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SCALAR MULTI–POINT BOUNDARY VALUE PROBLEMS AT RESONANCE

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Abstract. In this paper we discuss the solvability of multi-point boundary value problems of the form

$$y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_0(t)y(t)=g(t,y(t))$$

subject to

$$\sum_{j=1}^{n} b_{ij}(0)y^{(j-1)}(t_0)+\sum_{j=1}^{n} b_{ij}(1)y^{(j-1)}(t_1)+\cdots+\sum_{j=1}^{n} b_{ij}(k)y^{(j-1)}(t_k)=0$$

for $i=1,\cdots,n$.

We improve upon existing results in the literature regarding multi-point boundary value problems. Our approach uses an alternative method along with Schaefer's fixed point theorem.

1. Introduction

In this paper we provide conditions for the existence of nth order scalar differential equations of the form

$$y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_0(t)y(t)=g(t,y(t)) \tag{1.1}$$

subject to

$$\sum_{j=1}^{n} b_{ij}(0)y^{(j-1)}(t_0)+\sum_{j=1}^{n} b_{ij}(1)y^{(j-1)}(t_1)+\cdots+\sum_{j=1}^{n} b_{ij}(k)y^{(j-1)}(t_k)=0 \tag{1.2}$$

for $i=1,\cdots,n$.

Throughout we will assume that the $t_i, i=0,\cdots,k$, are fixed with $0=t_0<t_1<t_2<\cdots<t_k=1$, $g:\mathbb{R}\to\mathbb{R}$ is continuous, the coefficients $b_{ij}(\cdot)$ and $a_0(\cdot),\cdots,a_{n-1}(\cdot)$ are real valued with $a_0(t)\neq 0$ for all t, and the boundary conditions are independent.

Multi-point boundary value problems occur naturally in applications to science and engineering. Such is the case for many problems arising from the analysis of elastic beams, vibrations of plates and shells, electric power networks, and telecommunication

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lines, to name a few. For those interested in concrete examples of these applications, we suggest [3, 9, 10, 16, 18] and the references therein.

Section 4 contains our main result. We obtain an existence theorem in the case where the nonlinearity, g, satisfies a growth condition in its second component. Crucial to the result is the end behavior of the nonlinearity and its interaction with the solution space of the associated linear homogeneous boundary value problem

$$y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_0(t)y(t)=0 \tag{1.3}$$

subject to the boundary conditions (1.2).

The main focus of this paper will the be solvability of problems at resonance; that is, systems where the associated linear homogeneous problem, (1.3), subject to the boundary conditions, (1.2), has nontrivial solutions. In particular, we will be concerned with the case in which this linear homogeneous problem has 1-dimensional solution space. The dimension of this solution space is directly related to the complexity involved in solving the nonlinear boundary value problem (1.1)-(1.2). Much can be said when this solution space is trivial, see [2, 4, 6, 7, 8, 9, 19, 20, 21] and the references therein. Very little has been said in the 1-dimensional case for general nth order scalar equations with very general multi-point boundary conditions as in (1.1)-(1.2). For some results, see [17]. Results when the dimension of the solution space to the linear homogeneous problem is greater than 1 may be found in [1, 5, 12, 13, 15].

Our results are of the Landesman-Lazer type, but we would like to point out that we do not require the nonlinearity to be bounded; in fact, we do not make any assumptions about the existence of $\lim_{x\to\pm\infty} g(t,x)$. In section 5, we discuss how previous results in the literature follow directly from our new result. In particular, we show how the results of [14, 17] are direct consequences our main result, Theorem 4.1. We also comment on the restrictiveness of the formulation of the operator problem in [17] and how this was alleviated using the ideas of impulsive differential equations. In section 6, we conclude the paper by giving a concrete example to show the applicability of our main result, Theorem 4.1.

2. Preliminaries

We rewrite the nth order scalar equation as an equivalent system

$$x'(t)=A(t)x(t)+f(t,x(t)), \tag{2.1}$$

subject to boundary conditions

$$\sum_{i=0}^{k} B_i x(t_i)=0, \tag{2.2}$$

where $x(t)$ denotes $\begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}$, by defining, for each $t \in [0,1]$,

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t) \end{pmatrix},$$

$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by $f(t,x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(t,x_1) \end{pmatrix}$ and $B_i, i = 0, \cdots, k$, by $(B_i)_{rs} = b_{rs}(i)$.

REMARK 2.1. It will be important to know that the independence of the boundary conditions, (1.2), is equivalent to both the matrix $[B_0|B_1|\cdots|B_k]$ having full row rank and $\bigcap_{i=0}^{k} Ker(B_i^T) = 0$.

The nonlinear boundary value problem (2.1)-(2.2) will be viewed as an operator equation. To do so, we introduce the following spaces and operators. $PC_{\{t_i\}}[0,1]$ will represent the set of $\mathbb{R}^n$-valued continuous functions on $[0,1] \setminus \{t_1, \cdots, t_{k-1}\}$ which have right and left-hand limits at each t_i, $i = 1, \cdots, k-1$. On $PC_{\{t_i\}}[0,1]$ we will use the supremum norm; that is,

$$\|\phi\| = \sup_{t \in [0,1] \setminus \{t_1, \cdots, t_{k-1}\}} |\phi(t)|,$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^n$. It is well known that when endowed with this norm, $PC_{\{t_i\}}[0,1]$ is a Banach space. The subset of $PC_{\{t_i\}}[0,1]$ consisting of continuously differentiable functions $\phi : [0,1] \setminus \{t_1, \cdots, t_{k-1}\} \to \mathbb{R}^n$ such that ϕ' has finite right and left-hand limits at each t_i, $i = 1, \cdots, k-1$, will be denoted by $PC^1_{\{t_i\}}[0,1]$. Finally, we define

$$X = \{\phi \in PC_{\{t_i\}}[0,1] \mid B_0\phi(0) + \sum_{i=1}^{k-1} B_i\phi(t_i^+) + B_k\phi(1) = 0\}.$$

The topologies on $PC^1_{\{t_i\}}[0,1]$ and X will be those inherited from $PC_{\{t_i\}}[0,1]$.

