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INFINITELY MANY SOLUTIONS FOR A FOURTH–ORDER NONLINEAR ELLIPTIC SYSTEM

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Abstract. In this paper we study the existence of solutions for the nonlinear elliptic system

$$\begin{cases} \Delta^2 u - \Delta u + V_1(x)u = f_u(x,u,v), \\ \Delta^2 v - \Delta v + V_2(x)v = f_v(x,u,v), \\ u,v \in H^2(\mathbb{R}^N) \qquad x \in \mathbb{R}^N, \end{cases}$$

where $V_1(x)$ and $V_2(x)$ are positive continue functions. Under some assumptions on $f_u(x,u,v)$ and $f_v(x,u,v)$, we prove the existence of many nontrivial high and small energy solutions by variant Fountain theorems. This generalizes the results by Y. Ye and C. Tang (J. Math. Anal. Appl. 394, 841-854, 2012) to fourth-order nonlinear elliptic system.

1. Introduction

This paper deals with the existence of infinitely many solutions for the fourth-order nonlinear elliptic system

$$\begin{cases} \Delta^2 u - \Delta u + V_1(x)u = f_u(x,u,v), \\ \Delta^2 v - \Delta v + V_2(x)v = f_v(x,u,v), \\ u,v \in H^2(\mathbb{R}^N),\ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geqslant 1$, $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $F = F(x,u,v)$, $f_u = \frac{\partial F}{\partial u}$, and $f_v = \frac{\partial F}{\partial v}$. $V_1(x)$, $V_2(x)$ and $F(x,u,v)$ are positive functions. We are interested in the existence of many nontrivial high and small energy solutions.

The study of fourth-order elliptic equations appears to be important in many areas including the study of travelling waves in suspension bridges and static deflection of an elastic plate in a fluid. We refer to [7] and their references. The fourth-order elliptic problems have been extensively studied in recent years, obtained numerous results on existence, multiplicity of the positive solutions, see for example [8, 10, 11, 12, 13, 14,

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17, 20, 19, 15, 18, 2, 5]. In [20], Zhou and Wu investigated the fourth-order nonlinear elliptic boundary value problems

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

they obtained the existence and multiplicity of sign-changing solutions by variational techniques. Yang and Zhang in [14] proved the existence of positive, negative and sign-changing solutions of (1.2) by invariant sets of the gradient flows of the corresponding variational functionals. In [10], Wang et al. showed that problem (1.2) exist at least three nontrivial solutions by linking approaches. In [12], Ye and Tang obtained the existence of infinitely many large-energy and small-energy solutions for the fourth-order elliptic equation:

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.3}$$

by Rabinowitz's symmetric mountain pass theorem, where $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Zhang and Tang in [18] established the existence of infinitely many small energy solutions of (1.3) by using the genus properties in critical point theory where the nonlinearity $f(x,u)$ is indefinite sign and sublinear at infinity. Cheng in [2] considered the existence of high energy solutions of (1.3) by using some special techniques. Under more relaxed assumptions on $V(x)$, Ye and Tang established the existence and multiplicity of solutions for a class of fourth-order elliptic equations with a parameter $\lambda \geqslant 1$ large enough.

In [6], Jung and Choi studied the fourth-order elliptic system with Dirichlet boundary condition:

$$\begin{cases} \Delta^2 u + c\Delta u = a\big((u+v+1)^+ - 1\big) & \text{in } \Omega, \\ \Delta^2 v + c\Delta v = a\big((u+v+1)^+ - 1\big) & \text{in } \Omega, \\ u = 0, \quad v = 0, \quad \Delta u = 0 \quad \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

they proved the existence of nontrivial solutions via linking method and the contraction mapping principle on the Banach space. In [1], Afrouzi et al. studied the existence and multiplicity of solutions for a class of nonlocal fourth-order systems by critical point theory.

To the best of our knowledge, the existence of infinitely many nontrivial solutions of (1.1) has not ever been considered by variational methods. Our main objective in this article is to study the existence of infinitely many nontrivial high and small energy solutions for (1.1). Unlike the Rabinowitz's symmetric mountain pass arguments in [12], our main idea is to use the variant Fountain theorems generalizes the results in [12] to fourth-order nonlinear elliptic system.

In this paper, we assume

(H1) $V_i(x)$ are continuous in $\mathbb{R}^N$, and $\inf_{x\in\mathbb{R}^N} V_i(x) > 0$ $(i = 1,2)$. For each $M > 0$, meas $\{x \in \mathbb{R}^N : V_i(x) < M\} < \infty$ $(i = 1,2)$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$;

(H2) $F(x,u,v) \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$;
(H3) There exist $a_1,\ a_2 > 0,\ r \in (2, 2_*)$ where $2_* = \infty$ for $N \leqslant 4$ and $2_* = \frac{2N}{N-4}$ for $N > 4$, such that

$$|f_u(x,u,v)| + |f_v(x,u,v)| \leqslant a_1(|u|^{r-1} + |v|^{r-1}) + a_2(|u| + |v|), \ \ \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2;$$

(H4) There exist $\delta,\ \sigma \in (1,2),\ r \in (2, 2_*)$ and $a_3 > 0,\ \eta > 0,\ \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$, such that

$$\zeta(x)(|u|^\delta + |v|^\delta) \leqslant f_u(x,u,v)u + f_v(x,u,v)v$$

and

$$|f_u(x,u,v)| + |f_v(x,u,v)| \leqslant \eta m(x)(|u|^{\sigma-1} + |v|^{\sigma-1}) + a_3(|u|^{r-1} + |v|^{r-1}),$$

where $\zeta(x) \in L^{\frac{2}{2-\delta}}(\mathbb{R}^N)$, $m(x) \in L^{\frac{2}{2-\sigma}}(\mathbb{R}^N)$, $\zeta(x) > 0$, $m(x) > 0$, for $x \in \mathbb{R}^N$;
(H5) There exists $\mu \in (2, r)$, such that

$$\lim_{(|u|^2+|v|^2)\to\infty} \frac{f_u(x,u,v)u + f_v(x,u,v)v}{(|u|^2+|v|^2)^{\frac{\mu}{2}}} \geqslant C > 0$$

uniformly for $x \in \mathbb{R}^N$;
(H6) There exists $\theta \geqslant 1$, such that $\forall s \in [0,1]$,

$$\theta \pounds(x,u,v) \geqslant \pounds(x,su,sv), \qquad \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $\pounds(x,u,v) = \frac{1}{2}(f_u(x,u,v)u + f_v(x,u,v)v) - F(x,u,v)$;
(H7) $F(x,u,v) = F(x,-u,-v) \quad$ for $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$.

REMARK 1. With these assumptions on F, we give the following examples of F:
(i)

$$F(x,u,v) = (2 + \varepsilon^{\frac{1}{1+|x|}})(|u|^p + |v|^p + |u|^2 + |v|^2),$$

(ii)

$$F(x,u,v) = \frac{1+\sin^2 x_1}{1+|x|^{\frac{N}{2}}}(|u|^\sigma + |v|^\sigma) + |u|^p + |v|^p.$$

where $x = \{x_1, x_2, \ldots, x_N\}$, $2 < p < 2_*$, $1 < \sigma < 2$. $\square$

We state the main theorems in this paper:

THEOREM 1. *Assume that* $(H1)-(H3)$, $(H5)-(H7)$ *hold, then problem* (1.1) *possesses infinitely many high energy solutions* (u^k, v^k) *for all* $k \geqslant k_0$ $(k_0 \in \mathbb{N})$, *in the sense that*

$$\begin{aligned} I(u^k,v^k) =& \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\Delta u^k|^2 + |\Delta v^k|^2 + |\nabla u^k|^2 + |\nabla v^k|^2 + V_1(x)(u^k)^2 + V_2(x)(v^k)^2\Big)dx \\ &- \int_{\mathbb{R}^N} F(x,u^k,v^k)dx \to +\infty, \qquad \textit{as } \ k \to \infty. \end{aligned}$$

THEOREM 2. *Assume that $(H1)$, $(H2)$, $(H4)-(H7)$ hold, there exists $\Lambda_0 > 0$, when $\eta < \Lambda_0$, then problem (1.1) possesses infinitely many small energy solutions (u^k, v^k) for $k \in \mathbb{N}$, in the sense that*

$$
\begin{aligned}
I(u^k, v^k) =& \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\Delta u^k|^2 + |\Delta v^k|^2 + |\nabla u^k|^2 + |\nabla v^k|^2 + V_1(x)(u^k)^2 + V_2(x)(v^k)^2\Big) dx \\
& - \int_{\mathbb{R}^N} F(x, u^k, v^k) dx \to 0-, \qquad as \;\; k \to \infty.
\end{aligned}
$$

The paper is organized as follows. In Section 2, we present some preliminary results and prove some lemmas. In Section 3, we prove our main Theorems.

