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DYNAMICAL BEHAVIOR OF A STAGE STRUCTURED ECO–EPIDEMIOLOGICAL MODEL

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Abstract. In this paper, a stage structured eco-epidemiological model with linear functional response is proposed and studied. The stages for both prey and predator have been considered. Infection occurs in the prey population only. The proposed mathematical model consists of five nonlinear ordinary differential equations to describe the interaction among juvenile prey, adult prey, infected prey, juvenile predator and adult predator populations. The model is analyzed by using linear stability analysis to obtain the conditions for which our model exhibits stability around the possible equilibrium points.

1. Introduction

Many of the 'old' mathematical equations resulting from the progress of mathematical ecology such as classical Volterra prey-predator model, Lotka-Volterra competetition equations and logistic growth equations[1, 46], have imposed a huge amount of influence on the growth of theortical and mathematical ecology. They provided the fundamental basis to many (if not almost) to the underlie subdisciplines of the subject. Without paying due credit to them is not possible to model any ecological problem. Truly speaking they are the father of the further development in the subject. The main motivation behind writing this paper is these old studies.

The basic goal in population dynamics is to study the dynamical relationship between predator and prey which has long been and will continue to be one of the important aspect in the subject. This relationship may be formulated by the term so called 'functional response', which means the change in the quantity of prey consumed by a single predator per unit time in relation to prey density. In literature there have been proposed number of functional responses. Few of them are enlisted for ready reference (without full detail):

- $g(x) = C(t)x$: Holling type I or linear function response [7, 8, 59].
- $g(x) = \frac{C(t)x}{m+x}$: Holling type II [7, 8, 59, 54].
- $g(x) = \frac{C(t)x^p}{1+mx^p}, 0 < p \leqslant 1$: Generalized type II Holling functional response [9].

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- $g(x) = \frac{C(t)x^2}{m+x^2}$: Holling type III [7, 8, 59].
- $g(x) = \frac{C(t)x}{a+x+\frac{x^2}{m}}$: Holling type IV [59].
- $g(x,y) = \frac{xy}{ay+x}$: Ratio dependent [60, 53, 35].
- $g(x,y) = \frac{Cxy}{(1+ax)(1+by)}$: Crowley-Martin type functional response [32, 40].
- Hassell-Verley type functional response [25, 40].
- $g(x,y) = \frac{Cx}{1+k_1x+k_2y}$:Beddington-De Anglis type [11, 40, 18].
- Function response of the type $g(x) = k(1-e^{-Cx})$ [Ivlev][15].

These functional responses has been extensively used. It is obvious that all the function responses convey their importance. It is also remarkable that Holling type functional responses are more frequently used as compare to other functional responses. Recently a very good study of Dawes JH and Souza MO (2013) [9] on Holling type functional responses is published. They derive Holling's type I,II,III responses and possible generalization of these responses. It is also important to mention that all these functional reponses somewhat generalize the Lotka-Volterra model (response).

A general prey-predator is of the form

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta g(x), \\ \frac{dy}{dt} = \delta g(x) - \gamma y, \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are prey and predator populations respectively at any time t; $\frac{dx}{dt}, \frac{dy}{dt}$ represents growth rate of two populations over time t and the constants $\alpha, \beta, \delta, \gamma$ govern interaction among two populations. Function $g(x)$ is the functional response.

If $g(x)$ follows the Holling type I functional response, then model (1) takes the form

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy, \\ \frac{dy}{dt} = \delta xy - \gamma y, \end{cases} \tag{2}$$

Holling type I is linear as used in Lotka-Volterra system, hence it is oldest among all the functional responses. Since the intake rate is constant in this case, hence the curve represented by the Holling type I functional response is linear, passing through origin and unbounded. In other words, the model (2) is the Lotka-Volterra functional response.