We now introduce the operators which will be used to analyze the problem. Let $dom(\mathscr{L})$ denote $PC^1_{\{t_i\}}[0,1] \cap X$.

We define a linear operator $\mathscr{L}$: dom$(\mathscr{L}) \subset X \to PC_{\{t_i\}}[0,1] \times \mathbb{R}^{n(k-1)}$ by

$$\mathscr{L}x = \begin{pmatrix} x'(\cdot) - A(\cdot)x(\cdot) \\ x(t_1^+) - x(t_1^-) \\ \vdots \\ x(t_{k-1}^+) - x(t_{k-1}^-) \end{pmatrix}$$

We also define a nonlinear operator
$\mathscr{F} : PC_{\{t_i\}}[0,1] \to PC_{\{t_i\}}[0,1] \times \mathbb{R}^{n(k-1)}$ by

$$\mathscr{F}(x) = \begin{pmatrix} f(\cdot, x(\cdot)) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It is now clear that solving the nonlinear boundary value problem (2.1)-(2.2) is equivalent to solving $\mathscr{L}x = \mathscr{F}(x)$.

We begin our study of the nonlinear boundary problem (2.1)-(2.2) by analyzing the linear nonhomogeneous problem

$$\begin{gathered} x'(t) = A(t)x(t) + h(t), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_{k-1}\} \\ x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k-1 \end{gathered} \tag{2.3}$$

subject to the boundary conditions

$$B_0 x(0) + \sum_{i=1}^{k-1} B_i x(t_i^+) + B_k x(1) = 0. \tag{2.4}$$

Here we assume $h \in PC_{\{t_i\}}[0,1]$ and each v_i, $i = 1, \cdots, k-1$, is an element of $\mathbb{R}^n$. The characterization of this problem will play an important role in our analysis of the nonlinear boundary value problem using an alternative method.

PROPOSITION 2.2. *The linear nonhomogeneous problem* (2.3) *subject to boundary conditions* (2.4) *has a solution if and only if for each* $c \in Ker\Big(\Big(\sum_{i=0}^{k} B_i \Phi(t_i)\Big)^T\Big)$,

$$\Big\langle c, \sum_{i=1}^{k} B_i \Phi(t_i) \Big(\int_0^{t_i} \Phi^{-1}(s) h(s)\, ds + \sum_{j=1}^{i} \Phi^{-1}(t_j) v_j \Big) \Big\rangle = 0.$$

Here $\langle \cdot, \cdot \rangle$ *denotes the standard inner product on* $\mathbb{R}^n$, Φ *is the principal fundamental matrix solution to* $x'(\cdot) = A(\cdot)x(\cdot)$, *and* $\sum_{j=1}^{k} \Phi^{-1}(t_j) v_j := \sum_{j=1}^{k-1} \Phi^{-1}(t_j) v_j$.

Proof.

Using the variation of parameters formula, we have $\mathscr{L}x = \begin{pmatrix} h \\ v \end{pmatrix} := \begin{pmatrix} h \\ v_1 \\ \vdots \\ v_{k-1} \end{pmatrix}$ if and only if x is given by

$$x(t) = \Phi(t)\left(x(0) + \int_0^t \Phi^{-1}(s)h(s)\,ds + \sum_{t_i<t} \Phi^{-1}(t_i)v_i\right)$$

and satisfies the boundary conditions (2.4).

Applying the boundary conditions, we get $\begin{pmatrix} h \\ v \end{pmatrix} \in Im(\mathscr{L})$ if and only if there exists $w \in \mathbb{R}^n$ such that

$$\sum_{i=0}^{k} B_i\Phi(t_i)w + \sum_{i=1}^{k} B_i\Phi(t_i)\Big(\int_0^{t_i} \Phi^{-1}(s)h(s)\,ds + \sum_{j=1}^{i} \Phi^{-1}(t_j)v_j\Big) = 0,$$

which clearly happens if and only if

$$\sum_{i=1}^{k} B_i\Phi(t_i)\Big(\int_0^{t_i} \Phi^{-1}(s)h(s)\,ds + \sum_{j=1}^{i} \Phi^{-1}(t_j)v_j\Big) \in Im\Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big).$$

The result now follows from the fact that

$$Im\Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big) = Ker\bigg(\Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big)^T\bigg)^{\perp}.$$

As a consequence, we get the following result.

COROLLARY 2.3. *The linear nonhomogeneous problem* (2.3) *subject to boundary conditions* (2.4) *has a unique solution for every h in $PC_{\{t_i\}}[0,1]$ and $v_i, i = 1,\cdots,k-1$, in $\mathbb{R}^n$ if and only if $\sum_{i=0}^{k} B_i\Phi(t_i)$ is invertible.*

Proof. If the linear nonhomogeneous problem has a unique solution, then from the proof of Proposition 2.2, $\mathscr{L}$ is invertible. It follows that

$$\sum_{i=0}^{k} B_i\Phi(t_i)$$

is one to one. Since $\sum_{i=0}^{k} B_i\Phi(t_i)$ is an $n\times n$ matrix, it is also onto.

Now if $\sum_{i=0}^{k} B_i\Phi(t_i)$ is invertible, then the unique solution is given by

$$x(t) = \Phi(t)\Big(x(0) + \int_0^t \Phi^{-1}(s)h(s)\,ds + \sum_{t_i<t} \Phi^{-1}(t_i)v_i\Big),$$

where

$$x(0) = \Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big)^{-1}\Big(\sum_{i=1}^{k} B_i\Phi(t_i)\Big(\int_0^{t_i}\Phi^{-1}(s)h(s)\,ds + \sum_{j=1}^{i}\Phi^{-1}(t_j)v_j\Big)\Big).$$

From Corollary 2.3 we have that the linear homogeneous problem

$$\begin{aligned} x'(t) &= A(t)x(t), \quad t\in[0,1]\setminus\{t_1,t_2,\cdots,t_{k-1}\} \\ x(t_i^+) - x(t_i^-) &= 0, \quad i=1,...,k-1 \end{aligned} \tag{2.5}$$

subject to the boundary conditions (2.4) has a nontrivial solution if and only if $\sum_{i=0}^{k} B_i\Phi(t_i)$ is singular. It will be useful in our construction of the alternative method projection scheme to have a description of the solution space for this case.

