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EXISTENCE OF HOMOCLINIC SOLUTIONS FOR SECOND ORDER HAMILTONIAN SYSTEMS UNDER LOCAL CONDITIONS

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Abstract. Under some local conditions on $V(t,x)$ with respect to x, the existence of homoclinic solutions is obtained for a class of the second order Hamiltonian systems $\ddot{u}(t)+\nabla V(t,u(t)) = f(t)$, $\forall t\in\mathbb{R}$.

1. Introduction

Let us consider the second order Hamiltonian systems

$$\ddot{u}(t)+\nabla V(t,u(t)) = f(t), \quad \forall t\in\mathbb{R}, \tag{1.1}$$

where $\nabla V(t,x) = \frac{\partial V}{\partial x}(t,x)$. As usual, we say that u is a nontrivial homoclinic solution (to 0) if $u\in C^2(\mathbb{R},\mathbb{R}^N), u\not\equiv 0$ and $u(t)\to 0$ as $|t|\to\infty$. In the following, $(\cdot,\cdot):\mathbb{R}^N\times\mathbb{R}^N\mapsto\mathbb{R}$ denotes the standard inner product in $\mathbb{R}^N$ and $|\cdot|$ is the induced norm.

If $V(t,x) = -(L(t)x,x)/2+W(t,x)$, then (1.1) reduces to the following second order Hamiltonian systems

$$\ddot{u}(t)-L(t)u(t)+\nabla W(t,u(t)) = f(t), \quad \forall t\in\mathbb{R}, \tag{1.2}$$

where $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$.

With the variational methods, the existence and multiplicity of homoclinic solutions of problem (1.1) have been obtained by many papers (see [1–6, 9–20]), mainly in the case that V satisfies some global assumptions for all t and x. For example, Izydorek and Janczewska [4] established the following theorem.

THEOREM A. (see [4]) *Assume that V and f satisfy the following conditions:*

(H_1) $V(t,x) = -K(t,x)+W(t,x)$, *where $V\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ is $T-$periodic with respect to $t, T>0$;*

(H_2) *there are constants $b_1,b_2>0$ such that for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$*

$$b_1|x|^2\leqslant K(t,x)\leqslant b_2|x|^2;$$

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(H_3) *$K(t,x) \leqslant (\nabla K(t,x),x) \leqslant 2K(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$;*

(H_4) *$\nabla W(t,x) = o(|x|)$ as $|x| \to 0$ uniformly with respect to t;*

(H_5) *there is a constant $\mu > 2$ such that for all $(t,x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\})$*

$$0 < \mu W(t,x) \leqslant (\nabla W(t,x),x);$$

(H_6) *$f : \mathbb{R} \to \mathbb{R}^N$ is a continuous and bounded function such that*

$$\left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} < \frac{\bar{b}_1 - 2M}{2C^*},$$

where C^* is a suitable positive constant and

$$M := \sup\{W(t,x) | t \in [0,T], x \in \mathbb{R}^N, |x| = 1\}, \ \bar{b}_1 := \min\{1, 2b_1\} > 2M.$$

Then problem (1.1) has a nontrivial homoclinic solution.

Later, Tang and Xiao [11] extended Theorem A by using more general conditions.

THEOREM B. (see [11]) *Assume that V and $f \not\equiv 0$ satisfy (H_1), (H_4), (H_5) and the following conditions:*

(H_2') *there are constants $b^* > 0$ and $\gamma \in (1,2]$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^N$*

$$K(t,0) = 0, K(t,x) \geqslant b^* |x|^\gamma;$$

(H_3') *there is a constant $\rho \in [2,\mu)$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^N$*

$$(\nabla K(t,x),x) \leqslant \rho K(t,x);$$

(H_6') *$f : \mathbb{R} \to \mathbb{R}^N$ is a continuous and bounded function such that*

$$\left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} < \sqrt{2} \min\{\delta/2, b^* \delta^{\gamma-1} - M\delta^{\mu-1}\},$$

where

$$M := \sup\{W(t,x) | t \in [0,T], x \in \mathbb{R}^N, |x| = 1\}$$

and $\delta \in (0,1]$ such that

$$b^* \delta^{\gamma-1} - M\delta^{\mu-1} = \max_{x \in [0,1]} (b^* x^{\gamma-1} - M x^{\mu-1}).$$

Then problem (1.1) has a nontrivial homoclinic solution.

Among other results, under some local conditions on W, Lv and Jiang [6] investigated the existence of homoclinic solutions of problem (1.2) as a limit of periodic solutions of a certain sequence of boundary-value problems. They presented the following assumption on L:

(L) $L(t)$ is a positive symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R},(0,\infty))$ such that for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$

$$(L(t)x,x) \geqslant l(t)|x|^2$$

and $\sup_{t\in\mathbb{R}} |L_{ij}(t)| < \infty$, where $L(t) = (L_{ij}(t))_{N\times N}$.

