SYMMETRIC POSITIVE SOLUTIONS FOR SECOND–ORDER SINGULAR DIFFERENTIAL SYSTEMS WITH MULTI–POINT COUPLED INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study the existence and multiplicity of symmetric positive solutions for a class of second-order singular nonlinear differential systems with multi-point coupled integral boundary conditions. By constructing a special cone and applying the fixed point theorem of cone expansion and compression of norm type, the existence results of single and multiple symmetric positive solutions are established. As applications, two examples are given to demonstrate the applicability of our results.

1. Introduction

In this paper, we consider the following second-order singular nonlinear differential system with multi-point coupled integral boundary conditions:

$$\begin{cases} u''(t) + a(t)f(t,u(t),v(t)) = 0, \quad v''(t) + b(t)g(t,u(t),v(t)) = 0, \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)v(r)dr, \quad u(1) = \sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} p(r)v(r)dr, \\ v(0) = \sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)u(r)dr, \quad v(1) = \sum_{i=1}^{m} \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} q(r)u(r)dr, \end{cases} \tag{1.1}$$

where $m \geqslant 1$ is an integer, the parameters $0 = \eta_0 < \eta_1 < \eta_2 < \cdots \eta_{m-1} < \eta_m \leqslant 1/2$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots \xi_{m-1} < \xi_m \leqslant 1/2$, $\eta_i + \overline{\eta}_{m-i} = 1$ and $\xi_i + \overline{\xi}_{m-i} = 1$, $\alpha_i, \beta_i > 0$, $\alpha_i = \alpha_{m+1-i}$, $\beta_i = \beta_{m+1-i}$, for $i = 0, 1, \ldots, m$. We assume $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and $f(\cdot,u,v), g(\cdot,u,v)$ are symmetric on $[0,1]$ for all $u, v \in [0,\infty)$, $a, b : (0,1) \to [0,+\infty)$ are continuous, symmetric on $(0,1)$ and may be singular at $t = 0$ and/or $t = 1$. By a symmetric positive solution of the system (1.1), we mean that $(u,v) \in C[0,1] \cap C^2(0,1) \times C[0,1] \cap C^2(0,1)$ satisfies (1.1), (u,v) are symmetric and $u(t) > 0$, $v(t) > 0$ for all $t \in [0,1]$. To the best knowledge of the author, there are few papers which deal with the multi-point coupled integral boundary value problems for systems of nonlinear second-order differential equations.

Recently, the existence and multiplicity of positive solutions for a variety of multi-point and integral boundary value problems for nonlinear ordinary differential equations/systems have received a great deal of attention due to their distinguished applications to different areas of applied mathematics and physics, see [8, 19, 35, 36, 3, 37] and the references therein. However, most papers only focus on paying attention to the differential systems with uncoupled boundary conditions, see [2, 28, 4, 12, 13]. Coupled boundary conditions arise in the study of reaction-diffusion equations and Sturm-Liouville problems due to the wide applications in various fields of sciences and engineering. There were many works available in the literature for a variety of nonlinear higher ordinary and partial differential systems with coupled boundary conditions. To identify a few, we refer the reader to [18, 26, 38, 6] and references therein. For example, by mixed monotone method, Cui *et al.* [7] established sufficient conditions for the existence and uniqueness of positive solutions to a singular differential system with coupled integral boundary value conditions. Asif and Khan [16] investigated the existence of at least one positive solution to a coupled singular system subject to four-point coupled boundary conditions by using Guo-Krasnosel'skii fixed point theorem.

At the same time, an increasing interest has been observed in investigating the existence of symmetric positive solutions for nonlinear boundary value problems of ordinary differential equations, see [32, 24, 25, 9, 14] and the references therein. Generally, some standard fixed point theorems, the upper and lower solutions method and the monotone iterative technique are used to proving the existence of symmetric positive solutions to boundary value problems. For example, multi-point boundary value problems for ordinary differential equations constitute a very interesting and important problems, one may refer to [27, 10, 17, 1, 31, 20]. For an overview of the literature on ordinary differential equations with integral boundary conditions, see [33, 29, 30, 22, 34].

In [21], Li and Zhang considered the following nonlinear higher order differential system with coupled integral boundary conditions

$$\begin{cases} u^{(n)}(t)+a(t)f(t,u(t),v(t))=0, \quad v^{(n)}(t)+b(t)g(t,u(t),v(t))=0, \quad t\in(0,1), \\ u^{(k)}(0)=v^{(k)}(0)=0, \quad k=0,1,\ldots,n-2, \quad u(1)=\alpha[v], \quad v(1)=\beta[u], \end{cases}$$

where $f,g\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, $a,b\in C([0,1],[0,\infty))$, $n\geqslant 3$, $\alpha[v]$ and $\beta[u]$ are bounded linear functions involving Stieltjes integrals. By using the fixed point theorem of cone expansion and compression of norm type, they studied the existence of positive solutions for the above nonlinear higher order differential system with coupled integral boundary conditions.

In [23], Pang and Tong concentrated on the following second-order differential equation with integral boundary conditions

$$\begin{cases} u''(t)+f(t,u(t),u'(t))=0, \quad t\in(0,1), \\ u(0)=u(1)=\int_0^1 p(s)u(s)ds, \end{cases} \tag{1.2}$$

where $p\in L^1[0,1]$. By constructing a specific form of the symmetric upper and lower solutions and applying monotone iterative techniques, they obtained successive iterative schemes for approximating solutions of the boundary value problem (1.2).

In [5], Cerdik and Hamal are concerned with second order multi-point integral boundary value problem

$$
\begin{cases}
(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0, \quad t \in (0,1), \\
u(0) = \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)u(r)dr, \quad u(1) = \sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} g(r)u(r)dr,
\end{cases}
\tag{1.3}
$$

where $f \in C([0,1] \times [0,\infty) \times \mathbb{R}, \mathbb{R})$, $0 = \eta_0 < \eta_1 < \eta_2 < \cdots \eta_{m-1} <= 1$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots \xi_{m-1} = 1$, $\eta_i + \overline{\eta}_{m-1-i} = 1$ and $\xi_i + \overline{\xi}_{m-1-i} = 1$, $\alpha_i, \beta_i > 0$, $\alpha_i = \alpha_{m-i}$, $\beta_i = \beta_{m-i}$, for $i = 0,1,\ldots,m-1$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and homomorphism with $\phi(0) = 0$. By applying monotone iterative techniques, they constructed successive iterative schemes for approximating solutions of the boundary value problem (1.3).

In [15], Jiang *et al.* discussed the following singular system subject to multi-point coupled boundary conditions of the type

$$
\begin{cases}
u''(t) + a(t)f(t,u(t),v(t)) = 0, \quad v''(t) + b(t)g(t,u(t),v(t)) = 0, \quad t \in (0,1), \\
u(0) = \sum_{i=1}^{m} \alpha_i v(\eta_i), \quad u(1) = \sum_{i=1}^{m} \alpha_i v(\overline{\eta}_i), \\
v(0) = \sum_{i=1}^{m} \beta_i u(\xi_i), \quad v(1) = \sum_{i=1}^{m} \beta_i u(\overline{\xi}_i),
\end{cases}
\tag{1.4}
$$

where $m \geqslant 1$ is an integer, the parameters $\alpha_i, \beta_i > 0$, $0 < \eta_1 < \eta_2 < \cdots \eta_{m-1} < \eta_m < 1/2$, $0 < \xi_1 < \xi_2 < \cdots \xi_{m-1} < \xi_m < 1/2$, $\eta_i + \overline{\eta}_i = 1$ and $\xi_i + \overline{\xi}_i = 1$, for $i = 0,1,\ldots,m$. Based upon a specially constructed cone and the fixed point index theorem in cones, they studied the existence and multiplicity of symmetric positive solutions for the nonlinear system (1.4).

Motivated by the results mentioned above and the wide applications of coupled boundary value problems, we give sufficient conditions for the existence and multiplicity of symmetric positive solutions to the singular system (1.1). In Sect. 2, we obtain the corresponding Green's function for the singular system (1.1) and and some of its properties and a positive cone, and present a fixed point theorem which will be used to prove the main results. In Sect. 3, we give main results of the paper by applying fixed point theorem in cones. Finally, two examples are provided to illustrate the application of our main results.

