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DETERMINATION OF A LINEAR DIFFERENTIAL EQUATION ON HALF–LINE AND ITS SPECTRAL DISTRIBUTION FUNCTION FROM THE OTHERS RELATED

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Abstract. Consider two problems with symmetrical boundary value problems and defined by for $j=1,2$ through: $-y''+q_j(x)y=s^2y$, $0<x<\infty$, $y'(0)-k_jy(0)=0$ where $k_j\in\{h_1,h_2\},h_1,h_2$ are different real numbers, $s\in\{\lambda(h_1),\mu(h_2)\}$, $\{\lambda(h_1),\mu(h_2)\}$ represents the same family of eigenvalues for both problems, $q_j(x)$ are continuous real valued functions. Their uniqueness is determined through their respective spectral distribution function R_j. The aim of the paper is to relate both previous problems in the following way. We will assume the uniqueness of the first problem and determine the uniqueness of the second problem by linking: both spectral distribution functions R_j, both boundary conditions $y'(0)-k_jy(0)=0$ and both potential q_j.

1. Introduction

The inverse Sturm-Liouville problem on the half-line consists in determinig the Sturm-Liouville operator from the spectral function, that is, considering the Sturm-Liouville equation

$$\begin{aligned} -y''+q(x)y&=\lambda y, \qquad 0<x<\infty, \\ y'(0)-hy(0)&=0, \end{aligned} \tag{1}$$

where $q(x)$ is a real continuous function, λ is a spectral parameter and h is a real number.

The objective of inverse problem in general is to find out the conditions for the spectral function ρ, determined by a symmetrical boundary value problem, which means that the potential $q(x)$ and the constant real h are uniquely determined. The fact that the spectral function ρ defines the equation uniquely was proven by Marchenko [10, 11]. The complete solution to the inverse problem and the precisely necessary and sufficient conditions to be satisfied by the spectral function was first given by Gelfand & Levitan [3] as a final complete version in the celebrated paper [8, chapter III, pages 39–40], in which they give a complete solution for the case of the singular Sturm-Liouville equation by two of its spectra.

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We name *problem* j for $j \in \{1,2\}$ to be the symmetrical boundary value problem

$$\begin{aligned} -y'' + q_j(x)y &= s^2 y, \qquad 0 \leqslant x < \infty, \\ y_j'(0) - h_j y_j(0) &= 0, \end{aligned} \tag{2}$$

where $s \in \{\lambda, \mu \colon \lambda, \mu \in \mathbb{R}\}$, and $q_j(x)$ are real and continuous, h_j are real. The aim of the paper consists in finding the sufficient conditions in the first symmetrical boundary value problem to determine the uniqueness of the second symmetrical boundary value problem. An essential condition is that both problems are isospectrals, that is, for each $j \in \{1,2\}$, consider this boundary condition: $y'(0) - k_j y(0) = 0$, where $k_j \in \{h_1, h_2\}, h_1, h_2$ are different real numbers, $s \in \{\lambda(h_1), \mu(h_2)\}$ and

$$\{\lambda(h_1), \mu(h_2)\} \tag{3}$$

represents the same family of eigenvalues for both problems $j = 1$ and $j = 2$.

The approach used in this paper is the one offered by Marchenko [12, Chapter II].

Next, we turn to a description of content of this paper. In Section 2, we establish the necessary preliminaries: basic notation and results given in Marchenko [12, Chapter II, pages 101–153] will be used in the paper. In Section 3, by Lemma 1, we establish the existence of the so-called transfer kernel and how such transfer kernel satisfies the Integral Gelfand-Levitan equation and derives its uniqueness. In Section 4, given the uniqueness of the kernel of transfer, we establish through Theorem 3, the uniqueness of the second problem with conditions that relate the spectral functions, potentials and boundary conditions respectively, between both problems. In Section 5, we show two examples of isospectrals boundary value problems with two potential $q_1(x)$ and $q_2(x)$ specifics, with the homogeneous Dirichlet boundary condition at the origin. In addition, we will show that they possess extensions self-adjointes by using: [16], [22], [23], [25]. Finally, in Section 6, we provide relevant information on Estimates $L_1 - L_\infty$ for the equation of Schröndinger on the half-line, given in [23]. In addition, we consider it to be an open problem, the possibility of an extension of such Estimates in the case of the one-dimensional Hamiltonian regular given in [23], adding the singularity of quadratic order on the half-line, shown in Section 5.

2. Background and notation

By using the information on distributions of [12, Chapter II, pages 100–146], we will keep the enumeration of [12] to the right side of each equation of reference by means of the symbol $(a.b.c)$ when we believe it suitable.

Consider the boundary value problem generated on the half-line $0 \leqslant x < \infty$ by the differential equation and the boundary condition

$$\begin{aligned} -y''(x) + q(x)y(x) &= \lambda^2 y(x), \qquad 0 < x < \infty, && (2.2.1) \\ y'(0) - hy(0) &= 0, && (2.2.2) \end{aligned} \tag{4}$$

where $q(x)$ is an arbitrary complex-value function and h is an arbitrary complex number.

Let $f\colon [0,\infty)\to\mathbb{R}$, the support of f is defined as $\{x\in[0,\infty)\colon f(x)\neq 0\}$. If the closure $\overline{\{x\in[0,\infty)\colon f(x)\neq 0\}}$ is compact then $f(x)$ has compact support. We denote:

$$K[0,\infty):=\{f\colon [0,\infty)\to\mathbb{R}\colon f(x) \textit{is continuous with compact support}\},$$

$$L^2[0,\infty):=\Big\{f\colon [0,\infty)\to\mathbb{R}\colon f(x) \text{ is summable or measurable and}$$

$$\|f\|_2:=\int_0^\infty \left|f(x)\right|^2dx<\infty\Big\},$$

$$L^1(-\infty,\infty):=\Big\{f\colon (-\infty,\infty)\to\mathbb{R}\colon f(x) \text{ is summable and}$$

$$\|f\|_1:=\int_0^\infty \left|f(x)\right|dx<\infty\Big\},$$

$$K^2[0,\infty):=K^2:=\{f\colon [0,\infty)\to\mathbb{R}\colon f(x)\in L^2[0,\infty) \text{ of } \textit{compact support}\}.$$

For $\sigma>0$

$$K^2(\sigma):=\{f\in K^2\colon f(x)=0 \text{ for } x>\sigma\}.$$

We can observe that $K^2\supsetneqq K^2(\sigma)$ since if

$$\chi_{[0,n]}(x):=\begin{cases}1, & \text{if } x\in[0,n],\\ 0, & \text{if } x>n,\end{cases}$$

then $\chi_{[0,n]}(x)\in K^2$ and $\left\|\chi_{[0,n]}(x)\right\|_2=\sqrt{n}$. But $\chi_{[0,n]}(x)\notin K^2(\sigma)$ if $\sigma<x<n$.