2. The variational framework and preliminary results

Consider the Sobolev space $X = H^2(\mathbb{R}^N)$ endowed with the norm

$$
\|u\|_X = \left(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2) dx\right)^{\frac{1}{2}}.
$$

Now, we define the subspaces

$$
W_1 = \left\{u \in X \,|\, \int_{\mathbb{R}^N} V_1(x) u^2 < \infty\right\},
$$

$$
W_2 = \left\{v \in X \,|\, \int_{\mathbb{R}^N} V_2(x) v^2 < \infty\right\}.
$$

Obviously, W_1 and W_2 are Hilbert spaces endowed with the norm respectly

$$
\|u\|_{W_1} = \left(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V_1(x)u^2) dx\right)^{\frac{1}{2}}, \qquad u \in W_1,
$$

$$
\|v\|_{W_2} = \left(\int_{\mathbb{R}^N} (|\Delta v|^2 + |\nabla v|^2 + V_2(x)v^2) dx\right)^{\frac{1}{2}}, \qquad v \in W_2.
$$

Problem (1.1) is posed in the framework of the Hilbert space $W = W_1 \times W_2$ with the standard norm

$$
\|(u,v)\|_W^2 = \|u\|_{W_1}^2 + \|v\|_{W_2}^2.
$$

In addition, we define $|u|_p = (\int_{\mathbb{R}^N} |u|^p dx)^{\frac{1}{p}}$, which is the usual norm in $L^p(\mathbb{R}^N)$ and $L_2^p(\mathbb{R}^N) = L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ with the norm $|(u,v)|_p = (|u|_p^p + |v|_p^p)^{\frac{1}{p}}$. It is well known that under assumption $(H1)$, the embedding $W_1 \hookrightarrow L^p(\mathbb{R}^N)$, $W_2 \hookrightarrow L^p(\mathbb{R}^N)$ are compact for $p \in [2, 2_*)$, where $2_* = +\infty$ for $N \leqslant 4$ and $2_* = \frac{2N}{N-4}$ for $N > 4$.

A pair of functions $(u,v) \in W$ is said to be a weak solution of problem (1.1) if

$$
\int_{\mathbb{R}^N} \big(\Delta u \Delta \varphi_1 + \Delta v \Delta \varphi_2 + \nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 + V_1(x) u \varphi_1 + V_2(x) v \varphi_2\big) dx
$$

$$-\int_{\mathbb{R}^N} f_u(x,u,v)\varphi_1 + f_v(x,u,v)\varphi_2 dx = 0$$

for all $(\varphi_1,\varphi_2) \in W$.

The corresponding energy functional of problem (1.1) is defined by

$$I(u,v) = \frac{1}{2}\|(u,v)\|_W^2 - \int_{\mathbb{R}^N} F(x,u,v)dx.$$

In order to verify $I(u,v) \in C^1(W,R)$, we need the following lemmas

DEFINITION 1. (Definition 3.2, [4]) On the space $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, we define the norm

$$|u|_{p\wedge q} = |u|_p + |u|_q,$$

on the space $L_2^p(\mathbb{R}^N) \cap L_2^q(\mathbb{R}^N)$, we define the norm

$$|(u,v)|_{p\wedge q} = |(u,v)|_p + |(u,v)|_q,$$

on the space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$, we define the norm

$$|u|_{p\vee q} = \inf\big\{|v|_p + |w|_q : v \in L^p(\mathbb{R}^N), w \in L^q(\mathbb{R}^N), u = v + w\big\}.$$

LEMMA 1. (Lemma 3.3, [4]) *Assume that* $1 \leqslant p,\ r,\ q,\ s < \infty,\ f \in C(\mathbb{R}^N \times \mathbb{R}^2)$ *and*

$$f(x,u,v) \leqslant C_1(|u|^{\frac{p}{r}} + |v|^{\frac{p}{r}}) + C_2(|u|^{\frac{q}{s}} + |v|^{\frac{q}{s}}),$$

then, for every $(u,v) \in L_2^p(\mathbb{R}^N)\bigcap L_2^q(\mathbb{R}^N)$, $f(\cdot,u,v) \in L^r(\mathbb{R}^N) + L^s(\mathbb{R}^N)$, *and the operator*

$$T : L_2^p(\mathbb{R}^N) \cap L_2^q(\mathbb{R}^N) \to L^r(\mathbb{R}^N) + L^s(\mathbb{R}^N) : (u,v) \to f(x,u,v)$$

is continuous.

Now we consider the functional $\psi(u,v) = \int_{\mathbb{R}^N} F(x,u,v)dx$, then we have the following result.

LEMMA 2. *Assume that* $(H1)-(H3)$ *hold, then* $\psi(u,v) \in C^1(W,R)$ *and*

$$\big\langle \psi'(u,v),(\varphi_1,\varphi_2)\big\rangle = \int_{\mathbb{R}^N} (f_u(x,u,v)\varphi_1 + f_v(x,u,v)\varphi_2)dx,$$

where (u,v), $(\varphi_1,\varphi_2) \in W$.

Proof. It follows from (H_3) that

$$\begin{aligned} F(x,u,v) &= \int_0^1 \frac{dF(x,tu,tv)}{dt}dt \\ &= \int_0^1 f_u(x,tu,tv)u + f_v(x,tu,tv)v dt \end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^1 (|f_u(x,tu,tv)| + |f_v(x,tu,tv)|)(|u|+|v|)dt \\
&\leqslant \int_0^1 (a_1(|tu|^{r-1} + |tv|^{r-1}) + a_2(|tu|+|tv|)(|u|+|v|))dt \\
&\leqslant 2a_1(|u|^r+|v|^r)\int_0^1 t^{r-1}dt + 2a_2(|u|^2+|v|^2)\int_0^1 tdt \\
&\leqslant a_1(|u|^r+|v|^r) + a_2(|u|^2+|v|^2). \qquad (2.1)
\end{aligned}
$$

Let $g(x) := a_1(|u(x)|^r + |v(x)|^r) + a_2(|u(x)|^2+|v(x)|^2)$, then $g(x) \in L^1(\mathbb{R}^N)$. So, $I(u,v)$ is well defined.

