

## OSCILLATION CRITERIA FOR ODD HIGHER ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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*Abstract.* In this paper, the authors study oscillatory and asymptotic behavior of solutions of a class of nonlinear higher order neutral differential equations with positive and negative coefficients of the form

$$(a(t)(b(t)(y(t) + p(t)y(\sigma(t)))')')^{(n-2)} + q(t)G(y(\alpha(t))) - h(t)H(y(\beta(t))) = 0 \quad (E)$$

for  $n \geq 3$ ,  $n$  is an odd integer,  $0 \leq p(t) \leq p_1 < 1$  and  $-1 < p_2 \leq p(t) \leq 0$ . The results in this paper generalize the results of Panigrahi and Basu [9] and various results in the literature. We establish new conditions which guarantees that every solutions of (E) either oscillatory or converges to zero. Examples are considered to illustrate the main results.

### 1. Introduction

In this paper, we are concerned with the oscillatory and asymptotic behavior of solutions of a higher order nonlinear neutral delay differential equations of the form

$$(a(t)(b(t)(y(t) + p(t)y(\sigma(t)))')')^{(n-2)} + q(t)G(y(\alpha(t))) - h(t)H(y(\beta(t))) = 0, \quad (1.1)$$

where  $a, b, q \in C([t_0, \infty), (0, \infty))$ ,  $h \in C([t_0, \infty), [0, \infty))$ ,  $p, \sigma, \alpha, \beta \in C([t_0, \infty), \mathbb{R})$ ,

$$\sigma(t) \leq t, \quad \alpha(t) \leq t, \quad \beta(t) \leq t,$$

$$\lim_{t \rightarrow \infty} \sigma(t) = \infty, \quad \lim_{t \rightarrow \infty} \alpha(t) = \infty, \quad \lim_{t \rightarrow \infty} \beta(t) = \infty,$$

$G$  and  $H \in C(\mathbb{R}, \mathbb{R})$  with  $uG(u) > 0$ ,  $vH(v) > 0$ , for  $u, v \neq 0$ ,  $n (\geq 3)$  is odd number,  $H$  is bounded,  $G$  is non-decreasing under the assumptions

$$\int_{t_0}^{\infty} \frac{1}{b(t)} \int_t^{\infty} \frac{1}{a(s)} \int_s^{\infty} u^{n-3} h(u) du ds dt < \infty, \quad (H_1)$$

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$$\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty, \quad \int_{t_0}^{\infty} \frac{dt}{b(t)} = \infty, \quad (H_2)$$

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty, \quad \int_{t_0}^{\infty} \frac{dt}{b(t)} = \infty, \quad (H_3)$$

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty, \quad \int_{t_0}^{\infty} \frac{dt}{b(t)} < \infty, \quad (H_4)$$

and

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty, \quad \int_{t_0}^{\infty} \frac{dt}{b(t)} < \infty \quad (H_5)$$

for the ranges  $0 \leq p(t) \leq p_1 < 1$  and  $-1 < p_2 \leq p(t) \leq 0$ .

Oscillation and non-oscillations of neutral differential equations have been studied by many authors since last two decades (see [1, 2, 6, 7, 8, 10, 11, 12]). However, there are only a few works have been done on the oscillatory behaviour of higher order ( $n \geq 2$ ) neutral delay differential equations with positive and negative coefficients.

In [5], Li, Zhang and Xing have studied the oscillatory and asymptotic behaviour of solutions of

$$(a(t)(b(t)(x(t) + p(t)x(\sigma(t))))')' + q(t)x(\tau(t)) = 0 \quad (1.2)$$

under the assumptions  $(H_2)$ ,  $(H_3)$  and  $(H_4)$ , where  $a, b, q \in C([t_0, \infty), (0, \infty))$ ,  $p, \sigma, \tau \in C([t_0, \infty), \mathbb{R})$ ,  $\sigma(t) \leq t$ ,  $\tau(t) \leq t$ ,  $\lim_{t \rightarrow \infty} \sigma(t) = \infty$ ,  $\lim_{t \rightarrow \infty} \tau(t) = \infty$  and  $0 \leq p(t) \leq p < 1$ . Moreover, they did not investigate the oscillatory and asymptotic behaviour of (1.2) for the case  $(H_5)$ .

Later on, Panigrahi and Basu [9] have studied the oscillatory and asymptotic behaviour of solutions of a class of nonlinear third order neutral differential equations with positive and negative coefficients of the form

$$(a(t)(b(t)(y(t) + p(t)y(\sigma(t))))')' + q(t)G(y(\alpha(t))) - h(t)H(y(\beta(t))) = 0 \quad (1.3)$$

under the assumptions

$$\int_{t_0}^{\infty} \frac{1}{b(t)} \int_t^{\infty} \frac{1}{a(s)} \int_s^{\infty} h(u) du ds dt < \infty,$$

$(H_2)$ ,  $(H_3)$ ,  $(H_4)$  and  $(H_5)$  for the ranges  $0 \leq p(t) \leq p_1 < 1$  and  $-1 < p_2 \leq p(t) \leq 0$ , where  $a, b, q \in C([t_0, \infty), (0, \infty))$ ,  $h \in C([t_0, \infty), [0, \infty))$ ,  $p, \sigma, \alpha, \beta \in C([t_0, \infty), \mathbb{R})$ ,  $\sigma(t) \leq t$ ,  $\alpha(t) \leq t$ ,  $\beta(t) \leq t$ ,  $\lim_{t \rightarrow \infty} \sigma(t) = \infty$ ,  $\lim_{t \rightarrow \infty} \alpha(t) = \infty$ ,  $\lim_{t \rightarrow \infty} \beta(t) = \infty$ ,  $G$  and  $H \in C(\mathbb{R}, \mathbb{R})$  with  $uG(u) > 0$ ,  $vH(v) > 0$ , for  $u, v \neq 0$ ,  $H$  is bounded,  $G$  is non-decreasing. Clearly, equations (1.2) and (1.3) are particular cases of equations (1.1).

Keeping in view of the above facts, the motivation of the present work has come from the recent work of Panigrahi and Basu [9]. We may note that a very few work is available in this direction. This work is the generalization of the earlier work of [9].

By a solution of (1.1) we understand a function  $y(t) \in C([T_y, \infty))$ ,  $T_y \geq t_0 \geq 0$  such that

$$\begin{aligned} &(y(t) + p(t)y(\sigma(t))) \in C^1([T_y, \infty)), \\ &b(t)(y(t) + p(t)y(\sigma(t)))' \in C^1([T_y, \infty)), \\ &a(t)(b(t)(y(t) + p(t)y(\sigma(t)))')' \in C^{(n-2)}([T_y, \infty)) \end{aligned}$$

and satisfies (1.1) on  $[T_y, \infty)$ . We consider only those solutions  $y(t)$  of (1.1) which satisfies  $\sup\{|y(t)|; t \geq T\} > 0$  for every  $T \geq T_y$ . We assume that (1.1) has such a solution. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros on  $[T_y, \infty)$ ; otherwise, it is called non-oscillatory.

