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POSITIVE SOLUTIONS FOR A FOURTH ORDER DIFFERENTIAL INCLUSION WITH BOUNDARY VALUES

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Abstract. An existence result for positive solutions of a fourth order differential inclusion is proven. This is accomplished by using a certain fixed point theorem on cones and a minor extension of the Ascoli theorem. This inclusion allows both the function and its derivative on the right-hand side.

1. Introduction

In this article the following will be proven:

THEOREM 1. *Let $F:[0,1]\times[0,\infty)\times[0,\infty)\to P(\mathbb{R})$ be compact and convex valued, measurable in t for all $(x,y)\in\mathbb{R}^2$, upper semicontinuous in (x,y) for almost all $t\in[0,1]$ and*

1) *for all $r>0$, there exists $h_r\in L^1([0,1],\mathbb{R})$ such that for almost all t, every $(x,y)\in\mathbb{R}^2$ with $\mid(x,y)\mid< r$ and every $\gamma\in F(t,x,y)$, $\mid\gamma\mid\leqslant h_r(t)$ holds;*

2) *there exist α,β, $\alpha<\beta$, $[\alpha,\beta]\subset(0,1)$ and $F_1\subset[\alpha,\beta]$ of full measure such that for every $k>0$, there exists $L>0$, such that $p>L$ implies that*

$$\inf_{x\in[0,\infty)}\frac{F(t,x,p)}{p}>k \quad \textit{for all } t\in F_1;$$

here $\inf_{x\in[0,\infty)}\frac{F(t,x,p)}{p}$ means $\inf_{x\in[0,\infty)}\inf_{y\in F(t,x,p)}\frac{y}{p}$;

3) *there exists $M>0$ such that for almost all t and all $(x,p)\in[0,\infty)\times[0,\infty)$, $y\in F(t,x,p)$ implies that $y\geqslant -M$.*

Then there exists a positive solution to problem (1) below for $\lambda>0$ sufficiently small:

$$\begin{cases} u'''(t)\in\lambda F(t,u(t),u'(t)) \ \ a.e. \\ u(0)=u'(0)=u''(1)=u'''(1)=0,\ \lambda>0. \end{cases} \tag{1}$$

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Basic definitions of properties of set valued functions, differential inclusions, and operators on set valued functions may be found in many sources such as [1], [3] , and [4].

An example of such a function F satisfying the three conditions in the theorem to be proven is the following: for all $(t,x,y) \in (0,1] \times [0,\infty) \times [0,\infty)$,

$$F(t,x,y) = \begin{cases} [2t(x^2+y^2-1), t(x^2+y^2-1)], & x^2+y^2 < 1, \\ [t(x^2+y^2-1), 2t(x^2+y^2-1)], & x^2+y^2 > 1 \\ 0, & t = 0 \text{ or } x^2+y^2 = 1. \end{cases}$$

Let $[\alpha,\beta]$ be any subinterval of $(0,1)$, F_1 any set of full measure in $[\alpha,\beta]$, and $M = 2$. It can be easily shown that in this case one can let $h_r(t) = 2t(r^2+1) \in L^1([0,1],\mathbb{R})$ and for $p > 1$,

$$\inf_{x\in[0,\infty)} \frac{F(t,x,p)}{p} = \Big(\frac{p^2-1}{p}\Big)t > \Big(\frac{p^2-1}{p}\Big)\alpha,$$

which clearly approaches infinity as p does.

Note that the theorem in [6] will not apply here since among other things F is a function of three variables.

In this theorem we use the following definition of a positive solution.

DEFINITION. $u : [0,1] \to \mathbb{R}$ is a positive solution to (1) if

i) $u \in AC^3([0,1],\mathbb{R})$ (by this we mean u', u'' and u''' are each absolutely continuous on $[0,1]$),

ii) $u''''(t) \in \lambda F(t,u(t),u'(t))$ for almost all $t \in [0,1]$,

iii) $u(0) = 0$, $u'(0) = 0$, $u''(1) = 0$, $u'''(1) = 0$,

iv) $u(t) > 0$ for all $t \in (0,1]$.

This theorem generalizes a result due to Ma [9], which proved existence of positive solutions to the problem

$$\begin{cases} u''''(t) = \lambda f(t,u(t),u'(t)),\ t \in [0,1], \\ u(0) = 0,\ u'(0) = 0,\ u''(1) = 0,\ u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times [0,\infty) \times [0,\infty)$ is continuous and satisfies a number of other conditions. Ma's result is also similar to one by Yang [13].

