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EXISTENCE OF MILD SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS IN SEPARABLE BANACH SPACE

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Abstract. This paper is concerned with the existence of mild solutions for fractional semilinear differential equations with non local conditions in separable Banach spaces. The result is established by using the technique of measures of noncompactness in Banach spaces of continuous functions and Schauder fixed point theorem.

1. Introduction

Our aim in this paper is to discuss the existence of the mild solution for fractional semilinear nonlocal initial value problem of the form:

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + f(t, x(t)), \quad t \in [0,T], \\ x(0) = g(x), \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $A : D(A) \subset E \to E$ is a closed linear operator generating a C_0-semigroup $\{U(t)\}_{t \geqslant 0}$ and E is a real separable Banach space E.
Recently, the theory of fractional differential equations has attracted much interest due to their applications in physics, chemistry, biology and so on see [4, 5, 8, 9, 10, 15, 16, 17, 18, 20, 21]. The semilinear evolution nonlocal Cauchy problem was initiated by Byszewski [2]. The nonlocal condition can be applied in physics with better effect in applications than the classical initial condition since nonlocal conditions are usually more precise for physical measurements than the classical initial condition. Lin, Liu and Jawahdou [7, 11] studied semilinear integrodifferential equations with nonlocal Cauchy problems under Lipschitz-type conditions. Ntouyas and Tsamatos [14] studied the global existence of solutions for semilinear evolution equations with nonlocal conditions via a fixed point analysis approach. Fu and Ezzinbi [19] studied the existence of mild and strong solutions of semilinear neutral functional differential evolution equations with nonlocal conditions.

The considerations of this paper are based on the notion of measure of noncompactness in the space of all functions continuous on $[0,T]$ see [1]. In order to prove the existence result, we shall rely on the Schauder fixed point theorem.

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The work is organized as follows. Section 2 contains some preliminaries about fractional calculus and the Hausdorff's measure of noncompactness. In Section 3 the existence result is given. In section 4 an application is provided to illustrate the results of this work.

2. Preliminary tools

In what follows, we will collect some definitions and results which will be needed later. First, assume that E is a real Banach space with the norm $\|.\|$. Let θ be the zero element of E. Denote by $B(x,r)$ the closed ball centred at x and with radius r and by B_r the ball $B(\theta,r)$. If X is a nonempty subset of E we denote by $\overline{X}$, $\mathrm{Conv}(X)$ the closure and convex closure of X, respectively. Finally, let us denote by $\mathscr{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathscr{N}_E$ its subfamily consisting of all relatively compact sets. Following [1] we accept the following definition of the concept of a measure of noncompactness:

DEFINITION 1. [1] A function $\mu : \mathscr{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1. The family $\ker\, \mu = \{X \in \mathscr{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathscr{N}_E$.
2. $X \subset Y \;\Rightarrow\; \mu(X) \leqslant \mu(Y)$.
3. $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$.
4. $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
5. If (X_n) is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$ for $n = 1,2,\dots$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

REMARK 1. Let us notice that the intersection set X_∞ described in axiom 5. satisfies the equality $\mu(X_\infty) = 0$. In fact, the inequality $\mu(X_\infty) \leqslant \mu(X_n)$ for $n = 1,2,...$ implies that $\mu(X_\infty) = 0$.

This property of the set X_∞ will be very important in our investigations. The most frequently applied measure of noncompactness is that called the Hausdorff measure of noncompactness which is defined in the following way

$$\chi(X) = \inf\{\lambda > 0 :\ X \text{ can be covered by finitely many balls of radius } \lambda\}.$$