First, we prove the existence of the Gateaux derivative. Given $(\varphi_1,\varphi_2) \in W$, $|t| \in [0,1]$, $\theta \in (0,1)$, then

$$
\begin{aligned}
&\left|\frac{F(x,u+t\varphi_1,v+t\varphi_2) - F(x,u,v)}{t}\right| \\
=&\left(|f_u(x,u+t\theta\varphi_1,v+t\theta\varphi_2)||t\varphi_1| + |f_v(x,u+t\theta\varphi_1,v+t\theta\varphi_2)||t\varphi_2|\right) \times \frac{1}{|t|} \\
\leqslant& a_1\left(|u+t\theta\varphi_1|^{r-1} + |v+t\theta\varphi_2|^{r-1}\right) + a_2\left(|u+t\theta\varphi_1| + |v+t\theta\varphi_2|\right)\left(|\varphi_1|+|\varphi_2|\right) \\
\leqslant& C\left(|u|^{r-1} + |\varphi_1|^{r-1} + |v|^{r-1} + |\varphi_2|^{r-1} + |u| + |\varphi_1| + |v| + |\varphi_2|\right)\left(|\varphi_1|+|\varphi_2|\right).
\end{aligned}
$$

The Hölder inequality and the sobolev imbedding theorem imply that

$$
\left(|u|^{r-1} + |\varphi_1|^{r-1} + |v|^{r-1} + |\varphi_2|^{r-1} + |u| + |\varphi_1| + |v| + |\varphi_2|\right)\left(|\varphi_1|+|\varphi_2|\right) \in L^1(\mathbb{R}^N).
$$

It follows from the Lebesgue theorem that

$$
\left\langle \psi'(u,v),(\varphi_1,\varphi_2)\right\rangle = \int_{\mathbb{R}^N}\left(f_u(x,u,v)\varphi_1 + f_v(x,u,v)\varphi_2\right)dx.
$$

Next, we prove the continuity of the Gateaux derivative. Assume that $(u_n,v_n) \to (u,v)$ in W. By the sobolev imbedding theorems, $(u_n,v_n) \hookrightarrow (u,v)$ in $L_2^p(\mathbb{R}^N)$ for $p \in [2,2_*)$. By the Lemma 2.2, we obtain $f_u(x,u_n,v_n) \to f_u(x,u,v)$ and $f_v(x,u_n,v_n) \to f_v(x,u,v)$ in $L^{r'}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$, where $r' := \frac{r}{r-1}$. By the Hölder inequality and the Sobolev imedding theorem, we get

$$
\begin{aligned}
&\left|\left\langle \psi'(u_n,v_n) - \psi'(u,v),(\varphi_1,\varphi_2)\right\rangle\right| \\
\leqslant&\left|\int_{\mathbb{R}^N}(f_u(x,u_n,v_n) - f_u(x,u,v))\varphi_1| + |(f_v(x,u_n,v_n) - f_v(x,u,v))\varphi_2 dx\right| \\
\leqslant&|f_u(x,u_n,v_n) - f_u(x,u,v)|_{r'\vee 2}|\varphi_1|_{r\wedge 2} + |f_v(x,u_n,v_n) - f_v(x,u,v)|_{r'\vee 2}|\varphi_2|_{r\wedge 2} \\
\to& 0. \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence, by the above lemmas, we have $I(u,v) \in C^1(W,R)$. The proof is completed. $\square$

LEMMA 3. *Assume that* $(H1)$, $(H2)$ *and* $(H4)$ *hold, then* $\psi(u,v)\in C^1(W,R)$, *and* $\langle\psi'(u,v),(\varphi_1,\varphi_2)\rangle=\int_{\mathbb{R}^N}(f_u(x,u,v)\varphi_1+f_v(x,u,v)\varphi_2)dx$, *where* (u,v), $(\varphi_1,\varphi_2)\in W$.

Proof. By $(H4)$, similarly as in the Lemma 2, it is deduced that

$$F(x,u,v)=\int_0^1\frac{dF(x,tu,tv)}{dt}dt\leqslant 2\eta m(x)(|u|^\sigma+|v|^\sigma)+a_3(|u|^r+|v^r|). \tag{2.2}$$

By the Höld inequality and sobolev imbedding theorem, it can be deduced that

$$2\eta m(x)(|u|^\sigma+|v|^\sigma)+a_3(|u|^r+|v^r|)\in L^1(\mathbb{R}^N),$$

then $I(u,v)$ is well defined.

Almost as the same as in Lemma 2, the Gateaux derivative of $\psi(u,v)$ exists, and

$$\langle\psi'(u,v),(\varphi_1,\varphi_2)\rangle=\int_{\mathbb{R}^N}(f_u(x,u,v)\varphi_1+f_v(x,u,v)\varphi_2)dx.$$

Let $(u_n,v_n)\to(u,v)$ in W, then

$$u_n\to u \quad \text{in } L^p(\mathbb{R}^N), \quad \text{for } p\in[2,2_*), \tag{2.3}$$

$$v_n\to v \quad \text{in } L^p(\mathbb{R}^N), \quad \text{for } p\in[2,2_*), \tag{2.4}$$

$$u_n\to u \quad \text{a.e. } x\in\mathbb{R}^N, \tag{2.5}$$

$$v_n\to v \quad \text{a.e. } x\in\mathbb{R}^N. \tag{2.6}$$

We claim that

$$\left|\int_{\mathbb{R}^N}\big(f_u(x,u_n,v_n)-f_u(x,u,v)\big)\varphi_1dx\right|\to 0, \quad \text{as } n\to\infty,\ \forall\,\varphi_1\in W_1. \tag{2.7}$$

Otherwise, $\exists\ \varepsilon_0>0$, and a subsequence, denoted by $\{(u_{nk},v_{nk})\}$, such that

$$\left|\int_{\mathbb{R}^N}\big(f_u(x,u_{nk},v_{nk})-f_u(x,u,v)\big)\varphi_1dx\right|>\varepsilon_0, \quad \text{as } k\to\infty. \tag{2.8}$$

Since $(u_n,v_n)\to(u,v)$ in $L_2^2(\mathbb{R}^N)$, it can be assumed that

$$|u_{n(k+1)}-u_{nk}|_2\leqslant 2^{-k}, \quad |v_{n(k+1)}-v_{nk}|_2\leqslant 2^{-k},\ \forall k\geqslant 1.$$

Let us define

$$(\omega_1(x),\omega_2(x)):=\Big(|u_{n1}(x)|+\sum_{k=1}^{+\infty}|u_{n(k+1)}(x)-u_{nk}(x)|,$$
$$|v_{n1}(x)|+\sum_{k=1}^{+\infty}|v_{n(k+1)}(x)-v_{nk}(x)|\Big).$$

And $\omega_1(x) \in L^2(\mathbb{R}^N)$, $\omega_2(x) \in L^2(\mathbb{R}^N)$. It is clear that

$$|u_{nk}(x)| \leqslant \omega_1(x), \quad |v_{nk}(x)| \leqslant \omega_2(x).$$

Note that, by $(H4)$

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \big(f_u(x,u_{nk},v_{nk}) - f_u(x,u,v)\big)\varphi_1 dx\right| \\
\leqslant &\int_{\mathbb{R}^N} \big(|f_u(x,u_{nk},v_{nk})| + |f_u(x,u,v)|\big)|\varphi_1| dx \\
\leqslant &\int_{\mathbb{R}^N} \big[\eta m(x)(|u_{nk}|^{\sigma-1} + |v_{nk}|^{\sigma-1} + |u|^{\sigma-1} + |v|^{\sigma-1}) \\
&+ a_3(|u_{nk}|^{r-1} + |v_{nk}|^{r-1} + |u|^{r-1} + |v|^{r-1})\big]|\varphi_1| dx \\
\leqslant &\int_{\mathbb{R}^N} \big[\eta m(x)(|\omega_1(x)|^{\sigma-1} + |\omega_2(x)|^{\sigma-1} + |u|^{\sigma-1} + |v|^{\sigma-1}) \\
&+ a_3(|\omega_1(x)|^{r-1} + |\omega_2(x)|^{r-1} + |u|^{r-1} + |v|^{r-1})\big]|\varphi_1| dx.
\end{aligned}$$

By the Lebesgue dominated convergence theorem and from (2.5), (2.6), it can be deduced that

$$\lim_{k\to\infty}\left|\int_{\mathbb{R}^N} \big(f_u(x,u_{nk},v_{nk}) - f_u(x,u,v)\big)\varphi_1 dx\right| \to 0, \tag{2.9}$$

which is contradict to (2.8). Therefore, (2.7) holds. Similarly,

$$\left|\int_{\mathbb{R}^N} \big(f_v(x,u_n,v_n) - f_v(x,u,v)\big)\varphi_2 dx\right| \to 0, \qquad \text{as } \ n\to\infty, \ \ \forall \ \ \varphi_2 \in W_2. \tag{2.10}$$