### 2. Oscillation properties of homogeneous equation

In this section, sufficient conditions are obtained for the oscillatory and asymptotic behavior of solutions of (1.1). We need the following conditions and lemma for our use in the sequel.

$$(H_6) \int_{t_0}^{\infty} q(t)dt = \infty;$$

$$(H_7) G(-u) = -G(u), H(-u) = -H(u) \text{ for } u \in \mathbb{R};$$

$$(H_8) \int_{t^*}^{\infty} \frac{1}{a(t)} \int_{t^*}^t (t-s)^{n-3} q(s)dsdt = \infty;$$

$$(H_9) \int_{t^*}^{\infty} \frac{1}{b(t)} \int_{t^*}^t \frac{1}{a(v)} \int_{t^*}^v (v-s)^{n-3} q(s)dsdvdt = \infty, t^* \geq t_0.$$

LEMMA 1. [3], ([4], p. 193) *Let  $y \in C^{(n)}([0, \infty), \mathbb{R})$  be of constant sign. Let  $y^{(n)}(t)$  be of constant sign and  $\neq 0$  in any interval  $[T, \infty)$ ,  $T \geq 0$ , and  $y^{(n)}(t)y(t) \leq 0$ . Then there exists a number  $t_0 \geq 0$  such that the functions  $y^{(j)}(t)$ ,  $j = 1, 2, \dots, n-1$  are of constant sign on  $[t_0, \infty)$  and there exists a number  $k \in \{1, 3, \dots, n-1\}$  when  $n$  is even or  $k \in \{0, 2, \dots, n-1\}$  when  $n$  is odd such that*

$$\begin{aligned} &y(t)y^{(j)}(t) > 0 \text{ for } j = 0, 1, 2, \dots, k, t \geq t_0, \\ &(-1)^{n+j-1}y(t)y^{(j)}(t) > 0 \text{ for } j = k+1, k+2, \dots, n-1, t \geq t_0. \end{aligned}$$

THEOREM 2. *Let  $0 \leq p(t) \leq p_1 < 1$ . Suppose  $(H_1)$ ,  $(H_2)$ ,  $(H_6)$  and  $(H_7)$  hold, then every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a non-oscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0$ ,  $y(\beta(t)) > 0$ ,  $y(\sigma(t)) > 0$  for  $t \geq t_1$ . Set

$$z(t) = y(t) + p(t)y(\sigma(t)) \tag{2.1}$$

and

$$k(t) = \frac{1}{(n-3)!} \int_t^\infty \frac{1}{b(s)} \int_s^\infty \frac{1}{a(\theta)} \int_\theta^\infty (u-\theta)^{n-3} h(u) H(y(\beta(u))) du d\theta ds. \quad (2.2)$$

Note that condition  $(H_1)$  and the fact that  $H$  is bounded function implies that  $k(t)$  exists for all  $t$ . Now if we let

$$v(t) = z(t) + k(t), \quad (2.3)$$

then

$$w^{(n-2)}(t) = -q(t)G(y(\alpha(t))) \leq 0 (\neq 0), \quad (2.4)$$

where

$$w(t) = a(t)(b(t)v'(t))' \quad (2.5)$$

for  $t \geq t_1$ . Here  $w^{(n-2)}(t)$  represents the  $(n-2)^{th}$  derivative of  $w$  w.r.t  $t$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty)$ ,  $t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , then in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ .

Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $b(t)v'(t)$  is eventually monotonic function. Since  $b(t) > 0$ , then either  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. If  $v'(t) > 0$  for  $t \geq t_3$ , then  $z'(t) > 0$  eventually. Therefore,

$$(1-p_1)z(t) < (1-p(t))z(t) < z(t) - p(t)z(\sigma(t)) = y(t) - p(t)p(\sigma(t))y(\sigma(\sigma(t))),$$

which implies

$$(1-p_1)z(t) < y(t)$$

for  $t \geq t_4 > t_3$ . From (2.4),  $z'(t) > 0$  and by using the last inequality, we obtain

$$w^{(n-2)}(t) \leq -q(t)G((1-p_1)z(t_4))$$

for  $t \geq t_5 > t_4$ . Then integrating the preceding inequality from  $t_5$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t_5) > -w^{(n-3)}(t) + w^{(n-3)}(t_5) \geq G((1-p_1)z(t_4)) \int_{t_5}^t q(s) ds.$$

Since  $\lim_{t \rightarrow \infty} w^{(n-3)}(t) < \infty$ , then taking  $t \rightarrow \infty$  in the last inequality we have

$$\int_{t_5}^{\infty} q(t) dt < \infty,$$

which is a contradiction to  $(H_6)$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , we may note that  $\lim_{t \rightarrow \infty} v(t)$  exists and equal to  $l$  (say). We will claim that  $l = 0$ . If it is not true, then for every  $\varepsilon > 0$ , there exists  $t_4 > t_3$  such that  $l < v(t) < l + \varepsilon$  for  $t \geq t_4$ . Choose  $0 < \varepsilon < \frac{l(1-p_1)}{1+p_1}$ . Since  $\lim_{t \rightarrow \infty} k(t) = 0$ , then for the same chosen  $\varepsilon$ ,  $k(t) < \varepsilon$  for  $t \geq t_5 \geq t_4$ . Thus,

$$\begin{aligned} y(t) &= v(t) - p(t)y(\sigma(t)) - k(t) \\ &> v(t) - p(t)v(\sigma(t)) - k(t) \\ &> l - p_1(l + \varepsilon) - \varepsilon \end{aligned}$$

for  $t \geq t_6 > t_5$ . Now,

$$y(t) > (l - \varepsilon) - p_1(l + \varepsilon) > k_2(l + \varepsilon) > k_2v(t) > k_2l. \tag{2.6}$$

By the choice of  $\varepsilon$ , we can show that  $k_2 > 0$ . Using (2.6) in (2.4), we obtain

$$w^{(n-2)}(t) \leq -q(t)G(k_2l) \tag{2.7}$$

for  $t \geq t_7 > t_6$ . Integrating (2.7) from  $t_7$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t_7) > -w^{(n-3)}(t) + w^{(n-3)}(t_7) \geq G(k_2l) \int_{t_7}^t q(s)ds.$$

Since  $\lim_{t \rightarrow \infty} w^{(n-3)}(t) < \infty$ , then taking  $t \rightarrow \infty$  in the last inequality we obtain

$$\int_{t_7}^{\infty} q(t)dt < \infty,$$

which is a contradiction to  $(H_6)$ . Therefore,  $\lim_{t \rightarrow \infty} v(t) = 0$  and hence  $\lim_{t \rightarrow \infty} z(t) = 0$ . Since  $y(t) \leq z(t)$ , then it implies  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $w(t) < 0$  for  $t \geq t_2$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually, then  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (i):* Suppose  $w^{(n-3)}(t) > 0$  eventually. Now  $v'(t) > 0$  and  $k'(t) < 0$  implies  $z'(t) > 0$ . Therefore,  $(1 - p_1)z(t) < (1 - p(t))z(t) < z(t) - p(t)z(\sigma(t)) = y(t) - p(t)p(\sigma(t))y(\sigma(\sigma(t)))$ , which implies

$$(1 - p_1)z(t) < y(t) \tag{2.8}$$

for  $t \geq t_3$ . From (2.4) and  $z'(t) > 0$ , we obtain

$$w^{(n-2)}(t) \leq -q(t)G((1 - p_1)z(t_3))$$

for  $t \geq t_4 > t_3$ . Then integrating the last inequality from  $t_4$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t_4) > -w^{(n-3)}(t) + w^{(n-3)}(t_4) \geq G((1 - p_1)z(t_3)) \int_{t_4}^t q(s)ds.$$

Since  $\lim_{t \rightarrow \infty} w^{(n-3)}(t) < \infty$ , then taking  $t \rightarrow \infty$  in the last inequality we have

$$\int_{t_4}^{\infty} q(t)dt < \infty,$$

which is a contradiction to  $(H_6)$ .