2. Preliminaries and lemmas

In the rest of the paper, we make the following assumptions:

(H1) $p,q : [0,1] \to [0,+\infty)$ are continuous, symmetric on $[0,1]$, $0 = \eta_0 < \eta_1 < \eta_2 < \cdots \eta_{m-1} < \eta_m \leqslant 1/2$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots \xi_{m-1} < \xi_m \leqslant 1/2$, $\eta_i + \overline{\eta}_{m-i} = 1$ and $\xi_i + \overline{\xi}_{m-i} = 1$, $\alpha_i, \beta_i > 0$, $\alpha_i = \alpha_{m+1-i}$, $\beta_i = \beta_{m+1-i}$, for $i = 0,1,\ldots,m$,

and

$$0 < \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) < 1.$$

(H2) $a, b : (0,1) \to [0, +\infty)$ are continuous, symmetric on $(0,1)$, and

$$0 < \int_0^1 s(1-s)a(s)ds < \infty, \quad 0 < \int_0^1 s(1-s)b(s)ds < \infty.$$

(H3) $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and $f(\cdot, u, v), g(\cdot, u, v)$ are symmetric on $[0,1]$ for all $u, v \in [0,\infty)$.

We recall that the function ψ is said to be concave on $[0,1]$ if

$$\psi(\tau t_1 + (1-\tau)t_2) \geqslant \tau \psi(t_1) + (1-\tau)\psi(t_2), \quad \tau, t_1, t_2 \in [0,1].$$

and the function ψ is said to be symmetric on $[0,1]$ if $\psi(1-t) = \psi(t)$, $[0,1]$.

REMARK 2.1. It follows from (H1) that

$$\Delta = \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) \right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr \right) \right)$$

$$\geqslant \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) \right)^2 > 0,$$

$$\Delta_1 = \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} rp(r)dr + \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right)$$

$$\times \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} rq(r)dr \right) \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr \right) > 0,$$

$$\Delta_2 = \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right)$$

$$\times \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr \right) \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr \right)$$

$$\geqslant \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right)$$

$$\times \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr \right)$$

$$= \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr \right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) \right) > 0,$$

$$\Delta_3 = \sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} rq(r)dr + \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right)$$

$$\times \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} rp(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr \right) > 0,$$

$$\Delta_4 = \sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr - \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right)$$

$$\times \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr \right)$$

$$\geqslant \sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr - \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right)$$

$$\times \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr \right)$$

$$= \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr \right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) \right) > 0,$$

$$\Delta_5 = \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} rp(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr \right)$$

$$+ \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} rq(r)dr \right) > 0,$$

$$\Delta_6 = 1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} rp(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} rq(r)dr \right)$$

$$- \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr \right)$$

$$\geqslant 1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} rq(r)dr \right)$$

$$-\left(\sum_{i=1}^{m}\alpha_i\int_{\eta_{i-1}}^{\eta_i}p(r)dr\right)\left(\sum_{i=1}^{m}\beta_i\int_{\xi_{i-1}}^{\xi_i}(1-r)q(r)dr\right)$$

$$=1-\left(\sum_{i=1}^{m}\alpha_i\int_{\eta_{i-1}}^{\eta_i}p(r)dr\right)\left(\sum_{i=1}^{m}\beta_i\int_{\xi_{i-1}}^{\xi_i}q(r)dr\right)>0.$$

Set

$$G(t,s)=\begin{cases} s(1-t), & 0\leqslant s\leqslant t\leqslant 1,\\ t(1-s), & 0\leqslant t\leqslant s\leqslant 1, \end{cases}$$

Obviously,

$$t(1-s)s(1-s)\leqslant G(t,s)\leqslant s(1-s)\ \text{ or }\ t(1-t),\quad \forall t,s\in[0,1], \tag{2.1}$$

and

$$G(1-t,1-s)=G(t,s),\quad \forall t,s\in[0,1]. \tag{2.2}$$

LEMMA 2.1. *Assume that* (H1) *holds. Then for any* $x,y\in L(0,1)\cap C(0,1)$, *the differential system*

$$-u''(t)=x(t),\quad -v''(t)=y(t),\quad t\in(0,1), \tag{2.3}$$

with the multi-point coupled integral boundary value conditions

$$\begin{cases} u(0)=\displaystyle\sum_{i=1}^{m}\alpha_i\int_{\eta_{i-1}}^{\eta_i}p(r)v(r)dr, & u(1)=\displaystyle\sum_{i=1}^{m}\alpha_i\int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i}p(r)v(r)dr,\\[3mm] v(0)=\displaystyle\sum_{i=1}^{m}\beta_i\int_{\xi_{i-1}}^{\xi_i}q(r)u(r)dr, & v(1)=\displaystyle\sum_{i=1}^{m}\beta_i\int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i}q(r)u(r)dr, \end{cases} \tag{2.4}$$

has the following integral representation

$$\begin{cases} u(t)=\displaystyle\int_0^1 H_1(t,s)x(s)ds+\int_0^1 K_1(t,s)y(s)ds,\\[3mm] v(t)=\displaystyle\int_0^1 H_2(t,s)y(s)ds+\int_0^1 K_2(t,s)x(s)ds, \end{cases} \tag{2.5}$$

where

$$H_1(t,s)=G(t,s)+\frac{t}{\Delta}\left(\Delta_1\sum_{i=1}^{m}\beta_i\int_{\xi_{i-1}}^{\xi_i}G(r,s)q(r)dr+\Delta_2\sum_{i=1}^{m}\beta_i\int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i}G(r,s)q(r)dr\right)$$

$$+\frac{1-t}{\Delta}\left(\Delta_2\sum_{i=1}^{m}\beta_i\int_{\xi_{i-1}}^{\xi_i}G(r,s)q(r)dr+\Delta_1\sum_{i=1}^{m}\beta_i\int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i}G(r,s)q(r)dr\right),$$

$$H_2(t,s) = G(t,s) + \frac{t}{\Delta}\left(\Delta_3 \sum_{i=1}^{m}\alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr + \Delta_4 \sum_{i=1}^{m}\alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr\right)$$

$$+ \frac{1-t}{\Delta}\left(\Delta_4 \sum_{i=1}^{m}\alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr + \Delta_3 \sum_{i=1}^{m}\alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr\right),$$

$$K_1(t,s) = \frac{t}{\Delta}\left(\Delta_5 \sum_{i=1}^{m}\alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr + \Delta_6 \sum_{i=1}^{m}\alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr\right)$$

$$+ \frac{1-t}{\Delta}\left(\Delta_6 \sum_{i=1}^{m}\alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr + \Delta_5 \sum_{i=1}^{m}\alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr\right),$$

$$K_2(t,s) = \frac{t}{\Delta}\left(\Delta_5 \sum_{i=1}^{m}\beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,s)q(r)dr + \Delta_6 \sum_{i=1}^{m}\beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,s)q(r)dr\right)$$

$$+ \frac{1-t}{\Delta}\left(\Delta_6 \sum_{i=1}^{m}\beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,s)q(r)dr + \Delta_5 \sum_{i=1}^{m}\beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,s)q(r)dr\right).$$

Proof. The differential system (2.3) reduces to the following equivalent integral equations

$$u(t) = \int_0^1 G(t,s)x(s)ds + c_1 t + c_3(1-t), \quad t \in (0,1)$$
$$v(t) = \int_0^1 G(t,s)y(s)ds + c_2 t + c_4(1-t), \quad t \in (0,1),$$

(2.6)

where c_1, c_2, c_3 and c_4 are constants to be determined. Substituting (2.6) into (2.4), we have

$$c_3 = \sum_{i=1}^{m}\alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)\left(\int_0^1 G(r,s)y(s)ds\right)dr$$
$$+ \left(\sum_{i=1}^{m}\alpha_i \int_{\eta_{i-1}}^{\eta_i} rp(r)dr\right)c_2 + \left(\sum_{i=1}^{m}\alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr\right)c_4,$$

$$c_1 = \sum_{i=1}^{m}\alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} p(r)\left(\int_0^1 G(r,s)y(s)ds\right)dr$$
$$+ \left(\sum_{i=1}^{m}\alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} rp(r)dr\right)c_2 + \left(\sum_{i=1}^{m}\alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} (1-r)p(r)dr\right)c_4,$$

$$c_4 = \sum_{i=1}^{m}\beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)\left(\int_0^1 G(r,s)x(s)ds\right)dr$$
$$+ \left(\sum_{i=1}^{m}\beta_i \int_{\xi_{i-1}}^{\xi_i} rq(r)dr\right)c_1 + \left(\sum_{i=1}^{m}\beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr\right)c_3,$$

(2.7)

$$c_2 = \sum_{i=1}^{m} \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} q(r) \left(\int_0^1 G(r,s)x(s)ds \right) dr$$
$$+ \left(\sum_{i=1}^{m} \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} rq(r)dr \right) c_1 + \left(\sum_{i=1}^{m} \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} (1-r)q(r)dr \right) c_3.$$

(2.8)