If $g(x)$ follows the Holling type II functional response, then model (1) takes the form

$$\begin{cases} \frac{dx}{dt} = \alpha x - \frac{\beta xy}{m+x}, \\ \frac{dy}{dt} = \frac{\delta xy}{m+x} - \gamma y, \end{cases} \tag{3}$$

this model may also written as

$$\begin{cases} \frac{dx}{dt} = \alpha x - \frac{\beta xy}{m+ahx}, \\ \frac{dy}{dt} = \frac{\delta xy}{m+ahx} - \gamma y, \end{cases} \tag{4}$$

or

$$\begin{cases} \frac{dx}{dt} = \alpha x - \frac{\beta xy}{1+ahx}, \\ \frac{dy}{dt} = \frac{\delta xy}{1+ahx} - \gamma y, \end{cases} \tag{5}$$

here a and h are attack rate and handling time of a predator respectively. In the case of Holling type II, the curve is rectangular hyperbola(for example Holling's disc equation)[7, 8]. Hence, it is clear that model (3-5) are more general as compare to (2). More detail about the Type I,II,III responses may be found in [7, 8].

Model (1) may be written by other functional responses as mentioned in the discussion above. For example;

Holling type IV model;

$$\begin{cases} \frac{dx}{dt} = \alpha x - \frac{\beta yx}{a+x+\frac{x^2}{m}}, \\ \frac{dy}{dt} = \frac{\beta yx}{a+x+\frac{x^2}{m}} \delta xy - \gamma y, \end{cases} \tag{6}$$

Ratio dependent model;

$$\begin{cases} \frac{dx}{dt} = \alpha x - \frac{\beta xy}{ay+x}, \\ \frac{dy}{dt} = \frac{\delta xy}{ay+x} - \gamma y, \end{cases} \tag{7}$$

Crowley-Martin model;

$$\begin{cases} \frac{dx}{dt} = \alpha x - \frac{Cxy}{(1+ax)(1+by)}, \\ \frac{dy}{dt} = \frac{Cxy}{(1+ax)(1+by)} - \gamma y, \end{cases} \tag{8}$$

and so on.

Till now, we have mentioned two dimensional models. Three dimensional models have also been proposed by incorporating some more concepts. We mention two such concepts (i) infection in species (ii) stages of life. If infection occurs in prey species then model (2) becomes three dimensional model and takes the form;

$$\begin{cases} \frac{dx}{dt} = \alpha x - \xi xy, \\ \frac{dy}{dt} = \xi xy - d_1 y - pyz, \\ \frac{dz}{dt} = qpyz - d_2 y, \end{cases} \tag{9}$$

where x, y and z denote, respectively, the population densities of suspectable prey species, infected prey species and predator species. The model (9) with logistic growth term takes the form

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x+y}{K}) - \xi xy, \\ \frac{dy}{dt} = \xi xy - d_1 y - pyz, \\ \frac{dz}{dt} = qpyz - d_2 y. \end{cases} \tag{10}$$

The dynamical behavior of system (10) has been studied in [16]. For more studies wherein infection in prey have been considered, we refer [3, 10, 12, 27, 30, 37, 58, 39]. If we consider infection in predator population then again we have the three dimensional prey-predator system. For such systems, we refer reader to ([33, 48, 13] and references therein). Similarly, if one consider the stages of prey i.e. juvenile and adult, we have a

three dimensional model. For prey-predator system with stage-structured for prey, we cite ([23, 41] and references therein). If we consider the stages for predator, we have a three dimensional system. For such models, we refer reader to [24, 31, 36, 50, 61, 20].

For four dimensional models, in literature two situations (i) infection in both species (ii) stages for both species have been proposed. We can refer to [22, 43] for such four dimensional models. High dimensional models (dimension 5 and more) are rare. Five dimensional models evolved when infection and stages have been considered [44]. Motivated by this fact, in the present study, a five dimensional prey-predator model is proposed and analyzed.

Rest of the paper is organized as follows. The next Section is dealing with the model formulation. Stability results are presented in Section 3. Paper ends with a brief discussion in Section 4.