*Subcase (ii):* If  $w^{(n-3)}(t) < 0$  eventually, then from (2.4) we can conclude that  $w^{(n-4)}(t) < 0, \dots, w'(t) < 0$  for large  $t$ . Since  $w'(t) < 0$  for  $t > t_4 (> t_3)$ , then  $w(t) < w(t_4)$ , that is,

$$a(t)(b(t)v'(t))' < a(t_4)(b(t_4)v'(t_4))'.$$

Integrating the preceding inequality from  $t_4$  to  $t$ , we obtain

$$b(t)v'(t) < b(t_4)v'(t_4) + a(t_4)(b(t_4)v'(t_4))' \int_{t_4}^t \frac{ds}{a(s)}.$$

Using  $(H_2)$  in the preceding inequality, we obtain  $b(t)v'(t) \rightarrow -\infty$  as  $t \rightarrow \infty$ , a contradiction to the fact that  $v'(t) > 0$ .

Case IV. Suppose  $v'(t) < 0$  for  $t \geq t_3$ . Then, integrating  $(b(t)v'(t))' < 0$  twice from  $t_3$  to  $t$ , we obtain

$$v(t) \leq v(t_3) + b(t_3)v'(t_3) \int_{t_3}^t \frac{ds}{b(s)}.$$

Using  $(H_2)$  in the preceding inequality, we obtain  $v(t) \rightarrow -\infty$  as  $t \rightarrow \infty$ , a contradiction to the fact that  $v(t) > 0$ .

Finally, we suppose that  $y(t) < 0$  for  $t \geq t_0$ . From  $(H_7)$ , we note that  $G(-u) = -G(u)$  and  $H(-u) = -H(u), u \in \mathbb{R}$ . Hence putting  $x(t) = -y(t)$  for  $t \geq t_0$ , we obtain  $x(t) > 0$  and

$$(a(t)(b(t)(x(t) + p(t)x(\sigma(t)))')^{(n-2)} + q(t)G(x(\alpha(t))) - h(t)H(x(\beta(t)))) = 0.$$

Proceeding as above, we can show that every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ . This completes the proof of the theorem.

**THEOREM 3.** *Let  $0 \leq p(t) \leq p_1 < 1$ . Suppose that  $(H_1), (H_3)$  and  $(H_6) - (H_8)$  hold, then every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a non-oscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. (The proof in case  $y(t) < 0$  eventually is similar and will be omitted.) Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$  for  $t \geq t_1$ . Setting  $z(t), k(t), v(t)$  as in (2.1), (2.2) and (2.3) respectively, we get (2.4) and (2.5) for  $t \geq t_1$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty), t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , then in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ . Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $v'(t)$  is monotonic function. Thus,  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. If  $v'(t) > 0$  for  $t \geq t_3$ , then proceeding as in Case I of Theorem 2, we obtain a contradiction due to  $(H_6)$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , then proceeding as in Case II of Theorem 2, we obtain  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $w(t) < 0$  for  $t \geq t_2$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually, then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (i):* If  $w^{(n-3)}(t) > 0$  eventually, then proceeding as in *Subcase (i)* of Case III of Theorem 2, we obtain a contradiction due to  $(H_6)$ .

*Subcase (ii):* If  $w^{(n-3)}(t) < 0$  eventually, then from (2.4) it implies that  $w^{(n-4)}(t) < 0, \dots, w'(t) < 0$  for large  $t$ . Now  $v'(t) > 0$  implies  $z'(t) > 0$  eventually. Therefore from (2.4) and (2.8), we obtain

$$0 \geq w^{(n-2)}(t) + q(t)G((1 - p_1)z(t_3))$$

for  $t \geq t_4 > t_3$ . Integrating the last inequality consecutively  $(n - 2)$  times from  $t_4$  to  $t$ , we obtain

$$0 > w(t_4) \geq w(t) + \frac{1}{(n - 3)!} \int_{t_4}^t (t - s)^{n-3} q(s)G((1 - p_1)z(t_3)) ds.$$

Hence,

$$0 > (b(t)v'(t))' + \frac{1}{(n - 3)!} \frac{1}{a(t)} \int_{t_4}^t (t - s)^{n-3} q(s)G((1 - p_1)z(t_3)) ds.$$

Further integrating the preceding inequality from  $t_4$  to  $t$ , we obtain

$$b(t_4)v'(t_4) \geq b(t)v'(t) + \frac{1}{(n - 3)!} \int_{t_4}^t \frac{1}{a(v)} \int_{t_4}^v (v - s)^{n-3} q(s)G((1 - p_1)z(t_3)) ds dv.$$

Since  $\lim_{t \rightarrow \infty} b(t)v'(t) < \infty$ , then from the last inequality for large  $t$ , we get

$$\frac{1}{(n - 3)!} \int_{t_4}^{\infty} \frac{1}{a(t)} \int_{t_4}^t (t - s)^{n-3} q(s)G((1 - p_1)z(t_3)) ds dt < \infty,$$

a contradiction to  $(H_8)$ .

Case IV. Since  $v'(t) < 0$  and  $(b(t)v'(t))' < 0$  for  $t \geq t_3$ , then using  $(H_3)$  and proceeding as in Case IV of Theorem 2, we obtain a contradiction to the fact that  $v(t) > 0$ . Hence the proof of the theorem is complete.

**THEOREM 4.** *Let  $0 \leq p(t) \leq p_1 < 1$ . Suppose that  $(H_1)$ ,  $(H_4)$  and  $(H_6) - (H_9)$  hold, then every solution of (1.1) either oscillates or tends to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a non-oscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. (The proof in case  $y(t) < 0$  eventually is similar and will be omitted.) Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$  for  $t \geq t_1$ . Setting  $z(t), k(t), v(t)$  as in (2.1), (2.2) and (2.3) respectively, we get (2.4) and (2.5) for  $t \geq t_1$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty)$ ,  $t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , then in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ . Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $v'(t)$  is monotonic function. Thus,  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. If  $v'(t) > 0$  for  $t \geq t_3$ , then  $z'(t) > 0$  eventually. Therefore,

$$(1 - p_1)z(t) < (1 - p(t))z(t) < z(t) - p(t)z(\sigma(t)) = y(t) - p(t)p(\sigma(t))y(\sigma(\sigma(t))),$$

which implies

$$(1 - p_1)z(t) < y(t)$$

for  $t \geq t_4 > t_3$ . From (2.4),  $z'(t) > 0$  and by using the last inequality, we obtain

$$w^{(n-2)}(t) \leq -q(t)G((1 - p_1)z(t_5))$$

for  $t \geq t_5 > t_4$ . Then integrating the preceding inequality from  $t_5$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t_5) > -w^{(n-3)}(t) + w^{(n-3)}(t_5) \geq G((1 - p_1)z(t_5)) \int_{t_5}^t q(s)ds.$$

Since  $\lim_{t \rightarrow \infty} w^{(n-3)}(t) < \infty$ , then taking  $t \rightarrow \infty$  in the last inequality we have

$$\int_{t_5}^{\infty} q(t)dt < \infty,$$

which is a contradiction to  $(H_6)$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , then proceeding as in Case II of Theorem 2, we obtain  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $w(t) < 0$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually, then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (i):* If  $w^{(n-3)}(t) > 0$  eventually, then proceeding as in *Subcase (i)* of Case III of Theorem 2, we obtain a contradiction due to  $(H_6)$ .