Condition 2) may seem a little unusual. In [9] Ma requires the following to be satisfied by f:

There exists a subinterval $[\alpha,\beta] \subset (0,1)$ *with* $\alpha < \beta$ *such that*

$$\lim p \to \infty \frac{f(x,u,p)}{p} = \infty \ \textit{uniformly for}\ (x,u) \in [\alpha,\beta] \times [0,\infty).$$

Thus condition 2) in theorem 1 is simply a set valued version of this.

Note also that the condition need only be satisfied on a proper subinterval $[\alpha,\beta]$ of $[0,1]$ rather than the entire interval as is often required with this type of assumption. See for example [8].

We will see that several of Ma's proofs in [9] carry over to the set valued case quite nicely.

The use of fixed point theorems for such problems with different boundary values than ours is quite common. For example in [2] the Covitz-Nadler fixed point theorem is used to obtain existence theorems for a fourth order differential inclusion which generalize results in [12] where the Ky Fan fixed point theorem is used. In [10] monotone methods, upper and lower solutions, and a maximum principle are used to obtain certain extremal solutions to a fourth order single valued problem which has the second derivative on the right hand side. In [7] a contraction mapping principle involving the Green's function is applied to obtain a Filippov type existence result.

In [6] the following theorem is proven:

THEOREM [6]: *Let $F:[0,1]\times\mathbb{R}\to\mathscr{P}([0,\infty))$ be compact- and convex-valued, Lebesgue measurable in t for each x, upper semicontinuous in x for almost all t and for each $r>0$ there exists a function $h_r\in L^1([0,1],\mathbb{R})$ such that $|y|\leqslant h_r(t)$ for almost all t, every $x\in\mathbb{R}$ with $|x|<r$ and every $y\in F(t,x)$. Also, assume certain technical assumptions hold. Then, there exists a positive solution to the problem*

$$\begin{cases} u''''(t)\in F(t,u(t)),\, t\in[0,1] \\ u(0)=0,\, u'(0)=0,\, u''(1)=0,\, u'''(1)=0. \end{cases} \tag{1}$$

The technical assumptions mentioned above require the knowledge of the Green's function which will not be needed here. Also, instead of the second condition in theorem 1, the corresponding assumptions in the theorem in [6] require that there exists $c\in L^1([0,1],[0,\infty))$ such that for all $\varepsilon>0$ there exists $H>0$ such that there is a set of full measure S such that for all $t\in S$, all $x\in(0,H)$ and all $y\in F(t,x)$, $y\leqslant[c(t)+\varepsilon]x$. There is a similar condition involving a measurable function $d(t)$ which serves as a type of lower bound for $\frac{\inf F(t,x)}{x}$ as x approaches infinity for almost all t. The current result will not need such complicated assumptions. The fact that the differential inclusion has λ which may be chosen to be as small as needed will allow some slight alterations in the right hand side which were not possible in [6]. The same fixed point theorem below was used in [6], however. Applying it will involve showing the complete continuity of a certain operator which in [6] was accomplished by relying on a well known theorem. In the current setting complete continuity will be proven using a generalization of the Ascoli theorem which is specified later in the paper. Following Ma in [9], F is also allowed to take on values that are in $\mathscr{P}([-M,\infty))$ for some positive M but only for almost all t. This is slightly more general than the result in [6] in which only subsets of $[0,\infty)$ are allowed. Obviously, the major difference between the result in [6] and the current theorem is the fact that F will now be allowed to depend on $u'(t)$ as well as t and $u(t)$.

We will need the following set-valued version of the Guo-Krasnoselskii fixed point theorem for cones which is theorem 3 in [6] and is a special case of theorem 5.5 in [5].