Other facts concerning measures of noncompactness may be found in [1].In the sequel, we will work in the space $C([0,T],E)$ consisting of all functions defined and continuous on $[0,T]$ with values in the Banach space E. The space $C([0,T],E)$ is furnished with the standard norm

$$\sup_{t\in[0,T]} \|x(t)\|.$$

Moreover, For any fixed number $r > 0$, let us denote

$$B(r) = \{x \in C([0,T],E) \ : \ \|x(t)\| \leqslant r, \ t \in [0,T]\}.$$

the closed ball in $C([0,T],E)$ centered at zero element θ and with radius r. Next, we recall some properties of the measure of noncompactness in the space $C([0,T],E)$ which will be used in our work (see [1]). Let X be a nonempty and bounded subset of the space $C([0,T],E)$. Fix a positive number $t \in [0,T]$. For an arbitrary function $x \in X$ and $\varepsilon > 0$ denote by $w^t(x,\varepsilon)$ the modulus of continuity of x on the interval $[0,t]$, i.e

$$w^t(x,\varepsilon) = \sup\{\|x(t_2) - x(t_1)\| \ : \ t_1,t_2 \in [0,t], \ |t_1 - t_2| \leqslant \varepsilon\}.$$

Further, let us put:

$$w^t(X,\varepsilon) = \sup\left\{w^t(x,\varepsilon) : \ x \in X\right\},$$
$$w_0^t(X) = \lim_{\varepsilon \to 0^+} w^t(X,\varepsilon).$$

Define

$$\overline{\chi}(X) = \sup\{\chi(X(t)) : \ t \in [0,T]\},$$

where χ denotes Hausdorff measure of noncompactness in E. Finally, we define the function μ on the family of all nonempty and bounded subsets of $C([0,T],E)$ by putting

$$\mu(X) = w_0^t(X) + \overline{\chi}(X).$$

It may be shown that the function μ is the measure of noncompactness in the space $C([0,T],E)$ (see [1]). The kernel $ker\mu$ is the family of all nonempty and bounded sets X such that functions belonging to X are equicontinuous on $[0,T]$ and the set $X(t)$ is relatively compact in E for $t \in [0,T]$. This property will be crucial in our further study. Next, for a given nonempty and bounded subset X of the space $C([0,T],E)$. Next, for a bounded set $X \in C([0,T],E)$, let us denote

$$\int_0^t X(s)ds = \left\{\int_0^t x(s)ds : \ x \in X\right\}.$$

LEMMA 1. *([6]) If E is a separable Banach space and $X \subset C([0,T],E)$ nonempty and bounded then the function $t \mapsto \chi(X(t))$ is measurable and*

$$\chi\Big(\int_0^t X(s)ds\Big) \leqslant \int_0^t \chi(X(s))ds, \text{ for each } t \in [0,T].$$

The following lemmas borrowed from [12] will be needed in the proof of our existence result of solution of (1.1).

LEMMA 2. *Assume that a set $X \subset C([0,T],E)$ is bounded. Then*

$$\chi(X([0,t])) \leqslant w_0^t(X) + \sup_{s \leqslant t} \chi(X(s)), \text{ for } t \in [0,T]. \tag{2.1}$$

LEMMA 3. [3] *If X is bounded subset of Banach space E, then for each $\varepsilon > 0$ there is a sequence $\{x_k\}_{k=1}^{\infty}$ such that*

$$\chi(X) \leqslant 2\chi(\{x_k\}_{k=1}^{\infty}) + \varepsilon.$$

Next, we recall the following known definitions from the theory of fractional calculus. For more details, see [9].

DEFINITION 2. The Riemann-Liouville fractional integral of $u : [0,b] \to X$ of order $\alpha \in (0,\infty)$ is defined by

$$I_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

The Riemann-Liouville fractional derivative of $u : [0,b] \to X$ of order $\alpha \in (0,1)$ is defined by

$$D_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds.$$

The Caputo fractional derivative of $u : [0,b] \to X$ of order $\alpha \in (0,1)$ is defined by

$${}^{C}D_t^{\alpha} u(t) = D_t^{\alpha}(u(t) - u(0)).$$

Now let Φ_α be the Mainardi function:

$$\Phi_\alpha(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)},$$

then

1. $\Phi_\alpha(t) \geqslant 0$, for all $t > 0$
2. $\displaystyle\int_0^{+\infty} \Phi_\alpha(t) dt = 1$
3. $\displaystyle\int_0^{+\infty} t^{\eta} \Phi_\alpha(t) dt = \frac{\Gamma(1+\eta)}{\Gamma(1+\alpha\eta)}$, $\eta \in [0,1]$.