As far as the authors know, there is no research concerning the existence and multiplicity of homoclinic solutions for the more general Hamiltonian system (1.1) under local conditions. Motivated by the above facts, in this note, we will consider problem (1.1) where $V(t,x)$ satisfies only some local conditions near the origin. The exact assumptions are as follows.

THEOREM 1. *Assume that V and f satisfy the following conditions:*

(V_1) $V(t,0) = \nabla V(t,0) = 0$ *and* $V \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ *uniformly in* $t \in \mathbb{R}$;

(V_2) *there exist constants* $\rho > 0$, $a_0 > 0$ *and an* $a \in C(\mathbb{R},(-\infty,-a_0])$ *such that*

$$V(t,x) \leqslant a(t)|x|^2, \text{ for all } t \in \mathbb{R} \text{ and } |x| \leqslant \sqrt{2}\rho;$$

(F) $f \not\equiv 0$ *is a continuous and bounded function such that* $\int_{\mathbb{R}} |f(t)|^2 dt < \infty$ *and*

$$\left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} < \min\{1/2, a_0\}\rho.$$

Then problem (1.1) *has a nontrivial homoclinic solution.*

REMARK 1. On one hand, (V_2) in Theorem 1 can be deduced from conditions in Theorem B. In fact, by (H_1), (H_2') and (H_5) we have

$$V(t,x) = -K(t,x) + W(t,x) \leqslant -b^*|x|^\gamma + c|x|^\mu, \text{ for } |x| \leqslant 1,$$

where $c = \sup_{t\in[0,T],|x|=1} W(t,x)$. Since $\gamma \in (1,2]$ and $\mu > 2$, there is a positive constant $\xi > 0$ such that

$$V(t,x) \leqslant -b^*|x|^2/2, \text{ for all } t \in \mathbb{R} \text{ and } |x| \leqslant \xi.$$

On the other hand, there exist V and f that satisfy our conditions (V_1), (V_2) and (F) but do not satisfy conditions in Theorem A and Theorem B. For example, let

$$V(t,x) = -\left(\tfrac{1}{1+t^2} + 1\right)|x|^{3/2} \text{ for } |x| \leqslant 1,$$

$$f(t) = \tfrac{1}{6\sqrt{1+t^2}} \text{ with } a_0 = 1,\ a(t) = -1 \text{ and } \rho = \sqrt{2}/2.$$

Since $V(t,x)$ is not periodic with respect to t, the condition (H_1) in Theorem A and Theorem B is not satisfied.

2. Proof of theorems

Consider the following boundary-value problem

$$\begin{cases} \ddot{u}(t)+\nabla V(t,u(t))=f(t),\ \forall t\in[-T,T] \\ u(-T)-u(T)=\dot{u}(-T)-\dot{u}(T)=0, \end{cases} \tag{2.1}$$

for $T\in\mathbb{R}^+$.

Define

$$\begin{aligned} E_T &= W^{1,2}([-T,T],\mathbb{R}^N) \\ &= \big\{u:[-T,T]\to\mathbb{R}^N \mid u \text{ is absolutely continuous, } u(-T)=u(T) \\ &\qquad \text{and } \dot{u}\in L^2([-T,T],\mathbb{R}^N)\big\}. \end{aligned}$$

Then E_T is a Hilbert space equipped with the following norm:

$$\|u\|_{E_T}=\left[\int_{-T}^{T}(|\dot{u}|^2+|u|^2)dt\right]^{1/2}.$$

For $u\in E_T$, let

$$I_T(u)=\int_{-T}^{T}\Big[|\dot{u}|^2/2-V(t,u)+(f(t),u)\Big]dt.$$

It is easy to see that $I_T\in C^1(E_T,\mathbb{R})$ is weakly lower semi-continuous as the sum of a convex continuous function and of a weakly continuous one and

$$\langle I_T'(u),v\rangle=\int_{-T}^{T}[(\dot{u},\dot{v})-(\nabla V(t,u),v)+(f(t),v)]dt$$

for all $u,v\in E_T$. Moreover, it is well known that the critical points of I_T in E_T are classical solutions of problem (2.1). The following lemmas are important to our proofs.