THEOREM 2. *Let* $(X, \|\cdot\|)$ *be a Banach space over the reals, and let* $P \subseteq X$ *be a cone in* X. *Let* H_1 *and* H_2 *be real numbers such that* $H_2 > H_1 > 0$ *and let* $\Omega_i = \{u \in X \mid \|u\| < H_i\}$ *for* $i = 1, 2$. *If the operator* $T : P \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to P(P)$ *is compact and convex valued and is completely continuous such that either*

1) $\|w\| \leqslant \|u\|$ *for* $u \in P \cap \partial\Omega_1, w \in T(u)$ *and* $\|w\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_2, w \in T(u)$, *or*

2) $\|w\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_1, w \in T(u)$ *and* $\|w\| \leqslant \|u\|$ *for* $u \in P \cap \partial\Omega_2, w \in T(u)$,

then T *has a fixed point.*

We will also need to define a special Banach space as in [9]. Let

$$C_0^1[0,1] = \{u \mid u \in C^1[0,1],\ \ u(0) = u'(0) = 0\}.$$

We will associate the following norm with this space: $\| u \| = \sup\{| u'(t) |,\ t \in [0,1]\}$.

That this is a Banach space can be easily seen by integration and the completeness of C[0,1]. In order to prove our theorem we will need an understanding of compactness in this Banach space. Here is an extension of the Ascoli theorem for $C_0^1[0,1]$.

THEOREM 3. *Suppose* $T \subset C_0^1[0,1]$ *is closed and has the following properties:*

1) $\sup_{f \in T} \| f' \|_0 < \infty$, *where by* $\| f' \|_0$ *we mean the supremum norm on* $C[0,1]$*;*

2) *for all* $\varepsilon > 0$, *and for all* $t \in [0,1]$, *there exists* $\delta = \delta(t,\varepsilon)$ *such that*

$$\textit{for any } y \in [0,1] \textit{ with } | t - y | < \delta \textit{ we have } | f'(t) - f'(y) | < \varepsilon$$

(i.e. $\{f' \mid f \in T\}$ *is equicontinuous.)*

Then T *is compact in* $C_0^1[0,1]$.

Proof.

$a)$ Note that for all $f \in T$,

$$| f(t) | = \left| \int_0^t f'(x)dx \right| \leqslant \int_0^t | f'(x) | \, dx \leqslant Y < \infty,$$

where $Y = \sup_{f \in T} \| f' \|_0$, so T is uniformly bounded in C[0,1].

$b)$ Let $\varepsilon > 0$ and $t \in [0,1]$. The existence of $f'(t)$ guarantees the existence of δ_t such that $| y - t | < \delta_t$ implies

$$\left| \frac{f(y) - f(t)}{y - t} \right| < Y + 1 \ \text{ or } \ | f(y) - f(t) | < (Y+1) \, | y - t | .$$

Now choose $\delta = \min\{\delta_t, \frac{\varepsilon}{Y+1}\}$ and if $y \in [0,1]$ with $| t - y | < \delta$, then

$$| f(y) - f(t) | < (Y+1) \frac{\varepsilon}{Y+1} = \varepsilon.$$

Thus T is equicontinuous in C[0,1].

c) Thus we have by the Ascoli theorem that both T and $\{f' \mid f \in T\}$ are precompact in C[0,1].

d) Let $\{f_n\}$, $n = 1,2,3\cdots$ be a sequence in T. Note that the closures in C[0,1] of $\{f_n\}$ and $\{f_n'\}$ are compact in C[0,1]. Thus by taking subsequences of subsequences one can find a subsequence which we will relabel $\{f_m\}$ for which there are two functions f and g in C[0,1] such that $f_m \to g$ and $f_m' \to f$ uniformly.

We need to show that $g' = f$.

Note that for all m, $f_m(t) = \int_0^t f_m'(x)dx \leqslant Y$. Thus by the bounded convergence theorem we know that

$$\int_0^t f_m'(x)dx \to \int_0^t f(x)dx, \text{ for all } t \in [0,1].$$

Since $f_m \to g$ it follows that $g' = f$ as desired. The closure of T in $C_0^1[0,1]$ guarantees that $g \in T$.

Therefore every sequence in T has a convergent subsequence which implies that T is compact in $C_0^1[0,1]$. $\square$

2. Basic Lemmas

The proofs of the lemmas below are virtually the same as those found in [9] except that $u''''(t)$ is only almost everywhere equal to $y(t)$ in lemma 1, $u''''(t)$ is only almost everywhere equal to 1 in lemma 3, $u \in AC^3([0,1],\mathbb{R})$, $y(t) \in L[0,1]$, and all integrals are understood to be Lebesgue integrals so expressions such as $\int_s^1 y(\nu)d\nu$ will, of course, be interpreted as $Q(s) = \int_{[s,1]} y$ for some function Q.