PROPOSITION 2.4. *A function x is a solution to the linear homogeneous problem* (2.5) *subject to the boundary conditions* (2.4) *if and only if $x(t) = \Phi(t)b$ for some $b \in Ker\Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big)$.*

Proof. From the variation of parameters formula, we have x is a solution to the linear homogeneous problem if and only if

$$x(t) = \Phi(t)x(0).$$

It is now clear that the boundary conditions are satisfied if and only if

$$\sum_{i=0}^{k} B_i\Phi(t_i)x(0) = 0.$$

Since we are assuming that linear homogeneous problem, (1.3), subject to the boundary conditions in (1.2) has 1-dimensional solution space, Proposition 2.4 implies

$$dim\Big(Ker\Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big)\Big) = 1.$$

Thus, we may choose a vector b such that $span\{b\} = Ker\Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big)$. We define $S(t)$ to be the $n\times 1$ matrix defined by $S(t) = \Phi(t)b$. We then have $x \in PC_{\{t_i\}}[0,1]$ is a solution to the linear homogeneous problem (2.5) subject to the boundary conditions (2.4) if and only if $x = S(\cdot)\alpha$ for some $\alpha\in\mathbb{R}$.

We also choose a vector c which forms a basis for $Ker\Big(\Big(\sum_{i=0}^{k} B_i\Phi(t_i)\Big)^T\Big)$ and define

$$\Psi^T(t) = \begin{cases} \sum_{j=1}^{k} c^T B_j \Phi(t_j)\Phi^{-1}(t) & 0 \leqslant t < t_1 \\ \sum_{j=2}^{k} c^T B_j \Phi(t_j)\Phi^{-1}(t) & t_1 < t < t_2 \\ \vdots & \\ c^T B_k \Phi(1)\Phi^{-1}(t) & t_{k-1} < t \leqslant 1 \end{cases}.$$

REMARK 2.5. We would like to remark that the representation of Ψ^T is the main reason for formulating the boundary value problem (2.1)-(2.2) as an impulsive differential equation in the space $PC_{\{t_i\}}[0,1]$.

Using the above definition, we get the following characterization of the $Im(\mathscr{L})$.

PROPOSITION 2.6. *The linear nonhomogeneous problem* (2.3) *with boundary conditions* (2.4) *has a solution if and only if* $\int_0^1 \Psi^T(s)h(s)ds + \sum_{i=1}^{k-1} \Psi^T(t_i^-)v_i = 0$.

3. Alternative Method

We now turn our attention to the main objective of this paper, the study of the nonlinear boundary value problem (2.1)-(2.2) at resonance. In this case, we choose to analyze (2.1)-(2.2) using a projection scheme known as the Lyapunov-Schmidt procedure. To do so we construct projections onto the $Ker(\mathscr{L})$ and $Im(\mathscr{L})$. Those interested in the nonresonant case may see [11, 15].

DEFINITION 3.1.

Define $P\colon X \to X$ by

$$[Px](t) = S(t)(b^T b)^{-1} b^T x(0)$$

From our characterization of the $Ker(\mathscr{L})$, we have that P is a projection onto $Ker(\mathscr{L})$.

DEFINITION 3.2. Define $E : PC_{\{t_i\}}[0,1] \times \mathbb{R}^{n(k-1)} \to PC_{\{t_i\}}[0,1] \times \mathbb{R}^{n(k-1)}$ by

$$E\begin{bmatrix} h \\ v \end{bmatrix} = \begin{bmatrix} h(\cdot) - \Psi(\cdot)\Big(\int_0^1 |\Psi(s)|^2 ds\Big)^{-1}\Big(\int_0^1 \Psi^T(s)h(s)ds + \sum_{i=1}^{k-1} \Psi^T(t_i^-)v_i\Big) \\ v \end{bmatrix}.$$

PROPOSITION 3.3. *E is a projection onto $Im(\mathscr{L})$.*

Proof. The fact that E is continuous, $E^2 = E$, and that $Im(E) = Im(\mathscr{L})$ follow easily once we show E is well defined. We therefore content ourselves with showing that

$$\int_0^1 |\Psi(s)|^2\, ds$$

is nonzero.

Now, suppose $\int_0^1 |\Psi(s)|^2\, ds = 0$. Then $\Psi(\cdot)$ must be the zero function. Taking $t = 1$ and using the definition of Ψ, we see that $B_k^T c = 0$. Thus, $c \in Ker(B_k^T)$. Further, since $\Psi(t) = 0$ for all $t \in (t_{k-2}, t_{k-1})$, we must also have that

$$B_{k-1}^T c + (\Phi^{-1}(t_{k-1}))^T \Phi^T(t_k) B_k^T c = \Psi(t_{k-1}^-) = 0.$$

Using the fact that $c \in Ker(B_k^T)$, we see that $c \in Ker(B_{k-1}^T)$. Continuing this process we get, $c \in Ker(B_j^T)$ for all $j = 1, \cdots, k$. Now by the choice of the c, we have $c \in Ker\Big(\sum_{i=0}^{k} \Phi^T(t_i) B_i^T\Big)$. We therefore conclude that $c \in Ker(B_0^T)$. It follows that

$$c \in \bigcap_{i=0}^{k} Ker(B_i^T).$$

Since the augmented matrix $[B_0|B_1|\cdots|B_k]$ has full row rank, we must have $c = 0$, which is not the case, and the proof is finished.

The following is the result of the Lyapunov-Schmidt projection scheme. We include the derivation for the convenience of the reader.