*Subcase (ii):* If  $w^{(n-3)}(t) < 0$  eventually, then proceeding as in *Subcase (ii)* of Case III of Theorem 3, we obtain a contradiction  $(H_8)$ .

Case IV. Since  $v'(t) < 0$  and  $(b(t)v'(t))' < 0$  for  $t \geq t_3$ . Since,  $w^{(n-2)}(t) \leq 0$  eventually, then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (iii):* If  $w^{(n-3)}(t) < 0$  eventually, then,  $w^{(n-4)}(t) < 0, \dots, w'(t) < 0, w(t) < 0$  eventually. Since  $\lim_{t \rightarrow \infty} v(t) < \infty$ , let  $0 < \lim_{t \rightarrow \infty} v(t) < \infty$ . Therefore from (2.4), (2.6), we obtain

$$0 \geq w^{(n-2)}(t) + q(t)G(k_2l) \tag{2.9}$$

for  $t \geq t_4 > t_3$ . Integrating (2.9) consecutively  $(n - 2)$  times from  $t_4$  to  $t$ , we obtain

$$0 > w(t_4) \geq w(t) + \frac{1}{(n-3)!} \int_{t_4}^t (t-s)^{n-3} q(s)G(k_2l) ds.$$

Hence,

$$0 > (b(t)v'(t))' + \frac{1}{a(t)} \frac{1}{(n-3)!} \int_{t_4}^t (t-s)^{n-3} q(s)G(k_2l) ds.$$

Further integrating the preceding inequality from  $t_4$  to  $t$  and considering the fact that  $v'(t) < 0$ , we obtain

$$0 > b(t_4)v'(t_4) \geq b(t)v'(t) + \frac{1}{(n-3)!} \int_{t_4}^t \frac{1}{a(\theta)} \int_{t_4}^\theta (\theta-s)^{n-3} q(s)G(k_2l) ds d\theta.$$

Again integrating the last inequality from  $t_4$  to  $t$ , we get

$$v(t_4) \geq v(t) + \frac{1}{(n-3)!} \int_{t_4}^t \frac{1}{b(u)} \int_{t_4}^u \frac{1}{a(\theta)} \int_{t_4}^\theta (\theta-s)^{n-3} q(s)G(k_2l) ds d\theta du.$$

Since  $\lim_{t \rightarrow \infty} v(t) < \infty$ , then it implies for large  $t$

$$\frac{G(k_2l)}{(n-3)!} \int_{t_4}^\infty \frac{1}{b(u)} \int_{t_4}^u \frac{1}{a(\theta)} \int_{t_4}^\theta (\theta-s)^{n-3} q(s) ds d\theta du < \infty,$$

a contradiction to  $(H_9)$ .

If  $\lim_{t \rightarrow \infty} v(t) = 0$ , then  $\lim_{t \rightarrow \infty} z(t) = 0$ . Hence,  $\lim_{t \rightarrow \infty} y(t) = 0$  as  $y(t) \leq z(t)$ .

*Subcase (iv):* Suppose  $w^{(n-3)}(t) > 0$  for  $t \geq t_4 > t_3$ . If  $0 < \lim_{t \rightarrow \infty} v(t) < \infty$ , then from (2.4) and (2.6), we have

$$\int_{t_4}^\infty q(t) dt < \infty,$$

a contradiction to  $(H_6)$ . Hence,  $\lim_{t \rightarrow \infty} v(t) = 0$ . Thus,  $\lim_{t \rightarrow \infty} y(t) = 0$ . Hence proof of the theorem is complete.

**THEOREM 5.** *Let  $0 \leq p(t) \leq p_1 < 1$ . Suppose that  $(H_1)$ ,  $(H_5)$ ,  $(H_6)$ ,  $(H_7)$  and  $(H_9)$  hold, then every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a non-oscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. (The proof in case  $y(t) < 0$  eventually is similar and will be omitted.) Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$  for  $t \geq t_1$ . Setting  $z(t), k(t), v(t)$  as in (2.1), (2.2) and (2.3) respectively, we get (2.4) and (2.5) for  $t \geq t_1$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty)$ ,  $t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , then in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ . Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $v'(t)$  is monotonic function. Thus,  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. If  $v'(t) > 0$  for  $t \geq t_3$ , then proceeding as in Case I of Theorem 4, we obtain a contradiction due to  $(H_6)$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , then proceeding as in Case II of Theorem 2, we obtain  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $w(t) < 0$ , for  $t \geq t_2$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually. then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (i):* If  $w^{(n-3)}(t) > 0$  eventually, then proceeding as in *Subcase (i)* of Case III of Theorem 2, we obtain a contradiction due to  $(H_6)$ .

*Subcase (ii):* If  $w^{(n-3)}(t) < 0$  eventually, then using  $(H_5)$  and proceeding as in *Subcase (ii)* of Case III of Theorem 2, we obtain a contradiction to the fact that  $v'(t) > 0$ .

Case IV. Suppose  $v'(t) < 0$  for  $t \geq t_3$ . Since,  $w^{(n-2)}(t) \leq 0$  eventually, then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (iii):* Suppose  $w^{(n-3)}(t) < 0$  eventually. If  $0 < \lim_{t \rightarrow \infty} v(t) < \infty$ , then proceeding as in *Subcase (iii)* of Case IV of Theorem 4, we obtain a contradiction due to  $(H_9)$ .

If  $\lim_{t \rightarrow \infty} v(t) = 0$ , then we obtain  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (iv):* If  $w^{(n-3)}(t) > 0$  eventually, then proceeding as in *Subcase (iv)* of Case IV of Theorem 4, we obtain  $\lim_{t \rightarrow \infty} y(t) = 0$ . Hence proof of the theorem is complete.

**THEOREM 6.** *Let  $-1 < p_2 \leq p(t) \leq 0$ . Suppose that  $(H_1)$ ,  $(H_2)$ ,  $(H_6)$  and  $(H_7)$  hold, then every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a non-oscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. (The proof in case  $y(t) < 0$  eventually is similar and will be omitted.) Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$  for  $t \geq t_1$ . Setting  $z(t), k(t), v(t)$  as in (2.1), (2.2) and (2.3) respectively, we get (2.4) and (2.5) for  $t \geq t_1$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty), t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , then in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ . Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $v'(t)$  is monotonic function. Thus,  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Now  $v'(t) > 0$  and  $k'(t) < 0$  implies that  $z'(t) > 0$  eventually. Hence,  $z(t) > 0$  or  $< 0$  for  $t \geq t_4 > t_3$ .

*Subcase (i):* If  $z(t) > 0$  for  $t \geq t_4$ , then

$$y(t) \geq z(t)$$

for  $t \geq t_4 > t_3$ . Using the last inequality in (2.4), we obtain

$$w^{(n-2)}(t) \leq -q(t)G(z(t_4)).$$

Thus integrating this from  $t_5 (> t_4)$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t) > -w^{(n-3)}(t_5) + w^{(n-3)}(t_5) \geq G(z(t_4)) \int_{t_5}^t q(s)ds.$$

Since  $\lim_{t \rightarrow \infty} w^{(n-3)}(t) < \infty$ , then taking the limit as  $t \rightarrow \infty$  in the last inequality, we obtain

$$\int_{t_5}^{\infty} q(t)dt < \infty,$$

a contradiction to  $(H_6)$ .