We denote *the Fourier cosine transform of a function* $f\in K^2(\sigma)$ by

$$C(\lambda,f):=\int_0^\infty f(x)\cos\lambda x\,dx, \tag{5}$$

and designate the space of the *Fourier cosine transforms* by

$$CK^2(\sigma):=\{C(\lambda,f)\colon f\in K^2(\sigma)\}. \tag{6}$$

For an arbitrary function $f(x)\in K^2$, the *Fourier ω-transform* is defined by the formula

$$\omega(\lambda,f):=\int_0^\infty f(x)\omega(\lambda,x)\,dx, \tag{7}$$

where $\omega(\lambda,x):=\omega(\lambda,x;h)$ is the solution of the equation (2.1.1) of (4), with initial data

$$\omega(\lambda,0;h)=1,\omega'(\lambda,0;h)=h.$$

And then designate the space of the *Fourier ω-transforms* by

$$\hat{W}(K^2):=\{\omega(\lambda,f)\colon f\in K^2\}. \tag{8}$$

From the existence of transfomation operators, it follows that

$$\int_0^\infty f(x)\omega(\lambda,x)\,dx=\int_0^\infty\left[f(x)+\int_x^\infty f(\xi)K(\xi,x)\right]\cos\lambda x\,dx, \tag{2.2.3}$$

$$\int_0^\infty g(x)\cos\lambda x\,dx = \int_0^\infty \left[g(x) + \int_x^\infty g(\xi)L(\xi,x)\right]\omega(\lambda,x)\,dx, \qquad (2.2.4)$$

where the integrals are actually taken over bounded intervals, since the functions $f(x)$, $g(x)$ have compact support.

And $K(\xi,x)$ becomes the kernel of the transformation operator of the solution

$$\omega(\lambda,x;h) = \cos\lambda x + \int_0^x K(x,t;h)\cos\lambda t\,dt, \qquad (1.2.10)$$

or in a more compact notation defines the transformation operator

$$\omega(\lambda,x;h) = (\mathbb{I}+\mathbb{K}_h)\cos\lambda x,$$

where

$$\mathbb{I}\big(f(x)\big) := f(x) \text{ and } \mathbb{K}_h\big(f(x)\big) := \int_0^x K(x,t;h)f(t)\,dt,$$

of the Sturm-Liouville differential equation

$$\begin{aligned} y'' - q(x)y + \lambda^2 y = 0, &\qquad x \in (-a,a), \quad a \leqslant \infty &\qquad (1.2.1)\\ \omega(\lambda,0;h) = 1, &\qquad \omega'(\lambda,0;h) = h, &\qquad (1.2.6)\end{aligned}$$

and

$$K(x,t;h) := h + K(x,t) + K(x,-t) + h\int_x^t \big\{K(x,\xi) - K(x,-\xi)\big\}\,d\xi. \qquad (1.2.7)$$

Now by [12, 1.2.10", page 10], $L(\xi,x)$ is the kernel of the transformation operator of

$$\cos\lambda x = \omega(\lambda,x;h) + \int_0^x L(x,t;h)\omega(\lambda,t;h)\,dt, \qquad (1.2.10'')$$

or in the compact notation defines the transformation operator

$$\cos\lambda x = (\mathbb{I}+\mathbb{L}_h)\omega(\lambda,x;h).$$

[12, Chapter 1, Corollary of Theorem 1.2.1, page 8]. Both kernels $K(x,t;h)$ and $L(x,t;h)$ are continuous solutions of the corresponding Volterra integral equations and the transformation operator $\mathbb{I}+\mathbb{L}_h$ is the inverse of the transformation operator $\mathbb{I}+\mathbb{K}_h$.

And the following relationships held between the Fourier ω-transform $\omega(\lambda,f)$ and the Fourier cosines transform $C(\lambda,f)$ of the functions $f(x),g(x) \in K^2(\sigma)$

$$\omega(\lambda,f) = C(\lambda,\hat{f}), C(\lambda,g) = \omega(\lambda,\overset{\vee}{g}), \qquad (9)$$

where the functions $\hat{f}(x), \overset{\vee}{g}(x)$ are defined by the formulas

$$\hat{f}(x) := f(x) + \int_x^\infty f(\xi)K(\xi,x)\,d\xi,$$

$$\overset{\vee}{g}(x) := g(x) + \int_x^\infty g(\xi)L(\xi,x)\,d\xi,$$

for $f(x), g(x) \in K^2(\sigma)$.

Then really

$$\hat{f}(x) := f(x) + \int_x^\sigma f(\xi)K(\xi,x)\,d\xi,$$
$$\overset{\vee}{g}(x) := g(x) + \int_x^\sigma g(\xi)L(\xi,x)\,d\xi.$$

Then from (9) we conclude with the isomorphism

$$CK^2(\sigma) \simeq \hat{W}\big(K^2(\sigma)\big), \tag{10}$$

[12, Chapter 2, page 118].

The space of test functions

$$Z(\sigma) := \Big\{ f\colon \mathbb{C} \to \mathbb{C}\colon f(\lambda) \textit{ is even entire and summable on the real line and satisfies } \big|f(\lambda)\big| \leqslant Ce^{\sigma|\operatorname{Im}\lambda|}, \quad C \textit{ and } \lambda \textit{ depend on } f \Big\}, \quad (DEF.2.1.1)$$

where $f(\lambda)$ *is entire if* $f\colon \mathbb{C} \to \mathbb{C}$ *and* $f(\lambda)$ *is even if* $f(\lambda) = f(-\lambda), \quad \forall\lambda \in \mathbb{C}$. And with the norm

$$\|f\|_1 := \int_{-\infty}^\infty \big|f(\lambda)\big|\,d\lambda.$$

The sequence $f_n(\lambda) \in Z(\sigma)$ converges to $f(\lambda)$ if

$$\lim_{n\to\infty} \int_{-\infty}^\infty \big|f_n(\lambda) - f(\lambda)\big|\,d\lambda = 0,$$

and the types σ_n of the functions $f_n(\lambda)$ are bounded: $\sigma := \sup \sigma_n < \infty$. [12, Chapter 2, pages 102–103].

Let $R\colon Z(\sigma) \to \mathbb{C}$ a functional of $Z(\sigma)$. R is additive if $R(f+g) = R(f) + R(g) \quad \forall f, g \in Z(\sigma)$ and R is homogeneous if $R(f) = \alpha R(f), \quad \forall f \in Z(\sigma)$ and for some $\alpha \in \mathbb{C}$.

