Differential
Equations
& Applications
Volume 7, Number 3 (2015), 385–400
doi:10.7153/dea-07-22

PROPERTY OF GROWTH DETERMINED BY SPECTRUM OF OPERATOR ASSOCIATED WITH THE TIMOSHENKO SYSTEM WITH WEAKLY DISSIPATION

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(*Communicated by Mokhtar Kirane*)

Abstract. In this work, we consider the Timoshenko system with weakly dissipation, one dissipation, φ_t, on the transverse displacement and another ψ_t, on the rotation angle of a filament of the beam

$$\rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x + \varphi_t = 0, \text{ in } (0,L)\times(0,t),$$
$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \psi_t = 0, \text{ in } (0,L)\times(0,t),$$

with initial conditions

$$\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x).$$

In [13] was proved the exponential stability of the semigroup associated with this system, and now we prove the property of growth determined by spectrum of operator associated, present the type of semigroup and also indicate the best constant for the exponential stability.

1. Introduction

We will use standard notation of Sobolev spaces and theory of semigroups as in [9], [1], [11]. Let A be the infinitesimal generator of the C_0-semigroup e^{At} on a Banach space X. As usual, we define the type or growth order of the semigroup by

$$w_0(A) = inf\{w\in\mathbb{R} : ||e^{At}|| \leqslant Me^{wt}, \forall 0 \leqslant t < \infty\} = \inf_{t>0} \frac{ln||e^{At}||}{t},$$

and the spectral bound by

$$w_\sigma(A) = sup\{Re\lambda : \lambda\in\sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of A. If X is finitely dimensional, it is well known that

$$w_\sigma(A) = w_0(A). \quad (1)$$

In the infinite dimensional case, in general, the above equality (1) may not hold. From the Hille-Yosida theorem we see that

$$w_\sigma(A) \leqslant w_0(A).$$

Mathematics subject classification (2010): 35B35, 35B40, 35L50.
Keywords and phrases: Timoshenko, C_0-semigroup, spectrum, property of growth, stability.

Paper DEA-07-22

We notice that $w_0(A)$ describes the growth order of e^{At}. From the definition of $w_0(A)$ if $w > w_0(A)$ then there exists a constant $M \geqslant 1$ such that

$$||e^{At}|| \leqslant M e^{wt}.$$

The condition (1) is important because it gives a practical criterion for exponential stability of e^{At}. The exponential stability of e^{At} is equivalent to the condition that $w_0(A) < 0$. If $-\infty < w_0(A) < 0$, then the C_0-semigroup e^{At} is exponentially stable. For see this fact, for $0 < \varepsilon < |w_0(A)|$, we have

$$w_0(A) + \varepsilon > ln||e^{At}||t \Rightarrow e^{(w_0(A)+\varepsilon)t} \leqslant ||e^{At}||,\ \forall t > N_\varepsilon.$$

Using the continuity of the operator e^{At} on the compact interval $[0, N_\varepsilon]$, we obtain $M_\varepsilon > 0$ such that

$$||e^{At}|| \leqslant M_\varepsilon e^{(w_0(A)+\varepsilon)t},\ \forall t > 0.$$

Choosing $-\mu = w_0(A) + \varepsilon < 0$, follows the exponential stability of e^{At}. In this direction the exponential stability of e^{At} is completely determined by the spectrum of A and the condition (1) is usually called the spectrum determined growth assumption. In this direction consider the result bellow duo to M. Renardy for Hilbert spaces.

THEOREM 1. *Let H be a Hilbert space and $A = A_0 + B$ the infinitesimal generator of the C_0-semigroup on H where A_0 is normal and B is bounded. Consider $M > 0$ and $n > 0$.*

- (a) If $\lambda \in \sigma(A_0)$ and $|\lambda| > M - 1$, then λ is a isolated eigenvalue of finite multiplicity,

- (b) If $|z| > M$, then the number of eigenvalues of A_0 in the unit disk centered z (containing multiplicities) does not exceed n.
So, we have

$$w_\sigma(A) = w_0(A).$$

Proof. See, [14], [12].

The property of growth determined for spectrum given by $w_\sigma(A) = w_0(A)$ says that the best constant for the exponential stability is the upper bound of the spectrum of operator A.