*Subcase (ii):* If  $z(t) < 0$  for  $t \geq t_4 > t_3$ , then  $\lim_{t \rightarrow \infty} z(t)$  exists. Note that  $y(t)$  is bounded. Hence,

$$\begin{aligned} 0 &\geq \limsup_{t \rightarrow \infty} z(t) \geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t). \end{aligned}$$

Since  $(1 + p_2) > 0$ , then it implies  $\limsup_{t \rightarrow \infty} y(t) = 0$ . So also  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , then two cases are possible:  $v(t) > 0$  or  $< 0$  for  $t \geq t_4 > t_3$ .

*Subcase (iii):* If  $v(t) > 0$  for  $t \geq t_4$ , then  $\lim_{t \rightarrow \infty} v(t)$  exists and equal to  $l_1$  (say). We will claim that  $l_1 = 0$ . If it is not true, then for every  $\varepsilon > 0$ , there exists  $t_5 > t_4$  such that  $l_1 < v(t) < l_1 + \varepsilon$  for  $t \geq t_5$ . Choose  $0 < \varepsilon < l_1$ . Since  $\lim_{t \rightarrow \infty} k(t) = 0$ , then for the same chosen  $\varepsilon$ ,  $k(t) < \varepsilon$  for  $t \geq t_6 > t_5$ . Thus,

$$v(t) - y(t) - k(t) = p(t)y(\sigma(t)) \leq 0$$

for  $t \geq t_7 > t_6$ . Hence,

$$l_1 - \varepsilon < v(t) - k(t) \leq y(t).$$

From (2.4), we obtain

$$w^{(n-2)}(t) \leq -q(t)G(l_1 - \varepsilon)$$

for  $t \geq t_8 > t_7$ . Thus integrating the last inequality from  $t_8$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t) > -w^{(n-3)}(t_8) + w^{(n-3)}(t_8) \geq G(l_1 - \varepsilon) \int_{t_8}^t q(s)ds.$$

Since  $\lim_{t \rightarrow \infty} w^{n-3}(t) < \infty$ , then taking the limit as  $t \rightarrow \infty$  in the preceding inequality we get a contradiction to  $(H_6)$ . Hence,  $\lim_{t \rightarrow \infty} v(t) = 0$  and  $\lim_{t \rightarrow \infty} z(t) = 0$ . Hence,  $z(t)$  is bounded. We can show that  $y(t)$  is bounded. Thus,

$$\begin{aligned} 0 = \limsup_{t \rightarrow \infty} z(t) &\geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t). \end{aligned}$$

Since  $(1 + p_2) > 0$ , then it implies  $\limsup_{t \rightarrow \infty} y(t) = 0$ . Hence,  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (iv):* Suppose  $v(t) < 0$  for  $t \geq t_4$ , as  $v'(t) < 0$  so  $-\infty \leq \lim_{t \rightarrow \infty} v(t) < 0$ . Thus,  $-\infty \leq \lim_{t \rightarrow \infty} z(t) (= l_2) < 0$ . If  $l_2 = -\infty$ , then we get a contradiction due to the boundedness of  $y(t)$ .

If  $-\infty < l_2 < 0$ , then

$$\begin{aligned} 0 > \limsup_{t \rightarrow \infty} z(t) &\geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t). \end{aligned}$$

Since  $(1 + p_2) > 0$ , then it implies  $\limsup_{t \rightarrow \infty} y(t) < 0$ , a contradiction to the fact that  $y(t) > 0$ .

If  $w(t) < 0$  for  $t \geq t_2$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually, then  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (v):* Suppose  $w^{(n-3)}(t) > 0$  eventually. Now  $v'(t) > 0$  and  $k'(t) < 0$  implies that  $z'(t) > 0$  eventually. Hence,  $z(t) > 0$  or  $< 0$  eventually.

If  $z(t) > 0$  eventually, then

$$y(t) \geq z(t) \tag{2.10}$$

for  $t \geq t_4 > t_3$ . Using (2.10) in (2.4), we obtain

$$w^{(n-2)}(t) \leq -q(t)G(z(t_4)).$$

Thus integrating this from  $t_4$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t_4) > -w^{(n-3)}(t) + w^{(n-3)}(t_4) \geq G(z(t_4)) \int_{t_4}^t q(s)ds.$$

Since  $\lim_{t \rightarrow \infty} w^{n-3}(t) < \infty$ , then taking the limit as  $t \rightarrow \infty$  in the last inequality we obtain

$$\int_{t_4}^{\infty} q(t)dt < \infty,$$

a contradiction to  $(H_6)$ .

If  $z(t) < 0$  for  $t \geq t_4 > t_3$ , then  $\lim_{t \rightarrow \infty} z(t)$  exists. Let it be  $l_3$ . So  $-\infty < l_3 \leq 0$ . We may note that  $y(t)$  is bounded. Hence,

$$\begin{aligned} 0 &\geq \limsup_{t \rightarrow \infty} z(t) \geq \limsup_{t \rightarrow \infty} (y(t) + p_2y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t). \end{aligned}$$

Since  $(1 + p_2) > 0$ , then it implies  $\limsup_{t \rightarrow \infty} y(t) = 0$ . So also  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (vi):* If  $w^{(n-3)}(t) < 0$  eventually, then proceeding as in Subcase (ii) of Case III of Theorem 2, we obtain a contradiction due to  $v'(t) > 0$ .

Case IV. Suppose  $v'(t) < 0$  and  $(b(t)v'(t))' < 0$  for  $t \geq t_3$ , then integrating  $(b(t)v'(t))' < 0$  twice consecutively from  $t_3$  to  $t$ , we obtain

$$v(t) \leq v(t_3) + b(t_3)v'(t_3) \int_{t_3}^t \frac{ds}{b(s)}.$$

Using  $(H_2)$  in the last inequality, we obtain  $v(t) \rightarrow -\infty$  as  $t \rightarrow \infty$ . Thus,  $v(t) < 0$  for large  $t$ . It is easy to show that  $y(t)$  is bounded, hence  $v(t)$  is bounded, a contradiction. Hence proof of the theorem is complete.

**THEOREM 7.** *Let  $-1 < p_2 \leq p(t) \leq 0$ . Suppose that  $(H_1)$ ,  $(H_3)$  and  $(H_6) - (H_8)$  hold, then every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a non-oscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. (The proof in case  $y(t) < 0$  eventually is similar and will be omitted.) Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$  for  $t \geq t_1$ . Setting  $z(t), k(t), v(t)$  as in (2.1), (2.2) and (2.3) respectively, we get (2.4) and (2.5) for  $t \geq t_1$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty)$ ,  $t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ . Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $v'(t)$  is monotonic function. Thus,  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. If  $v'(t) > 0$  for  $t \geq t_3$ , then  $z'(t) > 0$  and hence two cases are possible:  $z(t) > 0$  or  $< 0$  for  $t \geq t_4 > t_3$ .

*Subcase (i):* If  $z(t) > 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (i)* of Case I of Theorem 6, we obtain a contradiction due to  $(H_6)$ .

*Subcase (ii):* If  $z(t) < 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (ii)* of Case I of Theorem 6, we obtain  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , we have two cases;  $v(t) > 0$  or  $v(t) < 0$  for  $t \geq t_4 > t_3$ .