2. Mathematical model and basic dynamical results

In this paper, we proposed a new prey-predator model

$$\begin{cases} \frac{dx_1}{dt} = rx_1 - \beta x_1 - d_1 x_1 - a x_1 x_5, \\ \frac{dx_2}{dt} = \beta x_1 - d_2 x_2 - b x_2 x_5 - \gamma x_2 x_3, \\ \frac{dx_3}{dt} = \gamma x_2 x_3 - d_3 x_3, \\ \frac{dx_4}{dt} = k_1 a x_1 x_5 + k_2 b x_2 x_5 - \delta x_4 - d_4 x_4, \\ \frac{dx_5}{dt} = \delta x_4 - d_5 x_5, \end{cases} \tag{11}$$

subjected to the initial conditions;

$$x_1(0) > 0, x_2(0) > 0, x_3(0) > 0, x_4(0) > 0, x_5(0) > 0. \tag{12}$$

In system (11), $x_1(t), x_2(t), x_3(t), x_4(t)$ and $x_5(t)$ stand for juvenile prey, adult prey, infected prey, juvenile predator and adult predator densities, at time t respectively. The constants $r, \beta, d_1, a, d_2, b, \gamma, d_3, k_1, k_2, \delta, d_4, d_5$ are positive. That stand for

r: birth rate of the juvenile prey;
β: the transmission rate of juvenile prey to adult one;
d_1: death rate of juvenile prey;
a: the capturing rate of juvenile prey by the adult predator;
d_2: death rate of adult prey;
b: the capturing rate of adult prey by the adult predator;
γ: infection coefficient of prey;
d_3: death rate of infected prey;
k_1, k_2: the coefficients of conversing prey to predator;
δ: the transmission rate of juvenile predator to adult one;
d_4: death rate of juvenile predator;
d_5: death rate of adult predator.

The model (11) is derived under the following ecological assumptions.

(A1) We assume that $x(t)$ is the total prey density at any time t. Prey density is divided into three parts viz. juvenile prey $(x_1(t))$, adult prey $(x_2(t))$ and infected prey $(x_3(t))$. Hence, at any time t, we have,

$$x(t) = (x_1(t) + x_2(t) + x_3(t)). \tag{13}$$

It is assumed that birth and death rates of all the species are linear.
(A2) We assume that $y(t)$ is the total predator density at any time t. Predator density is divided into two parts viz. juvenile predator $(x_4(t))$ and adult predator $(x_5(t))$. Hence, at any time t, we have,

$$y(t) = (x_4(t) + x_5(t)). \tag{14}$$

(A3) We assume that infection occurs in prey species only. Infection is not communicable to predator population. To do so, we assume that predator consume juvenile and adult prey only (infected prey are safe from predation). The infected prey population neither recover nor immune. The infection process follows simple mass action law action $\gamma x_2 x_3$, γ is called the transmission rate.
(A4) We also assume that only adult predator can hunt the prey population. Juvenile predator population depend on adult predator population for food and safety etc. We also assume that predator predate the juvenile and adult prey at different rates.

LEMMA 1. *Solutions of model (11) corresponding to the initial conditions (12) are defined on the interval* $[0, +\infty)$ *and remain positive for all time* $t \geqslant 0$.

Proof. The system of equations (11) can be written in the vector notation

$$\frac{dX(t)}{dt} = \mathbb{A}(X(t)), \tag{15}$$

where,

$$X(t) = col(x_1, x_2, x_3, x_4, x_5) \in R_+^3$$
$$X(0) = col(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) \in R_+^3$$

and

$$\mathbb{A}(X(t)) = \begin{pmatrix} \mathbb{A}_1(X(t)) \\ \mathbb{A}_2(X(t)) \\ \mathbb{A}_3(X(t)) \\ \mathbb{A}_4(X(t)) \\ \mathbb{A}_5(X(t)) \end{pmatrix}$$
$$= \begin{pmatrix} rx_1 - \beta x_1 - d_1 x_1 - a x_1 x_5 \\ \beta x_1 - d_2 x_2 - b x_2 x_5 - \gamma x_2 x_3 \\ \gamma x_2 x_3 - d_3 x_3 \\ k_1 a x_1 x_5 + k_2 b x_2 x_5 - \delta x_4 - d_4 x_4 \\ \delta x_4 - d_5 x_5 \end{pmatrix}$$

with $\mathbb{A} : \mathbb{R}^5 \to \mathbb{R}_+^5$ and $\mathbb{A} \in C^\infty(\mathbb{R}^5)$.