Then,

$$\begin{aligned}
\left|\left\langle \psi'(u_n,v_n) - \psi'(u,v),(\varphi_1,\varphi_2)\right\rangle\right| \leqslant &\left|\int_{\mathbb{R}^N} \big(f_u(x,u_n,v_n) - f_u(x,u,v)\big)\varphi_1 dx\right| \\
&+\left|\int_{\mathbb{R}^N} \big(f_v(x,u_n,v_n) - f_v(x,u,v)\big)\varphi_2 dx\right| \\
&\to 0,
\end{aligned}$$

as $n\to\infty$. Hence, $\psi(u,v) \in C^1(W,R)$. The proof is completed. □

From the above lemma, replace the $(H3)$ with $(H4)$, $I(u,v) \in C^1(W,R)$. Let W be a banach space with the norm $\|.\|$ and let W_j be a sequences of subspaces of W with dim $W_j < \infty$ for each $j \in \mathbb{N}$. Further, $W = \overline{\oplus_{j\in N} W_j}$, the closure of the direct sum of all W_j. Set

$$Y_k = \oplus_{j=0}^{k} W_j, \qquad Z_k = \overline{\oplus_{j=k}^{\infty} W_j}$$

and

$$B_k = \{u \in Y_k : \|u\| \leqslant \rho_k\} \qquad S_k = \{u \in Z_k : \|u\| = r_k\},$$

for $\rho_k > r_k > 0$. Consider a familarly of C^1-functionals $I_\lambda : W \to \mathbb{R}$ defined by

$$I_\lambda = A(u) - \lambda B(u).$$

The following two variant fountain theorems were established in [21].

THEOREM 3. *Assume that the functional I_λ defined above satisfies*
(A1) I_λ *maps bounded sets into bounded sets uniformly for* $\lambda \in [1,2]$, *and* $I_\lambda(-u) = I_\lambda(u)$ *for all* $\lambda \in [1,2]$, $u \in W$;
(A2) $B(u) \geqslant 0$ *for all* $u \in W$, $A(u) \to \infty$ *or* $B(u) \to \infty$ *as* $\|u\| \to \infty$;
or
(A3) $B(u) \leqslant 0$ *for all* $u \in W, B(u) \to -\infty$ *as* $\|u\| \to \infty$;
(A4) *There exists* $\rho_k > r_k > 0$ *such that*

$$b_k(\lambda) = \lim_{u \in Z_k, \|u\| = r_k} I_\lambda(u) > a_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u), \qquad \forall \lambda \in [1,2],$$

Then

$$b_k(\lambda) \leqslant c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)), \qquad \forall \lambda \in [1,2],$$

where $\Gamma_k = \{\gamma \in C(B_k, W) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$ $(k \geqslant 2)$. *Moreover, for almost every* $\lambda \in [1,2]$, *there exists a sequence* $\{u_n^k(\lambda)\}$ *such that*

$$\sup_n \|u_n^k(\lambda)\| < \infty, \qquad I_\lambda'(u_n^k(\lambda)) \to 0 \qquad \textit{and} \qquad I_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \qquad \text{as } n \to \infty.$$

THEOREM 4. *Assume that I_λ defined above satisfies*
(B1) I_λ *maps bounded sets into bounded sets uniformly for* $\lambda \in [1,2]$, *and* $I_\lambda(-u) = I_\lambda(u)$ *for all* $(\lambda, u) \in [1,2] \times W$;
(B2) $B(u) \geqslant 0$ *for all* $u \in W$ *and* $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of W;*
(B3) *There exists* $\rho_k > r_k > 0$ *such that*

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geqslant 0, \qquad b_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) < 0, \qquad \forall \lambda \in [1,2]$$

and

$$d_k(\lambda) = \inf_{u \in Z_k, \|u\| \leqslant \rho_k} I_\lambda(u) \to 0, \qquad k \to \infty, \qquad uniformly \ \ for \ \ \lambda \in [1,2].$$

Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ *such that*

$$I_{\lambda_n}'|_{Y_n}(u(\lambda_n)) = 0 \qquad \textit{and} \qquad I_{\lambda_n}(u(\lambda_n)) \to c_k \qquad \textit{as} \qquad n \to \infty,$$

where $c_k \in [d_k(2), b_k(1)]$. *In particular, if* $u(\lambda_n)$ *has a convergent subsequence for every* k, *then* I_1 *has infinitely many nontrivial critical points* $u_k \in W \backslash \{0\}$ *satisfying* $I_1(u_k) \to 0^-$ *as* $k \to \infty$.

In order to apply the above two theorems to prove our main results, we define

$$A(u,v) = \frac{1}{2}\|(u,v)\|^2, \qquad B(u,v) = \int_{\mathbb{R}^N} F(x,u,v)dx,$$

and

$$I_\lambda(u,v) = A(u,v) - \lambda B(u,v) = \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\mathbb{R}^N} F(x,u,v)dx,$$

for all $(u,v) \in W$ and $\lambda \in [1,2]$.

3. Existence results

LEMMA 4. *For the any $k-$dimensional subspace $\widetilde{W}$ of W, there exists $\varepsilon_k^p > 0$, $\varepsilon_k^\delta > 0$ such that*

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |u|^p + |v|^p \geqslant \varepsilon_k^p \|(u,v)\|^p\right\} \geqslant \varepsilon_k^p,\ (u,v) \in \widetilde{W}\backslash\{(0,0)\},\ p \in [2,2_*), \tag{3.1}$$

and

$$\operatorname{meas}\{x \in \mathbb{R}^N : \zeta(x)(|u|^\delta + |v|^\delta) \geqslant \varepsilon_k^\delta \|(u,v)\|^\delta\} \geqslant \varepsilon_k^\delta \qquad \forall (u,v) \in \widetilde{W}\backslash\{(0,0)\}, \tag{3.2}$$

where $\delta \in (1,2)$ and $\zeta(x) : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous functional such that $\zeta(x) \in L^{\frac{2}{2-\delta}}(\mathbb{R}^N)$.

The verification of Lemma 4 is almost the same as in [12].

LEMMA 5. *Let*

$$\alpha_k(p) = \sup_{(u,v)\in Z_k, \|(u,v)\|_W = 1} \|(u,v)\|_p \to 0 \qquad k \to \infty, \qquad \text{for } p \in [2,2_*), \tag{3.3}$$

where the Z_k is defined in Theorem 3.

Proof. Suppose that this is not the case, then there exists an $\varepsilon_0 > 0$ and $\{(u_j,v_j)\} \subset W$ with $\{(u_j,v_j)\} \perp W_{k_j-1}$ such that

$$\|u_j,v_j\|_W = 1, \quad \|(u_j,v_j)\|_{L_2^p(\mathbb{R}^N)} \geqslant \varepsilon_0,$$

where $k_j \to \infty$ as $j \to \infty$. For any $(u,v) \in W$, we may find $\{(\overline{u}_j,\overline{v}_j)\} \in W_{k_j-1}$, such that $(\overline{u}_j,\overline{v}_j) \to (u,v)$ as $j \to \infty$. Hence

$$\begin{aligned}
\left|\left\langle (u_j,v_j),(u,v)\right\rangle\right| &= \left|\left\langle (u_j,v_j),(\overline{u}_j,\overline{v}_j)-(u,v)\right\rangle\right| \\
&\leqslant \|(\overline{u}_j - u, \overline{v}_j - v)\| \\
&\to 0 \qquad \text{as } j \to \infty.
\end{aligned}$$

Thus, $(u_j,v_j) \rightharpoonup 0$ in W. By the sobolev theorem, $(u_j,v_j) \to (0,0)$ in $L_2^p(\mathbb{R}^N)$. This is a contradiction. The proof is completed. $\square$

LEMMA 6. *Let $(H1)$, $(H2)$, $(H3)$, and $(H5)$ hold, then there exist $\rho_k > r_k > 0$ such that for all $\lambda \in [1,2]$,*

$$b_k(\lambda) = \lim_{(u,v)\in Z_k, \|(u,v)\| = r_k} I_\lambda(u,v) > a_k(\lambda) = \max_{(u,v)\in Y_k, \|(u,v)\| = \rho_k} I_\lambda(u,v). \tag{3.4}$$

Proof. From (2.1) in Lemma 2, we have

$$F(x,u,v) \leqslant a_1(|u|^r+|v|^r)+a_2(|u|^2+|v|^2) \qquad \forall x \in \mathbb{R}^N, \ \ (u,v) \in \mathbb{R}^2.$$