In this paper we will study the following Timoshenko system

$$\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x + \varphi_t = 0, \text{ in } (0, L) \times (0, t), \tag{2}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \psi_t = 0, \text{ in } (0, L) \times (0, t), \tag{3}$$

with initial conditions

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x),\ \psi(x, 0) = \psi_0(x),\ \psi_t(x, 0) = \psi_1(x). \tag{4}$$

To facilitate our analysis, we consider Dirichlet-Neumann boundary condition given by

$$\varphi(0,t)= \varphi(L,t) = \psi_x(0,t)= \psi_x(L,t)=0,\ t>0. \tag{5}$$

For other different boundary conditions (5), what will change will be the domain of the operator A.

The energy space associated to system (2)-(3) is $H=[H_0^1(0,L)\times L^2(0,L)]^2$. The inner product in the energy space is defined for $U_j=(u^j,v^j,w^j,y^j)^T\in H,\ j=1,2$, as follows:

$$(U_1,U_2)=\int_0^L [k(u_x^1-w^1)(u_x^2-w^2)+\rho_1 v^1v^2+\rho_2 y^1y^2+bw^1w^2]dx.$$

In the sequel we will denote by $\|U\|^2=(U,U)$, the norm in the energy space. The system (2)-(3) cam be written as

$$\frac{d}{dt}U(t)-AU(t)=0,$$

where

$$U(t)=\begin{pmatrix}\varphi\\ \varphi_t\\ \psi\\ \psi_t\end{pmatrix},\quad A=\begin{pmatrix}0 & I & 0 & 0\\ \frac{k}{\rho_1}\partial_x^2 & \frac{-1}{\rho_1} & \frac{k}{\rho_1}\partial_x & 0\\ 0 & 0 & 0 & I\\ -\frac{k}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2-\frac{k}{\rho_2} & \frac{-1}{\rho_2}\end{pmatrix},$$

with $D(A)=[(H^2(0,L)\cap H_0^1(0,L))\times H_0^1(0,L)]^2$.

The operator A generates a C_0-semigroup of contractions $(e^{At})_{t>0}$ on H, see [16], and the C_0-semigroup of contractions $(e^{At})_{t>0}$, generated by A, is exponentially stable, see [13].

This work is organized as follows. In the section 2 we present the preliminary results, section 3 deals with the property of growth determined by spectrum of the infinitesimal generator of the C_0-semigroup and finally in the section 4 we calculate the best constant for a exponential stability.

2. Preliminar results

We denote H a Hilbert space and $A:D(A)\subset H\to H$ infinitesimal generator of the C_0-semigroup of contractions $S(t)=e^{At}$. Then $U(t)=S(t)U_0$ is solution of the equation

$$U_t-AU=0,\quad U(0)=U_0,$$

that can be extended to

$$U_{tt}-AU_t=0,\quad U(0)=U_0, U_1(0)=U_1.$$

For this, is equivalent to prove that the operator

$$\mathscr{A}_1 = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$$

generates a C_0-semigroup of contractions in the space

$$\mathscr{H}_1 = \{(U,V) \in D(A) \times H;\ AU - V \in D(A)\},$$

provided the inner product

$$\left\langle \begin{pmatrix} U^1 \\ V^1 \end{pmatrix}, \begin{pmatrix} U^2 \\ V^2 \end{pmatrix} \right\rangle_{\mathscr{H}_1} = (V^1, V^2)_H + (AU^1 - V^1, AU^2 - V^2)_H \\ + (A^2U^1 - AV^1, A^2U^2 - AV^2)_H,$$

with norm

$$\left\| \begin{pmatrix} U \\ V \end{pmatrix} \right\|_{\mathscr{H}_1}^2 = \|V\|_H^2 + \|AU - V\|_H^2 + \|A^2U - AV\|_H^2.$$

In Hilbert space $\mathscr{H}_1$ we take the domain of $\mathscr{A}_1$, as

$$D(\mathscr{A}_1) = \left\{\mathscr{U} = (U,V) \in \mathscr{H}_1;\ \ (U,V) \in D(A^2) \times D(A)\right\}.$$

We have

LEMMA 1. *The operator $\mathscr{A}_1$ is a infinitesimal generator of a C_0-semigroup of contractions on $\mathscr{H}_1$.*