It is clear that in the Eqs. (15), $\mathbb{A}_i(X_i)\,|_{X_i=0} \geqslant 0$, for $i = 1, 2, ..5$. Due to the general classical theorem introduced by Nagumo[29], the solution of (15) with initial conditions $\mathbb{A}(0) \in \mathbb{R}^5$, say $\mathbb{A} = \mathbb{A}(t; \mathbb{A}_0)$, such that $\mathbb{A} \in \mathbb{R}^5, \forall t \geqslant 0$ that is for all finite time.

LEMMA 2. *Solutions of model (11) corresponding to the initial conditions (12) which initiate in R^5_+ are uniformly bounded within the region Ω.*

Proof. Case I. $k_1 = k_2 = 1$.
Let

$$W = x_1 + x_2 + x_3 + x_4 + x_5, \tag{16}$$

differentiation of (16) and using Eq. (11), we get

$$\frac{dW}{dt} = (r - d_1)x_1 - d_2x_2 - d_3x_3 - d_4x_4 - d_5x_5. \tag{17}$$

For a constant η, by Eq. (17), we have

$$\begin{cases} \frac{dW}{dt} + \eta W = (r - d_1 + \eta)x_1 - (d_2 - \eta)x_2 - (d_3 - \eta)x_3 \\ -(d_4 - \eta)x_4 - (d_5 - \eta)x_5. \end{cases} \tag{18}$$

If we choose $\eta = \min\{d_1, d_2, d_3, d_4, d_5\}$, Eq. (18) gives

$$\frac{dW}{dt} + \eta W \leqslant (r + \eta + \varepsilon), \varepsilon > 0, \tag{19}$$

If we denote $K' = (r + \eta + \varepsilon)$, we have

$$\frac{dW}{dt} + \eta W \leqslant K', \tag{20}$$

integrating both sides, due to [5, 14, 21, 42], the above inequality (20), we get

$$0 < \leqslant \frac{\eta W(0) - K'}{\eta} e^{-\eta t} + \frac{K'}{\eta}, \tag{21}$$

taking $t \to \infty$, we have from (21),

$$0 < W(x_1, x_2, x_3, x_4, x_5) \leqslant \frac{K'}{\eta}. \tag{22}$$

Hence, from Eq. (22) it can be concluded that, all the solutions of system (11), initiating from $\{R^5_+ \backslash 0\}$ are confined in the region $\Omega = \{(x_1, x_2, x_3, x_4, x_5) \in R^5_+, W = \frac{K'}{\eta} + \varepsilon_1\}$ for any $\varepsilon_1 > 0$ and $t \to \infty$, hence the proof completed.

The two cases viz. (i) $k_1 = k_2 \neq 1$ (ii) $k_1 \neq k_2$ may also be proceeded similarly.

2.1. Equilibria and their existence

Infection free equilibrium $(E_{x_1x_2x_4x_5})$ may be obtained by solving the following system;

$$\begin{cases} rx_1 - \beta x_1 - d_1x_1 - ax_1x_5 = 0, \\ \beta x_1 - d_2x_2 - bx_2x_5 = 0, \\ k_1ax_1x_5 + k_2bx_2x_5 - \delta x_4 - d_4x_4 = 0, \\ \delta x_4 - d_5x_5 = 0, \end{cases} \tag{23}$$

The components of equilibrium point $E_{x_1x_2x_4x_5}$ are given by

$$\begin{cases} x_1 = \frac{x_2(d_2+bx_5)}{\beta}, \\ x_2 = \frac{\frac{(\delta+d_4)x_4}{x_5}}{\frac{(d_2+bx_5)k_1a}{\beta}+k_2b}, \\ x_4 = \frac{d_5(r-\beta-d_1)}{\delta a}, \\ x_5 = \frac{(r-\beta-d_1)}{a}, \end{cases} \tag{24}$$