For the details we refer to [15]. We set

$$S_\alpha(t) = \int_0^{+\infty} \Phi_\alpha(s) U(st^{\alpha}) ds \tag{2.2}$$

and

$$P_\alpha(t) = \int_0^{+\infty} \alpha s \Phi_\alpha(s) U(st^{\alpha}) ds. \tag{2.3}$$

In what follows, we consider The C_0-semigroup $\{U(t)\}_{t>0}$ generated by A is continuous and there exists a constant $M > 0$ such that$M = \sup\{U(t) : t \geqslant 0\} < +\infty$. Then we have the following result.

LEMMA 4. ([15]) *Let S_α and P_α be the operators defined respectively by (2.2) and (2.3). Then*

i. $\|S_\alpha(t)x\| \leqslant M\|x\|$*;* $\|P_\alpha(t)x\| \leqslant \frac{M}{\Gamma(\alpha)}\|x\|$*, for all* $x \in E$ *and* $t \geqslant 0$.

ii. The operators $S_\alpha(t)(t \geqslant 0)$ *and* $P_\alpha(t)(t \geqslant 0)$ *are strongly continuous.*

DEFINITION 3. ([15]) Let S_α and P_α be the operators defined respectively by (2.2) and (2.3). Then a continuous function $x : \mathbb{R}_+ \to E$ satisfying for any $t \geqslant 0$ the equation

$$x(t) = S_\alpha(t)g(x) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s,x(s))ds. \tag{2.4}$$

is called a mild solution of the equation (1.1)

In what follows, consider the operators

$$(Hx)(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s,x(s))ds,$$

$$(Gx)(t) = S_\alpha(t)g(x)$$

and

$$(Fx)(t) = (Gx)(t) + (Hx)(t).$$

THEOREM 1. *(Schauder's fixed point theorem) Let K be a closed convex subset of a Banach space E. If $F : K \to K$ continuous and $F(K)$ is relatively compact, then F has a fixed point in K.*

3. Main results

In this section by using the usual technique of measure of noncompactness and its application in differential equations in Banach space(see [12]), we give an existence result for the problem (1.1). The following hypotheses well be needed in the sequel.

(A_f) (i) $(t,x) \mapsto f(t,x)$ satisfies the Carathéodory condition, i.e. $f(.,x)$ is measurable for $x \in E \times E$ and $f(t,.)$ is continuous for a.e. $t \in [0,T]$,

(ii) $m : \mathbb{R}_+ \to \mathbb{R}_+$ continuous such that $\|f(t,x)\| \leqslant m(t)\phi(\|x(t)\|)$ for a.e. $t \in [0,T]$ and all $x \in E$, where $\phi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing.

(iii) there exists a positive constant k_f such that for any bounded set $X \subset C([0,T],E)$, one has

$$\chi(f([0,t] \times X)) \leqslant k_f \chi(X([0,t])),$$

where $f([0,T] \times X) = \{f(s,x(s)) : 0 \leqslant s \leqslant t, x \in X\}$.

(A_g) (i) The function $g : C([0,T],E) \to E$ is continuous,

(ii) there exists a positive constant k_g such that

$$\chi(g(X)) \leqslant k_g \chi(X([0,T])),$$

for each bounded set $X \in C([0,T],E)$.

(A_1) There exists a constant $r > 0$ such that for any $t \in [0,T]$

$$M \sup_{x\in B(r)} \|g(x)\| + \frac{M}{\Gamma(\alpha)} \phi(r) \sup_{t\in[0,T]} \int_0^t \frac{m(s)}{(t-s)^{1-\alpha}} ds < r.$$

(A_2) $k_g + \dfrac{T^\alpha}{\Gamma(\alpha+1)} k_f < \dfrac{1}{3M}$.