The space of distributions Z'

$$Z' := \Big\{ R\colon Z(\sigma) \to \mathbb{C}\colon R\big[f(\lambda)\big] := \big(f(\lambda), R\big) \text{ and } R \textit{ is additive, homogeneous and continuous functionals} \Big\}$$

Let $R \in Z'$ *be regular if it is given by the formula*

$$\big(f(\lambda), R\big) = \int_0^\infty f(\lambda)R(\lambda)\,d\lambda,$$

where $R(\lambda)$ is an arbitrary *bounded measurable function* on $0 \leqslant \lambda < \infty$. That is to say, $R \in Z'$ is regular if it defines a *functional which is continuous* in $L^1(-\infty,\infty)$. Moreover the distributions $R \in Z'$ can be multiplied by the multipliers $\varphi(\lambda) \in Z(\sigma)$ as

$$\big(f(\lambda), R\cdot\varphi(\lambda)\big) := \big(f(\lambda)\varphi(\lambda), R\big).$$

[12, Chapter 2, page 104].

To each such pair of functions $\omega(\lambda,f)$, $\omega(\lambda,g)$, we assign the number

$$R\big[\omega(\lambda,f),\omega(\lambda,g)\big] = \int_0^\infty f(x)g(x)\,dx,$$

that is, given the isomorphism (10), we obtain the functional

$$R\colon \hat{W}\big(K^2(\sigma)\big)\times\hat{W}\big(K^2(\sigma)\big) \to \mathbb{R}.$$

Actually, this functional depends only on the product $\omega(\lambda,f)\omega(\lambda,g)$. [12, Chapter 2, page 118].

It is shown in [12, Chapter 2, Theorem 2.2.1., pages 118–123] the existence of $R \in Z'$ and we give the full version of the theorem.

THEOREM 2.2.1. *To the boundary value problem* (4) *there corresponds a distribution* $R \in Z'$ *such that*

$$\big(\omega(\lambda,f)\omega(\lambda,g),R\big) = \int_0^\infty f(x)g(x)\,dx, \qquad (2.2.8)$$

where $f(x)$ *and* $g(x)$ *are arbitrary elements of* $L^2[0,\infty)$ *and* $\omega(\lambda,f)$, $\omega(\lambda,g)$ *designate their Fourier* ω*-tranforms as defined in* (7).

The distribution R *is connected with the kernel* $L(x,t)$ *of the transformation operator taking* $\omega(\lambda,x)$ *into* $\cos\lambda x$ *by the formula*

$$R = \frac{2}{\pi}\big(1+C(L)\big), \qquad (2.2.9)$$

where $C(L)$ *is the Fourier cosine transform of the function* $L(x,0)$.

Then, through the existence of a distribution $R \in Z'$ of (2.2.8) and the dependence of the product only $\omega(\lambda,f)\omega(\lambda,g)$ by R, we can conclude

$$\big(\omega(\lambda,f)\omega(\lambda,g),R\big) = R\big[\omega(\lambda,f),\omega(\lambda,g)\big] = \int_0^\infty f(x)g(x)\,dx, \qquad (11)$$

where $f(x), g(x) \in L^2[0,\infty)$ and $\omega(\lambda,f), \omega(\lambda,g)$ are their Fourier ω-tranforms as defined in (7).

3. The transference kernel and the integral Gelfand-Levitan equation

LEMMA 1. *Consider two symmetrical Sturm-Liouville boundary value problems given in* (2) *for* $j \in \{1,2\}$ *and* $s := \lambda$, *and the corresponding transformation operators:* $\mathbb{I}+\mathbb{L}_1$, $\mathbb{I}+\mathbb{K}_2$ *where*

$$\cos\lambda x := (\mathbb{I}+\mathbb{L}_1)\,\omega_1(\lambda,x;h_1) \qquad \textit{and} \qquad \omega_2(\lambda,x;h_2) := (\mathbb{I}+\mathbb{K}_2)\cos\lambda x$$

and $\omega_1(\lambda,x;h_1), \omega_2(\lambda,x;h_2)$ *are corresponding solutions to boundary value problems* $j=1$ *and* $j=2$*. We are to define*

$$\mathbb{I}+\mathbb{K}_{2,1} := (\mathbb{I}+\mathbb{K}_2)(\mathbb{I}+\mathbb{L}_1). \tag{12}$$

If we denote by $K_{2,1}(x,y)$ *the kernel of the operator* $\mathbb{K}_{2,1}$*, then* $K_{2,1}(x,y)$ *satisfies the equation*

$$F_{x,y}(x,y)+K_{2,1}(x,y)+\int_0^x K_{2,1}(x,t)f(x,t)\,dt = 0 \qquad (0\leqslant y\leqslant x), \tag{13}$$

which is analogous to [12, Formula (2.3.6)]*, where*

$$F(x,y) = \left(\int_0^x \omega_1(\lambda,t)\,dt \int_0^y \omega_1(\lambda,t)\,dt, R_2-R_1\right), \tag{14}$$

and $F_{xy}(x,y) := \partial F/\partial x\partial y(x,y)$ *and* R_j *(*$j\in\{1,2\}$*) are the corresponding distribution spectral functions.*

Proof. The operators $\mathbb{I}+\mathbb{L}_1$ transform the solution $\omega_1(\lambda,x;h_1)$ of the first boundary value problem into $\cos\lambda x$, whereas $\mathbb{I}+\mathbb{K}_2$ transform $\cos\lambda x$ into the solution $\omega_2(\lambda,x;h_2)$ of the second boundary value problem. Hence, the operator $\mathbb{I}+\mathbb{K}_{2,1} = (\mathbb{I}+\mathbb{K}_2)(\mathbb{I}+\mathbb{L}_1)$ transforms $\omega_1(\lambda,x;h_1)$ into $\omega_2(\lambda,x;h_2)$; moreover, it is also a Volterra integral operator. And of the equation

$$\cos\lambda x = (\mathbb{I}+\mathbb{L}_1)\omega_1(\lambda,x),$$

we obtain $(\mathbb{I}+\mathbb{K}_2)\cos\lambda x = (\mathbb{I}+\mathbb{K}_2)(\mathbb{I}+\mathbb{L}_1)\omega_1(\lambda,x) = \omega_2(\lambda,x)$, if

$$(\mathbb{I}+\mathbb{K}_{2,1}) := (\mathbb{I}+\mathbb{K}_2)(\mathbb{I}+\mathbb{L}_1),$$

since these are operators of Volterra, $(\mathbb{I}+\mathbb{K}_{2,1})^{-1}$ exists, and if

$$(\mathbb{I}+\mathbb{K}_{2,1})^{-1} := \mathbb{I}+\mathbb{K}_{1,2} \tag{15}$$

we obtain the following commutative diagram:

$$\begin{tikzcd}
\omega_1(\lambda,x;h_1) \arrow[r, "\mathbb{I}+\mathbb{L}_1"] \arrow[dr, shift right, "\mathbb{I}+\mathbb{K}_{2,1}"'] & \cos\lambda x \arrow[d, "\mathbb{I}+\mathbb{K}_2"] \\
 & \omega_2(\lambda,x;h_2) \arrow[ul, shift right, "\mathbb{I}+\mathbb{K}_{1,2}"']
\end{tikzcd}$$

then

$$\omega_1(\lambda,x) = (\mathbb{I}+\mathbb{K}_{1,2})\omega_2(\lambda,x) = \omega_2(\lambda,x)+\int_0^x K_{1,2}(x,t)\omega_2(\lambda,t)\,dt \text{ or}$$
$$\omega_2(\lambda,x) = (\mathbb{I}+\mathbb{K}_{2,1})\omega_1(\lambda,x) = \omega_1(\lambda,x)+\int_0^x K_{2,1}(x,t)\omega_1(\lambda,t)\,dt. \tag{16}$$