Note that $\eta_i + \overline{\eta}_{m-i} = 1$, $\alpha_i > 0$, $\alpha_i = \alpha_{m+1-i}$ $(i = 1, 2, \ldots, m)$, and let $r = 1 - \widehat{r}$, we obtain

$$\sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} rp(r)dr = \sum_{i=1}^{m} \alpha_i \int_{1-\overline{\eta}_i}^{1-\overline{\eta}_{i-1}} (1-\widehat{r})p(1-\widehat{r})d\widehat{r} = \sum_{i=1}^{m} \alpha_i \int_{\eta_{m-i}}^{\eta_{m+1-i}} (1-r)p(r)dr$$

$$= \alpha_1 \int_{\eta_{m-1}}^{\eta_m} (1-r)p(r)dr + \cdots + \alpha_m \int_{\eta_0}^{\eta_1} (1-r)p(r)dr$$

$$= \alpha_m \int_{\eta_{m-1}}^{\eta_m} (1-r)p(r)dr + \cdots + \alpha_1 \int_{\eta_0}^{\eta_1} (1-r)p(r)dr$$

$$= \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-r)p(r)dr := d_2$$

(2.9)

$$\sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} (1-r)rp(r)dr = \sum_{i=1}^{m} \alpha_i \int_{1-\overline{\eta}_i}^{1-\overline{\eta}_{i-1}} \widehat{r}p(1-\widehat{r})d\widehat{r} = \sum_{i=1}^{m} \alpha_i \int_{\eta_{m-i}}^{\eta_{m+1-i}} rp(r)dr$$

$$= \alpha_1 \int_{\eta_{m-1}}^{\eta_m} rp(r)dr + \cdots + \alpha_m \int_{\eta_0}^{\eta_1} rp(r)dr$$

$$= \alpha_m \int_{\eta_{m-1}}^{\eta_m} rp(r)dr + \cdots + \alpha_1 \int_{\eta_0}^{\eta_1} rp(r)dr = \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} rp(r)dr := d_1,$$

Similarly, $\xi_i + \overline{\xi}_{m-i} = 1$, $\beta_i > 0$ and $\beta_i = \beta_{m+1-i}$ $(i = 1, 2, \ldots, m)$, we have

$$\sum_{i=1}^{m} \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} rq(r)dr = \sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-r)q(r)dr := d_4$$
$$\sum_{i=1}^{m} \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} (1-r)q(r)dr = \sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} rq(r)dr := d_3.$$

(2.10)

From (2.9) and (2.10), then equations (2.7) and (2.8) can be rewrite as follows.

$$c_1 - d_2c_2 - d_1c_4 = \int_0^1 \left(\sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr \right) y(s)ds,$$

$$-d_1c_2 + c_3 - d_2c_4 = \int_0^1 \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr \right) y(s)ds,$$

(2.11)

$$-d_3c_1 - d_4c_3 + c_4 = \int_0^1 \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,s)q(r)dr \right) x(s)ds,$$

$$-d_4c_1 + c_2 - d_3c_3 = \int_0^1 \left(\sum_{i=1}^{m} \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,s)q(r)dr \right) x(s)ds.$$

Note that

$$
\begin{vmatrix}
1 & -d_2 & 0 & -d_1 \\
0 & -d_1 & 1 & -d_2 \\
-d_3 & 0 & -d_4 & 1 \\
-d_4 & 1 & -d_3 & 0
\end{vmatrix} = (1-(d_1+d_2)(d_3+d_4))(1-(d_2-d_1)(d_4-d_3)) = \Delta \neq 0.
$$

Thus, the system (2.11) has a unique solution for c_i $(i=1,2,3,4)$. By Cramer's rule and simple calculations, it follows that

$$
\begin{aligned}
c_1 = & \frac{\Delta_5}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr \right) y(s)ds \\
& + \frac{\Delta_6}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr \right) y(s)ds \\
& + \frac{\Delta_1}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,s)q(r)dr \right) x(s)ds \\
& + \frac{\Delta_2}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,s)q(r)dr \right) x(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
c_2 = & \frac{\Delta_3}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr \right) y(s)ds \\
& + \frac{\Delta_4}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr \right) y(s)ds \\
& + \frac{\Delta_5}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,s)q(r)dr \right) x(s)ds \\
& + \frac{\Delta_6}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,s)q(r)dr \right) x(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
c_3 = & \frac{\Delta_6}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr \right) y(s)ds \\
& + \frac{\Delta_5}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr \right) y(s)ds \\
& + \frac{\Delta_2}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,s)q(r)dr \right) x(s)ds \\
& + \frac{\Delta_1}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,s)q(r)dr \right) x(s)ds,
\end{aligned}
$$

$$c_4 = \frac{\Delta_4}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} G(r,s)p(r)dr \right) y(s)ds$$

$$+ \frac{\Delta_3}{\Delta} \int_0^1 \left(\sum_{i=1}^m \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} G(r,s)p(r)dr \right) y(s)ds$$

$$+ \frac{\Delta_6}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,s)q(r)dr \right) x(s)ds$$

$$+ \frac{\Delta_5}{\Delta} \int_0^1 \left(\sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,s)q(r)dr \right) x(s)ds,$$

Then from (2.6), it is obvious that (2.5) holds. This completes the proof of the lemma.

LEMMA 2.2. Assume that (H1) holds. Then the functions $H_j(t,s)$ and $K_j(t,s)$ $(j = 1,2)$ are continuous on $[0,1] \times [0,1]$ and

$$H_j(t,s) > 0, \quad K_j(t,s) > 0, \quad for \ t,s \in (0,1);$$
$$H_j(t,s) \geqslant 0, \quad K_j(t,s) \geqslant 0, \quad for \ t,s \in [0,1], \ j = 1,2.$$

Proof. It follows from 2.1 and Remark 2.1 that the results of Lemma 2.2 are true.

LEMMA 2.3. Assume that (H1) holds. For $t,s \in [0,1]$, we have

$$H_j(1-t,1-s) = H_j(t,s), \quad K_j(1-t,1-s) = K_j(t,s), \quad j = 1,2.$$

Proof. Note that $\xi_i + \overline{\xi}_{m-i} = 1$, $\beta_i > 0$ and $\beta_i = \beta_{m+1-i}$ $(i = 1,2,\ldots,m)$, and let $r = 1 - \widehat{r}$, by (2.2), we get

$$H_1(1-t,1-s) = G(1-t,1-s)$$

$$+ \frac{1-t}{\Delta} \left(\Delta_1 \sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,1-s)q(r)dr + \Delta_2 \sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,1-s)q(r)dr \right)$$

$$+ \frac{t}{\Delta} \left(\Delta_2 \sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} G(r,1-s)q(r)dr + \Delta_1 \sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} G(r,1-s)q(r)dr \right)$$

$$= G(t,s) + \frac{1-t}{\Delta} \left(\Delta_1 \sum_{i=1}^m \beta_i \int_{1-\xi_i}^{1-\xi_{i-1}} G(1-\widehat{r},1-s)q(1-\widehat{r})d\widehat{r} \right.$$

$$\left. + \Delta_2 \sum_{i=1}^m \beta_i \int_{1-\overline{\xi}_i}^{1-\overline{\xi}_{i-1}} G(1-\widehat{r},1-s)q(1-\widehat{r})d\widehat{r} \right)$$

$$+ \frac{t}{\Delta} \left(\Delta_2 \sum_{i=1}^m \beta_i \int_{1-\xi_i}^{1-\xi_{i-1}} G(1-\widehat{r},1-s)q(1-\widehat{r})d\widehat{r} \right.$$

$$\left. + \Delta_1 \sum_{i=1}^m \beta_i \int_{1-\overline{\xi}_i}^{1-\overline{\xi}_{i-1}} G(1-\widehat{r},1-s)q(1-\widehat{r})d\widehat{r} \right)$$

$$=G(t,s)+\frac{1-t}{\Delta}\left(\Delta_1\sum_{i=1}^{m}\beta_i\int_{\underline{\xi}_{m-i}}^{\overline{\xi}_{m+1-i}}G(r,s)q(r)dr+\Delta_2\sum_{i=1}^{m}\beta_i\int_{\xi_{m-i}}^{\xi_{m+1-i}}G(r,s)q(r)dr\right)$$

$$+\frac{t}{\Delta}\left(\Delta_2\sum_{i=1}^{m}\beta_i\int_{\underline{\xi}_{m-i}}^{\overline{\xi}_{m+1-i}}G(r,s)q(r)dr+\Delta_1\sum_{i=1}^{m}\beta_i\int_{\xi_{m-i}}^{\xi_{m+1-i}}G(r,s)q(r)dr\right)$$

$$=G(t,s)+\frac{1-t}{\Delta}\left(\Delta_1\sum_{i=1}^{m}\beta_i\int_{\underline{\xi}_{i-1}}^{\overline{\xi}_{i}}G(r,s)q(r)dr+\Delta_2\sum_{i=1}^{m}\beta_i\int_{\xi_{i-1}}^{\xi_{i}}G(r,s)q(r)dr\right)$$

$$+\frac{t}{\Delta}\left(\Delta_2\sum_{i=1}^{m}\beta_i\int_{\underline{\xi}_{i-1}}^{\overline{\xi}_{i}}G(r,s)q(r)dr+\Delta_1\sum_{i=1}^{m}\beta_i\int_{\xi_{i-1}}^{\xi_{i}}G(r,s)q(r)dr\right)$$

$$=H_1(t,s),\quad t,s\in[0,1].$$

Similarly, combining with $\eta_i+\overline{\eta}_{m-i}=1$, $\alpha_i>0$, $\alpha_i=\alpha_{m+1-i}$ $(i=1,2,\ldots,m)$, we can prove that $H_2(1-t,1-s)=H_1(t,s)$ for $t,s\in[0,1]$.