Therefore, with Lemma 5, $\exists\ k_0$, for $(u,v) \in Z_k$, $k \geqslant k_0$, we have

$$\begin{aligned} I_\lambda(u,v) &= \frac{1}{2}\|(u,v)\|_W^2 - \lambda \int_{\mathbb{R}^N} F(x,u,v)dx \\ &\geqslant \frac{1}{2}\|(u,v)\|_W^2 - \lambda \int_{\mathbb{R}^N} a_1(|u|^r+|v|^r)+a_2(|u|^2+|v|^2)dx \\ &\geqslant \frac{1}{2}\|(u,v)\|_W^2 - 2\alpha_k^r(r)a_1\|(u,v)\|_W^r - 2\alpha_k^2(2)a_2\|(u,v)\|_W^2 \\ &\geqslant \frac{1}{4}\|(u,v)\|_W^2 - \alpha_k^r(r)c\|(u,v)\|_W^r. \end{aligned}$$

If we choose $r_k = (8\alpha_k^r(r)c)^{\frac{1}{2-r}}$, where $c = 2a_1$, then for any $(u,v) \in Z_k$ with $\|(u,v)\| = r_k$, we get

$$I_\lambda(u,v) \geqslant \frac{1}{8}(8\alpha_k^r(r)c)^{\frac{2}{2-r}} > 0,$$

which implies that

$$b_k(\lambda) = \inf_{(u,v)\in Z_k, \|(u,v)\|=r_k} I_\lambda(u,v) \geqslant \frac{1}{8}(8\alpha_k^r(r)c)^{\frac{2}{2-r}} > 0, \qquad \forall \lambda \in [1,2]. \tag{3.5}$$

By (H_5), $\exists\ L > 0$, $|u|^2+|v|^2 \geqslant L$, then

$$f_u(x,u,v)u + f_v(x,u,v)v \geqslant c_1(|u|^2+|v|^2)^{\frac{\mu}{2}} \geqslant c_2(|u|^\mu+|v|^\mu),$$

then

$$\begin{aligned} F(x,u,v) &= \int_0^1 f_u(x,tu,tv)u + f_v(x,tu,,tv)vdt \\ &= \int_0^1 \frac{f_u(x,tu,tv)tu + f_v(x,tu,tv)tv}{t}dt \\ &\geqslant \int_0^1 \frac{c_2(|tu|^\mu + |tv|^\mu)}{t}dt \\ &\geqslant c_3(|u|^\mu + |v|^\mu), \end{aligned} \tag{3.6}$$

where $c_3 = \frac{c_2}{\mu}$.

Note $\Omega_k(\mu) = \{x \in \mathbb{R}^N : |u|^\mu + |v|^\mu \geqslant \varepsilon_k^\mu \|(u,v)\|^\mu\}$. Since $|u|^2+|v|^2 \geqslant (|u|^\mu+|v|^\mu)^{\frac{2}{\mu}}$ with $\frac{2}{\mu} \in (0,1)$ and (3.1) in Lemma 4, there exists R_k with $(\varepsilon_k^\mu R_k^\mu)^{\frac{2}{\mu}} \geqslant L$, $\|(u,v)\| \geqslant R_k$, $(u,v) \in Y_k$, then $|u|^2+|v|^2 \geqslant L$ when $x \in \Omega_k(\mu)$. Hence, for any $(u,v) \in Y_k$,with $\|(u,v)\|_W \geqslant R_k$, we have

$$I_\lambda(u,v) \leqslant \frac{1}{2}\|(u,v)\|_W^2 - \int_{\mathbb{R}^N} F(x,u,v)dx$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}\|(u,v)\|_W^2 - \int_{\Omega_k(\mu)} c_3(|u|^\mu + |v|^\mu)dx \\
&\leqslant \frac{1}{2}\|(u,v)\|_W^2 - c_3(\varepsilon_k^\mu)^2\|(u,v)\|_W^\mu \to -\infty \quad \text{ as } \|(u,v)\| \to \infty.
\end{aligned}
$$

Then, $\exists\ \rho_0 > R_k$, such that $I_\lambda(u,v)|_{\partial B(\rho_0)} < 0$. Then, we choose $\rho_k > \max\{r_k, \rho_0\}$

$$
a_k(\lambda) = \max_{(u,v)\in Y_k, \|(u,v)\|=\rho_k} I_\lambda(u,v) < 0 \qquad \forall k \in \mathbb{N},\ \ \lambda \in [1,2].
$$

The proof is completed. $\square$

LEMMA 7. *Let $(H1)$, $(H2)$, $(H4)$ and $(H5)$ hold, $\exists\ \Lambda_0 > 0$, when $\eta < \Lambda_0$, there exist $\rho_k > r_k > 0$, such that*

$$
a_k(\lambda) = \inf_{(u,v)\in Z_k, \|(u,v)\|=\rho_k} I_\lambda(u,v) \geqslant 0,
$$

$$
b_k(\lambda) = \max_{(u,v)\in Y_k, \|(u,v)\|=r_k} I_\lambda(u,v) < 0, \qquad \forall\ \lambda \in [1,2],
$$

and

$$
d_k(\lambda) = \inf_{(u,v)\in Z_k, \|(u,v)\|\leqslant\rho_k} I_\lambda(u,v) \to 0 \qquad k \to \infty, \qquad \text{uniformly for } \lambda \in [1,2].
$$

Proof. By (2.2) in Lemma 3, for $(u,v) \in Z_k$, it follows that

$$
\begin{aligned}
I_\lambda(u,v) &= \frac{1}{2}\|(u,v)\|_W^2 - \lambda\int_{\mathbb{R}^N} F(x,u,v)dx \\
&\geqslant \frac{1}{2}\|(u,v)\|_W^2 - 4\int_{\mathbb{R}^N} \eta m(x)(|u|^\sigma + |v|^\sigma) + a_3(|u|^r + |v^r|)dx \\
&\geqslant \frac{1}{2}\|(u,v)\|_W^2 - 4\eta|m(x)|_{\frac{2}{2-\sigma}}(|u|_2^\sigma + |v|_2^\sigma) - 2a_3(|u|_r^r + |v|_r^r) \\
&\geqslant \frac{1}{2}\|(u,v)\|_W^2 - 8\eta|m(x)|_{\frac{2}{2-\sigma}}(|u|^2 + |v|^2)^{\frac{\sigma}{2}} - 2a_3(|u|_r^r + |v|_r^r) \\
&\geqslant \frac{1}{2}\|(u,v)\|_W^2 - 8\eta|m(x)|_{\frac{2}{2-\sigma}}\alpha_k^\sigma(2)\|(u,v)\|_W^\sigma - 2c_3\alpha_k^r(r)\|(u,v)\|_W^r \\
&\geqslant \|(u,v)\|_W^\sigma\Big(\frac{1}{2}\|(u,v)\|_W^{2-\sigma} - 8\eta\varepsilon_0|m(x)|_{\frac{2}{2-\sigma}} - c_4\|(u,v)\|_W^{r-\sigma}\Big),
\end{aligned}
$$

where $\varepsilon_0 = \sup_{k\in\mathbb{N}} \alpha_k^\sigma(2)$, $c_4 = 2c_3 \sup_{k\in\mathbb{N}} \alpha_k^r(r)$.