LEMMA 5. *If our assumptions (A_f) and (A_g) are satisfied and a set $X \subset C([0,T],E)$ is bounded. Then*

$$w_0^t(HX) \leqslant 2M \frac{t^\alpha}{\Gamma(\alpha+1)} \chi\Big(f([0,t]\times X)\Big),$$

for $t \in [0,T]$

Proof. Fix $T > 0$ and denote $Z = f\Big([0,t]\times X\Big)$. First, we will show that

$$\lim_{\varepsilon\to 0} \sup_{z\in Z} \{\|P_\alpha(t_2-s) - P_\alpha(t_1-s)\| \|z\| : 0 \leqslant s \leqslant t_1 \leqslant t_2 \leqslant T,\ t_2 - t_1 \leqslant \varepsilon\} \leqslant \frac{2M}{\Gamma(\alpha)} \chi(Z). \quad (3.1)$$

Suppose contrary. Then there exists a number d such that

$$\lim_{\varepsilon\to 0} \sup_{z\in Z} \{\|P_\alpha(t_2-s) - P_\alpha(t_1-s)\| \|z\| : 0 \leqslant s \leqslant t_1 \leqslant t_2 \leqslant t,\ t_2 - t_1 \leqslant \varepsilon,\} > d. \quad (3.2)$$

Fix $\delta > 0$ such that

$$d > \frac{2M}{\Gamma(\alpha)} (\chi(Z) + \delta). \quad (3.3)$$

Condition (3.2) yields that there exist sequences $(t_{2,n}),(t_{1,n}),(s_n) \in [0,t]$ and $(z_n) \in Z$, such that $t_{2,n} \to t, t_{1,n} \to t, s_n \to s$ and

$$\|P_\alpha(t_{2,n} - s_n) - P_\alpha(t_{1,n} - s_n)\| \|z_n\| > d. \quad (3.4)$$

Suppose that the points $l_1, l_2, ..., l_k \in E$ are such that $Z \subset \bigcup_{i=1}^k B(l_i, \chi(Z)+\delta)$. Then there exist a point z_j and a subsequence of (y_n), (which is further denoted by (z_n)) such that $z_n \in B(l_j, \chi(Z)+\delta)$, for $n = 1,2,...$ Hence we have

$$\|z_n - l_j\| \leqslant \chi(Z) + \delta.$$

Further, we obtain

$$\begin{aligned}
&\|[P_\alpha(t_{2,n}-s_n)-P_\alpha(t_{1,n}-s_n)]z_n\| \\
&\leqslant \|P_\alpha(t_{2,n}-s_n)(z_n-l_j)\|+\|P_\alpha(t_{2,n}-s_n)l_j-P_\alpha(t_{1,n}-s_n)l_j\| \\
&+\|P_\alpha(t_{1,n}-s_n)l_j-P_\alpha(t_{1,n}-s_n)z_n\| \\
&\leqslant \frac{2M}{\Gamma(\alpha)}\|z_n-l_j\|+\|[P_\alpha(t_{2,n}-s_n)-P_\alpha(t_{1,n}-s_n)]l_j\|. \qquad (3.5)
\end{aligned}$$

Letting $n\to\infty$ and using the properties of the semigroup $\{U(s)\}_{\{0\leqslant s\leqslant t\}}$, from the above estimate we get

$$\limsup_{n\to\infty}\|[P_\alpha(t_{2,n}s_n)-P_\alpha(t_{1,n}-s_n)]z_n\|\leqslant\chi(Z)+\delta.$$

This contradicts (3.3) and (3.4).
Now, fix $\varepsilon>0$ and $t_1,t_2\in[0,T],\ 0\leqslant t_2-t_1\leqslant\varepsilon$. Applying the assumption (H_f) we get

$$\begin{aligned}
\|Hx(t_2)-Hx(t_1)\| \leqslant & \int_0^{t_1}\frac{\|[P_\alpha(t_2-s)-P_\alpha(t_1-s)]f(s,x(s))\|}{(t_2-s)^{1-\alpha}}ds \\
&+\int_0^{t_1}\|P_\alpha(t_1-s)f(s,x(s))\|\Big(\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}\Big)ds \\
&+\int_{t_1}^{t_2}\frac{\|P_\alpha(t_2-s)f(s,x(s))\|}{(t_2-s)^{1-\alpha}}ds.
\end{aligned}$$