Now since $f \in K^2(\sigma)$ and (9) yields

$$\begin{aligned}
\omega_1(\lambda,f) &= \int_0^\infty f(x)\omega_1(\lambda,x)\,dx = \int_0^\sigma f(x)\omega_1(\lambda,x)\,dx \\
&= \int_0^\sigma f(x)\left[\omega_2(\lambda,x) + \int_0^x K_{1,2}(x,t)\omega_2(\lambda,t)\right]dt\,dx \\
&= \int_0^\sigma f(x)\omega_2(\lambda,x)\,dx + \int_0^\sigma \int_0^x f(x)K_{1,2}(x,t)\omega_2(\lambda,t)\,dt\,dx \\
&= \int_0^\sigma f(x)\omega_2(\lambda,x)\,dx + \int_0^x \int_0^\sigma f(x)\omega_2(\lambda,t)K_{1,2}(x,t)\,dx\,dt
\end{aligned}$$

i.e. $$\omega_1(\lambda,f) = \int_0^\sigma f(x)\omega_2(\lambda,x)\,dx + \int_0^x \int_0^\sigma f(x)\omega_2(\lambda,t)K_{1,2}(x,t)\,dx\,dt. \tag{17}$$

By the geometric properties of the kernel $K_{1,2}$

$$\begin{cases} K_{1,2}(x,t) = 0 & \text{if } |t| > |x| \\ K_{1,2}(x,t) \neq 0 & \text{if } |t| \leqslant |x|, \end{cases}$$

and (10),

$$\widehat{f}(x) = f(x) + \int_x^\infty f(t)K_{1,2}(t,x)\,dt, \tag{18}$$

where $0 \leqslant x \leqslant t \leqslant \sigma < \infty$. Therefore $K_{1,2}(t,x) \neq 0$, and by (9)

$$\omega_2(\lambda,\widehat{f}) = \int_0^\infty \widehat{f}(x)\omega_2(\lambda,x)\,dx = \int_0^\sigma \left(f(x) + \int_x^\infty f(t)K_{1,2}(t,x)\,dt\right)\omega_2(\lambda,x)\,dx,$$

and by (17) yields

$$\omega_1(\lambda,f) = \omega_2(\lambda,\widehat{f}). \tag{19}$$

Now, for $j = 1,2$ there correspond distributions R_1 and $R_2 \in Z'$, respectively, and, in accordance with (11) we yield

$$\big(\omega(\lambda,f)\omega(\lambda,g),R_i\big) = \int_0^\infty f(x)g(x)\,dx, \qquad \text{for } i = 1,2. \tag{20}$$

With the formulas: (18), (19) and (20) we can compute:

$$\begin{aligned}
&\big(\omega_2(\lambda,f)\omega_2(\lambda,g),R_2 - R_1\big) \\
&\quad = \big(\omega_1(\lambda,\widehat{f})\omega_1(\lambda,\widehat{g}),R_2\big) - \big(\omega_2(\lambda,f)\omega_2(\lambda,g),R_1\big) \\
&\quad = \int_0^\infty \left[\widehat{f}(x)\widehat{g}(x) - f(x)g(x)\right]dx \\
&\quad = \int_0^\infty \Bigg[\left(f(x) + \int_x^\infty f(t)K_{1,2}(t,x)\,dt\right) \\
&\qquad\quad \times \left(g(x) + \int_x^\infty g(t)K_{1,2}(t,x)\,dt\right) - f(x)\Bigg]g(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \Big\{ f(x)g(x) + g(x)\int_x^\infty f(t)K_{1,2}(t,x)\,dt + f(x)\int_x^\infty g(t)K_{1,2}(t,x)\,dt \\
&\qquad + \Big(\int_x^\infty f(t)K_{1,2}(t,x)\,dt\Big)\Big(\int_x^\infty g(t)K_{1,2}(t,x)\,dt\Big) - f(x)g(x) \Big\}\,dx \\
&= \int_0^\infty \Big\{ g(x)\int_x^\infty f(t)K_{1,2}(t,x)\,dt + f(x)\int_x^\infty g(t)K_{1,2}(t,x)\,dt \\
&\qquad + \Big(\int_x^\infty f(t)K_{1,2}(t,x)\,dt\Big)\Big(\int_x^\infty g(t)K_{1,2}(t,x)\,dt\Big) \Big\}\,dx \\
&= \int_0^\infty\int_x^\infty f(t)g(x)K_{1,2}(t,x)\,dt\,dx + \int_0^\infty\int_x^\infty f(x)g(t)K_{1,2}(t,x)\,dt\,dx \\
&\qquad + \int_0^\infty \Big(\int_x^\infty f(t)K_{1,2}(t,x)\,dt\Big)\Big(\int_x^\infty g(t)K_{1,2}(t,x)\,dt\Big)\,dx.
\end{aligned}
$$

We remark that $f,g \in K^2(\sigma)$, then

$$
\begin{aligned}
&\int_0^\infty\int_x^\infty f(t)g(x)K_{1,2}(t,x)\,dt\,dx + \int_0^\infty\int_x^\infty f(x)g(t)K_{1,2}(t,x)\,dt\,dx \\
&\qquad + \int_0^\infty \Big(\int_x^\infty f(t)K_{1,2}(t,x)\,dt\Big)\Big(\int_x^\infty g(t)K_{1,2}(t,x)\,dt\Big)\,dx \\
&\quad = \int_0^\sigma\int_x^\sigma f(t)g(x)K_{1,2}(t,x)\,dt\,dx + \int_0^\sigma\int_x^\sigma f(x)g(t)K_{1,2}(t,x)\,dt\,dx \\
&\qquad + \int_0^\sigma \Big(\int_x^\sigma f(t)K_{1,2}(t,x)\,dt\Big)\Big(\int_x^\sigma g(t)K_{1,2}(t,x)\,dt\Big)\,dx.
\end{aligned}
$$