Thus, we will not repeat many of the details here.

LEMMA 1. *Let $y(t) \in L[0,1]$. The solution to the problem*

$$\begin{cases} u''''(t) = y(t) \ \text{ a.e. on } (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \\ u \in AC^3([0,1],\mathbb{R}), \end{cases} \tag{2}$$

is given by

$$u(t) = \int_0^t \left(\int_0^\tau \left\{ \int_r^1 \left[\int_s^1 y(\nu)d\nu \right] ds \right\} dr \right) d\tau.$$

Moreover if $y \geqslant 0$ a.e., then

$$u \geqslant 0,\ u' \geqslant 0,\ u'' \geqslant 0,\ \text{ and } \ u''' \leqslant 0, \ \text{ on } [0,1].$$

Proof. That u is in the form above is simply a matter of repeated integration. Note that

$$u'''(1) = \int_t^1 y(\nu)d\nu + u'''(t)$$

and since $u'''(1)=0$ we have that

$$u'''(t) = -\int_t^1 y(\nu)d\nu \leqslant 0 \text{ for } t\in[0,1]. \ \square$$

LEMMA 2. *If $y(t)\in L[0,1]$, $y \geqslant 0$ a.e., then the solution to problem (2) above satisfies:*

$$u'(t) \geqslant t \parallel u' \parallel_0,\ t\in[0,1].$$

LEMMA 3. *The boundary value problem*

$$\begin{cases} u''''(t) = 1 \text{ a.e. on } (0,1), \\ u(0)=u'(0)=u''(1)=u'''(1)=0, \\ u\in AC^3([0,1],\mathbb{R}), \end{cases} \tag{3}$$

has a solution

$$w_0(t) = \frac{t^2}{4} - \frac{t^3}{6} + \frac{t^4}{24}.$$

Moreover $w_0(t) \leqslant \frac{1}{8}t$, $w_0'(t) \leqslant \frac{1}{2}t$ for $t\in[0,1]$.

3. Proof of Theorem 1.

The proof follows that in [9], but must be altered in order to handle the set valued case.

Define $G:[0,1]\times R\times R \to P([0,\infty))$ by:

$$G(t,u,p) = \begin{cases} F(t,u,p)+M, & (t,u,p)\in[0,1]\times[0,\infty)\times[0,\infty), \\ F(t,u,0)+M, & (t,u,p)\in[0,1]\times[0,\infty)\times(-\infty,0], \\ F(t,0,p)+M, & (t,u,p)\in[0,1]\times(-\infty,0]\times[0,\infty), \\ F(t,0,0)+M, & (t,u,p)\in[0,1]\times(-\infty,0]\times(-\infty,0], \end{cases}$$

where by $F(t,u,p)+M$, we mean $\{y+M \mid y\in F(t,u,p)\}$ etc.

Clearly G is compact and convex valued because F is and $F+M$ is merely a translation. Also if $|(u,p)|<r$, then clearly $|(0,p)|<r$, $|(u,0)|<r$, and $0=|(0,0)|<r$. Since $\int_0^1 M dt = M$ we see that G is integrably bounded because whenever $|(u,p)|<r$, then for almost all t, $|\gamma| \leqslant h_r(t)+M$ for any $\gamma\in G(t,u,p)$.

By considering a collection of cases one can also show that G is upper semicontinuous. This is accomplished by proving the result for (t,u,p) on each coordinate axis and then in each quadrant. Since each case is proven similarly I will only show the argument for the case where $u<0$ and $p=0$.

First let t be such that $F(t,u,p)$ is upper semicontinuous in (u,p) and let V be an open subset such that $G(t,u,0)=F(t,0,0)+M\subset V$. Construct an open ball, $B_r(u,0)$, of radius r about $(u,0)$. Choose r so that it satisfies the following properties:

1) $r < |u|$;

2) the ball of radius r about $(0,0)$, $B_r(0,0)$, will have the property that: for each $(a,b) \in B_r(0,0) \cap ([0,\infty)\ x\ [0,\infty))$, $F(t,a,b)+M \subset V$. This is possible because of the upper semicontinuity of F and the fact that a translation of an open set is an open set.