Next we show that $K_1(1-t,1-s)=K_1(t,s)$ holds for $t,s\in[0,1]$. In fact, noticing that $\eta_i+\overline{\eta}_{m-i}=1$, $\alpha_i>0$, $\alpha_i=\alpha_{m+1-i}$ $(i=1,2,\ldots,m)$, and letting $r=1-\widehat{r}$ by (2.2), we obtain

$$K_1(1-t,1-s)$$

$$=\frac{1-t}{\Delta}\left(\Delta_5\sum_{i=1}^{m}\alpha_i\int_{\eta_{i-1}}^{\eta_i}G(r,1-s)p(r)dr+\Delta_6\sum_{i=1}^{m}\alpha_i\int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i}G(r,1-s)p(r)dr\right)$$

$$+\frac{t}{\Delta}\left(\Delta_6\sum_{i=1}^{m}\alpha_i\int_{\eta_{i-1}}^{\eta_i}G(r,1-s)p(r)dr+\Delta_5\sum_{i=1}^{m}\alpha_i\int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i}G(r,1-s)p(r)dr\right)$$

$$=\frac{1-t}{\Delta}\left(\Delta_5\sum_{i=1}^{m}\alpha_i\int_{1-\eta_i}^{1-\eta_{i-1}}G(1-\widehat{r},1-s)p(1-\widehat{r})d\widehat{r}\right.$$

$$\left.+\Delta_6\sum_{i=1}^{m}\alpha_i\int_{1-\overline{\eta}_i}^{1-\overline{\eta}_{i-1}}G(1-\widehat{r},1-s)p(1-\widehat{r})d\widehat{r}\right)$$

$$+\frac{t}{\Delta}\left(\Delta_6\sum_{i=1}^{m}\alpha_i\int_{1-\eta_i}^{1-\eta_{i-1}}G(1-\widehat{r},1-s)p(1-\widehat{r})d\widehat{r}\right.$$

$$\left.+\Delta_5\sum_{i=1}^{m}\alpha_i\int_{1-\overline{\eta}_i}^{1-\overline{\eta}_{i-1}}G(1-\widehat{r},1-s)p(1-\widehat{r})d\widehat{r}\right)$$

$$=\frac{1-t}{\Delta}\left(\Delta_5\sum_{i=1}^{m}\alpha_i\int_{\overline{\eta}_{m-i}}^{\overline{\eta}_{m+1-i}}G(r,s)p(r)dr+\Delta_6\sum_{i=1}^{m}\alpha_i\int_{\eta_{m-i}}^{\eta_{m+1-i}}G(r,s)p(r)dr\right)$$

$$+\frac{t}{\Delta}\left(\Delta_6\sum_{i=1}^{m}\alpha_i\int_{\overline{\eta}_{m-i}}^{\overline{\eta}_{m+1-i}}G(r,s)p(r)dr+\Delta_5\sum_{i=1}^{m}\alpha_i\int_{\eta_{m-i}}^{\eta_{m+1-i}}G(r,s)p(r)dr\right)$$

$$=\frac{1-t}{\Delta}\left(\Delta_5\sum_{i=1}^{m}\alpha_i\int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i}G(r,s)p(r)dr+\Delta_6\sum_{i=1}^{m}\alpha_i\int_{\eta_{i-1}}^{\eta_i}G(r,s)p(r)dr\right)$$

$$+\frac{t}{\Delta}\left(\Delta_6\sum_{i=1}^m\alpha_i\int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i}G(r,s)p(r)dr+\Delta_5\sum_{i=1}^m\alpha_i\int_{\eta_{i-1}}^{\eta_i}G(r,s)p(r)dr\right)$$
$$=K_1(t,s),\quad t,s\in[0,1].$$

Similarly, combining with $\xi_i+\overline{\xi}_{m-i}=1$, $\beta_i>0$, $\beta_i=\beta_{m+1-i}$ $(i=1,2,\dots,m)$, we can prove that $K_2(1-t,1-s)=K_1(t,s)$ for $t,s\in[0,1]$. The proof is complete.

LEMMA 2.4. Assume that (H1) holds. For $t,s\in[0,1]$, we have

$$\rho\left(\sum_{i=1}^m\alpha_i\int_{\eta_{i-1}}^{\eta_i}p(r)dr\right)\left(\sum_{i=1}^m\beta_i\int_{\xi_{i-1}}^{\xi_i}r(1-r)q(r)dr\right)s(1-s)$$
$$\leqslant H_1(t,s)\leqslant\rho s(1-s),\quad(2.12)$$

and

$$\rho\left(\sum_{i=1}^m\alpha_i\int_{\eta_{i-1}}^{\eta_i}r(1-r)p(r)dr\right)s(1-s)$$
$$\leqslant K_1(t,s)\leqslant\rho\left(\sum_{i=1}^m\alpha_i\int_{\eta_{i-1}}^{\eta_i}p(r)dr\right)s(1-s),\quad(2.13)$$

where

$$\rho=\left(1-\left(\sum_{i=1}^m\alpha_i\int_{\eta_{i-1}}^{\eta_i}p(r)dr\right)\left(\sum_{i=1}^m\beta_i\int_{\xi_{i-1}}^{\xi_i}q(r)dr\right)\right)^{-1}.$$

Proof. First, we will show the following fact: let $0=\eta_0<\eta_1<\eta_2<\cdots\eta_{m-1}<\eta_m\leqslant\frac{1}{2}$, $0=\xi_0<\xi_1<\xi_2<\cdots\xi_{m-1}<\xi_m\leqslant\frac{1}{2}$, $\eta_i+\overline{\eta}_{m-i}=1$ and $\xi_i+\overline{\xi}_{m-i}=1$, $\alpha_i,\beta_i>0$, $\alpha_i=\alpha_{m+1-i}$, $\beta_i=\beta_{m+1-i}$, for $i=0,1,\dots,m$, and let ψ be symmetric on $[0,1]$, then

$$\sum_{i=1}^m\beta_i\int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i}\psi(r)dr=\sum_{i=1}^m\beta_i\int_{\xi_{i-1}}^{\xi_i}\psi(r)dr,$$
$$\sum_{i=1}^m\alpha_i\int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i}\psi(r)dr=\sum_{i=1}^m\alpha_i\int_{\eta_{i-1}}^{\eta_i}\psi(r)dr.$$
(2.14)

In fact, let $r=1-\widehat{r}$, then

$$\sum_{i=1}^m\beta_i\int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i}\psi(r)dr=\sum_{i=1}^m\beta_i\int_{1-\overline{\xi}_i}^{1-\overline{\xi}_{i-1}}\psi(1-\widehat{r})d\widehat{r}=\sum_{i=1}^m\beta_i\int_{\xi_{m-i}}^{\xi_{m+1-i}}\psi(r)dr$$
$$=\beta_1\int_{\xi_{m-1}}^{\xi_m}\psi(r)dr+\cdots+\beta_m\int_{\xi_0}^{\xi_1}\psi(r)dr$$
$$=\beta_m\int_{\xi_{m-1}}^{\xi_m}\psi(r)dr+\cdots+\beta_1\int_{\xi_0}^{\xi_1}\psi(r)dr=\sum_{i=1}^m\beta_i\int_{\xi_{i-1}}^{\xi_i}\psi(r)dr,$$

$$\sum_{i=1}^m \alpha_i \int_{\xi_{i-1}}^{\overline{\xi}_i} \psi(r)dr = \sum_{i=1}^m \alpha_i \int_{1-\overline{\eta}_i}^{1-\overline{\eta}_{i-1}} \psi(1-\widehat{r})d\widehat{r} = \sum_{i=1}^m \alpha_i \int_{\eta_{m-i}}^{\eta_{m+1-i}} \psi(r)dr$$

$$=\alpha_1 \int_{\eta_{m-1}}^{\eta_m} \psi(r)dr + \cdots + \alpha_m \int_{\eta_0}^{\eta_1} \psi(r)dr$$

$$=\alpha_m \int_{\eta_{m-1}}^{\eta_m} \psi(r)dr + \cdots + \alpha_1 \int_{\eta_0}^{\eta_1} \psi(r)dr = \sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} \psi(r)dr.$$