If the coordinates are interchanged: $t := y,$ we obtain

$$
\begin{aligned}
\int_0^\sigma\int_x^\sigma f(t)g(x)K_{1,2}(t,x)\,dt\,dx &= \int_0^\sigma\int_0^t f(x)g(t)K_{1,2}(x,t)\,dx\,dt, \\
\int_0^\sigma\int_x^\sigma f(t)g(x)K_{1,2}(t,x)\,dt\,dx &= \int_0^\sigma\int_0^y f(x)g(y)K_{1,2}(x,y)\,dx\,dy, \\
\int_0^\sigma\int_x^\sigma f(x)g(t)K_{1,2}(t,x)\,dt\,dx &= \int_0^\sigma\int_x^\sigma f(x)g(y)K_{1,2}(y,x)\,dy\,dx,
\end{aligned}
\tag{21}
$$

and the reparametrization of the domain of integration

$$\big\{(x,y)\colon 0 \leqslant x \leqslant \sigma, x \leqslant y \leqslant \sigma\big\} = \big\{(x,y)\colon 0 \leqslant y \leqslant \sigma, 0 \leqslant x \leqslant y\big\},$$

yields us

$$\int_0^\sigma\int_x^\sigma f(x)g(y)K_{1,2}(y,x)\,dy\,dx = \int_0^\sigma\int_0^y f(x)g(y)K_{1,2}(y,x)\,dx\,dy \tag{22}$$

Now

$$\Big(\int_x^\sigma f(t)K_{1,2}(t,x)\,dt\Big)\Big(\int_x^\sigma g(t)K_{1,2}(t,x)\,dt\Big)$$

$$= \left(\int_x^\sigma f(t)K_{1,2}(t,x)\,dt\right)\left(\int_x^\sigma g(s)K_{1,2}(s,x)\,ds\right),$$

we remark that

$$K_{1,2}(t,x) \neq 0 \neq K_{1,2}(s,x) \quad \text{for} \quad x \leqslant s,t \leqslant \sigma,$$

then

$$\begin{aligned}
\int_0^\sigma & \left(\int_x^\sigma f(t)K_{1,2}(t,x)\,dt\right)\left(\int_x^\sigma g(t)K_{1,2}(t,x)\,dt\right)dx \\
&= \int_0^\sigma \left(\int_x^\sigma f(t)K_{1,2}(t,x)\,dt\right)\left(\int_x^\sigma g(s)K_{1,2}(s,x)\,ds\right)dx \\
&= \int_0^\sigma \int_x^\sigma \int_x^\sigma f(t)g(s)K_{1,2}(t,x)K_{1,2}(s,x)\,dt\,ds\,dx \\
&= \int_x^\sigma \int_x^\sigma \int_0^\sigma f(t)g(s)K_{1,2}(t,x)K_{1,2}(s,x)\,dx\,dt\,ds \\
&= \int_x^\sigma g(s)\,ds \int_x^\sigma \int_0^\sigma f(t)K_{1,2}(t,x)K_{1,2}(s,x)\,dx\,dt \\
&= \int_x^\sigma g(s)\,ds \int_0^\sigma \int_x^\sigma f(t)K_{1,2}(t,x)K_{1,2}(s,x)\,dt\,dx \\
&= \int_x^\sigma g(s)\,ds \int_0^\sigma \int_0^t f(x)K_{1,2}(x,t)K_{1,2}(s,t)\,dx\,dt \\
&= \int_x^\sigma \int_0^\sigma \int_0^t g(s)f(x)K_{1,2}(x,t)K_{1,2}(s,t)\,dx\,dt\,ds \\
&= \int_x^\sigma \int_0^\sigma \int_x^\sigma g(s)K_{1,2}(s,t)\,ds\left(\int_0^t f(x)K_{1,2}(x,t)\,dx\right)ds\,dt \\
&= \int_0^\sigma \int_0^t g(t)K_{1,2}(t,s)\left(\int_0^s f(x)K_{1,2}(x,s)\,dx\right)dt\,ds \\
&= \int_0^\sigma \int_0^t \int_0^s g(t)K_{1,2}(t,s)f(x)K_{1,2}(x,s)\,dx\,dt\,ds.
\end{aligned}$$

If $t := y$, then

$$\begin{aligned}
\int_0^\sigma & \int_0^t \int_0^s g(t)K_{1,2}(t,s)f(x)K_{1,2}(x,s)\,dx\,dt\,ds \\
&= \int_0^\sigma \int_0^s \int_0^y g(y)K_{1,2}(y,s)f(x)K_{1,2}(x,s)\,dx\,dy\,ds \\
&= \int_0^s \int_0^y \int_0^\sigma g(y)K_{1,2}(y,s)f(x)K_{1,2}(x,s)\,ds\,dx\,dy.
\end{aligned}$$

Since

$$K_{1,2}(y,s) \neq 0 \neq K_{1,2}(x,s)$$

then $0 \leqslant s \leqslant x,y$, we will assume that $0 \leqslant s \leqslant y < x$, and

$$K_{1,2}(y,s) = 0 = K_{1,2}(x,s), \quad \text{for} \quad s > y \geqslant x.$$

In these conditions we can write

$$\int_0^s \int_0^y \int_0^\sigma g(y)K_{1,2}(y,s)f(x)K_{1,2}(x,s)\,ds\,dx\,dy = \int_0^\sigma \int_0^\sigma \int_0^x g(y)K_{1,2}(y,s)f(x)K_{1,2}(x,s)\,ds\,dx\,dy,$$

then for $s > y \geqslant x$, we get

$$\int_0^s \int_0^y \int_0^\sigma g(y)K_{1,2}(y,s)f(x)K_{1,2}(x,s)\,ds\,dx\,dy = \int_0^\sigma \int_0^\sigma g(y)\Big\{\int_0^x K_{1,2}(y,s)K_{1,2}(x,s)\,ds\Big\}f(x)\,dx\,dy. \quad (23)$$

Similarly, for the condition $0 \leqslant s \leqslant y < x$, the formulas (21) and (22) transform themselves into

$$\int_0^\sigma \int_x^\sigma f(t)g(x)K_{1,2}(t,x)\,dt\,dx = \int_0^\sigma \int_0^\sigma f(x)g(y)K_{1,2}(x,y)\,dx\,dy, \quad (24)$$

and

$$\int_0^\sigma \int_x^\sigma f(x)g(y)K_{1,2}(y,x)\,dy\,dx = \int_0^\sigma \int_0^\sigma f(x)g(y)K_{1,2}(y,x)\,dx\,dy. \quad (25)$$

Finally we can write by substituting the formulae: (23), (24) and (25) for $0 \leqslant s \leqslant y < x$,

$$\begin{aligned}\Big(\omega_1(\lambda,f)\omega_1(\lambda,g),R_2-R_1\Big) &= \int_0^\sigma \int_0^\sigma f(x)g(y)K_{1,2}(x,y)\,dx\,dy \\ &\quad + \int_0^\sigma \int_0^\sigma f(x)g(y)K_{1,2}(y,x)\,dx\,dy \\ &\quad + \int_0^\sigma \int_0^\sigma g(y)\Big\{\int_0^x K_{1,2}(y,s)K_{1,2}(x,s)\,ds\Big\}f(x)\,dx\,dy \\ &= \int_0^\sigma \int_0^\sigma f(x)\Big\{K_{1,2}(x,y)+K_{1,2}(y,x) \\ &\qquad + \int_0^x K_{1,2}(y,s)K_{1,2}(x,s)ds\Big\}g(y)\,dx\,dy\end{aligned}$$

i.e., for $0 \leqslant s \leqslant y < x$,

$$\Big(\omega_1(\lambda,f)\omega_1(\lambda,g),R_2-R_1\Big) = \int_0^\sigma \int_0^\sigma f(x)\Big\{K_{1,2}(x,y)+K_{1,2}(y,x)+\int_0^x K_{1,2}(y,s)K_{1,2}(x,s)\,ds\Big\}g(y)\,dx\,dy.$$