Proof. $D(\mathscr{A}_1)$ is a closed and dense space on $\mathscr{H}_1$. We will show that $\mathbb{R}^+ \subset \rho(\mathscr{A}_1)$. For $\mathscr{F} = (F_1, F_2) \in \mathscr{H}_1$. We will show there exists a unique solution $\mathscr{U} \in D(\mathscr{A}_1)$ such that

$$\lambda\mathscr{U} - \mathscr{A}_1\mathscr{U} = \mathscr{F}.$$

In terms of its components

$$\lambda U - V = F_1, \tag{6}$$

$$\lambda V - AV = F_2. \tag{7}$$

As A is infinitesimal generator of a C_0-semigroup of contractions, we have that for all $F_2 \in H$ there exists a unique solution $V \in D(A)$ such that

$$\lambda V - AV = F_2.$$

Then

$$V = (\lambda I - A)^{-1}F_2, \tag{8}$$

and,

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{1}{\lambda}.$$

Replacing (8) into (6) we find

$$\lambda U - (\lambda I - A)^{-1} F_2 = F_1.$$

Applying $(\lambda I - A)$ in the above equation we have

$$\lambda U - AU = F_1 + \frac{1}{\lambda} F_2 - \frac{1}{\lambda} AF_1 \in D(A).$$

Since A is generator infinitesimal of a C_0-semigroup, we have that $U \in D(A^2)$.

Using (6) and (7) we have

$$\lambda AU - AV = AF_1,$$

$$\lambda V - AV = F_2.$$

Taking the difference of these equations

$$\lambda (AU - V) = AF_1 - F_2.$$

Similarly

$$\lambda (A^2 U - AV) = A^2 F_1 - AF_2.$$

Taking the norms, follows

$$\lambda \|AU - V\| \leqslant \|AF_1 - F_2\|, \qquad \lambda \|A^2 U - AV\| \leqslant \|A^2 F_1 - AF_2\|. \tag{9}$$

Taking the inner product of (7) with V we obtain

$$\lambda \|V\|^2 - \langle AV, V \rangle = \langle F_2, V \rangle.$$

As $\langle AV, V \rangle \leqslant 0$ for all $V \in D(A)$, by Holder's inequality, follows

$$\lambda \|V\| \leqslant \|F_2\|. \tag{10}$$

Squaring (9)-(10) we get

$$\lambda^2 \|\mathscr{U}\|^2 \leqslant \|\mathscr{F}\|^2,$$

thats implies

$$\|(\lambda I - \mathscr{A})^{-1}\| \leqslant \frac{1}{\lambda}.$$

By theorem of Hille - Yosida the conclusion hold.
We can extend this result to

$$U_{ttt} - AU_{tt} = 0, \quad U(0) = U_0, \quad U_t(0) = U_1, \quad U_{tt}(0) = U_2,$$

i.e., the operator

$$\mathscr{A}_2 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & A \end{pmatrix}$$

generates a C_0-semigroup of contractions in the space

$$\mathscr{H}_2 = \left\{(U,V,W) \in D(A^2) \times D(A) \times H;\ AU - V \in D(A^2);\ AV - W \in D(A)\right\},$$

with inner product

$$\left\langle \begin{pmatrix} U^1 \\ V^1 \\ W^1 \end{pmatrix}, \begin{pmatrix} U^2 \\ V^2 \\ W^2 \end{pmatrix} \right\rangle_{\mathscr{H}_2} = (W^1, W^2)_H + \sum_{j=1}^{3} (A^j U^1 - A^{j-1} V^1, A^j U^2 - A^{j-1} V^2)_H + \sum_{j=1}^{2} (A^j V^1 - A^{j-1} W^1, A^j V^2 - A^{j-1} W^2)_H,$$

and domain

$$D(\mathscr{A}_2) = \left\{\mathscr{U} = (U,V,W) \in \mathscr{H}_2;\ \ (U,V,W) \in D(A^3) \times D(A^2) \times D(A)\right\}.$$

