

## INFINITELY MANY SOLUTIONS FOR A FOURTH-ORDER NONLINEAR ELLIPTIC SYSTEM

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*Abstract.* In this paper we study the existence of solutions for the nonlinear elliptic system

$$\begin{cases} \Delta^2 u - \Delta u + V_1(x)u = f_u(x, u, v), \\ \Delta^2 v - \Delta v + V_2(x)v = f_v(x, u, v), \\ u, v \in H^2(\mathbb{R}^N) \quad x \in \mathbb{R}^N, \end{cases}$$

where  $V_1(x)$  and  $V_2(x)$  are positive continue functions. Under some assumptions on  $f_u(x, u, v)$  and  $f_v(x, u, v)$ , we prove the existence of many nontrivial high and small energy solutions by variant Fountain theorems. This generalizes the results by Y. Ye and C. Tang (J. Math. Anal. Appl. 394, 841-854, 2012) to fourth-order nonlinear elliptic system.

### 1. Introduction

This paper deals with the existence of infinitely many solutions for the fourth-order nonlinear elliptic system

$$\begin{cases} \Delta^2 u - \Delta u + V_1(x)u = f_u(x, u, v), \\ \Delta^2 v - \Delta v + V_2(x)v = f_v(x, u, v), \\ u, v \in H^2(\mathbb{R}^N), \quad x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $N \geq 1$ ,  $\Delta^2 := \Delta(\Delta)$  is the biharmonic operator,  $F = F(x, u, v)$ ,  $f_u = \frac{\partial F}{\partial u}$ , and  $f_v = \frac{\partial F}{\partial v}$ .  $V_1(x)$ ,  $V_2(x)$  and  $F(x, u, v)$  are positive functions. We are interested in the existence of many nontrivial high and small energy solutions.

The study of fourth-order elliptic equations appears to be important in many areas including the study of travelling waves in suspension bridges and static deflection of an elastic plate in a fluid. We refer to [7] and their references. The fourth-order elliptic problems have been extensively studied in recent years, obtained numerous results on existence, multiplicity of the positive solutions, see for example [8, 10, 11, 12, 13, 14,

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17, 20, 19, 15, 18, 2, 5]. In [20], Zhou and Wu investigated the fourth-order nonlinear elliptic boundary value problems

$$\begin{cases} \Delta^2 u + c\Delta u = f(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

they obtained the existence and multiplicity of sign-changing solutions by variational techniques. Yang and Zhang in [14] proved the existence of positive, negative and sign-changing solutions of (1.2) by invariant sets of the gradient flows of the corresponding variational functionals. In [10], Wang et al. showed that problem (1.2) exist at least three nontrivial solutions by linking approaches. In [12], Ye and Tang obtained the existence of infinitely many large-energy and small-energy solutions for the fourth-order elliptic equation:

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x, u) & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (1.3)$$

by Rabinowitz's symmetric mountain pass theorem, where  $V \in C(\mathbb{R}^N, \mathbb{R})$  and  $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ . Zhang and Tang in [18] established the existence of infinitely many small energy solutions of (1.3) by using the genus properties in critical point theory where the nonlinearity  $f(x, u)$  is indefinite sign and sublinear at infinity. Cheng in [2] considered the existence of high energy solutions of (1.3) by using some special techniques. Under more relaxed assumptions on  $V(x)$ , Ye and Tang established the existence and multiplicity of solutions for a class of fourth-order elliptic equations with a parameter  $\lambda \geq 1$  large enough.

In [6], Jung and Choi studied the fourth-order elliptic system with Dirichlet boundary condition:

$$\begin{cases} \Delta^2 u + c\Delta u = a((u+v+1)^+ - 1) & \text{in } \Omega, \\ \Delta^2 v + c\Delta v = a((u+v+1)^+ - 1) & \text{in } \Omega, \\ u = 0, v = 0, \Delta u = 0, \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

they proved the existence of nontrivial solutions via linking method and the contraction mapping principle on the Banach space. In [1], Afrouzi et al. studied the existence and multiplicity of solutions for a class of nonlocal fourth-order systems by critical point theory.

To the best of our knowledge, the existence of infinitely many nontrivial solutions of (1.1) has not ever been considered by variational methods. Our main objective in this article is to study the existence of infinitely many nontrivial high and small energy solutions for (1.1). Unlike the Rabinowitz's symmetric mountain pass arguments in [12], our main idea is to use the variant Fountain theorems generalizes the results in [12] to fourth-order nonlinear elliptic system.

In this paper, we assume

(H1)  $V_i(x)$  are continuous in  $\mathbb{R}^N$ , and  $\inf_{x \in \mathbb{R}^N} V_i(x) > 0$  ( $i = 1, 2$ ). For each  $M > 0$ ,  $\text{meas} \{x \in \mathbb{R}^N : V_i(x) < M\} < \infty$  ( $i = 1, 2$ ), where  $\text{meas}$  denotes the Lebesgue measure in  $\mathbb{R}^N$ ;

(H2)  $F(x, u, v) \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$ ;

(H3) There exist  $a_1, a_2 > 0, r \in (2, 2_*)$  where  $2_* = \infty$  for  $N \leq 4$  and  $2_* = \frac{2N}{N-4}$  for  $N > 4$ , such that

$$|f_u(x, u, v)| + |f_v(x, u, v)| \leq a_1(|u|^{r-1} + |v|^{r-1}) + a_2(|u| + |v|), \quad \forall (x, u, v) \in \mathbb{R}^N \times \mathbb{R}^2;$$

(H4) There exist  $\delta, \sigma \in (1, 2), r \in (2, 2_*)$  and  $a_3 > 0, \eta > 0, \forall (x, u, v) \in \mathbb{R}^N \times \mathbb{R}^2$ , such that

$$\zeta(x)(|u|^\delta + |v|^\delta) \leq f_u(x, u, v)u + f_v(x, u, v)v$$

and

$$|f_u(x, u, v)| + |f_v(x, u, v)| \leq \eta m(x)(|u|^{\sigma-1} + |v|^{\sigma-1}) + a_3(|u|^{r-1} + |v|^{r-1}),$$

where  $\zeta(x) \in L^{\frac{2}{2-\delta}}(\mathbb{R}^N), m(x) \in L^{\frac{2}{2-\sigma}}(\mathbb{R}^N), \zeta(x) > 0, m(x) > 0$ , for  $x \in \mathbb{R}^N$ ;

(H5) There exists  $\mu \in (2, r)$ , such that

$$\lim_{(|u|^2 + |v|^2) \rightarrow \infty} \frac{f_u(x, u, v)u + f_v(x, u, v)v}{(|u|^2 + |v|^2)^{\frac{\mu}{2}}} \geq C > 0$$

uniformly for  $x \in \mathbb{R}^N$ ;

(H6) There exists  $\theta \geq 1$ , such that  $\forall s \in [0, 1]$ ,

$$\theta \mathcal{L}(x, u, v) \geq \mathcal{L}(x, su, sv), \quad \forall (x, u, v) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where  $\mathcal{L}(x, u, v) = \frac{1}{2}(f_u(x, u, v)u + f_v(x, u, v)v) - F(x, u, v)$ ;

(H7)  $F(x, u, v) = F(x, -u, -v)$  for  $(x, u, v) \in \mathbb{R}^N \times \mathbb{R}^2$ .