If

$$F(x,y) := \int_0^x \int_0^y \Big\{K_{1,2}(\xi,\eta)+K_{1,2}(\eta,\xi)+\int_0^x K_{1,2}(\xi,\eta)K_{1,2}(\eta,\xi)\,ds\Big\}\,d\eta\,d\xi,$$

then

$$F_{x,y} := f(x,y) = K_{1,2}(x,y)+K_{1,2}(y,x)+\int_0^x K_{1,2}(y,s)K_{1,2}(x,s)\,ds,$$

where $F_{x,y} := \partial F/\partial x\partial y$ then

$$f(x,y) = K_{1,2}(x,y) + K_{1,2}(y,x) + \int_0^x K_{1,2}(y,s)K_{1,2}(x,s)\,ds.$$

In terms of Transformation Theory, if

$$(\mathbb{I})(f) := f,$$

then

$$\mathbb{I} = (\mathbb{I}+\mathbb{K}_{1,2})(\mathbb{I}+\mathbb{K}_{1,2}^*),$$

where $\mathbb{K}_{1,2}^*$ is the transpose of $\mathbb{K}_{1,2}$, then

$$\mathbb{I}+\mathbb{K}_{1,2}^* = (\mathbb{I}+\mathbb{K}_{1,2})^{-1},$$

since

$$(\mathbb{I}+\mathbb{K}_{1,2})^{-1} = \mathbb{I}+\mathbb{K}_{2,1},$$

in terms of the kernels

$$K_{1,2}(y,x) = K_{2,1}(x,y) + \int_0^x K_{2,1}(x,s)f(s,y)\,ds + f(x,y),$$

for $s \leqslant y < x$

$$K_{1,2}(y,x) = 0,$$

then we obtain finally (13)

$$K_{2,1}(x,y) + \int_0^x K_{2,1}(x,s)f(s,y)\,ds + f(x,y) = 0, \qquad \text{for} \quad y < x.$$

In particular: for $x,y \in [0,\infty)$ and $0 \leqslant t \leqslant x,y$, we defined the following functions

$$f(t) := \omega_1(\lambda,t)\chi_{[0,x]} \quad \text{and} \quad g(t) := \omega_1(\lambda,t)\chi_{[0,y]},$$

where

$$\chi_A(t) := \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A, \end{cases}$$

and, by (9) and (11), we have

$$\begin{aligned} F(x,y) &= \Big(\omega\big(\lambda,\omega_1(\lambda,\cdot)\chi_{[0,x]}\big)\omega\big(\lambda,\omega_1(\lambda,\cdot)\chi_{[0,y]}\big), R_2 - R_1\Big) \\ &= \Big(\int_0^x \omega_1(\lambda,t)\,dt \int_0^y \omega_1(\lambda,t)\,dt, R_2 - R_1\Big). \end{aligned} \tag{26}$$

REMARK 1. Truly the Functions $f(t)$ and $g(t)$ denote restrictions in the same $\omega_1(\lambda,\cdot)$.

4. The uniqueness of the second problem

THEOREM 1. *Let R_1 be a spectral distribution function of the boundary value problem (2) with $j=1$ and for*

$$s := \lambda, \mu \tag{27}$$

the corresponding solution of the equation: $j=1$

$$\omega_1(\lambda,x) := \omega_1(\lambda,x,h_1), \qquad \omega_1(\mu,y) := \omega_1(\mu,y,h_1) \tag{28}$$

respectively. We will assume the following hypothesis:

$$1 + c\int_0^x \omega_1^2(\mu,t)\,dt \neq 0 \quad \forall x \in [0,\infty), \tag{29}$$

and

$$\begin{aligned} q_2(x) &:= q_1(x) - 2K_{2,1}(x,x),\\ h_2 &:= h_1 - c,\\ R_2 &:= R_1 - c\delta(\mu-\lambda), \end{aligned} \tag{30}$$

where kernel $K_{2,1}(x,x)$ was defined in Lemma 1. Then R_2 is a spectral distribution function of the boundary value problem $j=2$

$$-y'' + q_2(x)y = \mu^2 y, \qquad y(0)h_2 - y'(0) = 0. \qquad (2.3.23) \tag{31}$$

REMARK 2. The constants

$$\int_0^x \omega_1^2(\mu,t)\,dt, \tag{32}$$

which represent the normalizing constant of the solutions $\omega_1(\mu,t)$ for the boundary value problem of $j=1$, $s=\mu$. $\delta(\mu-\lambda)$ is the Dirac distribution.

Proof. Since $\omega_2(\lambda,x)$ is the solution for the boundary value problem $j=2$, $s=\lambda$ and $R_2 \in Z'$ the corresponding distribution, now we can use (26) for $s=\lambda$

$$F(x,y) = \Big(\int_0^x \omega_1(\lambda,t)\,dt \int_0^y \omega_1(\lambda,t)\,dt, c\delta(\mu-\lambda)\Big), \tag{33}$$

since

$$\Big(\int_0^x \omega_1(\lambda,t)\,dt \int_0^y \omega_1(\lambda,t)\,dt, c\delta(\mu-\lambda)\Big) := c\int_0^x \omega_1(\lambda,t)\,dt \int_0^y \omega_1(\lambda,t)\,dt$$

and

$$f(x,y) = F_{xy}(x,y) := c\omega_1(\lambda,x)\omega_1(\lambda,y).$$

Then the equation (30) of kernel $K_{2,1}(x,y)$ yields: $c\omega_1(\lambda,x)\omega_1(\lambda,y)$, which satisfies (13) of Lemma 1

$$c\omega_1(\lambda,x)\omega_1(\lambda,y) + K_{2,1}(x,y) + c\int_0^x K_{2,1}(x,t)\omega_1(\lambda,t)\omega_1(\lambda,y)\,dt = 0 \tag{34}$$

and (16) implies

$$\omega_2(\lambda,x) = \omega_1(\lambda,x) + \int_0^x K_{2,1}(x,t)\omega_1(\lambda,t)\,dt, \tag{35}$$

in this case, for (37) we can find an explicit solution

$$K_{2,1}(x,y) := -\frac{c\omega_1(\mu,x)\omega_1(\mu,y)}{1+c\int_0^x \omega_1(\mu,t)^2 dt}. \tag{36}$$