Now consider the following result

LEMMA 2. *Let A be the infinitesimal generator of a C_0-semigroup of contractions, then*

$$\mathscr{A}_2 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & A \end{pmatrix}$$

verifies

$$\rho(\mathscr{A}_2) = \rho(A) \cup \{0\}.$$

Proof. Suppose $\lambda \in \rho(A)$ then exist a unique solution of the resolvent equation

$$\lambda U - AU = F,$$

where $F \in H$. We shall that $\lambda \in \rho(\mathscr{A}_2)$. Consider the problem

$$\lambda \mathscr{U} - \mathscr{A}_2 \mathscr{U} = \mathscr{F}.$$

We have

$$\lambda U - V = F_1 \quad \in \quad D(A^2), \tag{11}$$

$$\lambda V - W = F_2 \quad \in \quad D(A), \tag{12}$$

$$\lambda W - AW = F_3 \quad \in \quad H. \tag{13}$$

By the hypothesis there exists a unique $W \in D(A)$ solution of (13). So (12) can be written as

$$\lambda V - (\lambda I - A)^{-1} F_3 = F_2,$$

and then

$$\lambda(\lambda I - A)V = (\lambda I - A)F_2 + F_3 \in D(A).$$

Using the hypothesis again found that there is a unique $V \in D(A^2)$ such that

$$\lambda V = F_2 + (\lambda I - A)^{-1}F_3 \in D(A).$$

Substituting in (11) and multiplying by λ we get

$$\lambda^2 U - F_2 - (\lambda I - A)^{-1}F_3 = \lambda F_1 \in D(A^2).$$

From where follows

$$\lambda^2(\lambda I - A)U - (\lambda I - A)F_2 - F_3 = \lambda(\lambda I - A)F_1 \in D(A^2).$$

Thus

$$\lambda^2(\lambda U - AU) = \lambda^2 F_1 + \lambda F_2 - \lambda A F_1 - AF_2 + F_3 \in D(A^2).$$

Follow that $U \in D(A^3)$ and show $\lambda \in \rho(\mathscr{A}_2)$. Reciprocally, take $\lambda \in \rho(\mathscr{A}_2)$, then we have for all $\mathscr{F} \in \mathscr{H}_2$ that there exists a unique $\mathscr{U} \in D(\mathscr{A})$ verifying (11)-(13). In particular, from (13) we have that for all $F_3 \in H$ there exists a unique $W \in D(A)$. So we concludes that $\lambda \in \rho(A)$.
As a consequence of the previous lemmas we have the following theorem

THEOREM 2. *Suppose that A is a normal operator for which there exist $M > 0$ and an integer n verifying*

- (a) *If $\lambda \in \sigma(A_0)$ and $|\lambda| > M - 1$, then λ is an isolated eigenvalue of finite multiplicity, and*

- (b) *If $|z| > M$, then the number of eigenvalues A_0 into of unit disc centered in z (containing multiplicities) does not exceed n.*

With this conditions the operator $\mathscr{A}$ defined by

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & A \end{pmatrix}$$

is a normal operator which also satisfies the properties (a) and (b).

Proof. As A is self-adjoint an normal, we have that $\mathscr{A}^* = \mathscr{A}$, and in particular $\mathscr{A}$ is normal. Moreover, the spectra of A and $\mathscr{A}$ are equal out of a unit ball and the result follows.

3. Property of growth determined for spectrum

In this section we prove the property of growth determined by the spectrum of the operator in three different situations.

3.1. Property of growth with dissipation on transverse displacement and on the rotation angle of a filament of the beam.

Consider

$$\rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x+\varphi_t=0,\ \text{in}\ (0,L)\times(0,t), \tag{14}$$

$$\rho_2\psi_{tt}-b\psi_{xx}+\kappa(\varphi_x+\psi)\phi_t+\psi_t=0,\ \text{in}\ (0,L)\times(0,t). \tag{15}$$

The initial conditions are given by

$$\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x), \tag{16}$$

and boundary conditions are

$$\varphi(0,t)=\ \varphi(L,t)\ =\ \psi_x(0,t)=\ \psi_x(L,t)=0. \tag{17}$$

Making $U=\begin{pmatrix}\varphi\\ \varphi_t\\ \psi\\ \psi_t\end{pmatrix}$, we get

$$U_t=\begin{pmatrix}\varphi_t\\ \varphi_{tt}\\ \psi_t\\ \psi_{tt}\end{pmatrix}=\underbrace{\begin{pmatrix}\varphi_t\\ \frac{\kappa}{\rho_1}(\varphi_x+\psi)_x\\ \psi_t\\ \frac{b}{\rho_2}\psi_{xx}-\frac{\kappa}{\rho_2}(\varphi_x+\psi)\end{pmatrix}}_{:=\mathscr{A}_0U}+\underbrace{\begin{pmatrix}0\\ -\frac{1}{\rho_1}\varphi_t\\ 0\\ -\frac{1}{\rho_2}\psi_t\end{pmatrix}}_{:=\mathscr{A}_1U},$$