Keeping in mind that $\dfrac{1}{(t_2-s)^{1-\alpha}}\leqslant\dfrac{1}{(t_1-s)^{1-\alpha}}$, we derive the following inequality

$$\begin{aligned}
&\|Hx(t_2)-Hx(t_1)\| \\
&\leqslant \sup\{\|P_\alpha(t_2-s)-P_\alpha(t_1-s)z\|:0\leqslant s\leqslant t_1\leqslant t_2\leqslant t,t_2-t_1\leqslant\varepsilon,z\in Z\} \\
&\quad\times\int_0^{t_1}\frac{ds}{(t_1-s)^{1-\alpha}} \\
&+\frac{M}{\Gamma(\alpha)}\sup\{\|f(s,x(s)\|:s\in[0,t],x\in X\}\int_0^{t_1}\Big(\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}\Big)ds \\
&+\frac{M}{\Gamma(\alpha)}\sup\{\|f(s,x(s)\|:\ s\in[0,t],x\in X\}\int_{t_1}^{t_2}\frac{ds}{(t_2-s)^{1-\alpha}}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\|Hx(t_2)-Hx(t_1)\| \\
&\leqslant \frac{t^\alpha}{\alpha}\sup\{\|(P_\alpha(t_2-s)-P_\alpha(t_1-s))z\|:0\leqslant s\leqslant t_1\leqslant t_2\leqslant T,t_2-t_1\leqslant\varepsilon,\ z\in Z\} \\
&+\frac{M}{\Gamma(\alpha)}\sup\{\|f(s,x(s))\|:\ s\in[0,t],x\in X\}\Big[(t_2-t_1)^\alpha+t_1^\alpha-t_2^\alpha\Big] \\
&+\frac{M}{\alpha\Gamma(\alpha)}\sup\{\|f(s,x(s))\|:\ s\in[0,t],x\in X\}(t_2^\alpha-t_1^\alpha).
\end{aligned}$$

Letting $\varepsilon \to 0^+$ and keeping in mind (3.1) we get

$$w_0^t(HX) \leqslant 2M\frac{t^\alpha}{\Gamma(\alpha+1)}\chi\Big(f([0,t]\times X)\Big).$$

LEMMA 6. [13] *Assume that assumptions (A_g) are satisfied and let $X \subset C([0,T],E)$ be a bounded set. Then*

$$w_0^t(GX) \leqslant 2M\chi(g(X)), \textit{ for } t \in [0,T].$$

THEOREM 2. *If the Banach space E is separable then under assumptions (A_f), (A_g), (A_1) and (A_2), Equation (1.1) with initial condition has at least one mild solution $x = x(t)$.*

Proof. For any arbitrarily fixed $x \in C([0,T],E)$ and $t \in [0,T]$, let r be a positive number satisfying to the inequality of assumption (A_1).

$$\begin{aligned}\|(Fx)(t)\| &\leqslant \|(Hx)(t)\| + \|(Gx)(t)\|\\ &\leqslant M\|g(x)\| + \frac{M}{\Gamma(\alpha)}\int_0^t \frac{m(s)}{(t-s)^{1-\alpha}}\phi(\|x(s)\|)ds\\ &\leqslant M\sup_{x\in B(r)}\|g(x)\| + \frac{M}{\Gamma(\alpha)}\phi(r)\sup_{t\in[0,T]}\int_0^t \frac{m(s)}{(t-s)^{1-\alpha}}ds < r. \qquad (3.6)\end{aligned}$$

The above inequality show that F is a self-mapping of $B(r)$. Next, we prove that operator F is continuous in $B(\theta,r)$. To do this, take arbitrary $x, x_n \in B(\theta,r)$ such that $x_n \to x \in C([0,T],E)$. Observe that

$$\frac{\|f(s,x_n(s)) - f(s,x(s))\|}{\Gamma(\alpha)(t-s)^{1-\alpha}} \leqslant 2\frac{\phi(r)}{\Gamma(\alpha)}\frac{m(s)}{(t-s)^{1-\alpha}}ds \in L^1[0,T].$$