REMARK 1. With these assumptions on  $F$ , we give the following examples of  $F$ :

(i)

$$F(x, u, v) = (2 + \varepsilon^{\frac{1}{1+|x|}})(|u|^p + |v|^p + |u|^2 + |v|^2),$$

(ii)

$$F(x, u, v) = \frac{1 + \sin^2 x_1}{1 + |x|^{\frac{N}{2}}}( |u|^\sigma + |v|^\sigma ) + |u|^p + |v|^p.$$

where  $x = \{x_1, x_2, \dots, x_N\}, 2 < p < 2_*, 1 < \sigma < 2$ .  $\square$

We state the main theorems in this paper:

THEOREM 1. Assume that (H1) – (H3), (H5) – (H7) hold, then problem (1.1) possesses infinitely many high energy solutions  $(u^k, v^k)$  for all  $k \geq k_0$  ( $k_0 \in \mathbb{N}$ ), in the sense that

$$I(u^k, v^k) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u^k|^2 + |\Delta v^k|^2 + |\nabla u^k|^2 + |\nabla v^k|^2 + V_1(x)(u^k)^2 + V_2(x)(v^k)^2) dx - \int_{\mathbb{R}^N} F(x, u^k, v^k) dx \rightarrow +\infty, \quad \text{as } k \rightarrow \infty.$$

**THEOREM 2.** *Assume that (H1), (H2), (H4) – (H7) hold, there exists  $\Lambda_0 > 0$ , when  $\eta < \Lambda_0$ , then problem (1.1) possesses infinitely many small energy solutions  $(u^k, v^k)$  for  $k \in \mathbb{N}$ , in the sense that*

$$I(u^k, v^k) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\Delta u^k|^2 + |\Delta v^k|^2 + |\nabla u^k|^2 + |\nabla v^k|^2 + V_1(x)(u^k)^2 + V_2(x)(v^k)^2 \right) dx \\ - \int_{\mathbb{R}^N} F(x, u^k, v^k) dx \rightarrow 0-, \quad \text{as } k \rightarrow \infty.$$

The paper is organized as follows. In Section 2, we present some preliminary results and prove some lemmas. In Section 3, we prove our main Theorems.

## 2. The variational framework and preliminary results

Consider the Sobolev space  $X = H^2(\mathbb{R}^N)$  endowed with the norm

$$\|u\|_X = \left( \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}.$$

Now, we define the subspaces

$$W_1 = \left\{ u \in X \mid \int_{\mathbb{R}^N} V_1(x)u^2 < \infty \right\},$$

$$W_2 = \left\{ v \in X \mid \int_{\mathbb{R}^N} V_2(x)v^2 < \infty \right\}.$$

Obviously,  $W_1$  and  $W_2$  are Hilbert spaces endowed with the norm respectively

$$\|u\|_{W_1} = \left( \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V_1(x)u^2) dx \right)^{\frac{1}{2}}, \quad u \in W_1,$$

$$\|v\|_{W_2} = \left( \int_{\mathbb{R}^N} (|\Delta v|^2 + |\nabla v|^2 + V_2(x)v^2) dx \right)^{\frac{1}{2}}, \quad v \in W_2.$$

Problem (1.1) is posed in the framework of the Hilbert space  $W = W_1 \times W_2$  with the standard norm

$$\|(u, v)\|_W^2 = \|u\|_{W_1}^2 + \|v\|_{W_2}^2.$$

In addition, we define  $|u|_p = \left( \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}$ , which is the usual norm in  $L^p(\mathbb{R}^N)$  and  $L_2^p(\mathbb{R}^N) = L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$  with the norm  $|(u, v)|_p = \left( |u|_p^p + |v|_p^p \right)^{\frac{1}{p}}$ . It is well known that under assumption (H1), the embedding  $W_1 \hookrightarrow L^p(\mathbb{R}^N)$ ,  $W_2 \hookrightarrow L^p(\mathbb{R}^N)$  are compact for  $p \in [2, 2_*)$ , where  $2_* = +\infty$  for  $N \leq 4$  and  $2_* = \frac{2N}{N-4}$  for  $N > 4$ .

A pair of functions  $(u, v) \in W$  is said to be a weak solution of problem (1.1) if

$$\int_{\mathbb{R}^N} (\Delta u \Delta \varphi_1 + \Delta v \Delta \varphi_2 + \nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 + V_1(x)u\varphi_1 + V_2(x)v\varphi_2) dx$$

$$-\int_{\mathbb{R}^N} f_u(x, u, v)\varphi_1 + f_v(x, u, v)\varphi_2 dx = 0$$

for all  $(\varphi_1, \varphi_2) \in W$ .

The corresponding energy functional of problem (1.1) is defined by

$$I(u, v) = \frac{1}{2} \|(u, v)\|_W^2 - \int_{\mathbb{R}^N} F(x, u, v) dx.$$

In order to verify  $I(u, v) \in C^1(W, R)$ , we need the following lemmas

DEFINITION 1. (Definition 3.2, [4]) On the space  $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ , we define the norm

$$|u|_{p \wedge q} = |u|_p + |u|_q,$$

on the space  $L_2^p(\mathbb{R}^N) \cap L_2^q(\mathbb{R}^N)$ , we define the norm

$$|(u, v)|_{p \wedge q} = |(u, v)|_p + |(u, v)|_q,$$

on the space  $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ , we define the norm

$$|u|_{p \vee q} = \inf \{ |v|_p + |w|_q : v \in L^p(\mathbb{R}^N), w \in L^q(\mathbb{R}^N), u = v + w \}.$$

LEMMA 1. (Lemma 3.3, [4]) Assume that  $1 \leq p, r, q, s < \infty$ ,  $f \in C(\mathbb{R}^N \times \mathbb{R}^2)$  and

$$f(x, u, v) \leq C_1(|u|^{\frac{p}{r}} + |v|^{\frac{p}{r}}) + C_2(|u|^{\frac{q}{s}} + |v|^{\frac{q}{s}}),$$

then, for every  $(u, v) \in L_2^p(\mathbb{R}^N) \cap L_2^q(\mathbb{R}^N)$ ,  $f(\cdot, u, v) \in L^r(\mathbb{R}^N) + L^s(\mathbb{R}^N)$ , and the operator

$$T : L_2^p(\mathbb{R}^N) \cap L_2^q(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N) + L^s(\mathbb{R}^N) : (u, v) \rightarrow f(x, u, v)$$

is continuous.

Now we consider the functional  $\psi(u, v) = \int_{\mathbb{R}^N} F(x, u, v) dx$ , then we have the following result.

LEMMA 2. Assume that (H1) – (H3) hold, then  $\psi(u, v) \in C^1(W, R)$  and

$$\langle \psi'(u, v), (\varphi_1, \varphi_2) \rangle = \int_{\mathbb{R}^N} (f_u(x, u, v)\varphi_1 + f_v(x, u, v)\varphi_2) dx,$$

where  $(u, v), (\varphi_1, \varphi_2) \in W$ .

Proof. It follows from (H3) that

$$\begin{aligned} F(x, u, v) &= \int_0^1 \frac{dF(x, tu, tv)}{dt} dt \\ &= \int_0^1 f_u(x, tu, tv)u + f_v(x, tu, tv)v dt \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 (|f_u(x, tu, tv)| + |f_v(x, tu, tv)|)(|u| + |v|)dt \\
&\leq \int_0^1 (a_1(|tu|^{r-1} + |tv|^{r-1}) + a_2(|tu| + |tv|)(|u| + |v|))dt \\
&\leq 2a_1(|u|^r + |v|^r) \int_0^1 t^{r-1}dt + 2a_2(|u|^2 + |v|^2) \int_0^1 tdt \\
&\leq a_1(|u|^r + |v|^r) + a_2(|u|^2 + |v|^2). \tag{2.1}
\end{aligned}$$

Let  $g(x) := a_1(|u(x)|^r + |v(x)|^r) + a_2(|u(x)|^2 + |v(x)|^2)$ , then  $g(x) \in L^1(\mathbb{R}^N)$ . So,  $I(u, v)$  is well defined.