So, we have

$$U_t=\mathscr{A}_0U+\mathscr{A}_1U.$$

Deriving we have

$$U_{tt}=\mathscr{A}_0U_t+\mathscr{A}_1U_t.$$

Then, we obtain

$$U_{tt}=\mathscr{A}_0U_t+B,$$

where $B=\mathscr{A}_1U_t$.

THEOREM 3. *The system* (14) – (15) *verifies the property of growth determined for spectrum.*

Proof. In this case we consider the infinitesimal generator of the Timoshenko system,

$$\rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x=0, \tag{18}$$

$$\rho_2\psi_{tt}-b\psi_{xx}+\kappa(\varphi_x+\psi)=0, \tag{19}$$

ie,

$$\mathscr{A}_0 = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\kappa}{\rho_1}\partial_x^2 & 0 & \frac{\kappa}{\rho_1}\partial_x & 0 \\ 0 & 0 & 0 & I \\ -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{\kappa}{\rho_2}I & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \varphi \\ \varphi_t \\ \psi \\ \psi_t \end{pmatrix}.$$

Note that $\mathscr{A}_0$ is a normal operator. So, the problem $(18)-(19)$ can be rewrite as

$$U_t - \mathscr{A}_0 U = 0.$$

Therefore, the model associate with the Cauchy problem $(14)-(15)$ is de defined by

$$\mathscr{U}_t - \mathscr{A}\mathscr{U} = \mathscr{F},$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & \mathscr{A}_0 \end{pmatrix}, \qquad \mathscr{F} = \begin{pmatrix} 0 \\ 0 \\ B_t \end{pmatrix} \text{ and } \mathscr{U} = \begin{pmatrix} U \\ U_t \\ U_{tt} \end{pmatrix}.$$

As $\mathscr{A}_0$ is normal, from theorem 2 follows that $\mathscr{A}$ is a normal operator that verifies the conditions (a) and (b).

The continuity of B_t is made through derived inequalities. Using the theorem 1, the result follows.

3.2. Property of growth with dissipation on transverse displacement of the beam.

Consider

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \varphi_t = 0, \text{ in } (0,L)\times(0,t), \tag{20}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \text{ in } (0,L)\times(0,t). \tag{21}$$

The initial conditions are given by

$$\varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \tag{22}$$

and the boundary conditions type Dirichlet-Neumann are

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0. \tag{23}$$

Making $U = \begin{pmatrix} \varphi \\ \varphi_t \\ \psi \\ \psi_t \end{pmatrix}$, we obtain

$$U_t = \begin{pmatrix} \varphi_t \\ \varphi_{tt} \\ \psi_t \\ \psi_{tt} \end{pmatrix} = \underbrace{\begin{pmatrix} \varphi_t \\ \frac{\kappa}{\rho_1}(\varphi_x + \psi)_x \\ \psi_t \\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi) \end{pmatrix}}_{:=\mathscr{A}_0 U} + \underbrace{\begin{pmatrix} 0 \\ -\frac{1}{\rho_1}\varphi_t \\ 0 \\ 0 \end{pmatrix}}_{:=\mathscr{A}_2 U}.$$

So we have

$$U_t = \mathscr{A}_0 U + \mathscr{A}_2 U.$$

Deriving,

$$U_{tt} = \mathscr{A}_0 U_t + \mathscr{A}_2 U_t.$$

So, we obtain

$$U_{tt} = \mathscr{A}_0 U_t + C,$$

where $C = \mathscr{A}_2 U_t$.