Ecologically feasible co-existing equilibrium point may be obtained by solving the following system;

$$\begin{cases} rx_1 - \beta x_1 - d_1x_1 - ax_1x_5 = 0, \\ \beta x_1 - d_2x_2 - bx_2x_5 - \gamma x_2x_3 = 0, \\ \gamma x_2x_3 - d_3x_3 = 0, \\ k_1ax_1x_5 + k_2bx_2x_5 - \delta x_4 - d_4x_4 = 0, \\ \delta x_4 - d_5x_5 = 0, \end{cases} \tag{25}$$

in $R^5_+ = \{(x_1,x_2,x_3,x_4,x_5) \in R^5 : x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0, x_5 \geqslant 0\}$. Let co-existing equilibrium point(s)be $E^* = (x_1^*,x_2^*,x_3^*,x_4^*,x_5^*)$. Mathematically, components of interior equilibrium point $E^* = (x_1^*,x_2^*,x_3^*,x_4^*,x_5^*)$ are given by

$$\begin{cases} x_1^* = \left(\frac{(\delta+d_4)d_5}{\delta k_1a} - \frac{k_2bd_3}{\gamma k_1a}\right), \\ x_2^* = \frac{d_3}{\gamma}, \\ x_3^* = \left(\frac{(\delta+d_4)\beta d_5}{d_3\delta k_1a} - \frac{k_2b\beta}{\gamma k_1a} - \frac{d_2}{\gamma} - \frac{b(r-\beta-d_1)}{\gamma a}\right), \\ x_4^* = \frac{d_5(r-\beta-d_1)}{\delta a}, \\ x_5^* = \frac{(r-\beta-d_1)}{a}, \end{cases} \tag{26}$$

It is also important to mention that, our model (11) has unique co-existing equilibrium point $E^* = (x_1^*,x_2^*,x_3^*,x_4^*,x_5^*)$. Hence, the co-existing equilibrium $E^* = (x_1^*,x_2^*,x_3^*,x_4^*,x_5^*)$ exists provided the following conditions are satisfied;

$$\begin{cases} (\delta+d_4)d_5\gamma > k_2b\delta d_3, \\ \left(\frac{(\delta+d_4)\beta d_5}{d_3\delta k_1a} - \frac{k_2b\beta}{\gamma k_1a} - \frac{d_2}{\gamma} - \frac{b(r-\beta-d_1)}{\gamma a}\right) > 0, \\ (r-\beta-d_1) > 0. \end{cases} \tag{27}$$

REMARK 1. Equilibrium points other then $E_{x_1x_2x_4x_5}$, E^* (as mentioned above) viz. $E_{x_2x_3x_4x_5}$, $E_{x_1x_3x_4x_5}$, $E_{x_1x_2x_3x_5}$, $E_{x_1x_2x_3x_4}$, $E_{x_3x_4x_5}$, $E_{x_1x_4x_5}$, $E_{x_1x_2x_5}$, $E_{x_1x_2x_3}$, $E_{x_2x_4x_5}$, $E_{x_2x_3x_5}$ etc. do not exist.

3. Stability Analysis

At any non zero point (x_1,x_2,x_3,x_4,x_5), Jacobian matrix of Eq. (11) is given by

$$J=\begin{pmatrix} a_{11} & 0 & 0 & 0 & a_{15} \\ a_{21} & a_{22} & a_{23} & 0 & a_{25} \\ 0 & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{pmatrix}, \tag{28}$$

where, $a_{11}=(r-\beta-d_1-ax_5),\ a_{15}=-ax_1,\ a_{21}=\beta,\ a_{22}=-d_2-bx_5-\gamma x_3,\ a_{23}=-\gamma x_2,\ a_{25}=-bx_2, a_{32}=\gamma x_3, a_{33}=\gamma x_2-d_3, a_{41}=k_1ax_5, a_{42}=k_2bx_5, a_{44}=-\delta-d_4, a_{45}=k_1ax_1+k_2bx_2, a_{54}=\delta, a_{55}=-d_5.$