Now, by (16) and $\partial/\partial x :='$

$$\omega_2'(\lambda,x) = \omega_1'(\lambda,x) + \int_0^x \omega_1(\lambda,t)\frac{\partial}{\partial x}K_{2,1}(x,t)\,dt + K_{2,1}(x,x)\omega_1(\lambda,x),$$

since

$$\int_0^x \omega_1(\lambda,t)\frac{\partial}{\partial x}K_{2,1}(x,t)\,dt = \int_0^x \frac{\partial}{\partial x}\left[K_{2,1}(x,t)\omega_1(\lambda,t)\right]dt = K_{2,1}(x,x)\omega_1(\lambda,x),$$

then

$$\omega_2'(\lambda,x) = \omega_1'(\lambda,x) + 2K_{2,1}(x,x)\omega_1(\lambda,x) \tag{37}$$

of [12, equality (1.2.28), page 16],

$$K_{2,1}(x,x) = \frac{1}{2}\int_0^x q_2(\xi)\,d\xi, \tag{38}$$

we have

$$\omega_2'(\lambda,0) = \omega_1'(\lambda,0).$$

These solutions also satisfy the boundary conditions $\omega_2'(\lambda,0) - h_2\omega_2(\lambda,0) = 0 = \omega_1'(\lambda,0) - h_1\omega_1(\lambda,0)$ and as a particular case this satisfies them when

$$\begin{aligned} &\omega_2'(\lambda,0) = h_2, \omega_2(\lambda,0) = 1,\\ &\omega_1'(\lambda,0) = h_1 - c, \omega_1(\lambda,0) = 1 - \frac{c}{h_1}, \end{aligned}$$

we finally get

$$\omega_2'(\lambda,0) = h_2 = \omega_1'(\lambda,0) = h_1 - c. \tag{39}$$

We find a relation between the potentials across (37) since

$$\frac{d}{dx}K_{2,1}(x,x) = \frac{1}{2}q_2(x), \quad \text{and} \quad \frac{d}{dx}K_{1,2}(x,x) = \frac{1}{2}q_1(x),$$

due to the symmetry of the kernels

$$K_{1,2}(x,x) = K_{2,1}(x,x). \tag{40}$$

Then

$$q_2(x) = 2\frac{d}{dx}K_{2,1}(x,x) = \frac{d}{dx}K_{1,2}(x,x) + \frac{d}{dx}K_{2,1}(x,x)$$
$$= \frac{1}{2}q_1 - c\left(\frac{\omega_1}{1 + c\int_0^x \omega_1(\mu,t)^2 dt}\right)',$$

$$\text{i.e.,} \qquad q_2(x) = \frac{1}{2}q_1 - c\left(\frac{\omega_1}{1 + c\int_0^x \omega_1(\mu,t)^2 dt}\right)'. \tag{41}$$

We can add the last results to the following table

$$\left.\begin{aligned} \omega_2(\lambda,x) &= \omega_1(\lambda,x) - \frac{c\omega_1(\mu,x)}{1 + c\int_0^x \omega_1(\mu,t)^2 dt}\int_0^x \omega_1(\mu,t)\omega_1(\lambda,t)\,dt, \\ h_2 &= \omega_2'(\lambda,0) = h_1 - c, \\ q_2 &= \tfrac{1}{2}q_1 - c\left\{\frac{\omega_1(\mu,x)}{1 + c\int_0^x \omega_1(\mu,t)^2 dt}\right\}'. \end{aligned}\right\} \qquad (2.3.23)$$

(42)

5. Two relevant examples for Theorem 3

Below are two examples of relevant isospectral boundary value problems , self-adjointes with homogeneous Dirichlet boundary condition at the origin, where Theorem 3 can be applied.

5.1. The Time-Dependent Approach to Inverse Scattering.

One objective of the Time-dependent approach to inverse Scattering theory is to use, in an essential way, the physical propagation aspects to solve the Inverse Scattering Problem and obtain mathematical proofs that closely follow physical intuition. It is hoped that a good physical understanding of the inversion mechanisms will be reflected in more transparent mathematical methods, [22].

The Time-Dependent Approach to Inverse Scattering found at [22] was considered as in [23, Theorem 2.1, page 237].

THEOREM 2.1. *Suppose that V is regular, then*

$$\|e^{-itH}P_c\|_{B(L_1,L_\infty)} \leqslant C\frac{1}{\sqrt{|t|}}. \tag{43}$$

If, furthermore, $V \in L^1$,

$$\|e^{-itH}P_c\|_{B(W_{1,1},W_{1,\infty})} \leqslant C\frac{1}{\sqrt{|t|}}, \tag{44}$$

where

$$i\frac{\partial}{\partial t}u(x,t) = Hu(x,t),$$

is the Schrödinger equation, $u(0,t) = 0,\ u(x,0) = \phi(x),\ x \in (0,\infty),\ t \in \mathbb{R}^+$.

5.2. First example

The Hamiltonian, H, is the following operator,

$$H := -\frac{d^2}{dx^2} + V(x), \tag{45}$$

with domain,

$$D := \left\{ \phi \in L^2 : \ \phi, \frac{d\phi}{dx} \text{ are absolutely continuous on } (0,\infty), \right.$$
$$\left. \left(-\frac{d^2}{dx^2} + V(x) \right)\phi \in L^2, \quad \phi(0) = 0 \right\}, \tag{46}$$

$L^2(0,\infty)$ is the Hilbert spaces of square-integrable functions on $\mathbb{R}^+$. Then, it follows from [25, Theorem 5.8], that there exists H wich is self-adjoint in $L^2(0,\infty)$, it is the self-adjoint realization of the differential expression $-(d^2/dx^2) + V(x)$ with homogeneous Dirichlet boundary condition at zero, that is,

$$H = -\frac{d^2}{dx^2} + V(x), \qquad \phi(0) = 0, \tag{47}$$

is our first example of the one proposed and in our notation used in (2): $(d^2/dx^2) :=''$, $V(x) := q_1(x)$, $\phi(x) := y(x)$, $H\phi(x) = s^2\phi(x)$, $\phi(0) = 0$. Where P_c is the orthogonal projector onto H_c, where H_c is the subspace of continuity of H, *i.e.*, H_c is the subspace of $L^2(0,\infty)$ orthogonal to all eigenvectors of H. $B(X,Y)$ is the Banach space of the bounded operators from X into Y. $W_{l,p}, \quad l = 0,1,2,\cdots, \quad 1 \leqslant p \leqslant \infty$, we denote the standard Sobolev spaces in $\mathbb{R}^+$ [1] and by

$$W_{l,p}^{(0)}, \qquad 1 \leqslant p < \infty, \tag{48}$$

the completion of $C_0^\infty(\mathbb{R}^+)$ in the norm of $W_{l,p}$, also for $l \geqslant 1$, $u \in W_{l,p}^{(0)}$ satisfies the homogeneous Dirichlet boundary condition at zero, $(d/dx^j)u(0) = 0$, $j = 0,1,2,\ldots,l-1$. The potential is a real-valued function $V(x)$ called regular if