Next, we will show that (2.12) is true. By (2.1) and (2.14), we obtain

$$H_1(t,s) \leqslant s(1-s)$$

$$+\frac{t}{\Delta}\left(\left(\Delta_1 \sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr\right)s(1-s) + \left(\Delta_2 \sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} q(r)dr\right)s(1-s)\right)$$

$$+\frac{1-t}{\Delta}\left(\left(\Delta_2 \sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr\right)s(1-s) + \left(\Delta_1 \sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} q(r)dr\right)s(1-s)\right)$$

$$=s(1-s)+\frac{t}{\Delta}\left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right)\left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr\right)$$

$$\times \left(1-\left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr\right)\left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr\right)\right)s(1-s)$$

$$+\frac{1-t}{\Delta}\left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right)\left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr\right)$$

$$\times \left(1-\left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr\right)\left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr\right)\right)s(1-s)$$

$$=\rho s(1-s), \quad t,s \in [0,1].$$

On the other hand, by (2.1) and (2.14), we also have

$$H_1(t,s) \geqslant \frac{t}{\Delta}\left(\left(\Delta_1 \sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr\right)s(1-s)\right.$$

$$+\left(\Delta_2 \sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} r(1-r)q(r)dr\right)s(1-s)\right)$$

$$+\frac{1-t}{\Delta}\left(\left(\Delta_2 \sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr\right)s(1-s)\right.$$

$$+\left(\Delta_1 \sum_{i=1}^m \beta_i \int_{\overline{\xi}_{i-1}}^{\overline{\xi}_i} r(1-r)q(r)dr\right)s(1-s)\right)$$

$$=\frac{t}{\Delta}\left(\sum_{i=1}^m \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right)\left(\sum_{i=1}^m \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr\right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr\right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr\right)\right) s(1-s)$$

$$+ \frac{1-t}{\Delta} \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr\right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr\right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr\right)\right) s(1-s)$$

$$= \rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr\right) s(1-s), \quad t,s \in [0,1].$$

Finally, we show that (2.13) holds for $t,s \in [0,1]$. In fact, using (2.1) and (2.14), we get

$$K_1(t,s) \leqslant \frac{t}{\Delta} \left(\left(\Delta_5 \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right) s(1-s) + \left(\Delta_6 \sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} p(r)dr\right) s(1-s)\right)$$

$$+ \frac{1-t}{\Delta} \left(\left(\Delta_6 \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right) s(1-s) + \left(\Delta_5 \sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} p(r)dr\right) s(1-s)\right)$$

$$= \frac{t}{\Delta} \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr\right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr\right)\right) s(1-s)$$

$$+ \frac{1-t}{\Delta} \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr\right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr\right)\right) s(1-s)$$

$$= \rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr\right) s(1-s), \quad t,s \in [0,1].$$

On the other hand, by (2.1) and (2.14), we also have

$$K_1(t,s) \geqslant \frac{t}{\Delta} \left(\left(\Delta_5 \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr\right) s(1-s)\right.$$

$$+ \left(\Delta_6 \sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} r(1-r)p(r)dr\right) s(1-s)\right)$$

$$+ \frac{1-t}{\Delta} \left(\left(\Delta_6 \sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr\right) s(1-s)\right.$$

$$+ \left(\Delta_5 \sum_{i=1}^{m} \alpha_i \int_{\overline{\eta}_{i-1}}^{\overline{\eta}_i} r(1-r)p(r)dr \right) s(1-s) \right)$$

$$= \frac{t}{\Delta} \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr \right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr \right) \right) s(1-s)$$

$$+ \frac{1-t}{\Delta} \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr \right)$$

$$\times \left(1 - \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} (1-2r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} (1-2r)q(r)dr \right) \right) s(1-s)$$

$$= \rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr \right) s(1-s), \quad t,s \in [0,1].$$

The proof is complete.

Similarly, we have

LEMMA 2.5. *Assume that* (H1) *holds. For* $t,s \in [0,1]$ *and* ρ *is defined as Lemma* 2.4, *we have*

$$\rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) s(1-s) \leqslant H_2(t,s) \leqslant \rho s(1-s),$$

$$\rho \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr \right) s(1-s) \leqslant K_2(t,s) \leqslant \rho \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) s(1-s).$$

REMARK 2.2. *From Lemmas* 2.4 *and* 2.5, *for* $t,s \in [0,1]$ *and* ρ *is defined as* Lemma 2.4, *we have*

$$vs(1-s) \leqslant H_j(t,s) \leqslant \mu s(1-s), \quad vs(1-s) \leqslant K_j(t,s) \leqslant \mu s(1-s), \quad j=1,2,$$

where

$$v = \min \left\{ \rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right) \rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr \right), \right.$$

$$\rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} r(1-r)p(r)dr \right) \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr \right),$$

$$\left. \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right), \rho \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} r(1-r)q(r)dr \right) \right\},$$

and

$$\mu = \max \left\{ \rho, \rho \left(\sum_{i=1}^{m} \alpha_i \int_{\eta_{i-1}}^{\eta_i} p(r)dr \right), \rho \left(\sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)dr \right) \right\}.$$

Employing Lemma 2.1, the singular system (1.1) can be expressed as

$$\begin{cases} u(t) = \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds \\ \qquad + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds, \quad t \in [0,1], \\ v(t) = \int_0^1 H_2(t,s)b(s)g(s,u(s),v(s))ds \\ \qquad + \int_0^1 K_2(t,s)a(s)f(s,u(s),v(s))ds, \quad t \in [0,1]. \end{cases} \tag{2.15}$$

By a solution of the singular system (1.1), we mean a solution of the corresponding system of integral equations (2.15). Define an operator $T : K \to E$ by

$$T(u,v) = (T_1(u,v), T_2(u,v)),$$

where operators $T_1, T_2 : K \to P$ are defined by

$$\begin{cases} T_1(u,v)(t) = \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds \\ \qquad + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds, \quad t \in [0,1], \\ T_2(u,v)(t) = \int_0^1 H_2(t,s)b(s)g(s,u(s),v(s))ds \\ \qquad + \int_0^1 K_2(t,s)a(s)f(s,u(s),v(s))ds, \quad t \in [0,1]. \end{cases} \tag{2.16}$$

Clearly, if (u,v) is a fixed point of T, then (u,v) is a solution of the singular system (1.1).

The basic space used in this paper is $E = C[0,1] \times C[0,1]$. Obviously, the space E is a Banach space if it is endowed with the norm as follows:

$$\|(u,v)\| = \|u\| + \|v\|, \quad \|u\| = \max_{0 \leqslant t \leqslant 1} |u(t)|, \quad \|v\| = \max_{0 \leqslant t \leqslant 1} |v(t)|$$

for any $(u,v) \in E$. Denote $P = \{u \in C[0,1] : u(t) \geqslant 0, t \in [0,1]\}$. Let

$$K = \Big\{ (u,v) \in P : x,y \text{ are symmetric and concave on } [0,1],$$

$$\min_{t \in [0,1]} u(t) \geqslant \gamma \|u\|, \ \min_{t \in [0,1]} v(t) \geqslant \gamma \|v\| \Big\},$$

where $\gamma = v/\mu$, μ, v are defined as Remark 2.2. Clearly $0 < \gamma < 1$. It is easy to see that K is a cone of E. For any real constant $r > 0$, define $K_r = \{(u,v) \in K : \|(u,v)\| < r\}$ and $\partial K_r = \{(u,v) \in K : \|(u,v)\| = r\}$.

LEMMA 2.6. Assume that (H1)-(H3) hold. Then $T : K \to E$ is well defined. Furthermore, $T(K) \subset K$ and $T : K \to K$ is a completely continuous operator.

Proof. For any fixed $(u,v) \in K$, there exists a constant $r > 0$ such that $\|(u,v)\| < r$. Thus, for any $t \in [0,1]$, it follows from (2.16) and Remark 2.2 that

$$|T_1(u,v)(t)| = \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds$$
$$\leqslant \mu \int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds + \mu \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds$$
$$\leqslant \mu M \left(\int_0^1 s(1-s)a(s)ds + \mu \int_0^1 s(1-s)b(s)ds \right) < \infty,$$

where

$$M = \max_{(t,u,v) \in [0,1] \times [0,r] \times [0,r]} f(t,u,v) + \max_{(t,u,v) \in [0,1] \times [0,r] \times [0,r]} g(t,u,v).$$

Similarly, we have

$$|T_2(u,v)(t)| \leqslant \mu M \left(\int_0^1 s(1-s)a(s)ds + \mu \int_0^1 s(1-s)b(s)ds \right) < \infty.$$

Thus $T : K \to E$ is well defined.