So by Lebesgue dominated convergence theorem and assumption $(A_g)(i)$ we derive that F is continuous on $B(r)$. Further, let us consider the sequence (Q_n) of subsets of $C([0,T],E)$, where $Q_0 = B(r)$ and $Q_n = Conv(FQ_{n-1})$ for $n \in \mathbb{N}$. Observe that all of this sequence are nonempty, closed and convex.
Moreover, $Q_{n+1} \subset Q_n$ for $n \in \mathbb{N}$. Further, let us put

$$u_n(t) = \chi(Q_n([0,t])), \; v_n(t) = w_0^t(Q_n).$$

Observe that each of functions $u_n(t)$ and $v_n(t)$ are nondecreasing, while sequences $(u_n(t))$ and $(v_n(t))$ are nonincreasing at any fixed $t \in [0,T]$. Then sequences $(u_n(t))$ and $(v_n(t))$ have limits. Let

$$u_\infty(t) = \lim_{n\to\infty} u_n(t) \text{ and } v_\infty(t) = \lim_{n\to\infty} v_n(t), \text{ for } t \in [0,T].$$

By Lemmas 6 and (A_g) we get

$$\chi(GQ_n([0,t])) \leqslant w_0^t(GQ_n) + \sup_{s\leqslant t}\chi(GQ_n(s))$$

$$
\begin{aligned}
&\leqslant 2M\chi(g(Q_n)) + \sup_{s\leqslant t}\chi(GQ_n(s)) \\
&\leqslant 3M\chi(g(Q_n([0,T]))) \\
&\leqslant 3Mk_g u_n(T).
\end{aligned} \tag{3.7}
$$

Moreover, taking into account Lemmas 5, 6, and $(A_f)(iii)$ we infer the following estimate

$$
\begin{aligned}
\chi(HQ_n([0,t])) &\leqslant w_0^t(HQ_n) + \sup_{s\leqslant t}\chi(HQ_n(s)) \\
&\leqslant \frac{2Mt^\alpha}{\Gamma(\alpha+1)}\chi(f([0,t]\times Q_n) \\
&+ \sup_{s\leqslant t}\chi\Big(\int_0^s \frac{P_\alpha(s,\tau)}{(s-\tau)^{1-\alpha}} f(\tau,Q_n(\tau))d\tau\Big) \\
&\leqslant \frac{2Mt^\alpha}{\Gamma(\alpha+1)}\chi(f([0,t]\times Q_n) \\
&+ \frac{Mk_f}{\Gamma(\alpha)}\sup_{s\leqslant t}\int_0^s \frac{d\tau}{(s-\tau)^{1-\alpha}}\chi(Q_n(\tau))d\tau \\
&\leqslant \frac{2Mk_f t^\alpha}{\Gamma(\alpha)}u_n(t) + \frac{Mk_f}{\Gamma(\alpha+1)}\sup_{s\leqslant t}\int_0^s \frac{d\tau}{(s-\tau)^{1-\alpha}}u_n(\tau)d\tau.
\end{aligned} \tag{3.8}
$$

Then

$$
\begin{aligned}
u_{n+1}(t) = \chi(Q_{n+1}([0,t])) = \chi(FQ_n([0,t])) &\\
\leqslant \chi(HQ_n[0,t]) + \chi(GQ_n[0,t]) &\\
\leqslant 3Mk_g u_n(T) &\\
+ \frac{2Mk_f t^\alpha}{\Gamma(\alpha)}u_n(t) + \frac{Mk_f}{\Gamma(\alpha)}\sup_{s\leqslant t}\int_0^s \frac{d\tau}{(s-\tau)^{1-\alpha}}u_n(\tau)d\tau.&
\end{aligned} \tag{3.9}
$$