First, we prove the existence of the Gateaux derivative. Given  $(\varphi_1, \varphi_2) \in W$ ,  $|t| \in [0, 1]$ ,  $\theta \in (0, 1)$ , then

$$\begin{aligned}
&\left| \frac{F(x, u + t\varphi_1, v + t\varphi_2) - F(x, u, v)}{t} \right| \\
&= (|f_u(x, u + t\theta\varphi_1, v + t\theta\varphi_2)| |t\varphi_1| + |f_v(x, u + t\theta\varphi_1, v + t\theta\varphi_2)| |t\varphi_2|) \times \frac{1}{|t|} \\
&\leq a_1(|u + t\theta\varphi_1|^{r-1} + |v + t\theta\varphi_2|^{r-1}) + a_2(|u + t\theta\varphi_1| + |v + t\theta\varphi_2|)(|\varphi_1| + |\varphi_2|) \\
&\leq C(|u|^{r-1} + |\varphi_1|^{r-1} + |v|^{r-1} + |\varphi_2|^{r-1} + |u| + |\varphi_1| + |v| + |\varphi_2|)(|\varphi_1| + |\varphi_2|).
\end{aligned}$$

The Hölder inequality and the sobolev imbedding theorem imply that

$$(|u|^{r-1} + |\varphi_1|^{r-1} + |v|^{r-1} + |\varphi_2|^{r-1} + |u| + |\varphi_1| + |v| + |\varphi_2|)(|\varphi_1| + |\varphi_2|) \in L^1(\mathbb{R}^N).$$

It follows from the Lebesgue theorem that

$$\langle \psi'(u, v), (\varphi_1, \varphi_2) \rangle = \int_{\mathbb{R}^N} (f_u(x, u, v)\varphi_1 + f_v(x, u, v)\varphi_2)dx.$$

Next, we prove the continuity of the Gateaux derivative. Assume that  $(u_n, v_n) \rightarrow (u, v)$  in  $W$ . By the sobolev imbedding theorems,  $(u_n, v_n) \hookrightarrow (u, v)$  in  $L_2^p(\mathbb{R}^N)$  for  $p \in [2, 2_*)$ . By the Lemma 2.2, we obtain  $f_u(x, u_n, v_n) \rightarrow f_u(x, u, v)$  and  $f_v(x, u_n, v_n) \rightarrow f_v(x, u, v)$  in  $L^{r'}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$ , where  $r' := \frac{r}{r-1}$ . By the Hölder inequality and the Sobolev imbedding theorem, we get

$$\begin{aligned}
&\left| \langle \psi'(u_n, v_n) - \psi'(u, v), (\varphi_1, \varphi_2) \rangle \right| \\
&\leq \left| \int_{\mathbb{R}^N} (f_u(x, u_n, v_n) - f_u(x, u, v))\varphi_1 + (f_v(x, u_n, v_n) - f_v(x, u, v))\varphi_2 dx \right| \\
&\leq |f_u(x, u_n, v_n) - f_u(x, u, v)|_{r'\sqrt{2}} |\varphi_1|_{r\wedge 2} + |f_v(x, u_n, v_n) - f_v(x, u, v)|_{r'\sqrt{2}} |\varphi_2|_{r\wedge 2} \\
&\rightarrow 0. \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Hence, by the above lemmas, we have  $I(u, v) \in C^1(W, R)$ . The proof is completed.  $\square$

LEMMA 3. Assume that (H1), (H2) and (H4) hold, then  $\psi(u, v) \in C^1(W, \mathbb{R})$ , and  $\langle \psi'(u, v), (\varphi_1, \varphi_2) \rangle = \int_{\mathbb{R}^N} (f_u(x, u, v)\varphi_1 + f_v(x, u, v)\varphi_2) dx$ , where  $(u, v), (\varphi_1, \varphi_2) \in W$ .

*Proof.* By (H4), similarly as in the Lemma 2, it is deduced that

$$F(x, u, v) = \int_0^1 \frac{dF(x, tu, tv)}{dt} dt \leq 2\eta m(x)(|u|^\sigma + |v|^\sigma) + a_3(|u|^r + |v|^r). \tag{2.2}$$

By the Höld inequality and sobolev imbedding theorem, it can be deduced that

$$2\eta m(x)(|u|^\sigma + |v|^\sigma) + a_3(|u|^r + |v|^r) \in L^1(\mathbb{R}^N),$$

then  $I(u, v)$  is well defined.

Almost as the same as in Lemma 2, the Gateaux derivative of  $\psi(u, v)$  exists, and

$$\langle \psi'(u, v), (\varphi_1, \varphi_2) \rangle = \int_{\mathbb{R}^N} (f_u(x, u, v)\varphi_1 + f_v(x, u, v)\varphi_2) dx.$$

Let  $(u_n, v_n) \rightarrow (u, v)$  in  $W$ , then

$$u_n \rightarrow u \quad \text{in } L^p(\mathbb{R}^N), \quad \text{for } p \in [2, 2_*], \tag{2.3}$$

$$v_n \rightarrow v \quad \text{in } L^p(\mathbb{R}^N), \quad \text{for } p \in [2, 2_*], \tag{2.4}$$

$$u_n \rightarrow u \quad \text{a.e. } x \in \mathbb{R}^N, \tag{2.5}$$

$$v_n \rightarrow v \quad \text{a.e. } x \in \mathbb{R}^N. \tag{2.6}$$

We claim that

$$\left| \int_{\mathbb{R}^N} (f_u(x, u_n, v_n) - f_u(x, u, v)) \varphi_1 dx \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \forall \varphi_1 \in W_1. \tag{2.7}$$

Otherwise,  $\exists \varepsilon_0 > 0$ , and a subsequence, denoted by  $\{(u_{nk}, v_{nk})\}$ , such that

$$\left| \int_{\mathbb{R}^N} (f_u(x, u_{nk}, v_{nk}) - f_u(x, u, v)) \varphi_1 dx \right| > \varepsilon_0, \quad \text{as } k \rightarrow \infty. \tag{2.8}$$

Since  $(u_n, v_n) \rightarrow (u, v)$  in  $L^2_2(\mathbb{R}^N)$ , it can be assumed that

$$\|u_{n(k+1)} - u_{nk}\|_2 \leq 2^{-k}, \quad \|v_{n(k+1)} - v_{nk}\|_2 \leq 2^{-k}, \quad \forall k \geq 1.$$

Let us define

$$\begin{aligned} (\omega_1(x), \omega_2(x)) := & \left( |u_{n1}(x)| + \sum_{k=1}^{+\infty} |u_{n(k+1)}(x) - u_{nk}(x)|, \right. \\ & \left. |v_{n1}(x)| + \sum_{k=1}^{+\infty} |v_{n(k+1)}(x) - v_{nk}(x)| \right). \end{aligned}$$

And  $\omega_1(x) \in L^2(\mathbb{R}^N)$ ,  $\omega_2(x) \in L^2(\mathbb{R}^N)$ . It is clear that

$$|u_{nk}(x)| \leq \omega_1(x), \quad |v_{nk}(x)| \leq \omega_2(x).$$

Note that, by (H4)

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} (f_u(x, u_{nk}, v_{nk}) - f_u(x, u, v)) \varphi_1 dx \right| \\ & \leq \int_{\mathbb{R}^N} (|f_u(x, u_{nk}, v_{nk})| + |f_u(x, u, v)|) |\varphi_1| dx \\ & \leq \int_{\mathbb{R}^N} [\eta m(x) (|u_{nk}|^{\sigma-1} + |v_{nk}|^{\sigma-1} + |u|^{\sigma-1} + |v|^{\sigma-1}) \\ & \quad + a_3 (|u_{nk}|^{r-1} + |v_{nk}|^{r-1} + |u|^{r-1} + |v|^{r-1})] |\varphi_1| dx \\ & \leq \int_{\mathbb{R}^N} [\eta m(x) (|\omega_1(x)|^{\sigma-1} + |\omega_2(x)|^{\sigma-1} + |u|^{\sigma-1} + |v|^{\sigma-1}) \\ & \quad + a_3 (|\omega_1(x)|^{r-1} + |\omega_2(x)|^{r-1} + |u|^{r-1} + |v|^{r-1})] |\varphi_1| dx. \end{aligned}$$

By the Lebesgue dominated convergence theorem and from (2.5), (2.6), it can be deduced that

$$\lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}^N} (f_u(x, u_{nk}, v_{nk}) - f_u(x, u, v)) \varphi_1 dx \right| \rightarrow 0, \quad (2.9)$$

which is contradict to (2.8). Therefore, (2.7) holds. Similarly,

$$\left| \int_{\mathbb{R}^N} (f_v(x, u_n, v_n) - f_v(x, u, v)) \varphi_2 dx \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \forall \varphi_2 \in W_2. \quad (2.10)$$

Then,

$$\begin{aligned} \left| \langle \psi'(u_n, v_n) - \psi'(u, v), (\varphi_1, \varphi_2) \rangle \right| & \leq \left| \int_{\mathbb{R}^N} (f_u(x, u_n, v_n) - f_u(x, u, v)) \varphi_1 dx \right| \\ & \quad + \left| \int_{\mathbb{R}^N} (f_v(x, u_n, v_n) - f_v(x, u, v)) \varphi_2 dx \right| \\ & \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . Hence,  $\psi(u, v) \in C^1(W, \mathbb{R})$ . The proof is completed.  $\square$

From the above lemma, replace the (H3) with (H4),  $I(u, v) \in C^1(W, \mathbb{R})$ . Let  $W$  be a banach space with the norm  $\|\cdot\|$  and let  $W_j$  be a sequences of subspaces of  $W$  with  $\dim W_j < \infty$  for each  $j \in \mathbb{N}$ . Further,  $W = \overline{\bigoplus_{j \in \mathbb{N}} W_j}$ , the closure of the direct sum of all  $W_j$ . Set

$$Y_k = \bigoplus_{j=0}^k W_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} W_j}$$

and

$$B_k = \{u \in Y_k : \|u\| \leq \rho_k\} \quad S_k = \{u \in Z_k : \|u\| = r_k\},$$

for  $\rho_k > r_k > 0$ . Consider a familarly of  $C^1$ -functionals  $I_\lambda : W \rightarrow \mathbb{R}$  defined by

$$I_\lambda = A(u) - \lambda B(u).$$

The following two variant fountain theorems were established in [21].