3.1. Infection free equilibrium $(E_{x_1x_2x_4x_5})$

At the point $(x_1,x_2,0,x_4,x_5)$, Jacobian matrix J (corresponding to $(E_{x_1x_2x_4x_5})$) reduced to the following form

$$J=\begin{pmatrix} a_{11} & 0 & 0 & 0 & a_{15} \\ a_{21} & a_{22} & a_{23} & 0 & a_{25} \\ 0 & 0 & a_{33} & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{pmatrix}, \tag{29}$$

where, $a_{11}=(r-\beta-d_1-ax_5),\ a_{15}=-ax_1,\ a_{21}=\beta,\ a_{22}=-d_2-bx_5,\ a_{23}=-\gamma x_2,\ a_{25}=-bx_2, a_{33}=\gamma x_2-d_3, a_{41}=k_1ax_5, a_{42}=k_2bx_5, a_{44}=-\delta-d_4, a_{45}=k_1ax_1+k_2bx_2, a_{54}=\delta, a_{55}=-d_5.$ And x_1,x_2,x_4,x_5 are listed at Eq. (24). One eigenvalue of Eq. (29) is a_{33} and rest four are the eigen values of the matrix

$$\begin{pmatrix} a_{11} & 0 & 0 & a_{15} \\ a_{21} & a_{22} & 0 & a_{25} \\ a_{41} & a_{42} & a_{44} & a_{45} \\ 0 & 0 & a_{54} & a_{55} \end{pmatrix}, \tag{30}$$

the characteristic equation of Eq. (30) is given by

$$(\lambda^3+A_1\lambda^2+A_2\lambda+A_3)\lambda=0, \tag{31}$$

where,

$$\begin{cases} A_1=-[a_{11}+a_{22}+a_{44}+a_{55}], \\ A_2=-a_{44}a_{54}+a_{11}a_{44}, \\ A_3=(a_{45}a_{54})a_{11}. \end{cases} \tag{32}$$

Hence, characteristic equation of Eq. (29) is given by

$$(\lambda^3+A_1\lambda^2+A_2\lambda+A_3)\lambda(\lambda-a_{33})=0, \tag{33}$$

Therefore, '0' is an eigenvalue, hence, the infection free equilibrium point $(E_{x_1x_2x_4x_5})$ is unstable.

3.2. The co-existing equilibrium point $E^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$

The Jacobian matrix corresponding to E^* is given by

$$J = \begin{pmatrix} a_{11}^* & 0 & 0 & 0 & a_{15}^* \\ a_{21}^* & a_{22}^* & a_{23}^* & 0 & a_{25}^* \\ 0 & a_{32}^* & a_{33}^* & 0 & 0 \\ a_{41}^* & a_{42}^* & 0 & a_{44}^* & a_{45}^* \\ 0 & 0 & 0 & a_{54}^* & a_{55}^* \end{pmatrix}, \tag{34}$$

where, $a_{11}^* = (r - \beta - d_1 - ax_5^*)$, $a_{15}^* = -ax_1^*$, $a_{21}^* = \beta$, $a_{22}^* = -d_2 - bx_5^* - \gamma x_3^*$, $a_{23}^* = -\gamma x_2^*$, $a_{25}^* = -bx_2^*, a_{32}^* = \gamma x_3^*, a_{33}^* = \gamma x_2^* - d_3, a_{41}^* = k_1 ax_5^*, a_{42}^* = k_2 bx_5^*, a_{44}^* = -\delta - d_4, a_{45}^* = k_1 ax_1^* + k_2 bx_2^*, a_{54}^* = \delta, a_{55}^* = -d_5$.