Letting $n\to\infty$ we get

$$
u_\infty(t) \leqslant 3Mk_g u_\infty(T) + \frac{2Mk_f t^\alpha}{\Gamma(\alpha)}u_\infty(t) + \frac{Mk_f}{\Gamma(\alpha)}\sup_{s\leqslant t}\int_0^t \frac{d\tau}{(s-\tau)^{1-\alpha}}u_\infty(\tau)d\tau \tag{3.10}
$$

keeping in mind that the functions $u_n(t)$ is nondecreasing , we get

$$
u_\infty(t) \leqslant 3Mk_g u_\infty(T) + \frac{3Mk_f t^\alpha}{\Gamma(\alpha+1)}u_\infty(t). \tag{3.11}
$$

Putting $t=T$,

$$
u_\infty(t) \leqslant 3M\Big(k_g + \frac{3Mk_f T^\alpha}{\Gamma(\alpha+1)}\Big)u_\infty(T). \tag{3.12}
$$

In view of (A_2) we conclude that

$$
u_\infty(T) = 0. \tag{3.13}
$$

Moreover, applying Lemmas (3.3), (3.2), $(A_g)(ii)$ and $(A_f)(iii)$ we derive

$$v_{n+1}(t) = w_0^t(FQ_n) \leqslant w_0^t(HQ_n) + w_0^t(GQ_n) \\ \leqslant 2k_g M u_n(T) + \frac{2Mk_f t^\alpha}{\Gamma(\alpha+1)} u_n(t). \tag{3.14}$$

Letting $n \to \infty$ we get

$$v_\infty(t) \leqslant 2k_g M u_\infty(T) + \frac{2Mk_f t^\alpha}{\Gamma(\alpha+1)} u_\infty(t).$$

Putting $t = T$, and keeping in mind (3.13) we conclude

$$v_\infty(T) = 0.$$

Then, on has proved that $\lim_{n\to\infty} \mu(Q_n) = 0$. Hence, in view of Remark 1, we deduce that the set $Q_\infty = \bigcap_{n\geqslant 0} Q_n$ is nonempty, compact and convex. Then, by Schauder theorem we conclude, that the operator $F : Q_\infty \to Q_\infty$ has at least one fixed point $x = x(t)$. This completes the proof.

4. Final remark

In this section we are going to discuss the assumptions of theorem 2.

PROPOSITION 1. *Assume that f satisfies the Lipschiz conditions i.e there exist a constant $k_f > 0$ such that $\|f(t,x) - f(t,y)\| \leqslant k_f \|x - y\|$ for any $t \in [0,T]$ and for all $x, y \in E$. Then f satisfies the hypothesis $(A_f)(iii)$.*

Proof. Let us take $X \subset C([0,T],E)$ be a nonempty and bounded subset and fix $t \in [0,T]$ nd $\varepsilon > 0$. Let $\rho = \chi(X[0,t])$, then by definition of χ, there exist $a_1, a_2, \ldots, a_n \in E$ such that $X([0,t]) \subset \cup_1^n B(a_i, \varepsilon)$. By the continuity of $t \mapsto f(t, a_i)$ on $[0,t]$ for $i = 1,2,\ldots,n$ we deduce that there exist a partition $0 = b_1 < b_2 < \ldots < b_n = t$ of the interval $[0,t]$ such that for each $s \in [b_{j-1}, b_j]$ for $j = 1,2,\ldots,p$

$$\|f(s_{j-1}, a_i) - f(s_j, a_i)\| \leqslant \varepsilon, \text{ for } i = 1,1,2,\ldots,n.$$

let $s \in [0,t]$ and $x \in X$. For i and j chooses such that $s \in [b_{j-1}, b_j]$ we get

$$\begin{aligned} \|f(s,x(s)) &- f(b_j,a_i)\| \\ &\leqslant \|f(s,x(s)) - f(b_j,a_i)\| + \|f(s,x(s)) - f(s,a_i)\| + \|f(s,a_i) - f(b_j,a_i)\| \\ &\leqslant k_f \|x(s) - a_i\| + \varepsilon \\ &\leqslant k_f(\rho + \varepsilon) + \varepsilon. \end{aligned} \tag{4.1}$$