**THEOREM 3.** Assume that the functional  $I_\lambda$  defined above satisfies

(A1)  $I_\lambda$  maps bounded sets into bounded sets uniformly for  $\lambda \in [1, 2]$ , and  $I_\lambda(-u) = I_\lambda(u)$  for all  $\lambda \in [1, 2]$ ,  $u \in W$ ;

(A2)  $B(u) \geq 0$  for all  $u \in W$ ,  $A(u) \rightarrow \infty$  or  $B(u) \rightarrow \infty$  as  $\|u\| \rightarrow \infty$ ;

or

(A3)  $B(u) \leq 0$  for all  $u \in W$ ,  $B(u) \rightarrow -\infty$  as  $\|u\| \rightarrow \infty$ ;

(A4) There exists  $\rho_k > r_k > 0$  such that

$$b_k(\lambda) = \lim_{u \in Z_k, \|u\|=r_k} I_\lambda(u) > a_k(\lambda) = \max_{u \in Y_k, \|u\|=\rho_k} I_\lambda(u), \quad \forall \lambda \in [1, 2],$$

Then

$$b_k(\lambda) \leq c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2],$$

where  $\Gamma_k = \{\gamma \in C(B_k, W) : \gamma \text{ is odd, } \gamma|_{\partial B_k} = id\}$  ( $k \geq 2$ ). Moreover, for almost every  $\lambda \in [1, 2]$ , there exists a sequence  $\{u_n^k(\lambda)\}$  such that

$$\sup_n \|u_n^k(\lambda)\| < \infty, \quad I'_\lambda(u_n^k(\lambda)) \rightarrow 0 \quad \text{and} \quad I_\lambda(u_n^k(\lambda)) \rightarrow c_k(\lambda) \quad \text{as } n \rightarrow \infty.$$

**THEOREM 4.** Assume that  $I_\lambda$  defined above satisfies

(B1)  $I_\lambda$  maps bounded sets into bounded sets uniformly for  $\lambda \in [1, 2]$ , and  $I_\lambda(-u) = I_\lambda(u)$  for all  $(\lambda, u) \in [1, 2] \times W$ ;

(B2)  $B(u) \geq 0$  for all  $u \in W$  and  $B(u) \rightarrow \infty$  as  $\|u\| \rightarrow \infty$  on any finite dimensional subspace of  $W$ ;

(B3) There exists  $\rho_k > r_k > 0$  such that

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0, \quad b_k(\lambda) = \max_{u \in Y_k, \|u\|=r_k} I_\lambda(u) < 0, \quad \forall \lambda \in [1, 2]$$

and

$$d_k(\lambda) = \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \rightarrow 0, \quad k \rightarrow \infty, \quad \text{uniformly for } \lambda \in [1, 2].$$

Then there exist  $\lambda_n \rightarrow 1$ ,  $u(\lambda_n) \in Y_n$  such that

$$I'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0 \quad \text{and} \quad I_{\lambda_n}(u(\lambda_n)) \rightarrow c_k \quad \text{as } n \rightarrow \infty,$$

where  $c_k \in [d_k(2), b_k(1)]$ . In particular, if  $u(\lambda_n)$  has a convergent subsequence for every  $k$ , then  $I_1$  has infinitely many nontrivial critical points  $u_k \in W \setminus \{0\}$  satisfying  $I_1(u_k) \rightarrow 0^-$  as  $k \rightarrow \infty$ .

In order to apply the above two theorems to prove our main results, we define

$$A(u, v) = \frac{1}{2} \|(u, v)\|^2, \quad B(u, v) = \int_{\mathbb{R}^N} F(x, u, v) dx,$$

and

$$I_\lambda(u, v) = A(u, v) - \lambda B(u, v) = \frac{1}{2} \|(u, v)\|^2 - \lambda \int_{\mathbb{R}^N} F(x, u, v) dx,$$

for all  $(u, v) \in W$  and  $\lambda \in [1, 2]$ .

### 3. Existence results

LEMMA 4. *For the any  $k$  – dimensional subspace  $\tilde{W}$  of  $W$ , there exists  $\varepsilon_k^p > 0$ ,  $\varepsilon_k^\delta > 0$  such that*

$$\text{meas}\left\{x \in \mathbb{R}^N : |u|^p + |v|^p \geq \varepsilon_k^p \|(u, v)\|^p\right\} \geq \varepsilon_k^p, \quad (u, v) \in \tilde{W} \setminus \{(0, 0)\}, \quad p \in [2, 2_*), \quad (3.1)$$

and

$$\text{meas}\{x \in \mathbb{R}^N : \zeta(x)(|u|^\delta + |v|^\delta) \geq \varepsilon_k^\delta \|(u, v)\|^\delta\} \geq \varepsilon_k^\delta \quad \forall (u, v) \in \tilde{W} \setminus \{(0, 0)\}, \quad (3.2)$$

where  $\delta \in (1, 2)$  and  $\zeta(x) : \mathbb{R}^N \rightarrow \mathbb{R}$  is a positive continuous functional such that  $\zeta(x) \in L^{\frac{2}{2-\delta}}(\mathbb{R}^N)$ .

The verification of Lemma 4 is almost the same as in [12].

LEMMA 5. *Let*

$$\alpha_k(p) = \sup_{(u,v) \in Z_k, \|(u,v)\|_W=1} \|(u, v)\|_p \rightarrow 0 \quad k \rightarrow \infty, \quad \text{for } p \in [2, 2_*), \quad (3.3)$$

where the  $Z_k$  is defined in Theorem 3.

*Proof.* Suppose that this is not the case, then there exists an  $\varepsilon_0 > 0$  and  $\{(u_j, v_j)\} \subset W$  with  $\{(u_j, v_j)\} \perp W_{k_j-1}$  such that

$$\|(u_j, v_j)\|_W = 1, \quad \|(u_j, v_j)\|_{L_2^p(\mathbb{R}^N)} \geq \varepsilon_0,$$

where  $k_j \rightarrow \infty$  as  $j \rightarrow \infty$ . For any  $(u, v) \in W$ , we may find  $\{(\bar{u}_j, \bar{v}_j)\} \in W_{k_j-1}$ , such that  $(\bar{u}_j, \bar{v}_j) \rightarrow (u, v)$  as  $j \rightarrow \infty$ . Hence

$$\begin{aligned} |\langle (u_j, v_j), (u, v) \rangle| &= |\langle (u_j, v_j), (\bar{u}_j, \bar{v}_j) - (u, v) \rangle| \\ &\leq \|(\bar{u}_j - u, \bar{v}_j - v)\| \\ &\rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Thus,  $(u_j, v_j) \rightarrow 0$  in  $W$ . By the sobolev theorem,  $(u_j, v_j) \rightarrow (0, 0)$  in  $L_2^p(\mathbb{R}^N)$ . This is a contradiction. The proof is completed.  $\square$

