gamma(\alpha)-\mu)}, \left(\frac{2b}{(\Gamma(\alpha)-\mu)}\right)^{\frac{1}{1-\rho}}\right\}. \tag{3.7}$$

Then problem (1.1)-(1.2) has a solution $u(t)$ satisfying

$$0 \leqslant u(t) \leqslant \left(1+t^{\alpha-1}\right) r \ \textit{, for } t \in J.$$

Proof. In this case, we take

$$r > \max\left\{\frac{2a}{(\Gamma(\alpha)-\mu)}, \left(\frac{2b}{(\Gamma(\alpha)-\mu)}\right)^{\frac{1}{1-\rho}}\right\}.$$

The rest of the proof is similar to that of Theorem 3. So we omit it.

THEOREM 5. *Assume that (H_1) and (H_2) with $\sigma > 1$ hold. If there exists $r > 0$ such that*

$$\frac{2a}{(\Gamma(\alpha)-\mu)} < r < \left(\frac{2b}{(\Gamma(\alpha)-\mu)}\right)^{\frac{1}{1-\rho}}. \tag{3.8}$$

Then problem (1.1)-(1.2) has a solution $u(t)$ satisfying

$$0 \leqslant u(t) \leqslant \left(1+t^{\alpha-1}\right) r \ \ , for\ t \in J.$$

Proof. In this case, we take

$$\frac{2a}{(\Gamma(\alpha)-\mu)} < r < \left(\frac{2b}{(\Gamma(\alpha)-\mu)}\right)^{\frac{1}{1-\rho}}.$$

The rest of the proof is similar to that of Theorem 3 . So we omit it.

4. Application

We consider the following fractional boundary value problem on an unbounded domaine

$$D_{0^+}^{\frac{5}{2}} u(t)+\frac{1}{1+t^2}+\frac{|u(t)|^\sigma}{(1+t^4)\left(1+t^{\frac{3}{2}}\right)}=0, t \in J=[0,+\infty), \tag{4.1}$$

$$u(0)=u'(0)=0,\ D_{0^+}^{\alpha-1} u(\infty)=\int_0^\infty \frac{\sqrt{t}\exp(t)}{1+t^2} u(t)dt, \tag{4.2}$$

where $\alpha=5/2$, $\varphi(t)=\sqrt{t}\exp(t)/(1+t^2)$ and

$$f(t,u)=1/(1+t^2)+|u(t)|^\sigma/\left(1+t^4\right)\left(1+t^{\frac{3}{2}}\right).$$

Then we can easily show that

$$\begin{aligned} |f(t,u(t))| &= \left|\frac{1}{1+t^2}+\frac{|u(t)|^\sigma}{(1+t^4)\left(1+t^{\frac{3}{2}}\right)}\right| \\ &\leqslant \frac{1}{1+t^2}+\frac{|u(t)|^\sigma}{(1+t^4)\left(1+t^{\frac{3}{2}}\right)}, \end{aligned}$$

obviously, for a.e. $t \in [0,+\infty)$, we have

$$\mu=\int_0^{+\infty} t^{\alpha-1}\varphi(t)dt=\int_0^\infty \frac{t^2\exp(t)}{t^2+1}dt=0.37855.$$

Then $\mu \in (0,\ 1.3293)$, where $\Gamma(5/2)=1.3293$. Hence (H_1) is satisfied.

Also, set

$$a(t)=1/(1+t^2) \text{ and } b(t)=1/\left(1+t^4\right)\left(1+\sqrt{t}\right) \text{ for } t \in [0,+\infty).$$

Then we can easily show that

$$f(t,u(t)) \leqslant a(t)+b(t)\,|u(t)|^\rho.$$

By simple calculations we have:

Case 1. For $\sigma = 1$, an easy computation shows that:

$$a = \int_0^{+\infty} a(t)dt = \int_0^{+\infty} \frac{1}{1+t^2}dt = 1.5708$$

and

$$b = \int_0^{+\infty} \left(1+t^{\alpha-1}\right)^{\rho} b(t)dt = \int_0^{+\infty} \frac{1+t^{\frac{3}{2}}}{(1+t^4)\left(1+t^{\frac{3}{2}}\right)}dt = 1.1107.$$

So, condition (H_2) hold. Then by an application of Theorem 3 the FBVP (4.1)-(4.1) has a solution $u(t)$ satisfying

$$0 \leqslant u(t) \leqslant \left(1+t^{\alpha-1}\right)r \ \text{, for } t \in J,$$

where

$$\begin{aligned}
&r(1-b)/\left(\Gamma(\alpha)-\mu\right) > a/\left(\Gamma(\alpha)-\mu\right),\\
&\left(1-b\right)/\left(\Gamma(\alpha)-\mu\right) = -0.16824,\\
&a/\left(\Gamma(\alpha)-\mu\right) = 1.5708(1.3293-0.37855) = 1.6522.
\end{aligned}$$

Case 2. For $\sigma = 0.5 < 1$. An easy computation shows that:

$$a = \int_0^{+\infty} a(t)dt = \int_0^{+\infty} \frac{1}{1+t^2}dt = 1.5708,$$

and

$$b = \int_0^{+\infty} \left(1+t^{\alpha-1}\right)^{\rho} b(t)dt = \int_0^{\infty} \frac{1}{(t^4+1)\sqrt{t^{\frac{3}{2}}+1}}dt = 0.90382.$$