The characteristic equation of Eq. (34) is given by

$$\lambda^5 + B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5 = 0, \tag{35}$$

where,

$$\begin{cases} B_1 = a_{11}^* + a_{22}^* + a_{33}^* + a_{44}^* + a_{55}^*, \\ B_2 = (a_{44}^* a_{55}^* - a_{45}^* a_{54}^*) + (a_{22}^* a_{33}^* - a_{23}^* a_{32}^*) + (a_{44}^* + a_{55}^*) \\ + a_{11}^*(a_{44}^* + a_{55}^*) + a_{11}^*(a_{22}^* + a_{33}^*), \\ B_3 = (a_{44}^* + a_{55}^* - 1)(a_{22}^* a_{33}^* - a_{32}^* a_{23}^*) + (a_{22}^* + a_{33}^* - 1)(a_{44}^* a_{55}^*, \\ - a_{45}^* a_{54}^*) - (a_{44}^* + a_{55}^*)(a_{22}^* + a_{33}^*) - a_{54}^* a_{15}^* a_{41}^*, \\ B_4 = a_{11}^*(a_{22}^* + a_{33}^*)(a_{44}^* a_{55}^* - a_{45}^* a_{54}^*) + a_{11}^*(a_{44}^* + a_{55}^*), \\ (a_{22}^* a_{33}^* - a_{23}^* a_{32}^*) - (a_{44}^* a_{55}^* - a_{45}^* a_{54}^*)(a_{22}^* a_{33}^* - a_{23}^* a_{32}^*) \\ - a_{25}^* a_{54}^* a_{42}^* a_{33}^* + a_{11}^* a_{25}^* a_{54}^* a_{42}^* - a_{15}^* a_{42}^* a_{21}^* + a_{15}^*(a_{22} + a_{33}^*) a_{41}^*, \\ B_5 = a_{54}^* a_{15}^*(a_{21}^* a_{33}^* a_{42}^* + a_{23}^* a_{41}^* a_{32}^* - a_{41}^* a_{22}^* a_{33}^*), \\ - a_{25}^* a_{54}^* a_{42}^* a_{33}^* - (a_{44}^* a_{55}^* - a_{45}^* a_{54}^*)(a_{22}^* a_{33}^* - a_{23}^* a_{32}^*). \end{cases} \tag{36}$$

The co-existing equilibrium point $E^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ for the system (11) is locally asymptotically stable (using the Routh-Hurwitz criteria) if the following conditions hold as follows:

$$\begin{cases} B_i > 0, i = 1, 2, 3, 4, 5, \\ B_1 B_2 B_3 > B_3^2 + B_1^2 B_4, \\ (B_1 B_4 - B_5)(B_1 B_2 B_3 - B_3^2 - B_1^2 B_4) > B_5(B_1 B_2 - B_3)^2 + B_1 B_5^2. \end{cases} \tag{37}$$

REMARK 2. The existence of co-existing equilibrium point $E^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ convey a message that all the five species viz. immature prey, mature prey, infected prey, immature predator and mature predator exist in the ecosystem. It means that existence of E^* implies that infection in the system also exists. Infection free equilibrium point $(E_{x_1 x_2 x_4 x_5})$ is not stable. It means that once the infection occurs in the system, it can not remove. Because the co-existing equilibrium point E^* may be stable if the conditions listed in (37) are satisfied.

4. Discussion

In this paper, a stage structured prey-predator model is proposed and studied. Stages for prey and predator both have been considered, therefore the prey population is bifurcated into two populations viz. immature prey and mature prey and similarly predator population is also bifurcated into two classes viz. immature predator and mature predator. We also considered that infection occurs in the prey population only. By remark 1, it is observed that only two equilibrium points for the model (11) viz. infection free equilibrium $(E_{x_1x_2x_4x_5})$ and the co-existing equilibrium point $E^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ exist. Local stability analysis have been investigated and results show that the infection free equilibrium point $(E_{x_1x_2x_4x_5})$ is not stable. The co-existing equilibrium point $E^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ is conditionally stable i.e. it is stable provided the set of conditions listed in the Eq. (37) are satisfied. In real life situations the parameters are changing with time. Hence, models with time dependent parameters may be included in the future scope. As a matter of fact, this study is not a case study hence real data/parameters are not available. Real parameters investigation is also a concern of future study.

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