For all $(u,v) \in K$, by (2.16), we have $T_1(u,v)''(t) = -a(t)f(t,u(t),v(t)) \leqslant 0$, which implies that $T_1(u,v)(t)$ is concave on $[0,1]$. On the other hand, by (2.16) and Lemma 2.2 we obtain $T_1(u,v)(0) \geqslant 0$, $T_1(u,v)(1) \geqslant 0$. It follows that $T_1(u,v)(t) \geqslant 0$ for $t \in [0,1]$. Noticing that $a(t), b(t)$ are symmetric on $[0,1]$, u,v are symmetric on $[0,1]$, $f(\cdot,u,v)$ and $g(\cdot,u,v)$ are symmetric on $[0,1]$, we have

$$T_1(u,v)(1-t) = \int_0^1 H_1(1-t,s)a(s)f(s,u(s),v(s))ds$$
$$+ \int_0^1 K_1(1-t,s)b(s)g(s,u(s),v(s))ds$$
$$= \int_1^0 H_1(1-t,1-s)a(1-s)f(1-s,u(1-s),v(1-s))d(1-s)$$
$$+ \int_1^0 K_1(1-t,1-s)b(1-s)g(1-s,u(1-s),v(1-s))d(1-s)$$
$$= \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds$$
$$= T_1(u,v)(t),$$

i.e., $T_1(u,v)(1-t) = T_1(u,v)(t)$ for $t \in [0,1]$. Therefore, $T_1(u,v)(t)$ is symmetric on $[0,1]$.

On the other hand, for $(u,v) \in K$, $t \in [0,1]$, using (2.16) and Remark 2.2, we obtain

$$|T_1(u,v)(t)| = \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds$$

$$\leqslant \mu \int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds + \mu \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds,$$

which implies that

$$\|T_1(u,v)\| \leqslant \mu \left(\int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds \right.$$

$$\left. + \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds \right), \quad t \in [0,1]. \quad (2.17)$$

Also, for $(u,v) \in K$, $t \in [0,1]$, using (2.16) and (2.17) and Remark 2.2, we have

$$T_1(u,v)(t) = \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds$$

$$\geqslant v \int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds + v \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds$$

$$= \gamma\mu \left(\int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds + \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds \right)$$

$$\geqslant \gamma\|T_1(u,v)\|,$$

which implies that $\min_{t\in[0,1]} T_1(u,v)(t) \geqslant \gamma\|T_1(u,v)\|$ for $(u,v) \in K$. Similarly, we also can show that $T_2(u,v)$ is concave on $[0,1]$, $T_2(u,v)(t) \geqslant 0$ for $t \in [0,1]$, $T_2(u,v)(t)$ is symmetric on $[0,1]$ and $\min_{t\in[0,1]} T_2(u,v)(t) \geqslant \gamma\|T_2(u,v)\|$ for $(u,v) \in K$.. Hence, $T(K) \subset K$. Then operator $T : K \to K$ is continuous since $H_j(t,s)$, $K_j(t,s)$ $(j=1,2)$, $f(t,u,v)$, $g(t,u,v)$, $a(t)$, $b(t)$ are continuous. According to standard applications of Arzelà-Ascoli theorem, it is easy to prove that operator $T : K \to K$ is completely continuous. The proof is complete.

LEMMA 2.7. ([11]) Suppose E is a real Banach space and P is cone in E, and let Ω_1, Ω_2 be bounded open sets in E such that $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let operator $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the following two conditions holds:

(i) $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_2$,

(ii) $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_2$.

Then the operator T at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, we apply Lemma 2.7 to establish the existence of positive solutions for the singular system (1.1). For convenience, we first introduce the following

notations:

$$f^\delta = \limsup_{(u,v)\to\delta} \max_{t\in[0,1]} \frac{f(t,u,v)}{u+v}, \quad f_\delta = \liminf_{(u,v)\to\delta} \min_{t\in[0,1]} \frac{f(t,u,v)}{u+v},$$

$$g^\delta = \limsup_{(u,v)\to\delta} \max_{t\in[0,1]} \frac{g(t,u,v)}{u+v}, \quad g_\delta = \liminf_{(u,v)\to\delta} \min_{t\in[0,1]} \frac{g(t,u,v)}{u+v},$$

where δ denotes 0 or ∞, $(u,v)\to 0 \Leftrightarrow u+v\to 0$, $(u,v)\to\infty \Leftrightarrow u+v\to\infty$, and

$$L_1^{-1} = \max\left\{ 4\mu \int_0^1 s(1-s)a(s)ds, 4\mu \int_0^1 s(1-s)b(s)ds \right\},$$

$$L_2^{-1} = \min\left\{ 2\gamma\nu \int_0^1 s(1-s)a(s)ds, 2\gamma\nu \int_0^1 s(1-s)b(s)ds \right\}.$$

THEOREM 3.1. Assume that (H1)-(H3) hold. In addition, suppose that

(H4) $0 \leqslant f^0 < L_1$, $0 \leqslant g^0 < L_1$, and $L_2 < f_\infty \leqslant \infty$ or $L_2 < g_\infty \leqslant \infty$.

Then the singular system (1.1) has at least one symmetric positive solution (u^*, v^*), i.e. $u^*(t) > 0$, $v^*(t) > 0$ for $t \in [0,1]$.

Proof. At first, it follows from the assumptions $0 \leqslant f^0 < L_1$, $0 \leqslant g^0 < L_1$, there exist $r > 0$ and $\varepsilon_1 > 0$ such that $L_1 - \varepsilon_1 > 0$ and for any $t \in [0,1]$, we have

$$\begin{aligned} f(t,u,v) &\leqslant (L_1 - \varepsilon_1)(u+v) \leqslant (L_1 - \varepsilon_1)r, \\ g(t,u,v) &\leqslant (L_1 - \varepsilon_1)(u+v) \leqslant (L_1 - \varepsilon_1)r, \quad 0 < u+v \leqslant r. \end{aligned} \tag{3.1}$$

Then for any $(u,v) \in \partial K_r$ and $t \in [0,1]$, by (3.1) and Remark 2.2, we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds \\ &\leqslant \mu \int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds + \mu \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds \\ &\leqslant \mu(L_1 - \varepsilon_1)r \left(\int_0^1 s(1-s)a(s)ds + \int_0^1 s(1-s)b(s)ds \right) < \frac{r}{2}, \end{aligned}$$

$$\begin{aligned} T_2(u,v)(t) &= \int_0^1 H_2(t,s)b(s)g(s,u(s),v(s))ds + \int_0^1 K_2(t,s)a(s)f(s,u(s),v(s))ds \\ &\leqslant \mu \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds + \mu \int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds \\ &\leqslant \mu(L_1 - \varepsilon_1)r \left(\int_0^1 s(1-s)b(s)ds + \int_0^1 s(1-s)a(s)ds \right) < \frac{r}{2}, \end{aligned}$$

Therefore, we have

$$\|T(u,v)\| = \|T_1(u,v)\| + \|T_2(u,v)\| < r = \|(u,v)\|, \quad \forall (u,v) \in \partial K_r. \tag{3.2}$$

On the other hand, suppose $L_2 < f_\infty \leqslant \infty$, then there exist $N > 0$ and $\varepsilon_2 > 0$ such that

$$f(t,u,v) > (L_2 + \varepsilon_2)(u+v), \quad \forall t \in [0,1], \quad u+v \geqslant N. \tag{3.3}$$

Let $R = \max\{2r, \gamma^{-1}N\}$. Then for any $(u,v) \in \partial K_R$, $u(t) + v(t) \geqslant \gamma\|u\| + \gamma\|v\| = \gamma R \geqslant N$, $t \in [0,1]$. So, for any $(u,v) \in \partial K_R$, by (3.3) and Remark 2.2, we obtain

$$
\begin{aligned}
\|T_1(u,v)\| &\geqslant T_1(u,v)(t) \\
&= \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds + \int_0^1 K_1(t,s)b(s)g(s,u(s),v(s))ds \\
&\geqslant \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds \\
&\geqslant v\int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds \\
&> v(L_2 + \varepsilon_2)\int_0^1 s(1-s)a(s)(u(s)+v(s))ds \\
&\geqslant v(L_2 + \varepsilon_2)\gamma(\|u\| + \|v\|)\int_0^1 s(1-s)a(s)ds \geqslant \frac{\|u\| + \|v\|}{2}.
\end{aligned}
\tag{3.4}
$$

Similarly, for all $(u,v) \in \partial K_R$, using (3.3) and Remark 2.2, we also get

$$
\begin{aligned}
\|T_2(u,v)\| &\geqslant T_2(u,v)(t) \\
&= \int_0^1 H_2(t,s)b(s)g(s,u(s),v(s))ds + \int_0^1 K_2(t,s)a(s)f(s,u(s),v(s))ds \\
&\geqslant \int_0^1 K_2(t,s)a(s)f(s,u(s),v(s))ds \\
&\geqslant v\int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds \\
&> v(L_2 + \varepsilon_2)\int_0^1 s(1-s)a(s)(u(s)+v(s))ds \\
&\geqslant v(L_2 + \varepsilon_2)\gamma(\|u\| + \|v\|)\int_0^1 s(1-s)a(s)ds \geqslant \frac{\|u\| + \|v\|}{2}.
\end{aligned}
\tag{3.5}
$$

From (3.4) and (3.5), it follows that

$$\|T(u,v)\| = \|T_1(u,v)\| + \|T_2(u,v)\| > \|(u,v)\|, \quad \forall(u,v) \in \partial K_R. \tag{3.6}$$

If $L_2 < g_\infty \leqslant \infty$, we can get a similar result. By applying Lemmas 2.6 and 2.7 to (3.2) and (3.6), it follows that the operator T has at least one fixed point $(u^*, v^*) \in K \cap (K_R \setminus \overline{K}_r)$ with $r \leqslant \|(u^*, v^*)\| \leqslant R$.