THEOREM 4. *The system* $(20)-(21)$ *verifies the property of linear stability.*

Proof. In this case, we consider the infinitesimal generate as,

$$\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \tag{24}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \tag{25}$$

ie,

$$\mathscr{A}_0 = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\kappa}{\rho_1}\partial_x^2 & 0 & \frac{\kappa}{\rho_1}\partial_x & 0 \\ 0 & 0 & 0 & I \\ -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{\kappa}{\rho_2}I & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \varphi \\ \varphi_t \\ \psi \\ \psi_t \end{pmatrix}.$$

Note que $\mathscr{A}_0$ is a normal operator. So, the problem $(24)-(25)$ can be rewrite as

$$U_t - \mathscr{A}_0 U = 0.$$

Then, the associated Cauchy problem $(20)-(21)$ us defined by

$$\mathscr{U}_t - \mathscr{A}\mathscr{U} = \mathscr{G},$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & \mathscr{A}_0 \end{pmatrix}, \qquad \mathscr{G} = \begin{pmatrix} 0 \\ 0 \\ C_t \end{pmatrix} \text{ and } \mathscr{U} = \begin{pmatrix} U \\ U_t \\ U_{tt} \end{pmatrix}.$$

As $\mathscr{A}_0$ is normal, from theorem 2 follows that $\mathscr{A}$ is a normal operator that verifies the conditions (a) and (b). The continuity of C_t is maid through derivative inequalities. Using the theorem 1, the result follows.

3.3. Property of growth with dissipation on the rotation angle of a filament of the beam.

Consider

$$\rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x=0, \text{ in } (0,L)\times(0,t), \tag{26}$$

$$\rho_2\psi_{tt}-b\psi_{xx}+\kappa(\varphi_x+\psi)+\psi_t=0, \text{ in } (0,L)\times(0,t). \tag{27}$$

The initial conditions are given in this form

$$\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x), \tag{28}$$

and boundary conditions type Dirichlet - Neumann are

$$\varphi(0,t)=\ \varphi(L,t)\ =\ \psi_x(0,t)=\ \psi_x(L,t)=0. \tag{29}$$

Making $U=\begin{pmatrix}\varphi\\ \varphi_t\\ \psi\\ \psi_t\end{pmatrix}$, we obtain

$$U_t=\begin{pmatrix}\varphi_t\\ \varphi_{tt}\\ \psi_t\\ \psi_{tt}\end{pmatrix}=\underbrace{\begin{pmatrix}\varphi_t\\ \frac{\kappa}{\rho_1}(\varphi_x+\psi)_x\\ \psi_t\\ \frac{b}{\rho_2}\psi_{xx}-\frac{\kappa}{\rho_2}(\varphi_x+\psi)\end{pmatrix}}_{:=\mathscr{A}_0U}+\underbrace{\begin{pmatrix}0\\ 0\\ 0\\ -\frac{1}{\rho_2}\psi_t)\end{pmatrix}}_{:=\mathscr{A}_3U}.$$

So, we have

$$U_t=\mathscr{A}_0U+\mathscr{A}_3U. \tag{30}$$

After to derive we obtain

$$U_{tt}=\mathscr{A}_0U_t+\mathscr{A}_3U_t.$$

So, we obtain

$$U_{tt}=\mathscr{A}_0U_t+D,$$

where $D=\mathscr{A}_3U_t$.

THEOREM 5. *The system* $(26)-(27)$ *verifies the linear stability.*

Proof. For this case we consider the following infinitesimal generator,

$$\rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x=0, \tag{31}$$

$$\rho_2\psi_{tt}-b\psi_{xx}+\kappa(\varphi_x+\psi)=0, \tag{32}$$

ie,

$$\mathscr{A}_0 = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\kappa}{\rho_1}\partial_x^2 & 0 & \frac{\kappa}{\rho_1}\partial_x & 0 \\ 0 & 0 & 0 & I \\ -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{\kappa}{\rho_2} I & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \varphi \\ \varphi_t \\ \psi \\ \psi_t \end{pmatrix}.$$

Note that $\mathscr{A}_0$ is a normal operator, So, the problem $(31)-(32)$ can be rewrite as

$$U_t - \mathscr{A}_0 U = 0.$$

Hence the model associated to Cauchy problem $(26)-(27)$ is defined by

$$\mathscr{U}_t - \mathscr{A}\mathscr{U} = \mathscr{J},$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & \mathscr{A}_0 \end{pmatrix}, \qquad \mathscr{J} = \begin{pmatrix} 0 \\ 0 \\ D_t \end{pmatrix} \text{ e } \mathscr{U} = \begin{pmatrix} U \\ U_t \\ U_{tt} \end{pmatrix}.$$

As $\mathscr{A}_0$ is normal, from theorem 2 follows that $\mathscr{A}$ is a normal operator that verifies the conditions (a) and (b). The continuity of D_t is made through derivative inequalities. Using the theorem 1, we obtain the result.