LEMMA 6. *Let (H1), (H2), (H3), and (H5) hold, then there exist  $\rho_k > r_k > 0$  such that for all  $\lambda \in [1, 2]$ ,*

$$b_k(\lambda) = \lim_{(u,v) \in Z_k, \|(u,v)\|=r_k} I_\lambda(u, v) > a_k(\lambda) = \max_{(u,v) \in Y_k, \|(u,v)\|=\rho_k} I_\lambda(u, v). \quad (3.4)$$

*Proof.* From (2.1) in Lemma 2, we have

$$F(x, u, v) \leq a_1(|u|^r + |v|^r) + a_2(|u|^2 + |v|^2) \quad \forall x \in \mathbb{R}^N, \quad (u, v) \in \mathbb{R}^2.$$

Therefore, with Lemma 5,  $\exists k_0$ , for  $(u, v) \in Z_k$ ,  $k \geq k_0$ , we have

$$\begin{aligned} I_\lambda(u, v) &= \frac{1}{2} \|(u, v)\|_W^2 - \lambda \int_{\mathbb{R}^N} F(x, u, v) dx \\ &\geq \frac{1}{2} \|(u, v)\|_W^2 - \lambda \int_{\mathbb{R}^N} a_1(|u|^r + |v|^r) + a_2(|u|^2 + |v|^2) dx \\ &\geq \frac{1}{2} \|(u, v)\|_W^2 - 2\alpha_k^r(r) a_1 \|(u, v)\|_W^r - 2\alpha_k^2(2) a_2 \|(u, v)\|_W^2 \\ &\geq \frac{1}{4} \|(u, v)\|_W^2 - \alpha_k^r(r) c \|(u, v)\|_W^r. \end{aligned}$$

If we choose  $r_k = (8\alpha_k^r(r)c)^{\frac{1}{2-r}}$ , where  $c = 2a_1$ , then for any  $(u, v) \in Z_k$  with  $\|(u, v)\| = r_k$ , we get

$$I_\lambda(u, v) \geq \frac{1}{8} (8\alpha_k^r(r)c)^{\frac{2}{2-r}} > 0,$$

which implies that

$$b_k(\lambda) = \inf_{(u,v) \in Z_k, \|(u,v)\|=r_k} I_\lambda(u, v) \geq \frac{1}{8} (8\alpha_k^r(r)c)^{\frac{2}{2-r}} > 0, \quad \forall \lambda \in [1, 2]. \quad (3.5)$$

By  $(H_5)$ ,  $\exists L > 0$ ,  $|u|^2 + |v|^2 \geq L$ , then

$$f_u(x, u, v)u + f_v(x, u, v)v \geq c_1(|u|^2 + |v|^2)^{\frac{\mu}{2}} \geq c_2(|u|^\mu + |v|^\mu),$$

then

$$\begin{aligned} F(x, u, v) &= \int_0^1 f_u(x, tu, tv)u + f_v(x, tu, tv)v dt \\ &= \int_0^1 \frac{f_u(x, tu, tv)tu + f_v(x, tu, tv)tv}{t} dt \\ &\geq \int_0^1 \frac{c_2(|tu|^\mu + |tv|^\mu)}{t} dt \\ &\geq c_3(|u|^\mu + |v|^\mu), \end{aligned} \quad (3.6)$$

where  $c_3 = \frac{c_2}{\mu}$ .

Note  $\Omega_k(\mu) = \{x \in \mathbb{R}^N : |u|^\mu + |v|^\mu \geq \varepsilon_k^\mu \|(u, v)\|^\mu\}$ . Since  $|u|^2 + |v|^2 \geq (|u|^\mu + |v|^\mu)^{\frac{2}{\mu}}$  with  $\frac{2}{\mu} \in (0, 1)$  and (3.1) in Lemma 4, there exists  $R_k$  with  $(\varepsilon_k^\mu R_k^\mu)^{\frac{2}{\mu}} \geq L$ ,  $\|(u, v)\| \geq R_k$ ,  $(u, v) \in Y_k$ , then  $|u|^2 + |v|^2 \geq L$  when  $x \in \Omega_k(\mu)$ . Hence, for any  $(u, v) \in Y_k$ , with  $\|(u, v)\|_W \geq R_k$ , we have

$$I_\lambda(u, v) \leq \frac{1}{2} \|(u, v)\|_W^2 - \int_{\mathbb{R}^N} F(x, u, v) dx$$

$$\begin{aligned}
&\leq \frac{1}{2} \|(u, v)\|_W^2 - \int_{\Omega_k(\mu)} c_3(|u|^\mu + |v|^\mu) dx \\
&\leq \frac{1}{2} \|(u, v)\|_W^2 - c_3(\varepsilon_k^\mu)^2 \|(u, v)\|_W^\mu \rightarrow -\infty \quad \text{as } \|(u, v)\| \rightarrow \infty.
\end{aligned}$$

Then,  $\exists \rho_0 > R_k$ , such that  $I_\lambda(u, v)|_{\partial B(\rho_0)} < 0$ . Then, we choose  $\rho_k > \max\{r_k, \rho_0\}$

$$a_k(\lambda) = \max_{(u,v) \in Y_k, \|(u,v)\| = \rho_k} I_\lambda(u, v) < 0 \quad \forall k \in \mathbb{N}, \lambda \in [1, 2].$$

The proof is completed.  $\square$

LEMMA 7. Let (H1), (H2), (H4) and (H5) hold,  $\exists \Lambda_0 > 0$ , when  $\eta < \Lambda_0$ , there exist  $\rho_k > r_k > 0$ , such that

$$a_k(\lambda) = \inf_{(u,v) \in Z_k, \|(u,v)\| = \rho_k} I_\lambda(u, v) \geq 0,$$

$$b_k(\lambda) = \max_{(u,v) \in Y_k, \|(u,v)\| = r_k} I_\lambda(u, v) < 0, \quad \forall \lambda \in [1, 2],$$

and

$$d_k(\lambda) = \inf_{(u,v) \in Z_k, \|(u,v)\| \leq \rho_k} I_\lambda(u, v) \rightarrow 0 \quad k \rightarrow \infty, \quad \text{uniformly for } \lambda \in [1, 2].$$

*Proof.* By (2.2) in Lemma 3, for  $(u, v) \in Z_k$ , it follows that

$$\begin{aligned}
I_\lambda(u, v) &= \frac{1}{2} \|(u, v)\|_W^2 - \lambda \int_{\mathbb{R}^N} F(x, u, v) dx \\
&\geq \frac{1}{2} \|(u, v)\|_W^2 - 4 \int_{\mathbb{R}^N} \eta m(x) (|u|^\sigma + |v|^\sigma) + a_3(|u|^r + |v|^r) dx \\
&\geq \frac{1}{2} \|(u, v)\|_W^2 - 4\eta |m(x)|_{\frac{2}{2-\sigma}} (|u|_2^\sigma + |v|_2^\sigma) - 2a_3(|u|_r^r + |v|_r^r) \\
&\geq \frac{1}{2} \|(u, v)\|_W^2 - 8\eta |m(x)|_{\frac{2}{2-\sigma}} (|u|^2 + |v|^2)^{\frac{\sigma}{2}} - 2a_3(|u|_r^r + |v|_r^r) \\
&\geq \frac{1}{2} \|(u, v)\|_W^2 - 8\eta |m(x)|_{\frac{2}{2-\sigma}} \alpha_k^\sigma(2) \|(u, v)\|_W^\sigma - 2c_3 \alpha_k^r(r) \|(u, v)\|_W^r \\
&\geq \|(u, v)\|_W^\sigma \left( \frac{1}{2} \|(u, v)\|_W^{2-\sigma} - 8\eta \varepsilon_0 |m(x)|_{\frac{2}{2-\sigma}} - c_4 \|(u, v)\|_W^{r-\sigma} \right),
\end{aligned}$$

where  $\varepsilon_0 = \sup_{k \in \mathbb{N}} \alpha_k^\sigma(2)$ ,  $c_4 = 2c_3 \sup_{k \in \mathbb{N}} \alpha_k^r(r)$ .