Let

$$
f(t) = \frac{1}{2}t^{2-\sigma} - c_4 t^{r-\sigma}, \ \ t \geqslant 0.
$$

$f_{max} = f(\overline{x})$ with $\overline{x} = (\frac{2-\sigma}{2c_4(r-\sigma)})^{\frac{1}{r-2}}$, such that

$$
f_{max} = (\frac{1}{2})^{\frac{r-\sigma}{r-2}}.c_4^{\frac{2-\sigma}{2-r}}.(\frac{2-\sigma}{r-\sigma})^{\frac{2-\sigma}{r-2}}.\frac{r-2}{r-\sigma} > 0.
$$

Denote $\Lambda_0 = \frac{1}{8}\varepsilon_0^{-1}|m(x)|^{-1}_{\frac{2}{2-\sigma}} f_{max}$, when $\eta < \Lambda_0$, $\rho_k = \overline{x}$. Then

$$a_k(\lambda) = \inf_{(u,v)\in Z_k, \|(u,v)\|=\rho_k} I_\lambda(u,v) \geqslant 0. \tag{3.7}$$

In addition, for all $\lambda \in [1,2]$, and $(u,v) \in Z_k$, with $\|(u,v)\| \leqslant \rho_k$, we have

$$I_\lambda(u,v) \geqslant -8\eta|m(x)|_{\frac{2}{2-\sigma}} \alpha_k^\sigma(2)\|(u,v)\|_W^\sigma - a_3\alpha_k^r(r)\|(u,v)\|_W^r \to 0^-, \quad \text{as } k\to\infty.$$

Therefore

$$d_k(\lambda) = \inf_{(u,v)\in Z_k, \|(u,v)\|\leqslant\rho_k} I_\lambda(u,v) \to 0, \quad \text{as } k\to\infty \quad \text{uniformly for } \lambda\in[1,2]. \tag{3.8}$$

From (H_4), it can be obtained that $\forall (u,v,x) \in \mathbb{R}^N \times \mathbb{R}^2$,

$$\begin{aligned} F(x,u,v) &= \int_0^1 f_u(x,tu,tv)u + f_v(x,tu,tv)v dt \\ &\geqslant \int_0^1 \frac{f_u(x,tu,tv)tu + f_v(x,tu,tv)tv}{t} dt \\ &\geqslant \int_0^1 \frac{\zeta(x)(|tu|^\delta + |tv|^\delta)}{t} dt \\ &\geqslant \frac{1}{2}\zeta(x)(|u|^\delta + |v|^\delta). \end{aligned}$$

So, if $(u,v) \in Y_k$, by Lemma 4, one can get

$$\begin{aligned} I_\lambda(u,v) &= \frac{1}{2}\|(u,v)\|_W^2 - \int_{\mathbb{R}^N} F(x,u,v)dx \\ &\leqslant \frac{1}{2}\|(u,v)\|_W^2 - \frac{1}{2}\int_{\mathbb{R}^N} \zeta(x)(|u|^\delta + |v|^\delta)dx \\ &\leqslant \frac{1}{2}\|(u,v)\|_W^2 - \frac{1}{2}(\varepsilon_k^\delta)^2\|(u,v)\|_W^\delta, \quad x\in\mathbb{R}^N, \ (u,v)\in\mathbb{R}^2. \end{aligned}$$

Hence, we choose $r_k > 0$, small enough satisfying $r_k < \rho_k$ such that

$$b_k(\lambda) = \max_{(u,v)\in Y_k, \|(u,v)\|_W = r_k} I_\lambda(u,v) < 0, \quad \text{for all } \lambda\in[1,2]. \tag{3.9}$$

The proof is completed. □

Proof of Theorem 1.1. From the assumption $(H2)$, we know that $B(u,v) \geqslant 0$ for all $(u,v) \in W$, and $A(u,v) \to \infty$ as $\|(u,v)\| \to \infty$. Moreover, $I_\lambda(-u,-v) = I_\lambda(u,v)$ for all $(u,v) \in W$, and $\lambda \in [1,2]$. It follows from the conditions $(H1)$, $(H2)$ and $(H3)$, $I_\lambda(u,v)$ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. Combining with Lemma 6, $(A1)$, $(A2)$, $(A4)$ of Theorem 3 are verified. Therefore, for $a.e.$ $\lambda\in[1,2]$, there exists a sequence $\{(u_n^k(\lambda), v_n^k(\lambda))\}_{n=1}^\infty$, such that

$$\sup_n \left\| \left(u_n^k(\lambda), v_n^k(\lambda)\right) \right\| < \infty, I_\lambda'\left(u_n^k(\lambda), v_n^k(\lambda)\right) \to 0, \mathrm{I}_\lambda\left(u_n^k(\lambda), v_n^k(\lambda)\right) \to c_k(\lambda) \tag{3.10}$$

as $n \to \infty$. By Theorem 3, (3.5) and the assumption that $r > 2$, it implies that

$$c_k(\lambda) \geqslant b_k(\lambda) = \inf_{(u,v)\in Z_k, \|(u,v)\|=r_k} I_\lambda(u,v) \geqslant \frac{1}{8}(8\alpha_k^r c)^{\frac{2}{2-r}} = \overline{b_k} \to +\infty, \tag{3.11}$$

as $k \to \infty$. Also since

$$c_k(\lambda) = \inf_{r\in\Gamma_k} \max_{(u,v)\in B_k} I_\lambda(u,v) \leqslant \max_{(u,v)\in B_k} I_\lambda(u,v) = \overline{c_k}.$$

Hence,

$$\overline{b_k} \leqslant c_k(\lambda) \leqslant \overline{c_k}, \qquad \text{for } k \geqslant k_0. \tag{3.12}$$

If we choose a sequence $\lambda_m \in [1,2]$, such that $\lambda_m \to 1$, it follows from (3.10) that the sequence $\left\{\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right)\right\}$ is bounded. If we can prove that the sequence $\left\{\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right)\right\}$ has a strong convergent subsequence as $n \to \infty$, we can assume that

$$\lim_{n\to\infty} \left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right) = \left(u^k(\lambda_m), v^k(\lambda_m)\right)$$

for every $m \in \mathbb{N}$ and $k \geqslant k_0$. By (3.10) and (3.12), we can get

$$I'_{\lambda_m}\left((u^k(\lambda_m), v^k(\lambda_m)\right) = (0,0) \text{ and } I_{\lambda_m}\left(u^k(\lambda_m), v^k(\lambda_m)\right) \in [\overline{b_k}, \overline{c_k}] \text{ for } k \geqslant k_0. \tag{3.13}$$

If we can prove that $\left\{\left(u^k(\lambda_m), v^k(\lambda_m)\right)\right\}_{m=1}^{\infty}$ possesses a strong convergent subsequence with the limit $\left(u^k, v^k\right) \in W$ for all $k \geqslant k_0$, the limit $\left(u^k, v^k\right)$ is a critical point of $I(u,v) = I_1(u,v)$ with $I(u^k, v^k) \in [\overline{b_k}, \overline{c_k}]$. Since $\overline{b_k} \to \infty$ as $k \to \infty$, we get infinitely many nontrivial critical points of $I(u,v)$. Consequently, problem (1.1) possesses infinitely many nontrivial solutions with high energy.

(1) We prove that the sequence $\left\{\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right)\right\}_{n=1}^{\infty}$ has a strong convergent subsequence. Since $\left\{\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right)\right\}_{n=1}^{\infty}$ is bounded, up to a subsequence, denoted by $\left\{\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right)\right\}_{n=1}^{\infty}$, there exists sequence $\left(u^k(\lambda_m), v^k(\lambda_m)\right)$, such that

$$\begin{aligned}
&\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right) \rightharpoonup \left(u^k(\lambda_m), v^k(\lambda_m)\right) \quad \text{in } W,\\
&\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right) \to \left(u^k(\lambda_m), v^k(\lambda_m)\right) \quad \text{in } L_2^p(\mathbb{R}^N), \;\; p \in [2, 2_*).
\end{aligned}$$

Again,

$$\begin{aligned}
&\Big\langle dI_{\lambda_m}\left(u_n^k(\lambda_m), v_n^k(\lambda_m)\right) - dI_{\lambda_m}\left(u^k(\lambda_m), v^k(\lambda_m)\right),\\
&\qquad\qquad\qquad \left(u_n^k(\lambda_m) - u^k(\lambda_m), v_n^k(\lambda_m) - v^k(\lambda_m)\right)\Big\rangle\\
&= \left\|\left(u_n^k(\lambda_m) - u^k(\lambda_m), v_n^k(\lambda_m - v^k(\lambda_m)\right)\right\|_W^2\\
&\quad - \lambda_m \int_{\mathbb{R}^N} \left(f_u\left(x, u_n^k(\lambda_m), v_n^k(\lambda_m)\right) - f_u\left(x, u^k(\lambda_m), v^k(\lambda_m)\right)\right)\left(u_n^k(\lambda_m) - u^k(\lambda_m)\right)dx
\end{aligned}$$

$$-\lambda_m\int_{\mathbb{R}^N}\Big(f_v\big(x,u_n^k(\lambda_m),v_n^k(\lambda_m)\big)-f_v\big(x,u^k(\lambda_m),v^k(\lambda_m)\big)\Big)\Big(v_n^k(\lambda_m)-v^k(\lambda_m)\Big)dx. \tag{3.14}$$