Finally, (u^*, v^*) is positive. In fact, from $\|(u^*, v^*)\| \geqslant r > 0$, we know that $\|u\| \geqslant r/2$ and/or $\|v\| \geqslant r/2$. Without loss of generality, we may suppose $\|u\| \geqslant r/2$, by concavity of u^* and by construction of the cone K, we have $\min_{t \in [0,1]} u^*(t) \geqslant \gamma\|u\| \geqslant \gamma r/2 > 0$, which implies that $u^*(t) > 0$ for all $t \in [0,1]$. From the boundary conditions of the singular system (1.1), keeping in mind that $q : [0,1] \to [0,+\infty)$

is continuous, symmetric on $[0,1]$, $\beta_i > 0$, $\beta_i = \beta_{m+1-i}$, for $i = 0,1,\ldots,m$, we have $v^*(0) = \sum_{i=1}^{m} \beta_i \int_{\xi_{i-1}}^{\xi_i} q(r)u^*(r)dr > 0$, this yields that $\|v\| \geqslant v^*(0) > 0$. Similarly, from $\|v^*\| > 0$, we also have $v^*(t) > 0$ for all $t \in [0,1]$. Hence, (u^*,v^*) is the desired symmetric positive solution for the singular system (1.1).

From the proof of Theorem 3.1, we can also obtain the following result.

THEOREM 3.2. Assume that (H1)-(H3) hold. In addition, suppose that

(H5) $0 \leqslant f^{\infty} < L_1$, $0 \leqslant g^{\infty} < L_1$, and $L_2 < f_0 \leqslant \infty$ or $L_2 < g_0 \leqslant \infty$.

Then the singular system (1.1) has at least one symmetric positive solution.

Next we discuss the multiplicity of positive solutions for the boundary value problem (1.1).

THEOREM 3.3. Assume that (H1)-(H3) hold. In addition, suppose that

(H6) $2L_2 < f_0 \leqslant \infty$ or $2L_2 < g_0 \leqslant \infty$;

(H7) $2L_2 < f_{\infty} \leqslant \infty$ or $2L_2 < g_{\infty} \leqslant \infty$;

(H8) There exists ρ_1 such that $f(t,u,v) < L_1\rho_1$, $g(t,u,v) < L_1\rho_1$, for $t \in [0,1]$, $0 < u,v \leqslant \rho_1$.

Then the singular system (1.1) has at least two symmetric positive solutions (u_1,v_1) and (u_2,v_2) satisfying $0 < \|(u_1,v_1)\| < \rho_1 < \|(u_2,v_2)\|$.

Proof. At first, it follows from the assumption $2L_2 < f_0 \leqslant \infty$ that there exists an $0 < \underline{\rho}_1 < \rho_1$ and a sufficiently small $\varepsilon_3 > 0$ such that

$$f(t,u,v) > (f_0 - \varepsilon_3)(u+v), \quad \forall t \in [0,1], \quad u+v \leqslant \underline{\rho}_1, \tag{3.7}$$

where ε_3 satisfies $f_0 - \varepsilon_3 \geqslant 2L_2$. For all $(u,v) \in \partial K_{\underline{\rho}_1}$, using (3.7) and Remark 2.2, we get

$$
\begin{aligned}
\|T(u,v)\| &\geqslant \min_{t \in [0,1]} T(u,v)(t) \geqslant \min_{t \in [0,1]} T_1(u,v)(t) \\
&\geqslant \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds \\
&\geqslant v \int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds \\
&> v(f_0 - \varepsilon_3)\gamma(\|u\| + \|v\|) \int_0^1 s(1-s)a(s)ds \geqslant \|(u,v)\|.
\end{aligned}
\tag{3.8}
$$

If $2L_2 < g_0 \leqslant \infty$, we can get a similar result. Further, by using $2L_2 < f_{\infty} \leqslant \infty$, there exists $\overline{\rho}_1 > \rho_1 > 0$ and a sufficiently small $\varepsilon_4 > 0$ such that

$$f(t,u,v) > (f_{\infty} - \varepsilon_4)(u+v), \quad \forall t \in [0,1], \quad u+v \geqslant \overline{\rho}_1, \tag{3.9}$$

where ε_4 satisfies $f_\infty - \varepsilon_4 \geqslant 2L_2$. For all $(u,v) \in \partial K_{\overline{\rho}_1}$, using (3.9) and Remark 2.2, we have

$$
\begin{aligned}
\|T(u,v)\| &\geqslant \min_{t \in [0,1]} T(u,v)(t) \geqslant \min_{t \in [0,1]} T_1(u,v)(t) \\
&\geqslant \int_0^1 H_1(t,s)a(s)f(s,u(s),v(s))ds \\
&\geqslant v \int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds \\
&> v(f_\infty - \varepsilon_4)\gamma(\|u\| + \|v\|) \int_0^1 s(1-s)a(s)ds \geqslant \|(u,v)\|.
\end{aligned}
\tag{3.10}
$$

If $2L_2 < g_\infty \leqslant \infty$, we can get a similar result. By assumption (H8), for for all $(u,v) \in \partial K_{\rho_1}$, and Remark 2.2, we obtain

$$
\begin{aligned}
\|T(u,v)\| &= \|T_1(u,v)\| + \|T_2(u,v)\| \\
&\leqslant 2\mu \left(\int_0^1 s(1-s)a(s)f(s,u(s),v(s))ds \right. \\
&\quad \left. + \int_0^1 s(1-s)b(s)g(s,u(s),v(s))ds \right) \\
&< 2L_1\mu\rho_1 \left(\int_0^1 s(1-s)a(s)ds + \int_0^1 s(1-s)b(s)ds \right) \leqslant \rho_1 = \|(u,v)\|.
\end{aligned}
\tag{3.11}
$$

From (3.8), (3.10) and (3.11), it is easy to know that two conditions of Lemma 2.7 are both satisfied. By applying Lemmas 2.6 and 2.7 to (3.8), (3.10) and (3.11), it follows that operator T has at least a fixed point $(u_1,v_1) \in K \cap (K_{\rho_1} \setminus K_{\underline{\rho}_1})$ and a fixed point $(u_2,v_2) \in K \cap (K_{\overline{\rho}_1} \setminus K_{\rho_1})$. Both are positive solutions of the singular system (1.1) and satisfy $\underline{\rho}_1 \leqslant \|(u_1,v_1)\| < \rho_1 < \|(u_2,v_2)\| \leqslant \overline{\rho}_1$. This means the singular system (1.1) has at least two positive solutions.

Using similar arguments as those used in the proof of Theorem 3.3, we also have the following result.

THEOREM 3.4. Assume that (H1)-(H3) hold. In addition, suppose that

(H9) $0 \leqslant f^0 < L_1$ and $0 \leqslant g^0 < L_1$;

(H10) $0 \leqslant f^\infty < L_1$ and $0 \leqslant g^\infty < L_1$;

(H11) There exists ρ_2 such that $f(t,u,v) > 2L_2\rho_2$ or $g(t,u,v) > 2L_2\rho_2$, for $t \in [0,1]$, $\gamma\rho_2 \leqslant u+v \leqslant \rho_2$.

Then the singular system (1.1) has at least two symmetric positive solutions (u_1,v_1) and (u_2,v_2) satisfying $0 < \|(u_1,v_1)\| < \rho_2 < \|(u_2,v_2)\|$.

4. Two examples

In order to illustrate our results, we consider two examples.