PROPOSITION 3.4. *Solving $\mathscr{L}x = \mathscr{F}x$ is equivalent to solving the system*

$$\begin{cases} x = M_p E \mathscr{F}(S(\cdot)\alpha + x) \\ \text{and} \\ \int_0^1 [\Psi(t)]_n g(t, [S(t)\alpha + x(t)]_1)\, dt = 0 \end{cases}$$

where M_p is $\left(\mathscr{L}_{|Ker(P)\cap dom(\mathscr{L})}\right)^{-1}$.

Proof. We have

$$\mathscr{L}x = \mathscr{F}x \Longleftrightarrow \begin{cases} E(\mathscr{L}x - \mathscr{F}x) = 0 \\ \text{and} \\ (I-E)(\mathscr{L}x - \mathscr{F}x) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \mathscr{L}x - E\mathscr{F}x = 0 \\ \text{and} \\ (I-E)\mathscr{F}x = 0 \end{cases}$$

$$\iff \begin{cases} M_p\mathscr{L}x - M_pE\mathscr{F}x = 0 \\ \text{and} \\ (I-E)\mathscr{F}x = 0 \end{cases}$$

$$\iff \begin{cases} (I-P)x - M_pE\mathscr{F}x = 0 \\ \text{and} \\ (I-E)\mathscr{F}x = 0 \end{cases}$$

$$\iff \begin{cases} (I-P)x = M_pE\mathscr{F}x \\ \text{and} \\ \int_0^1 \Psi^T(t)f(t,x(t))\,dt = 0 \end{cases}$$

$$\iff \begin{cases} w = M_pE\mathscr{F}(S(\cdot)\alpha + w) \\ \text{and} \\ \int_0^1 [\Psi(t)]_n g(t,[S(t)\alpha + w(t)]_1)\,dt = 0 \end{cases}$$

4. Main Result

We now come to our main result. Before giving the statement of the theorem, we introduce some notation that will be useful in the proof. We let $s(t)$ denote the first component of $S(t)$, $\psi(t)$ denote the nth component of $\Psi(t)$ and we define $p : \mathbb{R} \times Im(I-P) \to Im(I-P)$ by

$$p(\alpha,x) = M_pE\mathscr{F}(S(\cdot)\alpha + x).$$

We also introduce the following sets

$$O_{+,+} = \{t \mid \psi(t) > 0 \text{ and } s(t) > 0\},$$

$$O_{+,-} = \{t \mid \psi(t) > 0 \text{ and } s(t) < 0\},$$

$$O_{-,+} = \{t \mid \psi(t) < 0 \text{ and } s(t) > 0\},$$

and

$$O_{-,-} = \{t \mid \psi(t) < 0 \text{ and } s(t) < 0\}.$$

THEOREM 4.1. *Suppose the following conditions hold:*

C1. $\lim_{r\to\infty} \frac{\|g\|_r}{r} = 0$, *where, for* $s > 0$, $\|g\|_s$ *denotes*

$$\sup\{|g(t,x)| \mid t \in [0,1], x \in [-s,s]\}.$$

C2. There exists a real number R and functions $W_1, U_1, W_2, U_2, w_1, u_1, w_2$ *and* u_2 *in*

$L^1[0,1]$ *such that*

$$\begin{aligned}
&\text{if } x > R, \text{ then } W_1(t) \leqslant g(t,x) \text{ for a.e. } t \in O_{+,+}\\
&\text{if } x < -R, \text{ then } g(t,x) \leqslant U_1(t) \text{ for a.e. } t \in O_{+,+}\\
&\text{if } x > R, \text{ then } g(t,x) \leqslant u_1(t) \text{ for a.e. } t \in O_{+,-}\\
&\text{if } x < -R, \text{ then } w_1(t) \leqslant g(t,x) \text{ for a.e. } t \in O_{+,-}\\
&\text{if } x > R, \text{ then } g(t,x) \leqslant W_2(t) \text{ for a.e. } t \in O_{-,+}\\
&\text{if } x < -R, \text{ then } U_2(t) \leqslant g(t,x) \text{ for a.e. } t \in O_{-,+}\\
&\text{if } x > R, \text{ then } u_2(t) \leqslant g(t,x) \text{ for a.e. } t \in O_{-,-}
\end{aligned}$$

and

$$\text{if } x < -R, \text{ then } g(t,x) \leqslant w_2(t) \text{ for a.e. } t \in O_{-,-}$$

C3. $J_2 < 0 < J_1$, *where*

$$J_1 = \int_0^1 \psi(t)K_1(t)dt,$$

$$J_2 = \int_0^1 \psi(t)K_2(t)dt,$$

and K_1 *and* K_2 *are defined by*

$$K_1(t) = \begin{cases} W_1(t) & t \in O_{+,+} \\ w_1(t) & t \in O_{+,-} \\ W_2(t) & t \in O_{-,+} \\ w_2(t) & t \in O_{-,-} \end{cases},$$

and

$$K_2(t) = \begin{cases} U_1(t) & t \in O_{+,+} \\ u_1(t) & t \in O_{+,-} \\ U_2(t) & t \in O_{-,+} \\ u_2(t) & t \in O_{-,-} \end{cases}.$$

Then, there exists a solution to to the nonlinear boundary value problem (1.1)-(1.2).

Proof. Without loss of generality, we will assume that $\|s\| = 1$. We start by making $\mathbb{R} \times Im(I-P)$ a Banach space using the max norm

$$\|(\alpha,x)\| = \max\{|\alpha|, \|x\|\},$$

and by defining $H : \mathbb{R} \times Im(I-P) \to \mathbb{R} \times Im(I-P)$ by

$$H(\alpha,x) = \begin{pmatrix} \alpha - \int_0^1 \psi(t)g(t,s(t)\alpha + x_1(t))dt \\ p(\alpha,x) \end{pmatrix}.$$

It is clear, from Proposition 3.4, that the solutions to (2.1)-(2.2) are the fixed points of H. We would like to remark here that since M_p is an integral operator (see Corollary 2.3) and $PC_{\{t_i\}}[0,1]$ has been given the supremum norm, M_p is compact by Arzelá-Ascoli applied to the subintervals of $[0,1]$. Further, since g is sublinear, so is $\mathscr{F}$. We then have that $\mathscr{F}$ maps bounded sets to bounded sets under the supremum norm. It follows easily that p is a compact mapping, and thus, so is H. We will show that (2.1)-(2.2) has a solution by showing that H satisfies Shaefer's fixed point theorem; that is, if

$$FP := \{(\alpha,x) \mid (\alpha,x) = \lambda H(\alpha,x) \text{ for some } \lambda \in (0,1)\}$$

is a priori bounded, then H has a fixed point.