*Subcase (iii):* If  $v(t) > 0$  for  $t \geq t_4$ , then  $\lim_{t \rightarrow \infty} v(t) < \infty$  and proceeding as in *Subcase (iii)* of Case II of Theorem 6, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (iv):* If  $v(t) < 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (iv)* of Case II of Theorem 6, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $w(t) < 0$  for  $t \geq t_2$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually, then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (v):* Suppose  $w^{(n-3)}(t) > 0$  eventually. Now  $v'(t) > 0$  and  $k'(t) < 0$  implies that  $z'(t) > 0$ . Hence,  $z(t) > 0$  or  $< 0$  eventually.

If  $z(t) > 0$  eventually, then proceeding as in *Subcase (v)* of Case III of Theorem 6, we obtain a contradiction due to  $(H_6)$ .

If  $z(t) < 0$  eventually, then proceeding as in *Subcase (v)* of Case III of Theorem 6, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (vi):* Suppose  $w^{(n-3)}(t) < 0$  eventually. Since  $z'(t) > 0$  eventually. Hence,  $z(t) >$  or  $< 0$  eventually.

If  $z(t) > 0$  eventually, then proceeding as in *Subcase (ii)* of Case III of Theorem 3, we get a contradiction due to  $(H_8)$ .

If  $z(t) < 0$  eventually, then  $\lim_{t \rightarrow \infty} z(t)$  exists. Note that  $y(t)$  is bounded. Hence,

$$\begin{aligned} 0 \geq \limsup_{t \rightarrow \infty} z(t) &\geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t) \end{aligned}$$

which implies  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Case IV. If  $v'(t) < 0$  for  $t \geq t_3$ , then using  $(H_3)$  and proceeding as in Case IV of Theorem 6, we get a contradiction to the fact that  $y(t)$  is bounded. Hence proof of the theorem is complete.

**THEOREM 8.** *Let  $-1 < p_2 \leq p(t) \leq 0$ . Suppose that  $(H_1)$ ,  $(H_4)$  and  $(H_6) - (H_9)$  hold, then every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a nonoscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. (The proof in case  $y(t) < 0$  eventually is similar and will be omitted.) Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$  for  $t \geq t_1$ . Setting  $z(t), k(t), v(t)$  as in (2.1), (2.2) and (2.3) respectively, we get (2.4) and (2.5) for  $t \geq t_1$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty)$ ,  $t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , then in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ . Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $v'(t)$  is monotonic function. Thus,  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. If  $v'(t) > 0$  for  $t \geq t_3$ , then  $z'(t) > 0$  eventually. Thus, we have two cases;  $z(t) > 0$  or  $z(t) < 0$  for  $t \geq t_4 > t_3$ .

*Subcase (i):* If  $z(t) > 0$  for  $t \geq t_4$ , then proceeding same as in *Subcase (i)* of Case I of Theorem 6, we obtain a contradiction due to  $(H_6)$ .

*Subcase (ii):* If  $z(t) < 0$  for  $t \geq t_4$ , then  $\lim_{t \rightarrow \infty} z(t)$  exists. Let it be  $l_4$ . Now  $-\infty < l_4 \leq 0$ . Note that  $y(t)$  is bounded. Hence,

$$\begin{aligned} 0 \geq \limsup_{t \rightarrow \infty} z(t) &\geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \end{aligned}$$

$$\begin{aligned}
&= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\
&= (1 + p_2) \limsup_{t \rightarrow \infty} y(t).
\end{aligned}$$

Since  $(1 + p_2) > 0$ , then it implies  $\limsup_{t \rightarrow \infty} y(t) = 0$ . So also  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , then we have two cases:  $v(t) > 0$  or  $v(t) < 0$  for  $t \geq t_4 > t_3$ .

*Subcase (iii):* If  $v(t) > 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (iii)* of Case II of Theorem 6, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (iv):* If  $v(t) < 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (iv)* of Case II of Theorem 6, we get a contradiction due to  $y(t) > 0$ .

If  $w(t) < 0$  for  $t \geq t_2$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually, Then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (v):* Suppose  $w^{(n-3)}(t) > 0$  eventually. Since  $v'(t) > 0$  for  $t \geq t_3$ , then  $z'(t) > 0$  eventually. Thus, we have two cases:  $z(t) > 0$  or  $z(t) < 0$  for  $t \geq t_4 > t_3$ .

If  $z(t) > 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (v)* of Case III of Theorem 6, we get a contradiction to  $(H_6)$ .

If  $z(t) < 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (v)* of Case III of Theorem 6, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (vi):* Suppose  $w^{(n-3)}(t) < 0$  eventually. Since  $v'(t) > 0$  for  $t \geq t_3$ , then  $z'(t) > 0$  eventually. Thus, we have two cases:  $z(t) > 0$  or  $z(t) < 0$  for  $t \geq t_4 > t_3$ .

If  $z(t) > 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (ii)* of Case III of Theorem 3, we get a contradiction due to  $(H_8)$ .

If  $z(t) < 0$  for  $t \geq t_4$ , then  $\lim_{t \rightarrow \infty} z(t)$  exists. Note that  $y(t)$  is bounded. Hence,

$$\begin{aligned}
0 &\geq \limsup_{t \rightarrow \infty} z(t) \geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\
&\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\
&= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\
&= (1 + p_2) \limsup_{t \rightarrow \infty} y(t)
\end{aligned}$$

which implies  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Case IV. Suppose  $v'(t) < 0$  for  $t \geq t_3$ . Since  $w^{(n-2)}(t) \leq 0$  eventually, then we have two cases  $w^{(n-3)}(t) > 0$  or  $w^{(n-3)}(t) < 0$  eventually.

*Subcase (vii):* Suppose  $w^{(n-3)}(t) > 0$  eventually. Since  $v'(t) < 0$ , then  $v(t) > 0$  or  $v(t) < 0$  for  $t \geq t_4 > t_3$ .



If  $v(t) > 0$  for  $t \geq t_4$ , then  $\lim_{t \rightarrow \infty} v(t)$  exists and equal to  $l_5$  (say). We will claim that  $l_5 = 0$ . If it is not true, then for every  $\varepsilon > 0$ , there exists  $t_5 > t_4$  such that  $l_5 < v(t) < l_5 + \varepsilon$  for  $t \geq t_5$ . Choose  $0 < \varepsilon < l_5$ . Since  $\lim_{t \rightarrow \infty} k(t) = 0$ , then for the same chosen  $\varepsilon$ ,  $k(t) < \varepsilon$  for  $t \geq t_6 > t_5$ . Thus,

$$v(t) - y(t) - k(t) = p(t)y(\sigma(t)) \leq 0$$

for  $t \geq t_7 > t_6$ . Hence,

$$l_5 - \varepsilon < v(t) - k(t) \leq y(t).$$

From (2.4), we obtain

$$w^{(n-2)}(t) \leq -q(t)G(l_5 - \varepsilon)$$

for  $t \geq t_8 > t_7$ . Thus integrating the last inequality from  $t_8$  to  $t$ , we obtain

$$\infty > w^{(n-3)}(t_8) > -w^{(n-3)}(t) + w^{(n-3)}(t_8) \geq G(l_5 - \varepsilon) \int_{t_8}^t q(s)ds.$$

Since  $\lim_{t \rightarrow \infty} w^{n-3}(t) < \infty$ , then taking the limit as  $t \rightarrow \infty$  in the preceding inequality we get a contradiction to  $(H_6)$ . Hence,  $\lim_{t \rightarrow \infty} v(t) = 0$  and  $\lim_{t \rightarrow \infty} z(t) = 0$ . Therefore,  $z(t)$  is bounded. We can show that  $y(t)$  is also bounded. Thus,

$$\begin{aligned} 0 = \limsup_{t \rightarrow \infty} z(t) &\geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t). \end{aligned}$$

Since  $(1 + p_2) > 0$ , then  $\limsup_{t \rightarrow \infty} y(t) = 0$  and hence  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Suppose  $v(t) < 0$  for  $t \geq t_4$  as  $v'(t) < 0$ . Thus,  $-\infty \leq \lim_{t \rightarrow \infty} v(t) < 0$ . Hence,  $-\infty \leq \lim_{t \rightarrow \infty} z(t) (= l_6) < 0$ . If  $l_6 = -\infty$ , then we get a contradiction due to the boundedness of  $y(t)$ .