Since $dI_{\lambda_m}\big(u_n^k(\lambda_m),v_n^k(\lambda_m)\big)\to 0$ and $\big(u_n^k(\lambda_m),v_n^k(\lambda_m)\big)\rightharpoonup\big(u^k(\lambda_m),v^k(\lambda_m)\big)$ as $n\to\infty$, then

$$\Big\langle dI_{\lambda_m}\big(u_n^k(\lambda_m),v_n^k(\lambda_m)\big)-dI_{\lambda_m}\big(u^k(\lambda_m),v^k(\lambda_m)\big),\\ \big(u_n^k(\lambda_m)-u^k(\lambda_m),v_n^k(\lambda_m)-v^k(\lambda_m)\big)\Big\rangle\to 0.$$

From $(H1),(H3)$, Lemma 1 and $\big\{\big(u_n^k(\lambda_m),v_n^k(\lambda_m)\big)\big\}_{n=1}^{\infty}$ is bounded, it implies

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\Big(f_u\big(x,u_n^k(\lambda_m),v_n^k(\lambda_m)\big)-f_u\big(x,u^k(\lambda_m),v^k(\lambda_m)\big)\Big)\Big(u_n^k(\lambda_m)-u^k(\lambda_m)\Big)dx\right|\\ &\leqslant\Big\|f_u\big(x,u_n^k(\lambda_m),v_n^k(\lambda_m)\big)-f_u\big(x,u^k(\lambda_m),v^k(\lambda_m)\big)\Big\|_{r'\vee 2}\Big\|u_n^k(\lambda_m)-u^k(\lambda_m)\Big\|_{r\wedge 2}\\ &\to 0,\ \text{as}\ \ n\to\infty.\end{aligned} \tag{3.15}$$

Similarly,

$$\left|\int_{\mathbb{R}^N}\Big(f_v\big(x,u_n^k(\lambda_m),v_n^k(\lambda_m)\big)-f_v\big(x,u^k(\lambda_m),v^k(\lambda_m)\big)\Big)\Big(v_n^k(\lambda_m)-v^k(\lambda_m)\Big)dx\right|\to 0. \tag{3.16}$$

then, it follows from (3.14), (3.15), (3.16) that

$$\Big\|\Big(u_n^k(\lambda_m)-u^k(\lambda_m),v_n^k(\lambda_m)-v^k(\lambda_m)\Big)\Big\|_W^2\to 0,\ \ n\to\infty.$$

Therefore,

$$\Big(u_n^k(\lambda_m),v_n^k(\lambda_m)\Big)\to\Big(u^k(\lambda_m),v^k(\lambda_m)\Big),\ \ n\to\infty,\ \ \text{for}\ \ k\geqslant k_0.$$

(2) We prove that $\big\{\big(u^k(\lambda_m),v^k(\lambda_m)\big)\big\}_{m=1}^{\infty}$ has a strong convergent subsequence in W. Since $\big\{\big(u^k(\lambda_m),v^k(\lambda_m)\big)\big\}_{m=1}^{\infty}$ satisfying (3.13). We claim that $\big\{\big(u^k(\lambda_m),v^k(\lambda_m)\big)\big\}_{m=1}^{\infty}$ is bounded in W. If it is not the case, we consider

$$\big(\widetilde{u}^k(\lambda_m),\widetilde{v}^k(\lambda_m)\big):=\Big(\frac{u^k(\lambda_m)}{\|(u^k(\lambda_m),v^k(\lambda_m))\|_W},\frac{v^k(\lambda_m)}{\|(u^k(\lambda_m),v^k(\lambda_m))\|_W}\Big),$$

then up to a subsequence, still denoted by $\big(\widetilde{u}^k(\lambda_m),\widetilde{v}^k(\lambda_m)\big)$, $\exists(\widetilde{u}^k,\widetilde{v}^k)$, such that

$$\begin{aligned}&\big(\widetilde{u}^k(\lambda_m),\widetilde{v}^k(\lambda_m)\big)\rightharpoonup(\widetilde{u}^k,\widetilde{v}^k), \quad \text{in}\ \ W,\\ &\big(\widetilde{u}^k(\lambda_m),\widetilde{v}^k(\lambda_m)\big)\to(\widetilde{u}^k,\widetilde{v}^k), \quad \text{in}\ \ L_2^p(\mathbb{R}^N),\ \ p\in[2,2_*),\\ &\widetilde{u}^k(\lambda_m)\to\widetilde{u}^k, \quad \text{a.e.}\ \ x\in\mathbb{R}^N,\\ &\widetilde{v}^k(\lambda_m)\to\widetilde{v}^k, \quad \text{a.e.}\ \ x\in\mathbb{R}^N.\end{aligned}$$

*Case*1 : if $(\widetilde{u}^k,\widetilde{v}^k) \neq (0,0)$ in W. Since

$$
\begin{aligned}
&1-\frac{\left\langle I'_{\lambda_m}\left(u^k(\lambda_m),v^k(\lambda_m)\right),\left(u^k(\lambda_m),v^k(\lambda_m)\right)\right\rangle}{\left\|(u^k(\lambda_m),v^k(\lambda_m))\right\|_W^2}\\
=&\lambda_m\int_{\mathbb{R}^N}\frac{f_u\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)u^k(\lambda_m)+f_v\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)v^k(\lambda_m)}{\left\|(u^k(\lambda_m),v^k(\lambda_m))\right\|_W^2}dx\\
=&\lambda_m\int_{\Omega_m}\left(|\widetilde{u}^k(\lambda_m)|^2+|\widetilde{v}^k(\lambda_m)|^2\right)\\
&\cdot\frac{f_u\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)u^k(\lambda_m)+f_v\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)v^k(\lambda_m)}{|u^k(\lambda_m)|^2+|v^k(\lambda_m)|^2}dx,
\end{aligned}
$$

where $\Omega_m=\left\{x\in\mathbb{R}^N:\left(\widetilde{u}^k(\lambda_m),\widetilde{v}^k(\lambda_m)\right)\neq(0,0)\right\}$. By (3.13), $(H5)$ and Fatou's Lemma, we can deduce a contradiction that

$$
\begin{aligned}
1=&\liminf_{m\to\infty}\lambda_m\int_{\Omega_m}\Big(\left(|\widetilde{u}^k(\lambda_m)|^2+|\widetilde{v}^k(\lambda_m)|^2\right)\\
&\cdot\frac{f_u\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)u^k(\lambda_m)+f_v\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)v^k(\lambda_m)}{|u^k(\lambda_m)|^2+|v^k(\lambda_m)|^2}\Big)dx\\
\to&\infty,\qquad \text{as } m\to\infty.
\end{aligned}
$$

Case 2: If $\left(\widetilde{u}^k,\widetilde{v}^k\right)=(0,0)$ in W. we can define

$$I_{\lambda_m}\left(t_mu^k(\lambda_m),t_mv^k(\lambda_m)\right)=\max_{t\in[0,1]}I_{\lambda_m}\left(tu^k(\lambda_m),tv^k(\lambda_m)\right).$$