$$\int_0^\infty x|V(x)|\,dx < \infty. \tag{49}$$

5.3. Second example

We propose as the second example the Hamiltonian prior (45), to which we have added a term with a singularity of quadratic, that is,

$$\mathscr{H} := -\Delta + V(x) + \frac{l(l+1)}{x^2}, \tag{50}$$

associated to the following problem of Sturm-Liouville

$$-y'' + \Big(V(x) + \frac{l(l+1)}{x^2}\Big)y = \lambda^2 y, \qquad (0 < x < \infty), \tag{51}$$

where Δ is the Laplacian, $V(x)$ is a potential regular defined as (49), λ is a number real, l is a fixed entire number with homogeneous Dirichlet boundary condition at zero,

$$y(0) = 0. \tag{52}$$

We can remark that the equation preceding it occurred while $\lambda := l(l+1)$, in the separation of variables of the equation of Schrödinger in the case 3D (the equation of Schrödinger in the stationary case for two particles) when the potential has radial symmetry.

Likewise, we have considered the context of Inverse Sturm-Liouville Problems, the case of angular fixed momentum and we have analyzed the case: $l > 0$, see [9, Chapter 4], [13] and [14].

The resulted Theorem 3, Section 4 above, continues hold with the homogeneous Dirichlet boundary condition at zero, since there exists a distribution spectral function for the same problem boundary except that we have the homogeneous Dirichlet boundary condition at zero, *i.e.*,

$$\begin{gathered} -y''(x) + q(x)y = \lambda^2 y, \qquad 0 \leqslant x < \infty, \\ y(0) = 0, \end{gathered} \tag{53}$$

where

$$q_1(x) := V(x) + \frac{l(l+1)}{x^2} \quad \text{and} \quad q_2(x) := V(x). \tag{54}$$

Adding the restriction that the potential $V(x)$ *will be continuous on* $[0,\infty)$ see [12, Chapter 2, Section Problems: problem 6 A, page 153].

REMARK 3. Self-adjointess of $\mathscr{H} := -\Delta + V(x) + \dfrac{l(l+1)}{x^2}$.

We united the notation of [23] and we adopted the notation of [12] hereafter: $y''(x) := (d^2y/dx^2)$, $V(x) := q_2(x)$. Moreover, remarking that, in our case, when the potential $q_2(x) := V(x)$ is regular with $V(x)$ continuous (47), there exists a spectral function ρ_2 and there exists the self-adjoint realization of the differential expression with homogeneous Dirichlet boundary condition at zero

$$-y''(x) + q_2(x)y(x) = \lambda^2 y, \qquad 0 \leqslant x < \infty, \tag{55}$$

$$y(0) = 0.$$

[25, Theorem 5.8, page 84] and [8, Chapter 2, Section Problems: problem 6 A, page 153].

Likewise, if $\tau := -(d^2/dx^2) + \overline{q}(x)$ in [25, Theorem 6.4 a), page 91], and $\overline{q}(x) := \big(l(l+1)/x^2\big) > (3/4)x^{-2}$ then τ is in the *limit point* at $x = 0$. That is, for the problem of Sturm-Liouville: $-y'' + \big((l(l+1)/x^2\big)y = \lambda^2 y$, $0 < x < \infty$, $x = 0$ is *limit point*. Similarily, for this problem of Sturm-Liouville, in [25, Theorem 6.6, page 96], we concluded since $\overline{q}(x) := \big(l(l+1)/x^2\big) \geqslant -(1/x^2)$, then τ is in the *limit point case at* ∞. Then of [16, Theorem X.7, page 152], there exists a self-adjoint realization of the differential expresion

$$\tau y := -y'' + \overline{q}(x)y, \qquad 0 < x < \infty, \tag{56}$$

with domain:

$$D' := \left\{ y \in L^2 : y, \frac{dy}{dx} \text{ are absolutely continuous on } (0,\infty), \right.$$
$$\left. \left(-\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} \right) y \in L^2, \qquad y(0) = 0 \right\}. \tag{57}$$

Then, as in the first example, there exists a self-adjoint realization of the differential expresion

$$\tau := -\frac{d^2}{dx^2} + V(x) + \frac{l(l+1)}{x^2}, \qquad 0 < x < \infty \tag{58}$$

with $W_{l,p}^{(0)}$, $1 \leqslant p < \infty$, as the completion of $C_0^\infty(\mathbb{R}^+)$ in the norm of $W_{l,p}$ (47).

6. The Estimates $L_1 - L_\infty$ for the Equation of Schröndinger on the Half-Line, [23]

Estimates of the type $L_1 - L_\infty$ express the dispersive nature of the Schröndinger equation and are the essential elements in the study of the problems of initial values, the asymptotic times for large solutions and Scattering Theory for the Schröndinger equation and non-linear in general; for other equations of Non-linear Evolution, see [20] and [23].

In general, the estimates $Lp - Lp'$ [23] express the dispersive nature of this equation. It is well known for his time, that his study plays an important role in the study of problems non-linear initial values. Likewise, it is well known at the same time, that his study plays an important role in the study of nonlinear initial values problems; see [2], [4] and [18].

In particular, the estimates $Lp - Lp'$ imply the famous Strichartz inequality for the linear Schrödinger equation with a potential; see [6], [7], [19], [20] and [23]. These estimates also play an important role in the construction of center manifold for the Schrödinger equation with non-linear potential; see [15], [17] and [21]. Finally, we can mention that the author could apply these estimates to the solution of the forward and inverse problem for the Schrödinger non-linear forced equation with a potential on the half-line in [24].

In [2, page 27], the importance of these estimations is mentioned, in the context of search of global solutions for the equations of non-linear Schrödinger. In particular, the obtained ones in [5] which perform the form

$$\|e^{it(\Delta-V)}\phi\|_\infty \leqslant C|t|^{-d/2}\|\phi\|_1, \qquad \text{for} \quad d \geqslant 3. \tag{59}$$

Also, in [2, page 27], it is mentioned that "*it would be most interesting to prove the analogue* (59) *in low dimension* $d = 1,2$. *This is certainly a project of independent importance*," In [20] and [23] these estimations were demonstrated for the case $d = 1$ on the line and half-line respectively; see (43) and (44) in the case of the half-line. Finally, as an open problem mentioned in the Introduction, we consider the possibility of extending the Estimates given in (43) and (44), in the case of the previous one-dimensional singular Hamiltonian given in (58). Where will be essential the spectral and scattering properties shown above of the two examples (47), (51) and (52) of the Theorem 3, Section 4 shown in Section 5, as well, the fundamental relationship found in both examples, as shown in Remark 5, (53) and (54) in Section 5.

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