To this end, first note that by the absolute continuity of the integral, we have that there is a $\delta > 0$ such that

$$\int_T \psi(t)K_1(t)dt > \frac{J_1}{2}$$

and

$$\int_T \psi(t)K_2(t)dt < \frac{J_2}{2}$$

whenever $m(T^c) < \delta$, where m denotes Lebesgue measure.

For $0 < \eta < 1$, let A_η denote $\{t \in [0,1] \mid |s(t)| \geqslant \eta\}$. Since $\{t \mid s(t) = 0\}$ has Lebesgue measure zero, it follows that $m(A_\eta) \to 1$ as $\eta \to 0$. We may therefore chose $\eta^* > 0$ such that if $0 < \eta \leqslant \eta^*$, then $m(A_\eta^c) < \delta$.

Let a be a positive real number with

$$D_1 := \frac{2\|M_pE\|a}{1 - \|M_pE\|a} < 1.$$

Using C1., we may then choose b such that $|g(t,x)| \leqslant a|x| + b$ all $x \in \mathbb{R}$ and every $t \in [0,1]$. We let D_2 denote

$$\frac{2\|M_pE\|b}{1 - \|M_pE\|a}$$

and choose $r^* > 1$ such that for $r \geqslant r^*$,

$$r - (D_1 r + D_2) > R. \tag{4.1}$$

Define Ω_η to be the closed ball of radius $\dfrac{r^*}{\eta}$. We will show that $\partial\Omega_\eta \cap FP = \emptyset$, for 'small' enough η. This will show that FP is a priori bounded and thus H will have a fixed point by Schaefer's fixed point theorem.

To see this, first suppose $(\alpha,x)\in\partial\Omega_\eta$, with $\|x\|=\dfrac{r^*}{\eta}$. We then have

$$\begin{aligned}\|p(\alpha,x)\| &= \left\|M_pE\mathscr{F}\Big(S(\cdot)\alpha+x\Big)\right\|\\ &\leqslant \left\|M_pE\right\|\sup_t|g(t,s(t)\alpha+x_1(t))|\\ &\leqslant \left\|M_pE\right\|(a(\sup_t|s(t)\alpha|+\sup_t|x_1(t)|)+b)\\ &\leqslant \left\|M_pE\right\|(a(|\alpha|+\|x\|)+b)\\ &\leqslant \frac{1}{2}(1-\left\|M_pE\right\|a)(|\alpha|+\|x\|) \qquad \text{(Using ((4.1)) and } \eta<1)\\ &\leqslant \frac{1}{2}(1-\left\|M_pE\right\|a)\frac{2r^*}{\eta}<\frac{r^*}{\eta}\end{aligned}$$

Thus, $x\neq\lambda p(\alpha,x)$ when $(\alpha,x)\in\partial\Omega_\eta$ and $\|x\|=\dfrac{r^*}{\eta}$.

Now suppose $(\alpha,x)\in\partial\Omega_\eta$, with $|\alpha|=\dfrac{r^*}{\eta}$. We may assume that there exists a $\lambda\in(0,1)$ such that $x=\lambda p(\alpha,x)$. From our above calculation, we have

$$\begin{aligned}|x(t)|=|\lambda p(\alpha,x)(t)|\leqslant |M_pE\mathscr{F}\Big(S(t)\alpha+x(t)\Big)| &\\ \leqslant \left\|M_pE\right\||g(t,s(t)\alpha+x_1(t))| &\\ \leqslant \left\|M_pE\right\|(a(|s(t)\alpha|+|x_1(t)|)+b) &\\ \leqslant \left\|M_pE\right\|(a(|s(t)\alpha|+|x(t)|)+b). &\end{aligned}$$

Rearranging, we get

$$|x(t)|\leqslant\frac{\left\|M_pE\right\|(a|s(t)\alpha|+b)}{1-\left\|M_pE\right\|a}\leqslant D_1|s(t)\alpha|+D_2.$$

Since

$$\begin{aligned}|s(t)\alpha+x_1(t)| &\geqslant |s(t)\alpha|-|x_1(t)|\\ &\geqslant |s(t)\alpha|-|x(t)|\\ &\geqslant |s(t)\alpha|-(D_1|s(t)\alpha|+D_2)\\ &= |s(t)|\frac{r^*}{\eta}-(D_1|s(t)|\frac{r^*}{\eta}+D_2),\end{aligned}$$

we have, by the choice of r^*, that for every $t\in A_\eta$, $|s(t)\alpha+x_1(t)|>R$. If $\alpha>0$, then we have

$$\begin{aligned}&W_1(t)\leqslant g(t,s(t)\alpha+x_1(t)) \text{ for } a.e.\ t\in O_{+,+}\cap A_\eta\\ &w_1(t)\leqslant g(t,s(t)\alpha+x_1(t)) \text{ for } a.e.\ t\in O_{+,-}\cap A_\eta\\ &g(t,s(t)\alpha+x_1(t))\leqslant W_2(t) \text{ for } a.e.\ t\in O_{-,+}\cap A_\eta\\ &\textit{and}\\ &g(t,s(t)\alpha+x_1(t))\leqslant w_2(t) \text{ for } a.e.\ t\in O_{-,-}\cap A_\eta.\end{aligned}$$

Thus, $\psi(t)g(t,s(t)\alpha+x_1(t)) \geqslant \psi(t)K_1(t)$ for *a.e.* $t\in A_\eta$. It follows that

$$\int_{A_\eta}\psi(t)g(t,s(t)\alpha+x_1(t))dt \geqslant \int_{A_\eta}\psi(t)K_1(t)dt.$$