4. Calculation of $\omega_0(\mathscr{A})$, the growth order of the semigroup.

In this section we will calculate the polynomial whose roots give us the estimate of $\omega_0(\mathscr{A})$. Identifying $(\varphi, \varphi_t, \psi, \psi_t)^T$ with $U = (\varphi, \Phi, \psi, \Psi)^T$, we denote, as in section 4.1, the operator $\mathscr{A}$ given by

$$\mathscr{A}U = \begin{bmatrix} \Phi \\ \frac{\kappa}{\rho_1}(\varphi_x + \psi)_x - \frac{1}{\rho_1}\Phi \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi) - \frac{1}{\rho_2}\Psi \end{bmatrix}. \tag{33}$$

We remember that the energy space associated is

$$V = [H_0^1(0,L) \times L^2(0,L)]^2$$

and the domain of the operator $\mathscr{A}$ is given by

$$D(\mathscr{A}) = [H^2(0,L) \cap H_0^1(0,L)] \times H_0^1(0,L)]^2.$$

The resolvent equation

$$\lambda U - \mathscr{A}U = F,$$

that in terms of the scalar components is given by

$$\lambda\varphi - \Phi = f^1 \in H_0^1(0,L), \tag{34}$$

$$\lambda\rho_1\Phi - \kappa(\varphi_x+\psi)_x - \Phi = \rho_1 f^2 \in L^2(0,L), \tag{35}$$

$$\lambda\psi - \Psi = f^3 \in H^1(0,L), \tag{36}$$

$$\lambda\rho_2\Psi - b\psi_{xx} + \kappa(\varphi_x+\psi) + \Psi = \rho_2 f^4 \in L^2(0,L). \tag{37}$$

We must find the elements of the spectrum of operator. It's simple verifies that $\lambda \in \sigma(\mathscr{A})$ if and only if,

$$\lambda U - \mathscr{A}U = 0.$$

From (34) and (36) for $f^i = 0, \forall i = \{1,\dots,4\}$, we find

$$\Phi = \lambda\varphi \text{ and } \Psi = \lambda\psi,$$

ie,

$$\lambda^2\rho_1\varphi - \kappa(\varphi_x+\psi)_x + \lambda\varphi = 0, \tag{38}$$

$$\lambda^2\rho_2\psi - b\psi_{xx} + \kappa(\varphi_x+\psi) + \lambda\psi = 0. \tag{39}$$

Differentiating (39) in relation x, we have

$$\lambda^2\rho_2\psi_x - b\psi_{xxx} + \kappa(\varphi_x+\psi)_x + \lambda\psi_x = 0. \tag{40}$$

From (38), folows that

$$\kappa(\varphi_x+\psi)_x = \lambda^2\rho_1\varphi + \lambda\varphi. \tag{41}$$

Replacing (41) in (40), we obtain

$$\lambda^2\rho_2\psi_x - b\psi_{xxx} + \lambda^2\rho_1\varphi + \lambda\varphi + \lambda\psi_x = 0. \tag{42}$$

By outer side, using the boundary conditions (17), ie,

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \; t>0,$$

we have φ and ψ of type

$$\varphi(x,t) = e^{\lambda t}\sin(\widetilde{\gamma}x) \;\text{ and }\; \psi(x,t) = e^{\lambda t}\cos(\widetilde{\gamma}x),$$

where $\widetilde{\gamma}$ is such that $\widetilde{\gamma}L = n\pi$, $n \in \mathbb{N}$.