If  $-\infty < l_6 < 0$ , then

$$\begin{aligned} 0 > \limsup_{t \rightarrow \infty} z(t) &\geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t). \end{aligned}$$

Since  $(1 + p_2) > 0$ , then  $\limsup_{t \rightarrow \infty} y(t) < 0$ , a contradiction to the fact that  $y(t) > 0$ .

*Subcase (viii):* Suppose  $w^{(n-3)}(t) < 0$  eventually, then from (2.4) we can conclude that  $w^{(n-4)}(t) < 0, \dots, w'(t) < 0$  for large  $t$ .

If  $v(t) > 0$  eventually, then  $\lim_{t \rightarrow \infty} v(t) < \infty$  and equal to  $l_7$  (say). We will claim  $l_7 = 0$ . If it is not true, then for every  $\varepsilon > 0$ , there exists  $t_4 > t_3$  such that  $l_7 < v(t) < l_7 + \varepsilon$  for  $t \geq t_4$ . Choose  $0 < \varepsilon < l_7$ . Since  $\lim_{t \rightarrow \infty} k(t) = 0$ , then for the same chosen  $\varepsilon$ ,  $k(t) < \varepsilon$  for  $t \geq t_5 > t_4$ . Thus,

$$v(t) - y(t) - k(t) = p(t)y(\sigma(t)) \leq 0$$

for  $t \geq t_6 > t_5$ . Hence,

$$l_7 - \varepsilon < v(t) - k(t) \leq y(t).$$

Therefore using the last inequality in (2.4), we obtain

$$0 \geq w^{(n-2)}(t) + q(t)G(l_7 - \varepsilon)$$

for  $t \geq t_7 > t_6$ . Integrating the last inequality consecutively  $(n-2)$  times from  $t_7$  to  $t$ , we obtain

$$0 > w(t_7) \geq w(t) + \frac{1}{(n-3)!} \int_{t_7}^t (t-s)^{n-3} q(s) G(l_7 - \varepsilon) ds.$$

Hence,

$$0 > (b(t)v'(t))' + \frac{1}{a(t)} \frac{1}{(n-3)!} \int_{t_7}^t (t-s)^{n-3} q(s) G(l_7 - \varepsilon) ds.$$

Further integrating the preceding inequality from  $t_7$  to  $t$  and considering the fact that  $v'(t) < 0$ , we obtain

$$0 > b(t_7)v'(t_7) \geq b(t)v'(t) + \frac{1}{(n-3)!} \int_{t_7}^t \frac{1}{a(\theta)} \int_{t_7}^{\theta} (\theta-s)^{n-3} q(s) G(l_7 - \varepsilon) ds d\theta.$$

Again integrating the last inequality from  $t_7$  to  $t$ , we obtain

$$v(t_7) \geq v(t) + \frac{1}{(n-3)!} \int_{t_7}^t \frac{1}{b(u)} \int_{t_7}^u \frac{1}{a(\theta)} \int_{t_7}^{\theta} (\theta-s)^{n-3} q(s) G(l_7 - \varepsilon) ds d\theta du.$$

Since  $\lim_{t \rightarrow \infty} v(t) < \infty$ , then it implies that

$$\frac{1}{(n-3)!} G(l_7 - \varepsilon) \int_{t_7}^{\infty} \frac{1}{b(u)} \int_{t_7}^u \frac{1}{a(\theta)} \int_{t_7}^{\theta} (\theta-s)^{n-3} q(s) ds d\theta du < \infty,$$

a contradiction to  $(H_9)$ . Hence,  $\lim_{t \rightarrow \infty} v(t) = 0$  and  $\lim_{t \rightarrow \infty} z(t) = 0$ . Thus,

$$0 = \limsup_{t \rightarrow \infty} z(t) \geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t)))$$

$$\begin{aligned} &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t). \end{aligned}$$

Since  $(1 + p_2) > 0$ , then  $\limsup_{t \rightarrow \infty} y(t) = 0$ . Hence,  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $v(t) < 0$  eventually, then  $-\infty \leq \lim_{t \rightarrow \infty} v(t) < 0$ . If  $-\infty < \lim_{t \rightarrow \infty} v(t) < 0$ , then  $-\infty < \lim_{t \rightarrow \infty} z(t) < 0$ . Hence,

$$\begin{aligned} 0 > \limsup_{t \rightarrow \infty} z(t) &\geq \limsup_{t \rightarrow \infty} (y(t) + p_2 y(\sigma(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\sigma(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t), \end{aligned}$$

which implies  $\limsup_{t \rightarrow \infty} y(t) < 0$ , a contradiction to the fact that  $y(t) > 0$ . If  $\lim_{t \rightarrow \infty} v(t) = -\infty$ , then we obtain a contradiction to the fact that  $y(t)$  is bounded. Hence the proof of the theorem is complete.

**THEOREM 9.** *Let  $-1 < p_2 \leq p(t) \leq 0$ . Suppose that  $(H_1)$ ,  $(H_5)$ ,  $(H_6)$ ,  $(H_7)$  and  $(H_9)$  hold, then every solution of (1.1) either oscillates or converges to zero as  $t \rightarrow \infty$ .*

*Proof.* Let  $y(t)$  be a non-oscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , say  $y(t)$  is an eventually positive solution. (The proof in case  $y(t) < 0$  eventually is similar and will be omitted.) Then there exists  $t_1 > t_0$  such that  $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$  for  $t \geq t_1$ . Setting  $z(t), k(t), v(t)$  as in (2.1), (2.2) and (2.3) respectively, we get (2.4) and (2.5) for  $t \geq t_1$ . Clearly,  $w^{(n-3)}(t), w^{(n-4)}(t), \dots, w'(t), w(t)$  are monotonic functions and of constant sign for  $t \in [t_2, \infty)$ ,  $t_2 \geq t_1$ .

If  $w(t) > 0$  for  $t \geq t_2$ , then in view of Lemma 1,  $w^{(n-3)}(t) > 0$  for  $t \geq t_2$ . Now  $w(t) > 0$  implies  $(b(t)v'(t))' > 0$  for  $t \geq t_2$ , which in turn implies  $v'(t)$  is monotonic function. Thus,  $v'(t) > 0$  or  $< 0$  for  $t \geq t_3 > t_2$ .

Case I. If  $v'(t) > 0$  for  $t \geq t_3$ , then  $z'(t) > 0$  eventually. Thus, we have two cases:  $z(t) > 0$  or  $z(t) < 0$  for  $t \geq t_4 > t_3$ .

*Subcase (i):* If  $z(t) > 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (i)* of Case I of Theorem 8, we get a contradiction due to  $(H_6)$ .