If $\eta<\eta^*$, then $m(A_\eta^c)<\delta$, so

$$\int_{A_\eta}\psi(t)K_1(t)dt > \frac{J_1}{2}.$$

We then have that

$$\begin{aligned}
&\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt = \\
&\int_{A_\eta}\psi(t)g(t,s(t)\alpha+x_1(t))dt + \int_{A_\eta^c}\psi(t)g(t,s(t)\alpha+x_1(t))dt \\
&\geqslant \int_{A_\eta}\psi(t)g(t,s(t)\alpha+x_1(t))dt - m(A_\eta^c)\,\|\psi\|\,(a\sup_{t\in A_\eta^c}|s(t)\alpha+x_1(t)|+b) \\
&\geqslant \int_{A_\eta}\psi(t)K_1(t)dt - m(A_\eta^c)\,\|\psi\|\,(a(\sup_{t\in A_\eta^c}|s(t)\alpha|+D_1\sup_{t\in A_\eta^c}|s(t)\alpha|+D_2)+b) \\
&\geqslant \int_{A_\eta}\psi(t)K_1(t)dt - m(A_\eta^c)\,\|\psi\|\,(a(\eta\frac{r^*}{\eta}+D_1\eta\frac{r^*}{\eta}+D_2)+b) \\
&= \int_{A_\eta}\psi(t)K_1(t)dt - m(A_\eta^c)\,\|\psi\|\,(a(r^*+D_1r^*+D_2)+b) \\
&\geqslant \frac{J_1}{2} - m(A_\eta^c)\,\|\psi\|\,(a(r^*+D_1r^*+D_2)+b).
\end{aligned}$$

Since $m(A_\eta^c)\to 0$ as $\eta\to 0$, we may choose η sufficiently ‘small’ so that

$$\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt > 0.$$

Similarly, if $\alpha<0$, then $\psi(t)g(t,s(t)\alpha+x_1(t)) \leqslant \psi(t)K_2(t)$ for *a.e.* $t\in A_\eta$, so that

$$\int_{A_\eta}\psi(t)g(t,s(t)\alpha+x_1(t))dt \leqslant \int_{A_\eta}\psi(t)K_2(t)dt < \frac{J_2}{2},$$

when $\eta<\eta^*$.

We then also have

$$
\begin{aligned}
&\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt = \\
&\quad \int_{A_\eta} \psi(t)g(t,s(t)\alpha+x_1(t))dt + \int_{A_\eta^c} \psi(t)g(t,s(t)\alpha+x_1(t))dt \\
&\quad \leqslant \int_{A_\eta} \psi(t)g(t,s(t)\alpha+x_1(t))dt + m(A_\eta^c)\,\|\psi\|\,(a \sup_{t\in A_\eta^c} |s(t)\alpha+x_1(t)|+b) \\
&\quad \leqslant \int_{A_\eta} \psi(t)K_1(t)dt + m(A_\eta^c)\,\|\psi\|\,(a(r^*+D_1r^*+D_2)+b) \\
&\quad < \frac{J_2}{2} + m(A_\eta^c)\,\|\psi\|\,(a(r^*+D_1r^*+D_2)+b).
\end{aligned}
$$

Thus, for ‘small’ enough η,

$$\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt < 0.$$

We conclude that in either case, for ‘small’ enough η, α and

$$\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt$$

have the same sign. If $(\alpha,x) = \lambda H(\alpha,x)$ for some $\lambda \in (0,1)$, then

$$\alpha = \lambda\alpha - \lambda\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt$$

or

$$(1-\lambda)\alpha + \lambda\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt = 0,$$

which is not the case since α and $\int_0^1 \psi(t)g(t,s(t)\alpha+x_1(t))dt$ have the same sign.

This shows that $FP\cap\partial\Omega_\eta = \emptyset$ for ‘small’ η and thus FP is a priori bounded. It follows from Schaefer’s fixed point theorem that H has a fixed point. This fixed point is a solution to (1.1)-(1.2).

REMARK 4.2. If the inequalities of Theorem 4.1 are reversed; that is,

$$
\begin{aligned}
&\text{if } x > R, \text{ then } W_1(t) \geqslant g(t,x) \text{ for } a.e.\ t \in O_{+,+} \\
&\text{if } x < -R, \text{ then } g(t,x) \geqslant U_1(t) \text{ for } a.e.\ t \in O_{+,+} \\
&\text{if } x > R, \text{ then } g(t,x) \geqslant u_1(t) \text{ for } a.e.\ t \in O_{+,-} \\
&\text{if } x < -R, \text{ then } w_1(t) \geqslant g(t,x) \text{ for } a.e.\ t \in O_{+,-} \\
&\text{if } x > R, \text{ then } g(t,x) \geqslant W_2(t) \text{ for } a.e.\ t \in O_{-,+} \\
&\text{if } x < -R, \text{ then } U_2(t) \geqslant g(t,x) \text{ for } a.e.\ t \in O_{-,+} \\
&\text{if } x > R, \text{ then } u_2(t) \geqslant g(t,x) \text{ for } a.e.\ t \in O_{-,-}
\end{aligned}
$$

and

$$\text{if } x < -R, \text{ then } g(t,x) \geqslant w_2(t) \text{ for } a.e.\ t \in O_{-,-},$$

then provided $J_1 < 0 < J_2$, (1.1)-(1.2) has a solution. The proof is essentially the same.

REMARK 4.3. The proof of Theorem 4.1 actually shows that when g has linear growth; that is, $|g(t,x)| \leqslant a|x| + b$ for all $t \in [0,1]$ and every $x \in \mathbb{R}$, then provided a is sufficiently 'small'

$$\left(\frac{2\left\|M_p E\right\| a}{1 - \left\|M_p E\right\| a} < 1 \right),$$

(1.1)-(1.2) will have a solution whenever C2. and C3. hold. We prefer the formulation in C1. $(\lim_{r\to\infty} \frac{\|g\|_r}{r} = 0)$ for its simplicity and 'ease' of calculation, as the relative 'smallness' of a may be something which is difficult to calculate.