EXAMPLE 4.1. Consider the following singular differential system multi-point coupled integral boundary conditions

$$
\begin{cases}
u''(t) + \dfrac{1}{\sqrt{t(1-t)}} \dfrac{9(u+v)e^{2(u+v)}}{45(1+4t(1-t))e^{u+v}+e^{2(u+v)}} = 0, \quad t \in (0,1), \\[3mm]
v''(t) + \dfrac{1}{\sqrt{t(1-t)}}(u+v)\left(\dfrac{1}{3}\left(t-\dfrac{1}{2}\right)^2 + 10\tanh(u+v)\right) = 0, \quad t \in (0,1), \\[3mm]
u(0) = 41\displaystyle\int_{\frac{1}{6}}^{\frac{1}{5}} r(1-r)v(r)dr + 41\int_{\frac{1}{4}}^{\frac{1}{3}} r(1-r)v(r)dr, \\[3mm]
u(1) = 41\displaystyle\int_{\frac{2}{3}}^{\frac{3}{4}} r(1-r)v(r)dr + 41\int_{\frac{4}{5}}^{\frac{5}{6}} r(1-r)v(r)dr, \\[3mm]
v(0) = 41\displaystyle\int_{\frac{1}{6}}^{\frac{1}{5}} r(1-r)u(r)dr + 41\int_{\frac{1}{4}}^{\frac{1}{3}} r(1-r)u(r)dr, \\[3mm]
v(1) = 41\displaystyle\int_{\frac{2}{3}}^{\frac{3}{4}} r(1-r)v(r)dr + 41\int_{\frac{4}{5}}^{\frac{5}{6}} r(1-r)u(r)dr,
\end{cases}
\tag{4.1}
$$

then the singular system (4.1) has at least one symmetric positive solution for $t \in [0,1]$.

Proof. Let $a(t) = b(t) = 1/\sqrt{t(1-t)}$, then

$$
\int_0^1 s(1-s)a(s)ds = \int_0^1 s(1-s)b(s)ds = \pi/8.
$$

Denote

$$
f(t,u,v) = \frac{9(u+v)e^{2(u+v)}}{45(1+4t(1-t))e^{u+v}+e^{2(u+v)}}, \quad t \in [0,1],
$$

and

$$
g(t,u,v) = (u+v)\left(\frac{1}{3}\left(t-\frac{1}{2}\right)^2 + 10\tanh(u+v)\right), \quad t \in [0,1].
$$

Clearly, the conditions (H1)-(H3) hold. Nextly, we prove that the condition (H4) of Theorem 3.1 is satisfied. By a simple computation, we have $\mu = \rho \approx 5.71995$, $\nu \approx 0.914533$, $\gamma \approx 0.159885$, $L_1 \approx 0.111298$, $L_2 \approx 8.7077$,

$$
\begin{aligned}
f^0 &= \limsup_{(u,v)\to 0} \max_{t\in[0,1]} \frac{f(t,u,v)}{u+v} \\
&= \limsup_{(u,v)\to 0} \max_{t\in[0,1]} \frac{9e^{2(u+v)}}{45(1+4t(1-t))e^{u+v}+e^{2(u+v)}} = \frac{9}{91} < 0.111298 \approx L_1,
\end{aligned}
$$

$$g^0 = \limsup_{(u,v)\to 0} \max_{t\in[0,1]} \frac{g(t,u,v)}{u+v}$$

$$= \limsup_{(u,v)\to 0} \max_{t\in[0,1]} \left(\frac{1}{3}\left(t-\frac{1}{2}\right)^2 + 6\tanh(u+v) \right) = \frac{1}{12} < 0.111298 \approx L_1,$$

$$f_\infty = \liminf_{(u,v)\to\infty} \min_{t\in[0,1]} \frac{f(t,u,v)}{u+v}$$

$$= \liminf_{(u,v)\to\infty} \min_{t\in[0,1]} \frac{9e^{2(u+v)}}{45(1+4t(1-t))e^{u+v}+e^{2(u+v)}} = 9 > 8.7077 \approx L_2,$$

or

$$g_\infty = \liminf_{(u,v)\to\infty} \min_{t\in[0,1]} \frac{g(t,u,v)}{u+v}$$

$$= \liminf_{(u,v)\to\infty} \min_{t\in[0,1]} \left(\frac{1}{3}\left(t-\frac{1}{2}\right)^2 + 10\tanh(u+v) \right) = 10 > 8.7077 \approx L_2.$$

Hence, by Theorem (3.1), the singular system (4.1) has at least one symmetric positive solution for $t \in [0,1]$.

EXAMPLE 4.2. Consider the following singular differential system multi-point coupled integral boundary conditions

$$
\begin{cases}
u''(t) + \dfrac{1}{8\sqrt{t^3(1-t)^3}} \dfrac{35(u+v)e^{2(u+v)}}{(1+t(1-t))e^{u+v}+e^{2(u+v)}} = 0, & t \in (0,1), \\[2mm]
v''(t) + \dfrac{1}{8\sqrt{t^3(1-t)^3}} \dfrac{35(u+v)e^{2(u+v)}}{(1+t(1-t))e^{u+v}+e^{2(u+v)}} = 0, & t \in (0,1), \\[2mm]
u(0) = 131\displaystyle\int_{\frac{1}{6}}^{\frac{1}{5}} \left(t-\frac{1}{2}\right)^2 v(r)dr + 131\displaystyle\int_{\frac{1}{4}}^{\frac{1}{3}} \left(t-\frac{1}{2}\right)^2 v(r)dr, \\[2mm]
u(1) = 131\displaystyle\int_{\frac{2}{3}}^{\frac{3}{4}} \left(t-\frac{1}{2}\right)^2 v(r)dr + 131\displaystyle\int_{\frac{4}{5}}^{\frac{5}{6}} \left(t-\frac{1}{2}\right)^2 v(r)dr, \\[2mm]
v(0) = 131\displaystyle\int_{\frac{1}{6}}^{\frac{1}{5}} \left(t-\frac{1}{2}\right)^2 u(r)dr + 131\displaystyle\int_{\frac{1}{4}}^{\frac{1}{3}} \left(t-\frac{1}{2}\right)^2 u(r)dr, \\[2mm]
v(1) = 131\displaystyle\int_{\frac{2}{3}}^{\frac{3}{4}} \left(t-\frac{1}{2}\right)^2 v(r)dr + 131\displaystyle\int_{\frac{4}{5}}^{\frac{5}{6}} \left(t-\frac{1}{2}\right)^2 u(r)dr,
\end{cases}
\tag{4.2}
$$

then the singular system (4.2) has at least two symmetric positive solutions (u_1,v_1) and (u_2,v_2) satisfying $0 < \|(u_1,v_1)\| < 379 < \|(u_2,v_2)\|$.

Proof. Let $a(t) = b(t) = 1/(8\sqrt{t^3(1-t)^3})$, then

$$\int_0^1 s(1-s)a(s)ds = \int_0^1 s(1-s)b(s)ds = \pi/8.$$

Denote

$$f(t,u,v) = g(t,u,v) = \frac{35(u+v)e^{2(u+v)}}{(1+t(1-t))e^{u+v} + e^{2(u+v)}}, \quad t \in [0,1].$$

Clearly, the conditions (H1)-(H3) hold. Now, we prove that the conditions (H6) and (H7) of Theorem 3.3 is satisfied. By a simple computation, we have $\mu = \rho \approx 6.38922$, $v \approx 0.957851$, $\gamma \approx 0.149917$, $L_1 \approx 0.0996397$, $L_2 \approx 8.86669$,

$$\begin{aligned} f_0 = g_0 &= \liminf_{(u,v)\to 0} \min_{t\in[0,1]} \frac{f(t,u,v)}{u+v} \\ &= \liminf_{(u,v)\to 0} \min_{t\in[0,1]} \frac{35e^{2(u+v)}}{(1+t(1-t))e^{u+v} + e^{2(u+v)}} = \frac{35}{2} > 17.7334 \approx 2L_2, \end{aligned}$$

and

$$\begin{aligned} f_\infty = g_\infty &= \liminf_{(u,v)\to \infty} \min_{t\in[0,1]} \frac{f(t,u,v)}{u+v} \\ &= \liminf_{(u,v)\to \infty} \min_{t\in[0,1]} \frac{35e^{2(u+v)}}{(1+t(1-t))e^{u+v} + e^{2(u+v)}} = 35 > 17.7334 \approx 2L_2, \end{aligned}$$

Furthermore, we choose $\rho_1 = 379$ such that $f(t,u,v) = g(t,u,v) < 37.7634 \approx L_1\rho_1$, for $t \in [0,1]$, $0 < u,v \leqslant 379 = \rho_1$. Hence, by Theorem (3.3), the singular system (4.2) has at least two symmetric positive solutions (u_1,v_1) and (u_2,v_2) satisfying $0 < \|(u_1,v_1)\| < 379 < \|(u_2,v_2)\|$.

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