Let

$$f(t) = \frac{1}{2} t^{2-\sigma} - c_4 t^{r-\sigma}, \quad t \geq 0.$$

$f_{\max} = f(\bar{x})$  with  $\bar{x} = \left(\frac{2-\sigma}{2c_4(r-\sigma)}\right)^{\frac{1}{r-2}}$ , such that

$$f_{\max} = \left(\frac{1}{2}\right)^{\frac{r-\sigma}{r-2}} \cdot c_4^{\frac{2-\sigma}{r-2}} \cdot \left(\frac{2-\sigma}{r-\sigma}\right)^{\frac{2-\sigma}{r-2}} \cdot \frac{r-2}{r-\sigma} > 0.$$

Denote  $\Lambda_0 = \frac{1}{8}\varepsilon_0^{-1}|m(x)|^{-\frac{1}{2}}f_{max}$ , when  $\eta < \Lambda_0$ ,  $\rho_k = \bar{x}$ . Then

$$a_k(\lambda) = \inf_{(u,v) \in Z_k, \|(u,v)\| = \rho_k} I_\lambda(u, v) \geq 0. \tag{3.7}$$

In addition, for all  $\lambda \in [1, 2]$ , and  $(u, v) \in Z_k$ , with  $\|(u, v)\| \leq \rho_k$ , we have

$$I_\lambda(u, v) \geq -8\eta|m(x)|^{\frac{2}{2-\sigma}}\alpha_k^\sigma(2)\|(u, v)\|_W^\sigma - a_3\alpha_k^r(r)\|(u, v)\|_W^r \rightarrow 0^-, \quad \text{as } k \rightarrow \infty.$$

Therefore

$$d_k(\lambda) = \inf_{(u,v) \in Z_k, \|(u,v)\| \leq \rho_k} I_\lambda(u, v) \rightarrow 0, \quad \text{as } k \rightarrow \infty \quad \text{uniformly for } \lambda \in [1, 2]. \tag{3.8}$$

From (H4), it can be obtained that  $\forall (u, v, x) \in \mathbb{R}^N \times \mathbb{R}^2$ ,

$$\begin{aligned} F(x, u, v) &= \int_0^1 f_u(x, tu, tv)u + f_v(x, tu, tv)tv \, dt \\ &\geq \int_0^1 \frac{f_u(x, tu, tv)tu + f_v(x, tu, tv)tv}{t} \, dt \\ &\geq \int_0^1 \frac{\zeta(x)(|tu|^\delta + |tv|^\delta)}{t} \, dt \\ &\geq \frac{1}{2}\zeta(x)(|u|^\delta + |v|^\delta). \end{aligned}$$

So, if  $(u, v) \in Y_k$ , by Lemma 4, one can get

$$\begin{aligned} I_\lambda(u, v) &= \frac{1}{2}\|(u, v)\|_W^2 - \int_{\mathbb{R}^N} F(x, u, v) \, dx \\ &\leq \frac{1}{2}\|(u, v)\|_W^2 - \frac{1}{2}\int_{\mathbb{R}^N} \zeta(x)(|u|^\delta + |v|^\delta) \, dx \\ &\leq \frac{1}{2}\|(u, v)\|_W^2 - \frac{1}{2}(\varepsilon_k^\delta)^2\|(u, v)\|_W^\delta, \quad x \in \mathbb{R}^N, \quad (u, v) \in \mathbb{R}^2. \end{aligned}$$

Hence, we choose  $r_k > 0$ , small enough satisfying  $r_k < \rho_k$  such that

$$b_k(\lambda) = \max_{(u,v) \in Y_k, \|(u,v)\|_W = r_k} I_\lambda(u, v) < 0, \quad \text{for all } \lambda \in [1, 2]. \tag{3.9}$$

The proof is completed.  $\square$

*Proof of Theorem 1.1.* From the assumption (H2), we know that  $B(u, v) \geq 0$  for all  $(u, v) \in W$ , and  $A(u, v) \rightarrow \infty$  as  $\|(u, v)\| \rightarrow \infty$ . Moreover,  $I_\lambda(-u, -v) = I_\lambda(u, v)$  for all  $(u, v) \in W$ , and  $\lambda \in [1, 2]$ . It follows from the conditions (H1), (H2) and (H3),  $I_\lambda(u, v)$  maps bounded sets into bounded sets uniformly for  $\lambda \in [1, 2]$ . Combining with Lemma 6, (A1), (A2), (A4) of Theorem 3 are verified. Therefore, for a.e.  $\lambda \in [1, 2]$ , there exists a sequence  $\{(u_n^k(\lambda), v_n^k(\lambda))\}_{n=1}^\infty$ , such that

$$\sup_n \left\| \left( u_n^k(\lambda), v_n^k(\lambda) \right) \right\| < \infty, I'_\lambda \left( u_n^k(\lambda), v_n^k(\lambda) \right) \rightarrow 0, I_\lambda \left( u_n^k(\lambda), v_n^k(\lambda) \right) \rightarrow c_k(\lambda) \tag{3.10}$$

as  $n \rightarrow \infty$ . By Theorem 3, (3.5) and the assumption that  $r > 2$ , it implies that

$$c_k(\lambda) \geq b_k(\lambda) = \inf_{(u,v) \in \mathcal{Z}_k, \|(u,v)\| = r_k} I_\lambda(u,v) \geq \frac{1}{8} (8\alpha_k^r c)^{\frac{2}{2-r}} = \overline{b}_k \rightarrow +\infty, \quad (3.11)$$

as  $k \rightarrow \infty$ . Also since

$$c_k(\lambda) = \inf_{r \in \Gamma_k} \max_{(u,v) \in B_k} I_\lambda(u,v) \leq \max_{(u,v) \in B_k} I_\lambda(u,v) = \overline{c}_k.$$

Hence,

$$\overline{b}_k \leq c_k(\lambda) \leq \overline{c}_k, \quad \text{for } k \geq k_0. \quad (3.12)$$

If we choose a sequence  $\lambda_m \in [1, 2]$ , such that  $\lambda_m \rightarrow 1$ , it follows from (3.10) that the sequence  $\{(u_n^k(\lambda_m), v_n^k(\lambda_m))\}$  is bounded. If we can prove that the sequence  $\{(u_n^k(\lambda_m), v_n^k(\lambda_m))\}$  has a strong convergent subsequence as  $n \rightarrow \infty$ , we can assume that

$$\lim_{n \rightarrow \infty} (u_n^k(\lambda_m), v_n^k(\lambda_m)) = (u^k(\lambda_m), v^k(\lambda_m))$$

for every  $m \in \mathbb{N}$  and  $k \geq k_0$ . By (3.10) and (3.12), we can get

$$I'_{\lambda_m}((u^k(\lambda_m), v^k(\lambda_m))) = (0, 0) \text{ and } I_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)) \in [\overline{b}_k, \overline{c}_k] \text{ for } k \geq k_0. \quad (3.13)$$

If we can prove that  $\{(u^k(\lambda_m), v^k(\lambda_m))\}_{m=1}^\infty$  possesses a strong convergent subsequence with the limit  $(u^k, v^k) \in W$  for all  $k \geq k_0$ , the limit  $(u^k, v^k)$  is a critical point of  $I(u, v) = I_1(u, v)$  with  $I(u^k, v^k) \in [\overline{b}_k, \overline{c}_k]$ . Since  $\overline{b}_k \rightarrow \infty$  as  $k \rightarrow \infty$ , we get infinitely many nontrivial critical points of  $I(u, v)$ . Consequently, problem (1.1) possesses infinitely many nontrivial solutions with high energy.