So, condition (H_2) hold. Then by an application of Theorem 4 the FBVP (4.1)-(4.1) has a solution $u(t)$ satisfying

$$0 \leqslant u(t) \leqslant \left(1+t^{\alpha-1}\right)r \ \text{, for } t \in J.$$

Here

$$r > \max\left\{\frac{2a}{(\Gamma(\alpha)-\mu)}, \left(\frac{2b}{(\Gamma(\alpha)-\mu)}\right)^{\frac{1}{1-\rho}}\right\} = 3.3043.$$

Case 3: For $\sigma = 2 > 1$. An easy computation shows that:

$$a = \int_0^{+\infty} a(t)dt = \int_0^{+\infty} \frac{1}{1+t^2}dt = 1.5708$$

and

$$b = \int_0^{+\infty} \left(1+t^{\alpha-1}\right)^{\rho} b(t)dt = \int_0^{\infty} \frac{1}{t^4+1}\left(t^{\frac{3}{2}}+1\right)dt = 1.9608.$$

So, condition (H_2) hold. Then by an application of Theorem 5 the FBVP (4.1)-(4.1) has a solution $u(t)$ satisfying

$$0 \leqslant u(t) \leqslant \left(1+t^{\alpha-1}\right) r \ \text{, for } t \in J.$$

Here

$$\frac{2a}{(\Gamma(\alpha)-\mu)} = 3.3043 < r < \left(\frac{2b}{(\Gamma(\alpha)-\mu)}\right)^{\frac{1}{1-\rho}} = 17.013$$

REFERENCES

[1] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, **141**, Cambridge University Press, Cambridge, UK, (2001).

[2] C. Bai, *Triple positive solutions for a boundary value problem of nonlinear fractional differential equation*, E. J. Qualitative Theory of Diff.Equ., **24**, (2008) 1–10.

[3] C. Bai, *Positive solutions for nonlinear fractional differential equations with coefficient that changes sign*, Nonlinear Analysis., **64**, (2006), 677–685.

[4] Z. Bai, H. Lü, *Positive solutions for a boundary value problem of nonlinear fractional differential equations*, J. Math. Anal. Appl., **311**, (2005), 495–505.

[5] M. Benchohra, J. Henderson, S. K. Ntoyuas, A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl., **338**, (2008), 1340–1350.

[6] D. Delbosco, L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. Appl., **204**, (1996) 609–625.

[7] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cone*, Academic Press, Saniego, 1988.

[8] W. Jiang, *Solvability for fractional differential equations atresonance on the half line*, Applied Mathematics and Computation., **61**, (2014), 90–99.

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier science B. V., Amsterdam, 2006.

[10] M. A. Krasnosel'skii, *Topological Methods in the Theory on Nonlinear Integral Equations, (English) Translated by A. H. Armstrong*, A Pergamon Press Book, MacMillan, New York, 1964.

[11] S. Liang, J. Zhang, *Existence of three positive solutions of m-point boundary value problems for some nonlinear fractional differential equations on an infinite interval*, Computers and Mathematics with Applications., **61**, (2011), 3343–3354.

[12] Y. Liu, *Existence and unboundedness of positive solutions for singular boundary value problems on half-line*, Applied Mathematics and Computation., **144**, (2003), 543–556.

[13] Y. Liu, B. Ahmad, R.P. Agarwal, *Existence of solutions for a coupled system of nonlinear fractional differential equations with fractional boundary conditions on the half-line*, Advances in Difference Equations., **2013**, 46, (2013).

[14] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[15] A. Saadi, M. Benbachir, *Positive solutions for three-point nonlinear fractional boundary value problems*, E. J. Qualitative Theory of Diff. Equ., **3**, (2011), 1–19.

[16] A. Saadi, A. Benmezai, M. Benbachir, *Positive solutions for three-point nonlinear fractional semi- positone boundary value problems*, PanAmerican Mathematical Journal., **22**, 4, (2012), 41–57.

[17] A. Saadi, A Benmezai, *Positive solutions for three-point nonlinear singular semi- positone fractional boundary value problems*, Journal of Advanced Research in Dynamical and Control Systems., **5**, 1, (2013), 1–17.

[18] X.Su, S. Zhang, *Unbounded solutions to a boundary value problem of fractional order on the half-line*, Computers and Mathematics with applications., **61**, (2011), 1079–1087.

[19] L. ZHANG, B. AHMAD, G, WANG, R. P. AGARWAL, M. AL-YAMI AND W. SHAMMAKH , *Nonlocal integrodifferential boundary value problem for nonliear fractional differential equations on an unbounded domain*, Abstract and Applied Analysis., Volume(2013), Article ID 813903, (2013), 5 page.

[20] G. WANG, A. CABADA AND L. ZHANG, *An integral boundary value problem for nonlinear differential equations of fractional order on an unbounded domain*, J. Integral Equations Appl., **26**, 1, (2014), 117–129.

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