For any $\beta>0$, letting $\left(\overline{u}^k(\lambda_m),\overline{v}^k(\lambda_m)\right)=\sqrt{4\beta}\left(\widetilde{u}^k(\lambda_m),\widetilde{v}^k(\lambda_m)\right)$, one can has

$$
\begin{aligned}
&\left(\overline{u}^k(\lambda_m),\overline{v}^k(\lambda_m)\right)\to(0,0),\qquad \text{in } L_2^p(\mathbb{R}^N),\ \ p\in[2,2_*),\\
&\overline{u}^k(\lambda_m)\to 0,\qquad \text{a.e. } x\in\mathbb{R}^{\mathbb{N}},\\
&\overline{v}^k(\lambda_m)\to 0,\qquad \text{a.e. } x\in\mathbb{R}^{\mathbb{N}}.
\end{aligned}
$$

Similarly as (2.9) in Lemma 3 and by (H_3), we can find $\omega(x)\in L^1(\mathbb{R}^N)$, such that

$$\left|F\left(x,\overline{u}^k(\lambda_m),\overline{v}^k(\lambda_m)\right)\right|\leqslant\omega(x),$$

hence, using the Lebesgue dominated convergence theorem, we have

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}F\left(x,\overline{u}^k(\lambda_m),\overline{v}^k(\lambda_m)\right)dx=\int_{\mathbb{R}^N}F(x,0,0)dx=0.$$

Thus, for m large enough, $\sqrt{4\beta}\left\|(u^k(\lambda_m),v^k(\lambda_m))\right\|^{-1}\in(0,1)$, we derive that

$$I_{\lambda_m}\left(t_mu^k(\lambda_m),t_mv^k(\lambda_m)\right)\geqslant I_{\lambda_m}\left(\overline{u}^k(\lambda_m),\overline{v}^k(\lambda_m)\right)$$

$$= \frac{1}{2}\left\|(\overline{u}^k(\lambda_m),\overline{v}^k(\lambda_m))\right\|_W^2 - \lambda_m\int_{\mathbb{R}^N} F\left(x,\overline{u}^k(\lambda_m),\overline{v}^k(\lambda_m)\right)dx,$$

which implies that

$$\liminf_{m\to\infty} I_{\lambda_m}\left(t_m u^k(\lambda_m), t_m v^k(\lambda_m)\right) \geqslant \beta \to \infty, \qquad \text{as } \beta\to\infty.$$

Since $I_{\lambda_m}(0,0)=0$, and $I_{\lambda_m}(u^k(\lambda_m),v^k(\lambda_m)) \in [\overline{b_k},\overline{c_k}]$, we see that, for m sufficiently large, $t_m\in(0,1)$ and

$$\begin{aligned}\left\langle I'_{\lambda_m}\left(t_m u^k(\lambda_m),t_m v^k(\lambda_m)\right),\left(t_m u^k(\lambda_m),t_m v^k(\lambda_m)\right)\right\rangle =&t_m\frac{d}{dt}|_{t=t_m} I_{\lambda_m}\left(tu^k(\lambda_m),tv^k(\lambda_m)\right)\\ =&0.\end{aligned}$$

Therefore, using $(H6)$, it can be deduced that

$$\begin{aligned}&\lambda_m\int_{\mathbb{R}^N}\frac{1}{2}\Big(f_u\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)u^k(\lambda_m)+f_v\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)v^k(\lambda_m)\Big)\\ &\quad -F\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)dx\\ &\geqslant \lambda_m\frac{1}{\theta}\int_{\mathbb{R}^N}\frac{1}{2}\Big(f_u\left(x,t_m u^k(\lambda_m),t_m v^k(\lambda_m)\right)t_m u^k(\lambda_m)\\ &\qquad +f_v\left(x,t_m u^k(\lambda_m),t_m v^k(\lambda_m)\right)t_m v^k(\lambda_m)-F\left(x,t_m u^k(\lambda_m),t_m v^k(\lambda_m)\right)\Big)dx\\ &=\frac{1}{\theta}I_{\lambda_m}\left(t_m u^k(\lambda_m),t_m v^k(\lambda_m)\right)\to\infty, \qquad m\to\infty.\end{aligned}$$

However, from (3.13), it implies that

$$\begin{aligned}&\lambda_m\int_{\mathbb{R}^N}\frac{1}{2}\Big(f_u\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)u^k(\lambda_m)+f_v\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)v^k(\lambda_m)\\ &\qquad\qquad -F\left(x,u^k(\lambda_m),v^k(\lambda_m)\right)\Big)dx\\ &=I_{\lambda_m}\left(u^k(\lambda_m),v^k(\lambda_m)\right)-\frac{1}{2}\left\langle I'_{\lambda_m}\left(u^k(\lambda_m),v^k(\lambda_m)\right),\left(u^k(\lambda_m),v^k(\lambda_m)\right)\right\rangle\\ &=I_{\lambda_m}\left(u^k(\lambda_m),v^k(\lambda_m)\right)\in\left[\overline{b_k},\overline{c_k}\right].\end{aligned}$$

This is a contradiction. similarly in (1), we can prove the sequence $\{(u^k(\lambda_m),v^k(\lambda_m))\}$ has a strongly convergent subsequence. Then there exists $(u^k,v^k)\in W$, such that $I(u^k,v^k)\in[\overline{b_k},\overline{c_k}]$ and $I'(u^k,v^k)=(0,0)$. The proof is completed. □

Proof of Theorem 1.2 From the (3.6) and the discussion in Lemma 6 on any finite dimensional subspace of W, we get

$$B(u,v)=\int_{\mathbb{R}^N}F(x,u,v)dx\to\infty, \qquad \text{as } \|(u,v)\|_W\to\infty.$$

By the assumption $(H2)$, we know that $B(u,v)\geqslant 0$, for all $(u,v)\in W$, and by the assumption $(H1)$, $(H4)$, $I_\lambda(u,v)$ maps bounded sets into bounded sets uniformly for

$\lambda \in [1,2]$. By $(H7)$, $I_\lambda(u,v) = I_\lambda(-u,-v)$, for all $\lambda \in [1,2]$ and $(u,v) \in W$. Combining with Lemma 7, we see that all the conditions of Theorem 4 are verified. Consequently, for each $k \in \mathbb{N}$, there exists $\lambda_n \to 1$, $(u(\lambda_n), v(\lambda_n)) \in Y_n$, such that

$$I'_{\lambda_n}|_{Y_n}(u(\lambda_n), v(\lambda_n)) = (0,0), \quad I_{\lambda_n} \to c_k \in [d_k(2), b_k(1)], \text{ as } n \to \infty. \tag{3.17}$$

Let $P_n : W \to Y_n$, is the orthogonal projection operator for all $n \in \mathbb{N}$, where Y_n is the $n-dimensional$ subspace of W. By (3.17), we have

$$\begin{aligned}\left\langle P_n I'_{\lambda_n}\big(u(\lambda_n), v(\lambda_n)\big), \big(u(\lambda_n), v(\lambda_n)\big)\right\rangle &= \left\langle I'_{\lambda_n}\big(u(\lambda_n), v(\lambda_n)\big), P_n\big(u(\lambda_n), v(\lambda_n)\big)\right\rangle \\ &= \left\langle I'_{\lambda_n}\big(u(\lambda_n), v(\lambda_n)\big), \big(u(\lambda_n), v(\lambda_n)\big)\right\rangle.\end{aligned}$$

So, as the proof in Theorem 1.1, $\big\{(u(\lambda_n), v(\lambda_n))\big\}$ is bounded in W. Up to a subsequence, $\exists$ (u,v) in W, such that $\big(u(\lambda_n), v(\lambda_n)\big) \rightharpoonup (u,v)$ in W. (3.17) can be changed into that

$$I'_{\lambda_n}\big(u(\lambda_n), v(\lambda_n)\big) \to (0,0), \ I_{\lambda_n} \to c_k \in [d_k(2), b_k(1)], \text{ as } n \to \infty.$$

Therefore, as (1) in Theorem 1.1, there exists (u,v), such that $(u(\lambda_n), v(\lambda_n)) \to (u,v)$ in W. So, for every $k \in \mathbb{N}$, we obtain the critical point of $I(u,v)$ with $I'(u,v) = (0,0)$ and $I(u,v) = c_k \in [d_k(2), b_k(1)]$. Therefore, from Theorem 2.6, problem (1.1) possesses infinitely many nontrivial critical points with small energy. The proof is completed. □

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