(1) We prove that the sequence  $\{(u_n^k(\lambda_m), v_n^k(\lambda_m))\}_{n=1}^\infty$  has a strong convergent subsequence. Since  $\{(u_n^k(\lambda_m), v_n^k(\lambda_m))\}_{n=1}^\infty$  is bounded, up to a subsequence, denoted by  $\{(u_n^k(\lambda_m), v_n^k(\lambda_m))\}_{n=1}^\infty$ , there exists sequence  $(u^k(\lambda_m), v^k(\lambda_m))$ , such that

$$\begin{aligned} (u_n^k(\lambda_m), v_n^k(\lambda_m)) &\rightharpoonup (u^k(\lambda_m), v^k(\lambda_m)) && \text{in } W, \\ (u_n^k(\lambda_m), v_n^k(\lambda_m)) &\rightarrow (u^k(\lambda_m), v^k(\lambda_m)) && \text{in } L_2^p(\mathbb{R}^N), \quad p \in [2, 2_*]. \end{aligned}$$

Again,

$$\begin{aligned} &\left\langle dI_{\lambda_m}(u_n^k(\lambda_m), v_n^k(\lambda_m)) - dI_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)), \right. \\ &\quad \left. (u_n^k(\lambda_m) - u^k(\lambda_m), v_n^k(\lambda_m) - v^k(\lambda_m)) \right\rangle \\ &= \left\| (u_n^k(\lambda_m) - u^k(\lambda_m), v_n^k(\lambda_m) - v^k(\lambda_m)) \right\|_W^2 \\ &\quad - \lambda_m \int_{\mathbb{R}^N} \left( f_u(x, u_n^k(\lambda_m), v_n^k(\lambda_m)) - f_u(x, u^k(\lambda_m), v^k(\lambda_m)) \right) (u_n^k(\lambda_m) - u^k(\lambda_m)) dx \end{aligned}$$

$$-\lambda_m \int_{\mathbb{R}^N} \left( f_v(x, u_n^k(\lambda_m), v_n^k(\lambda_m)) - f_v(x, u^k(\lambda_m), v^k(\lambda_m)) \right) \left( u_n^k(\lambda_m) - v^k(\lambda_m) \right) dx. \tag{3.14}$$

Since  $dI_{\lambda_m}(u_n^k(\lambda_m), v_n^k(\lambda_m)) \rightarrow 0$  and  $(u_n^k(\lambda_m), v_n^k(\lambda_m)) \rightarrow (u^k(\lambda_m), v^k(\lambda_m))$  as  $n \rightarrow \infty$ , then

$$\left\langle dI_{\lambda_m}(u_n^k(\lambda_m), v_n^k(\lambda_m)) - dI_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)), (u_n^k(\lambda_m) - u^k(\lambda_m), v_n^k(\lambda_m) - v^k(\lambda_m)) \right\rangle \rightarrow 0.$$

From (H1), (H3), Lemma 1 and  $\{(u_n^k(\lambda_m), v_n^k(\lambda_m))\}_{n=1}^\infty$  is bounded, it implies

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \left( f_u(x, u_n^k(\lambda_m), v_n^k(\lambda_m)) - f_u(x, u^k(\lambda_m), v^k(\lambda_m)) \right) \left( u_n^k(\lambda_m) - u^k(\lambda_m) \right) dx \right| \\ & \leq \left\| f_u(x, u_n^k(\lambda_m), v_n^k(\lambda_m)) - f_u(x, u^k(\lambda_m), v^k(\lambda_m)) \right\|_{r', \sqrt{2}} \left\| u_n^k(\lambda_m) - u^k(\lambda_m) \right\|_{r \wedge 2} \\ & \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.15}$$

Similarly,

$$\left| \int_{\mathbb{R}^N} \left( f_v(x, u_n^k(\lambda_m), v_n^k(\lambda_m)) - f_v(x, u^k(\lambda_m), v^k(\lambda_m)) \right) \left( v_n^k(\lambda_m) - v^k(\lambda_m) \right) dx \right| \rightarrow 0. \tag{3.16}$$

then, it follows from (3.14), (3.15), (3.16) that

$$\left\| (u_n^k(\lambda_m) - u^k(\lambda_m), v_n^k(\lambda_m) - v^k(\lambda_m)) \right\|_W^2 \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore,

$$(u_n^k(\lambda_m), v_n^k(\lambda_m)) \rightarrow (u^k(\lambda_m), v^k(\lambda_m)), \quad n \rightarrow \infty, \text{ for } k \geq k_0.$$

(2) We prove that  $\{(u^k(\lambda_m), v^k(\lambda_m))\}_{m=1}^\infty$  has a strong convergent subsequence in  $W$ . Since  $\{(u^k(\lambda_m), v^k(\lambda_m))\}_{m=1}^\infty$  satisfying (3.13). We claim that  $\{(u^k(\lambda_m), v^k(\lambda_m))\}_{m=1}^\infty$  is bounded in  $W$ . If it is not the case, we consider

$$(\tilde{u}^k(\lambda_m), \tilde{v}^k(\lambda_m)) := \left( \frac{u^k(\lambda_m)}{\|(u^k(\lambda_m), v^k(\lambda_m))\|_W}, \frac{v^k(\lambda_m)}{\|(u^k(\lambda_m), v^k(\lambda_m))\|_W} \right),$$

then up to a subsequence, still denoted by  $(\tilde{u}^k(\lambda_m), \tilde{v}^k(\lambda_m))$ ,  $\exists(\tilde{u}^k, \tilde{v}^k)$ , such that

$$\begin{aligned} & (\tilde{u}^k(\lambda_m), \tilde{v}^k(\lambda_m)) \rightharpoonup (\tilde{u}^k, \tilde{v}^k), \quad \text{in } W, \\ & (\tilde{u}^k(\lambda_m), \tilde{v}^k(\lambda_m)) \rightarrow (\tilde{u}^k, \tilde{v}^k), \quad \text{in } L_2^p(\mathbb{R}^N), \quad p \in [2, 2_*), \\ & \tilde{u}^k(\lambda_m) \rightarrow \tilde{u}^k, \quad \text{a.e. } x \in \mathbb{R}^N, \\ & \tilde{v}^k(\lambda_m) \rightarrow \tilde{v}^k, \quad \text{a.e. } x \in \mathbb{R}^N. \end{aligned}$$

Case 1 : if  $(\tilde{u}^k, \tilde{v}^k) \neq (0, 0)$  in  $W$ . Since

$$\begin{aligned}
& 1 - \frac{\left\langle I'_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)), (u^k(\lambda_m), v^k(\lambda_m)) \right\rangle}{\|(u^k(\lambda_m), v^k(\lambda_m))\|_W^2} \\
&= \lambda_m \int_{\mathbb{R}^N} \frac{f_u(x, u^k(\lambda_m), v^k(\lambda_m))u^k(\lambda_m) + f_v(x, u^k(\lambda_m), v^k(\lambda_m))v^k(\lambda_m)}{\|(u^k(\lambda_m), v^k(\lambda_m))\|_W^2} dx \\
&= \lambda_m \int_{\Omega_m} (|\tilde{u}^k(\lambda_m)|^2 + |\tilde{v}^k(\lambda_m)|^2) \\
&\quad \cdot \frac{f_u(x, u^k(\lambda_m), v^k(\lambda_m))u^k(\lambda_m) + f_v(x, u^k(\lambda_m), v^k(\lambda_m))v^k(\lambda_m)}{|u^k(\lambda_m)|^2 + |v^k(\lambda_m)|^2} dx,
\end{aligned}$$

where  $\Omega_m = \{x \in \mathbb{R}^N : (\tilde{u}^k(\lambda_m), \tilde{v}^k(\lambda_m)) \neq (0, 0)\}$ . By (3.13), (H5) and Fatou's Lemma, we can deduce a contradiction that

$$\begin{aligned}
1 &= \liminf_{m \rightarrow \infty} \lambda_m \int_{\Omega_m} \left( (|\tilde{u}^k(\lambda_m)|^2 + |\tilde{v}^k(\lambda_m)|^2) \right. \\
&\quad \left. \cdot \frac{f_u(x, u^k(\lambda_m), v^k(\lambda_m))u^k(\lambda_m) + f_v(x, u^k(\lambda_m), v^k(\lambda_m))v^k(\lambda_m)}{|u^k(\lambda_m)|^2 + |v^k(\lambda_m)|^2} \right) dx \\
&\rightarrow \infty, \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

Case 2: If  $(\tilde{u}^k, \tilde{v}^k) = (0, 0)$  in  $W$ . we can define

$$I_{\lambda_m}(t_m u^k(\lambda_m), t_m v^k(\lambda_m)) = \max_{t \in [0, 1]} I_{\lambda_m}(t u^k(\lambda_m), t v^k(\lambda_m)).$$