5. Comparision to previous results

In this section we show how Theorem 4.1 improves upon existing results in the literature.

5.1. General Multi-point

In [17] the authors look at the existence of solutions to (1.1)-(1.2). They obtain results by placing conditions on the nonlinearity, g, which are much more restrictive than Theorem 4.1. Their main result, written in terms of the notation of this paper, is the following:

THEOREM 5.1. *Suppose* (1.3) *subject to boundary conditions* (1.2) *has a* 1 *- dimensional solution space. If*

H1. g *is independent of* t*,*

H2. g *is Lipschitz continuous,*

H3. $g(\pm\infty) := \lim_{x\to\pm\infty} g(x)$ *exist,*

and

H4. $L_1 L_2 < 0$*, where*

$$L_1 = g(+\infty) \int_{\{s(t)>0\}} \psi(t)dt + g(-\infty) \int_{\{s(t)<0\}} \psi(t)dt$$

and

$$L_2 = g(-\infty) \int_{\{s(t)>0\}} \psi(t)dt + g(+\infty) \int_{\{s(t)<0\}} \psi(t)dt,$$

then, there exists a solution to the nonlinear boundary value problem (1.1)-(1.2).

THEOREM 5.2. *If the assumptions of Theorem* 5.1 *hold, then so do those of Theorem* 4.1.

Proof. Suppose the conditions of Theorem 5.1 hold and assume $L_2 < 0 < L_1$. Since $g(\pm\infty)$ exist, we must have that g is bounded and thus clearly $\lim_{r\to\infty} \frac{\|g\|_r}{r} = 0$. Let $\varepsilon > 0$ and define the functions $W_1, U_1, W_2, U_2, w_1, u_1, w_2$ and u_2 in Theorem 4.1 as follows: $W_1(t) = g(+\infty) - \varepsilon$, $U_1(t) = g(-\infty) + \varepsilon$, $W_2(t) = g(+\infty) + \varepsilon$, $U_2(t) = g(-\infty) - \varepsilon$, $w_1(t) = g(-\infty) - \varepsilon$, $u_1(t) = g(+\infty) + \varepsilon$, $w_2(t) = g(-\infty) + \varepsilon$, $u_2(t) = g(+\infty) - \varepsilon$. It is clear that for these functions there exists an R, depending on ε, such that $C2$. of Theorem 4.1 holds.

Now, if we calculate $J_1 = \int_0^1 \psi(t)K_1(t)dt$, we get

$$\int_{O_{+,+}} \psi(t)(g(+\infty)-\varepsilon)dt + \int_{O_{+,-}} \psi(t)(g(-\infty)-\varepsilon)dt + \int_{O_{-,+}} \psi(t)(g(+\infty)+\varepsilon)dt + \int_{O_{-,-}} \psi(t)(g(-\infty)+\varepsilon)dt,$$

or

$$g(+\infty)\int_{O_{+,+}\cup O_{-,+}} \psi(t)dt + g(-\infty)\int_{O_{+,-}\cup O_{-,+}} \psi(t)dt - \int_0^1 |\psi(t)|\varepsilon dt.$$

However, this is equal to $L_1 - \int_0^1 |\psi(t)|\varepsilon dt$. Similarly, $J_2 = L_2 + \int_0^1 |\psi(t)|\varepsilon dt$. Since we are assuming $L_2 < 0 < L_1$, it is easy to see that for small enough ε, $J_2 < 0 < J_1$. The case where $L_1 < 0 < L_2$ follows from Remark (4.2) by a similar argument.

REMARK 5.3. Theorem 5.2 shows that Theorem 4.1 is a substantial improvement of the result found in [17]. Firstly, Theorem 4.1 allows for functions which depend on time. Secondly, it shows that the Lipschitz condition placed on g was superficial; it was needed only because of the authors formulation of the problem in $L^2[0,1]$, something we overcome by formulating the problem as an impulsive differential equation in the space $PC_{\{t_i\}}[0,1]$. Finally, it does not require the existence of $g(\pm\infty)$, an assumption much more restrictive than C1. of Theorem 4.1.

5.2. Sturm-Liouville

In [14] the authors prove the existence of solutions to regular Sturm-Liouville problems of the form

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t)) \tag{5.1}$$

subject to

$$ax(0) + bx'(0) = 0 \text{ and } cx(1) + dx'(1) = 0, \tag{5.2}$$

where throughout it is assumed that $f : \mathbb{R} \to \mathbb{R}, p : [0,1] \to \mathbb{R}$ and $q : [0,1] \to \mathbb{R}$ are continuous, $p(t) > 0$ for all $t \in [0,1]$, $a^2 + b^2, c^2 + d^2 > 0$, and λ is an eigenvalue of the associated linear Sturm-Liouville problem.

Their main result is the following:

THEOREM 5.4. *Suppose* $f:\mathbb{R}\to\mathbb{R}$ *satisfies* $|f(x)|\leqslant M_1|x|^{\beta}+M_2$, *where* M_1 *and* M_2 *are nonnegative constants and* $\beta\in[0,1)$. *If there exist* $z^*,J>0$ *such that*

$$\forall z>z^*, f(z)>J \text{ and } \forall z<-z^*, f(z)<-J,$$

then there exists a solution to (5.1)-(5.2).