Estimating the derivatives of ψ, we get

$$\psi_x = -\widetilde{\gamma}\varphi, \;\; \psi_{xx} = -\widetilde{\gamma}^2\psi \text{ and } \psi_{xxx} = \widetilde{\gamma}^3\varphi.$$

Replacing in (42) and, simplifying by $\varphi \neq 0$, follows that

$$-\widetilde{\gamma}\lambda^2\rho_2 - b\widetilde{\gamma}^3 + \lambda^2\rho_1 + \lambda - \lambda\widetilde{\gamma} = 0,$$

ie,

$$(\rho_1 - \widetilde{\gamma}\rho_2)\lambda^2 + (1-\widetilde{\gamma})\lambda - b\widetilde{\gamma}^3 = 0.$$

Using $\widetilde{\gamma}L = n\pi$, we have

$$\left(\rho_1 - \frac{\rho_2 n\pi}{L}\right)\lambda^2 + \left(1 - \frac{n\pi}{L}\right)\lambda - \frac{bn^3\pi^3}{L^3} = 0,$$

that can be rewrite as

$$\left(\frac{\rho_1 L - \rho_2 n\pi}{L}\right)\lambda^2 + \left(\frac{L - n\pi}{L}\right)\lambda - \frac{bn^3\pi^3}{L^3} = 0.$$

Denoting

$$a_2(n) = \frac{\rho_1 L - \rho_2 n\pi}{L}, \ \ a_1(n) = \frac{L - n\pi}{L}, \ \ a_0(n) = -\frac{bn^3\pi^3}{L^3},$$

then, for $n \in \mathbb{N}$, the polynomial whose roots give us the estimate of $\omega_0(\mathscr{A})$ is the polynomial of degree 2 given by

$$P_1(\lambda) = a_2(n)\lambda^2 + a_1(n)\lambda + a_0(n). \tag{43}$$

We have

$$\sigma(\mathscr{A}) = \{\lambda \in \mathbb{C}/P_1(\lambda) = 0\}.$$

Solving (43), we obtain

$$\lambda = -\frac{a_1(n)}{2a_2(n)} \pm \frac{\sqrt{a_1^2(n) - 4a_2(n)a_0(n)}}{2a_2(n)}.$$

First we note that

$$\left(-\frac{a_1(n)}{2a_2(n)}\right)_{n\in\mathbb{N}} = \left(\frac{n\pi - L}{2(\rho_1 L - \rho_2 n\pi)}\right)_{n\in\mathbb{N}}$$

and for $n = 1$ we have

$$\frac{\pi - L}{2(\rho_1 L - \rho_2\pi)} < 0 \text{ if } \frac{\pi(1 + 2\rho_2)}{1 + 2\rho_1} < L,$$

and for $n \geqslant 1$

$$\left(-\frac{a_1(n)}{2a_2(n)}\right)_{n\in\mathbb{N}} \quad \text{is increasing if } \frac{\rho_1}{\rho_2} < L.$$

Now observe that

$$a_1^2(n) - 4a_2(n)a_0(n) = \frac{L^4 - 2L^3\pi n + \pi^2 n^2 + 4\rho_1 Lbn^3 - 4\rho_2\pi^4 bn^4}{L^4}$$

is a polynomial of degree 4 and so for $n > n_0$, $a_1^2(n) - 4a_2(n)a_0(n) < 0$ and then

$$\sup \mathscr{R}e\,\lambda = \lim_{n\to\infty} -\frac{a_1(n)}{2a_2(n)} = -\frac{1}{2\rho_2}.$$

Finally, for

$$L \geqslant max\left\{\frac{\rho_1}{\rho_2}, \frac{\pi(1+2\rho_2)}{1+2\rho_1}\right\}$$

the growth order of the semigroup is given by

$$w_0(A) = \begin{cases} -\frac{1}{2\rho_2}, & \text{if } n > n_0, \\ max\,\lambda, & \text{if } n \leqslant n_0. \end{cases} \tag{44}$$

For the case in which the dissipation is just on transverse displacement or is just on the rotation angle of a filament of the beam, we applies the same technique, and obtain respectively, the polynomials,

$$P_2(\lambda) = \left(\rho_1 - \frac{\rho_2 n\pi}{L}\right)\lambda^2 + \lambda - b\frac{n^3\pi^3}{L^3},\ n \in \mathbb{N},$$

and

$$P_3(\lambda) = \left(\rho_1 - \frac{\rho_2 n\pi}{L}\right)\lambda^2 - \frac{n\pi}{L}\lambda - b\frac{n^3\pi^3}{L^3},\ n \in \mathbb{N}.$$

Acknowledgements. We would like to express our gratitude to the anonymous referees for a number of valuable comments that led to the clarification of various points.

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(Received May 16, 2015)
(Revised August 24, 2015)

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