For any  $\beta > 0$ , letting  $(\bar{u}^k(\lambda_m), \bar{v}^k(\lambda_m)) = \sqrt{4\beta}(\tilde{u}^k(\lambda_m), \tilde{v}^k(\lambda_m))$ , one can has

$$\begin{aligned}
& (\bar{u}^k(\lambda_m), \bar{v}^k(\lambda_m)) \rightarrow (0, 0), \quad \text{in } L_2^p(\mathbb{R}^N), \quad p \in [2, 2_*), \\
& \bar{u}^k(\lambda_m) \rightarrow 0, \quad \text{a.e. } x \in \mathbb{R}^N, \\
& \bar{v}^k(\lambda_m) \rightarrow 0, \quad \text{a.e. } x \in \mathbb{R}^N.
\end{aligned}$$

Similarly as (2.9) in Lemma 3 and by (H3), we can find  $\omega(x) \in L^1(\mathbb{R}^N)$ , such that

$$|F(x, \bar{u}^k(\lambda_m), \bar{v}^k(\lambda_m))| \leq \omega(x),$$

hence, using the Lebesgue dominated convergence theorem, we have

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^N} F(x, \bar{u}^k(\lambda_m), \bar{v}^k(\lambda_m)) dx = \int_{\mathbb{R}^N} F(x, 0, 0) dx = 0.$$

Thus, for  $m$  large enough,  $\sqrt{4\beta} \|(u^k(\lambda_m), v^k(\lambda_m))\|^{-1} \in (0, 1)$ , we derive that

$$I_{\lambda_m}(t_m u^k(\lambda_m), t_m v^k(\lambda_m)) \geq I_{\lambda_m}(\bar{u}^k(\lambda_m), \bar{v}^k(\lambda_m))$$



$$= \frac{1}{2} \|(\bar{u}^k(\lambda_m), \bar{v}^k(\lambda_m))\|_W^2 - \lambda_m \int_{\mathbb{R}^N} F(x, \bar{u}^k(\lambda_m), \bar{v}^k(\lambda_m)) dx,$$

which implies that

$$\liminf_{m \rightarrow \infty} I_{\lambda_m}(t_m u^k(\lambda_m), t_m v^k(\lambda_m)) \geq \beta \rightarrow \infty, \quad \text{as } \beta \rightarrow \infty.$$

Since  $I_{\lambda_m}(0, 0) = 0$ , and  $I_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)) \in [\bar{b}_k, \bar{c}_k]$ , we see that, for  $m$  sufficiently large,  $t_m \in (0, 1)$  and

$$\begin{aligned} \left\langle I'_{\lambda_m}(t_m u^k(\lambda_m), t_m v^k(\lambda_m)), (t_m u^k(\lambda_m), t_m v^k(\lambda_m)) \right\rangle &= t_m \frac{d}{dt} \Big|_{t=t_m} I_{\lambda_m}(t u^k(\lambda_m), t v^k(\lambda_m)) \\ &= 0. \end{aligned}$$

Therefore, using (H6), it can be deduced that

$$\begin{aligned} &\lambda_m \int_{\mathbb{R}^N} \frac{1}{2} (f_u(x, u^k(\lambda_m), v^k(\lambda_m)) u^k(\lambda_m) + f_v(x, u^k(\lambda_m), v^k(\lambda_m)) v^k(\lambda_m)) \\ &\quad - F(x, u^k(\lambda_m), v^k(\lambda_m)) dx \\ &\geq \lambda_m \frac{1}{\theta} \int_{\mathbb{R}^N} \frac{1}{2} (f_u(x, t_m u^k(\lambda_m), t_m v^k(\lambda_m)) t_m u^k(\lambda_m) \\ &\quad + f_v(x, t_m u^k(\lambda_m), t_m v^k(\lambda_m)) t_m v^k(\lambda_m) - F(x, t_m u^k(\lambda_m), t_m v^k(\lambda_m))) dx \\ &= \frac{1}{\theta} I_{\lambda_m}(t_m u^k(\lambda_m), t_m v^k(\lambda_m)) \rightarrow \infty, \quad m \rightarrow \infty. \end{aligned}$$

However, from (3.13), it implies that

$$\begin{aligned} &\lambda_m \int_{\mathbb{R}^N} \frac{1}{2} (f_u(x, u^k(\lambda_m), v^k(\lambda_m)) u^k(\lambda_m) + f_v(x, u^k(\lambda_m), v^k(\lambda_m)) v^k(\lambda_m) \\ &\quad - F(x, u^k(\lambda_m), v^k(\lambda_m))) dx \\ &= I_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)) - \frac{1}{2} \left\langle I'_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)), (u^k(\lambda_m), v^k(\lambda_m)) \right\rangle \\ &= I_{\lambda_m}(u^k(\lambda_m), v^k(\lambda_m)) \in [\bar{b}_k, \bar{c}_k]. \end{aligned}$$

This is a contradiction. similarly in (1), we can prove the sequence  $\{(u^k(\lambda_m), v^k(\lambda_m))\}$  has a strongly convergent subsequence. Then there exists  $(u^k, v^k) \in W$ , such that  $I(u^k, v^k) \in [\bar{b}_k, \bar{c}_k]$  and  $I'(u^k, v^k) = (0, 0)$ . The proof is completed.  $\square$

*Proof of Theorem 1.2* From the (3.6) and the discussion in Lemma 6 on any finite dimensional subspace of  $W$ , we get

$$B(u, v) = \int_{\mathbb{R}^N} F(x, u, v) dx \rightarrow \infty, \quad \text{as } \|(u, v)\|_W \rightarrow \infty.$$

By the assumption (H2), we know that  $B(u, v) \geq 0$ , for all  $(u, v) \in W$ , and by the assumption (H1), (H4),  $I_\lambda(u, v)$  maps bounded sets into bounded sets uniformly for

$\lambda \in [1, 2]$ . By (H7),  $I_\lambda(u, v) = I_\lambda(-u, -v)$ , for all  $\lambda \in [1, 2]$  and  $(u, v) \in W$ . Combining with Lemma 7, we see that all the conditions of Theorem 4 are verified. Consequently, for each  $k \in \mathbb{N}$ , there exists  $\lambda_n \rightarrow 1$ ,  $(u(\lambda_n), v(\lambda_n)) \in Y_n$ , such that

$$I'_{\lambda_n}|_{Y_n}(u(\lambda_n), v(\lambda_n)) = (0, 0), \quad I_{\lambda_n} \rightarrow c_k \in [d_k(2), b_k(1)], \quad \text{as } n \rightarrow \infty. \quad (3.17)$$

Let  $P_n : W \rightarrow Y_n$ , is the orthogonal projection operator for all  $n \in \mathbb{N}$ , where  $Y_n$  is the  $n$ -dimensional subspace of  $W$ . By (3.17), we have

$$\begin{aligned} \langle P_n I'_{\lambda_n}(u(\lambda_n), v(\lambda_n)), (u(\lambda_n), v(\lambda_n)) \rangle &= \langle I'_{\lambda_n}(u(\lambda_n), v(\lambda_n)), P_n(u(\lambda_n), v(\lambda_n)) \rangle \\ &= \langle I'_{\lambda_n}(u(\lambda_n), v(\lambda_n)), (u(\lambda_n), v(\lambda_n)) \rangle. \end{aligned}$$

So, as the proof in Theorem 1.1,  $\{(u(\lambda_n), v(\lambda_n))\}$  is bounded in  $W$ . Up to a subsequence,  $\exists (u, v)$  in  $W$ , such that  $(u(\lambda_n), v(\lambda_n)) \rightharpoonup (u, v)$  in  $W$ . (3.17) can be changed into that

$$I'_{\lambda_n}(u(\lambda_n), v(\lambda_n)) \rightarrow (0, 0), \quad I_{\lambda_n} \rightarrow c_k \in [d_k(2), b_k(1)], \quad \text{as } n \rightarrow \infty.$$

Therefore, as (1) in Theorem 1.1, there exists  $(u, v)$ , such that  $(u(\lambda_n), v(\lambda_n)) \rightarrow (u, v)$  in  $W$ . So, for every  $k \in \mathbb{N}$ , we obtain the critical point of  $I(u, v)$  with  $I'(u, v) = (0, 0)$  and  $I(u, v) = c_k \in [d_k(2), b_k(1)]$ . Therefore, from Theorem 2.6, problem (1.1) possesses infinitely many nontrivial critical points with small energy. The proof is completed.  $\square$

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