LEMMA 1. (see [7]) *Let X be a real reflexive Banach space and $\Omega\subset X$ be a closed bounded convex subset of X. Suppose that $\varphi: X\to\mathbb{R}$ is a weakly lower semi-continuous functional. If there exists a point $x_0\in\Omega\setminus\partial\Omega$ such that*

$$\varphi(x)>\varphi(x_0),\ \forall x\in\partial\Omega.$$

Then there must be a $x^*\in\Omega\setminus\partial\Omega$ such that

$$\varphi(x^*)=\inf_{u\in\Omega}\varphi(u).$$

LEMMA 2. (see [6]) *Let $u\in E_T$, then the following inequality holds*

$$\|u\|_{L^\infty_{[-T,T]}}\leqslant\sqrt{2}\|u\|_{E_T}.$$

LEMMA 3. *Under the conditions of Theorem 1, problem* (2.1) *possesses a solution* $u_T \in E_T$ *such that*

$$\int_{-T}^{T}(|\dot{u}_T|^2+|u_T|^2)dt<\rho^2,\ \forall T\in\mathbb{R}^+. \tag{2.2}$$

Proof. For any $T\in\mathbb{R}^+$, let

$$\Omega_T=\left\{u\in E_T \mid \int_{-T}^{T}(|\dot{u}|^2+|u|^2)dt\leqslant\rho^2\right\},$$

where ρ is a constant given in condition (V_2). Clearly, Ω_T is a closed bounded convex subset of E_T.

For any $u\in\partial\Omega_T$, we have

$$\int_{-T}^{T}(|\dot{u}|^2+|u|^2)dt=\rho^2.$$

By (V_1), (V_2), (F) and Lemma 2, we get

$$\begin{aligned}
I_T(u)&=\int_{-T}^{T}[|\dot{u}|^2/2-V(t,u)+(f(t),u)]dt\\
&\geqslant(1/2)\int_{-T}^{T}|\dot{u}|^2dt+a_0\int_{-T}^{T}|u|^2dt-\left(\int_{-T}^{T}|f(t)|^2dt\right)^{1/2}\left(\int_{-T}^{T}|u|^2dt\right)^{1/2}\\
&\geqslant\min\{1/2,a_0\}\rho^2-\rho\left(\int_{-T}^{T}|f(t)|^2dt\right)^{1/2}\\
&>0=I_T(0)
\end{aligned}$$

for all $u\in\partial\Omega_T$. Then by Lemma 1, for any $T\in\mathbb{R}^+$, there exists a point

$$u_T\in\Omega_T\setminus\partial\Omega_T=\left\{u\in E_T \mid \int_{-T}^{T}(|\dot{u}|^2+|u|^2)dt<\rho^2\right\}$$

such that

$$I_T(u_T)=\inf_{u\in\Omega_T}I_T(u).$$

Now by Theorem 1.3 in [8] and the fact that $\Omega_T\setminus\partial\Omega_T$ is an open subset of E_T, we have

$$I_T'(u_T)=0.$$

Since $u_T\in\Omega_T\setminus\partial\Omega_T$, we have

$$\int_{-T}^{T}(|\dot{u}_T|^2+|u_T|^2)dt<\rho^2.$$

Therefore, (2.2) holds. The proof is complete. $\square$

Proof. [Proof of Theorem 1] Let $\{T_n\} \to \infty$ as $n \to \infty$ and consider problem (2.1) on the interval $[-T_n, T_n]$. By Lemma 3, problem (2.1) has a solution u_n and $\|u_n\|_{E_{T_n}}$ is bounded uniformly in n. As in the proof of Theorem 2.1 in [5], by the fact that

$$|u_n(t_1) - u_n(t_2)| \leqslant \int_{t_1}^{t_2} |\dot{u}_n| dt \leqslant \sqrt{t_2 - t_1} \left(\int_{t_1}^{t_2} |\dot{u}_n|^2 dt \right)^{1/2},$$

we claim that the sequence $\{u_n\}$ is equicontinuous and uniformly bounded on every interval $[-T_n, T_n]$ and we can select a subsequence $\{u_{n_k}\}$ such that it converges uniformly on any bounded interval to a function u. Since $\|u_n\|_{E_{T_n}}$ is bounded uniformly in n, we conclude that $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ and thus $u(t) \to 0$ as $t \to \infty$.

Expressing $\ddot{u}_{n_k}$ using (2.1), we get that the sequence $\ddot{u}_{n_k}$, and then also $\dot{u}_{n_k}$ converges uniformly on bounded intervals. Writing

$$u_{n_k}(t) = \int_0^t (t-s)\ddot{u}_{n_k}(s)ds + t\dot{u}_{n_k}(0) + u_{n_k}(0),$$

we have that $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $\ddot{u}_{n_k} \to \ddot{u}$ uniformly on bounded intervals. Now consider problem (2.1) on interval $[-m, m]$ for $m \in N$. Then by the diagonal process and let $m \to \infty$, we can get that u satisfies problem (1.1), that is, u is a classical solution of problem (1.1). By (V_1) and (F), we get $\nabla V(t, 0) = 0$ and $f \not\equiv 0$. Thus u is a nontrivial homoclinic solution of problem (1.1). The proof is complete. □

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