Theorem 5.4 is also a consequence of Theorem 4.1. This follows from the fact that in the case of the Sturm-Liouville problem, because of the self-adjointness associated with it, $\psi(t)$ and $s(t)$ (Theorem 4.1), may be chosen to be equal. In this case, $O_{+,-}$ and $O_{-,+}$ are empty. With $g(t,x)=f(x)$, C2. of Theorem 4.1 then simplifies to

(NC2.) There exists a real number R and functions W_1,U_1,w_2 and u_2 in $L^1[0,1]$ such that

$$\text{if } x>R, \text{ then } W_1(t)\leqslant g(t,x) \text{ for } a.e.\ t\in O_{+,+}$$
$$\text{if } x<-R, \text{ then } g(t,x)\leqslant U_1(t) \text{ for } a.e.\ t\in O_{+,+}$$
$$\text{if } x>R, \text{ then } u_2(t)\leqslant g(t,x) \text{ for } a.e.\ t\in O_{-,-}$$

and

$$\text{if } x<-R, \text{ then } g(t,x)\leqslant w_2(t) \text{ for } a.e.\ t\in O_{-,-}.$$

If we take $R=z^*$, $W_1=J=u_2$ and $U_1=-J=w_2$, then

$$J_1=\int_0^1\psi(t)K_1(t)dt=\int_0^1|\psi(t)|Jdt \text{ and } \int_0^1\psi(t)K_2(t)dt=-\int_0^1|\psi(t)|Jdt=J_2,$$

so that clearly $J_2<0<J_1$. It is now evident that C1.-C3. of Theorem 4.1 are satisfied.

6. Example

In what follows, we give a concrete example of the application of our main result, Theorem 4.1. We note that the results of Theorem 4.1 remain valid for multi-point conditions in any interval $[a,b]$, so we do not restrict our example to $[0,1]$.

Consider

$$y''(t)+y(t)=g(t,y(t)) \tag{6.1}$$

subject to

$$y(0)-y(\pi/6)-y'(\pi/3)=0 \text{ and } y(\pi/6)-y(\pi/3)=0. \tag{6.2}$$

Looking at equations (1.1) and (1.2), we see that $n=k=2$. Writing this in system form, we have

$$x'(t)=Ax(t)+f(x(t))$$

subject to

$$B_0x(0)+B_1x(\pi/6)+B_2x(\pi/3)=0,$$

where

$$x(t)=\begin{bmatrix}y(t)\\y'(t)\end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(t,x) = \begin{pmatrix} 0 \\ g(t,x_1) \end{pmatrix}$. For completeness, we point out that it is clear that $[B_0|B_1|B_2]$ has full row rank.

From the basic theory of second-order linear differential equations, it follows that

$$\Phi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Calculating $B_0\Phi(0) + B_1\Phi(\pi/6) + B_2\Phi(\pi/3)$, we get

$$\begin{bmatrix} 1 & -1 \\ a & -a \end{bmatrix},$$

where $a = \dfrac{\sqrt{3}-1}{2}$. Thus, $Ker(B_0\Phi(0) + B_1\Phi(\pi/6) + B_2\Phi(\pi/3))$ is 1-dimensional and we may take

$$S(t) = \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) - \sin(t) \end{bmatrix}.$$

It follows that $s(t) = \cos(t) + \sin(t)$.

Further, using the definition of $\Psi(t)^T$, it follows that we may take

$$\psi(t) = [\Psi(t)]_2 = \begin{cases} \sin(t) & 0 < t < \pi/6 \\ b\cos(t) - c\sin(t) & \pi/6 < t < \pi/3 \end{cases},$$

where $b = \dfrac{\sqrt{3}}{\sqrt{3}-1} - \dfrac{1}{2}$ and $c = \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{3}-1}$.

From the descriptions of s and ψ, we get the following:

$$O_{+,+} = \{t \mid \psi(t) > 0 \text{ and } s(t) > 0\} = (0, \tan^{-1}(b/c)),$$
$$O_{+,-} = \{t \mid \psi(t) > 0 \text{ and } s(t) < 0\} = \emptyset,$$
$$O_{-,+} = \{t \mid \psi(t) < 0 \text{ and } s(t) > 0\} = (\tan^{-1}(b/c), \pi/3)$$
and
$$O_{-,-} = \{t \mid \psi(t) < 0 \text{ and } s(t) < 0\} = \emptyset.$$

C2. of Theorem 4.1 then simplifies to

(NNC2.) There exists a real number R and functions W_1, U_1, W_2 and U_2 in $L^1[0,1]$ such that

$$\text{if } x > R, \text{ then } W_1(t) \leqslant g(t,x) \text{ for } a.e.\ t \in O_{+,+}$$
$$\text{if } x < -R, \text{ then } g(t,x) \leqslant U_1(t) \text{ for } a.e.\ t \in O_{+,+}$$
$$\text{if } x > R, \text{ then } g(t,x) \leqslant W_2(t) \text{ for } a.e.\ t \in O_{-,+}$$

and

$$\text{if } x < -R, \text{ then } U_2(t) \leqslant g(t,x) \text{ for } a.e.\ t \in O_{-,+}.$$

If we define

$$g(t,x) = ((\tan^{-1}(b/c) - t)\left(\left(\frac{1}{\ln(2+|x|)}\right)x + \frac{x|x|^{\beta}}{1+|x|}\ln(1+|x|) + M\right),$$

where $\beta \in [0,1)$ and M is any positive constant, then clearly $\frac{\|g\|_r}{r} \to 0$ as $r \to \infty$, so that C1. of Theorem 4.1 holds.

Further, if we define $W_1(t) = (\tan^{-1}(b/c) - t) = -U_1(t)$ and $W_2(t) = 0 = U_2(t)$, then there certainly exists and R such that NNC2. holds.

Finally,

$$\begin{aligned} J_1 = \int_0^1 \psi(t)K_1(t)dt &= \int_{O_{++}} \psi(t)W_1(t)dt + \int_{O_{-+}} \psi(t)W_2(t)dt \\ &= \int_{O_{++}} \psi(t)W_1(t)dt > 0 \end{aligned}$$

and

$$\begin{aligned} J_2 = \int_0^1 \psi(t)K_2(t)dt &= \int_{O_{++}} \psi(t)U_1(t)dt + \int_{O_{-+}} \psi(t)U_2(t)dt \\ &= \int_{O_{++}} \psi(t)U_1(t)dt < 0. \end{aligned}$$

Thus, C3. of Theorem 4.1 holds. It now follows from Theorem 4.1 that the nonlinear multi-point boundary value problem (6.1) subject to (6.2) has a solution.

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