This prove that $f([0,t] \times X) \subset \cup_1^n \cup_1^p B(f(b_j,a_i), k_f(\rho+\varepsilon)+\varepsilon)$. Therefore

$$\chi(f([0,t] \times X)) \leqslant k_f \chi(X[0,t]).$$

PROPOSITION 2. *Assume that g is compact, then the hypothesis $(A_g)(ii)$ is satisfies and A_2 can be replaced by*

$$(A_2')\ \ k_f < \frac{\Gamma(\alpha+1)}{3MT^{\alpha}}.$$

5. Application

In what follows we investigate some particular cases. Let $X = L^2(\mathbb{R}^n)$. Consider the following fractional parabolic nonlocal Cauchy problem:

$$\begin{cases} D^{\alpha}u(t,z) = (\Delta u)(t,z) + f(t,u(t,z)) & t \in [0,1],\ z \in \mathbb{R}^n, \\ u(0,z) = g(z) \end{cases} \tag{5.1}$$

where D^{α} is the Caputo fractional partial derivative of order $0 < \alpha < 1$, f is a given function. Moreover,

$$(\Delta u)(t,z) = \Sigma_{i,j=1}^{n} a_{i,j}(z)\frac{\partial u}{\partial z_i \partial z_j}(t,z) + \Sigma_{i=1}^{n} b_i(z)\frac{\partial u}{\partial z_i}(t,z) + \overline{c}(z)u(t,z), \tag{5.2}$$

where the coefficients $a_{i,j}$, b_i, $\overline{c}$, $i,j = 1,2,...,n$ satisfy the usual uniformly ellipticity conditions.
We define an operator A by $A = L$ with the domain

$$D(A) = \{v(.) \in X : H^2(\mathbb{R}^n)\}.$$

From [19], we know that A generates an analytic, noncompact semigroup $\{U(t)\}_{t \geqslant 0}$ on $L^2(\mathbb{R}^n.)$ In addition, there exists a constant $M > 0$ such that

$$M = \sup\{\|U(t)\|\ ;\ t \geqslant\} < \infty.$$

Let's take $\alpha = \frac{1}{2}$ and $f(t,x(t)) = t^{-\frac{1}{2}} \tan x(t)$. Then from $\|f(t,x(t))\| \leqslant \frac{\pi}{2} t^{-\frac{1}{2}}$, we get $(A_f)(i)$ and $(A_f)(ii)$ holds with $\phi(\|x\|) = 1$. From

$$\|f(t,x(t)) - f(t,y(t))\| \leqslant t^{-\frac{1}{2}} \|x-y\|_{\infty}$$

and the proposition 1 we get that $(A_f)(iii)$ is satisfied.

Now, we estimate the constant k_g from assumption A_g in two cases.

(1) If the function $g : C([0,1],E) \to E$ is given by formula

$$g(x) = \sum_{i=1}^{p} c_i x(t_i),$$

where $x(t_i) = u(t_i,.)$, that is $x(t_i)z = u(t_i,z)$, $z \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$, $t_i \in [0,1]$, $i = 1,2,...,p$. Then it is easy to show that

$$\chi(g(X)) \leqslant \Big(\sum_{i=1}^{p} |c_i|\Big)\chi(X), \text{ for } X \subset C([0,1],E).$$

Take $k_g = \sum_{i=1}^{p} |c_i|$, then the assumption A_g is satisfied.

(2) Let us take now

$$g(x)=\int_0^1 l(s,x(s))ds,$$

wher $l:[0,1]\times E\to E$ is Carathéodory function and there exists a function $\psi\in L^1([0,1],\mathbb{R}_+)$ such that for any bounded $X\subset E$

$$\chi(l(t,X))\leqslant\psi(t)\chi(X),\ \text{for a.e}\ t\in[0,1]$$

Using Lemma 3 we obtain

$$\chi(g(X))\leqslant 2\Big(\int_0^1\psi(t)dt\Big)\chi(X).$$

Hence, (A_g) is satisfied with constant $k_g=2\int_0^1\psi(t)dt$.

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