*Subcase (ii):* If  $z(t) < 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (ii)* of Case I of Theorem 8, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Case II. If  $v'(t) < 0$  for  $t \geq t_3$ , then we have two cases:  $v(t) > 0$  or  $v(t) < 0$  for  $t \geq t_4 > t_3$ .

*Subcase (iii):* If  $v(t) > 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (iii)* of Case II of Theorem 6,  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (iv):* If  $v(t) < 0$  for  $t \geq t_4$ , then proceeding as in *Subcase (iv)* of Case II of Theorem 6, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $w(t) < 0$  for  $t \geq t_2$ , then  $(b(t)v'(t))' < 0$  for  $t \geq t_2$ . Thus,  $v'(t) > 0$  or  $v'(t) < 0$  for  $t \geq t_3 > t_2$ .

Case III. Suppose  $v'(t) > 0$  for  $t \geq t_3$ . Since,  $w^{(n-2)}(t) \leq 0$  eventually, then either  $w^{(n-3)}(t) > 0$  or  $< 0$  eventually.

*Subcase (v):* Suppose  $w^{(n-3)}(t) > 0$  eventually. Now  $v'(t) > 0$  and  $k'(t) < 0$  implies that  $z'(t) > 0$  eventually. Hence,  $z(t) > 0$  or  $< 0$  eventually.

If  $z(t) > 0$  eventually, then proceeding as in *Subcase (v)* of Case III of Theorem 6, we get a contradiction to  $(H_6)$ .

If  $z(t) < 0$  for  $t \geq t_4 > t_3$ , then proceeding as in *Subcase (v)* of Case III of Theorem 6, we get  $\lim_{t \rightarrow \infty} y(t) = 0$ .

*Subcase (vi):* If  $w^{(n-3)}(t) < 0$  eventually, then using  $(H_5)$  and proceeding same as in *Subcase (ii)* of Case III of Theorem 2, we get a contradiction due to  $v'(t) > 0$ .

Case IV. Suppose  $v'(t) < 0$  eventually. Since  $w^{(n-2)}(t) \leq 0$  eventually, then we have two cases;  $w^{(n-3)}(t) > 0$  or  $w^{(n-3)}(t) < 0$  eventually.

*Subcase (vii):* If  $w^{(n-3)}(t) > 0$  eventually, then proceeding as in *Subcase (vii)* of Case IV of Theorem 8 for  $v(t) > 0$  part, we get  $\lim_{t \rightarrow \infty} y(t) = 0$  and for  $v(t) < 0$  part we get a contradiction due to  $y(t) > 0$ .

*Subcase (viii):* If  $w^{(n-3)}(t) < 0$  eventually, then first we consider:

If  $v(t) > 0$  eventually, then  $\lim_{t \rightarrow \infty} v(t) < \infty$ .

If  $0 < \lim_{t \rightarrow \infty} v(t) < \infty$ , then proceeding as in *Subcase (viii)* of Case IV of Theorem 8, we get a contradiction due to  $(H_9)$ .

If  $\lim_{t \rightarrow \infty} v(t) = 0$ , then  $\lim_{t \rightarrow \infty} y(t) = 0$ .

If  $v(t) < 0$  eventually, then proceeding as in *Subcase (viii)* of Case IV of Theorem 8, we get a contradiction due to  $y(t) > 0$ .

Hence proof of the theorem is complete.

### 3. Examples

EXAMPLE 1. Consider the fifth order differential equation

$$\begin{aligned} \left(y(t) + \frac{1}{2}y(t - \pi)\right)^{(v)} + \left(\frac{1}{2} + e^{-t}\right)y\left(t - \frac{\pi}{2}\right) \\ - e^{-t}\left(1 + \sin^2\left(t - \frac{\pi}{2}\right)\right)\frac{y\left(t - \frac{\pi}{2}\right)}{1 + y^2\left(t - \frac{\pi}{2}\right)} = 0, \quad t \geq 4. \end{aligned} \quad (3.1)$$

It is easy to verify that the hypothesis of Theorem 2 are satisfied. Thus, every solution of (3.1) either oscillates or tends to zero as  $t \rightarrow \infty$ . Indeed,  $y(t) = \sin t$  is such an oscillatory solution of (3.1).

EXAMPLE 2. Consider the third order equation

$$\left( e^{-\frac{t}{8}} \left( e^{\frac{t}{4}} \left( y(t) + \frac{1}{2e^\pi} y(t - \pi) \right) \right)' \right)' + \left( \frac{63}{64} e^{\frac{7t}{8}} + e^{-\frac{5t}{4}} \right) y^7 \left( \frac{t}{4} \right) - \frac{e^{-2t} (1 + e^{-2t + \pi})}{e^{\frac{\pi}{2}}} \frac{y \left( t - \frac{\pi}{2} \right)}{1 + y^2 \left( t - \frac{\pi}{2} \right)} = 0, \quad t \geq 4. \quad (3.2)$$

It is easy to verify that the conditions  $(H_1), (H_5), (H_6), (H_7)$  and  $(H_9)$  are satisfied, so equation (3.2) satisfies the hypothesis of Theorem 5. Thus, every solution of (3.2) either oscillates or tends to zero as  $t \rightarrow \infty$ . Indeed,  $y(t) = e^{-t}$  is such a solution of (3.2).

EXAMPLE 3. Consider the fifth order equation

$$\left( y(t) - \frac{1}{2} y(t - 2\pi) \right)^{(v)} + \left( \frac{1}{2} + e^{-t} \right) y \left( t - \frac{\pi}{2} \right) - e^{-t} \left( 1 + \sin^2 \left( t - \frac{\pi}{2} \right) \right) \frac{y \left( t - \frac{\pi}{2} \right)}{1 + y^2 \left( t - \frac{\pi}{2} \right)} = 0, \quad t \geq 4. \quad (3.3)$$

It is easy to verify that the conditions  $(H_1), (H_2), (H_6)$  and  $(H_7)$  are satisfied, so equation (3.3) satisfies the hypothesis of Theorem 6. Thus, every solution of (3.3) either oscillates or tends to zero as  $t \rightarrow \infty$ . Indeed,  $y(t) = \sin t$  is such an oscillatory solution of (3.3).

EXAMPLE 4. Consider the third order differential equation

$$\left( e^{-\frac{t}{8}} \left( e^{\frac{t}{4}} \left( y(t) - \frac{1}{2e^\pi} y(t - \pi) \right) \right)' \right)' + \left( \frac{21}{64} e^{\frac{7t}{8}} + e^{-\frac{5t}{4}} \right) y^7 \left( \frac{t}{4} \right) - \frac{e^{-2t} (1 + e^{-2t + \pi})}{e^{\frac{\pi}{2}}} \frac{y \left( t - \frac{\pi}{2} \right)}{1 + y^2 \left( t - \frac{\pi}{2} \right)} = 0, \quad t \geq 4. \quad (3.4)$$

It is easy to verify that the conditions  $(H_1), (H_5), (H_6), (H_7)$  and  $(H_9)$  are satisfied, so equation (3.4) satisfies the hypothesis of Theorem 9. Thus, every solution of (3.4) either oscillates or tends to zero as  $t \rightarrow \infty$ . Indeed,  $y(t) = e^{-t}$  is such a solution of (3.4).

REMARK 1. It would be interesting to study the qualitative behavior of solutions of (1.1) with  $n \geq 3$ ,  $n$  is an odd integer for the ranges  $1 \leq p(t) < \infty$  and  $-\infty < p(t) \leq -1$  under the hypothesis  $(H_2) - (H_5)$ .

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