The lower half of $B_r(u,0)$, namely, $\{(x,y) \in B_r(u,0) \mid y \leqslant 0\}$ is mapped to $F(t,0,0)+M \subset V$. If (x^*,p^*) lies in the upper portion, $\{(x,y) \in B_r(u,0) \mid y > 0\}$, then it will map to $F(t,0,p^*)+M$ for some $0 < p^* < r$. Since $(0,p^*) \in B_r(0,0) \cap ([0,\infty)\ x\ [0,\infty))$ we have from property 2) above that $F(t,0,p^*)+M \subset V$.

Thus we have found an open neighborhood of (u,p) which maps into V which is what we need to prove that G is upper semicontinuous.

The other cases are similar.

Now let $z(t) = \lambda M w_0(t)$ where $w_0(t)$ has been defined above.

LEMMA 4. *Problem* (1) *has a positive solution* u *if* $u+z=\widetilde{u}$ *is a solution of*

$$\begin{cases} \widetilde{u}'''' \in \lambda G(t,\widetilde{u}-z,\widetilde{u}'-z'), \textit{for almost all } t \in (0,1), \\ \widetilde{u}(0) = \widetilde{u}'(0) = \widetilde{u}''(1) = \widetilde{u}'''(1) = 0, \\ \widetilde{u}'(t) > z'(t)\ \textit{for } t \in (0,1). \end{cases} \tag{4}$$

Proof. Since lemma 3 shows that $w_0''''(t) = 1$ we have that $z'''' = \lambda M$. Since $z(0) = z'(0) = z''(1) = z'''(1) = 0$ we know that $u(0) = u'(0) = u''(1) = u'''(1) = 0$.

Thus u satisfies the boundary conditions for problem (1). Also we know that since z' is nonnegative on $[0,1]$ and $\widetilde{u}'(t) > z'(t)$, then $u' > 0$ on $(0,1)$ which means that $u = \widetilde{u}-z$ is increasing and thus positive on $(0,1]$.

Now we have for almost all $t \in (0,1)$:

$$\widetilde{u}'''' = u'''' + z'''' = u'''' + \lambda M \in \lambda G(t,\widetilde{u}-z,\widetilde{u}'-z').$$

Since $\widetilde{u}-z$ and $\widetilde{u}'-z'$ are positive,

$$\lambda G(t,\widetilde{u}-z,\widetilde{u}'-z') = \lambda F(t,\widetilde{u}-z,\widetilde{u}'-z') + \lambda M = \lambda F(t,u,u') + \lambda M.$$

Thus we have that $u'''' = \lambda F(t,u,u')$ for almost all $t \in (0,1)$ and u is a positive solution of problem (1). $\square$

Now let us consider the following cone, K, in $C_0^1[0,1]$,

$$K = \{u \mid u \in C_0^1[0,1],\ u(t) \geqslant 0, u'(t) \geqslant 0, \text{ and } u'(t) \geqslant \| u \|\, t \text{ for all } t \in [0,1]\}.$$

For $v \in K$ define Av by

$$Av = \{w \in AC^3 \mid w \text{ is a solution to problem (2) with } y(t) = f(t), \\ \text{where } f(t) \in G(t,v-z,v'-z') \text{ and } f \in L[0,1]\}.$$

Note that a fixed point of A would satisfy the first two requirements of a solution to problem (4). That such integrable selections, f, exist is guaranteed by theorem 6.45 in [11]. Thus by lemma 1 $Av \neq \varnothing$.

Let us consider $w \in Av$ with its associated integrable selection f.

Note that property 3) of theorem 1 and the definition of G guarantee that $f \geqslant 0$ a.e. Thus we can apply lemma 2 to show that $w'(t) \geqslant t \parallel w \parallel$ for all $t \in [0,1]$ and therefore $AK \subset K$.

Now let $\lambda \in (0,\Lambda)$ where $\Lambda < \min\{\frac{4}{M_1}, \frac{1}{M}\}$ where $M_1 = \int_0^1 (h_{2\sqrt{2}}(t)+M)dt$. Let $\Omega_1 = \{u \in C_0^1[0,1], \parallel u \parallel < 2\}$.

Note that since $u(t) = \int_0^t u'(x)dx$, for $u \in \Omega_1$ we have that both $\mid u(t) \mid$ and $\mid u'(t) \mid$ are less than 2 for all t and thus $\mid (u(t),u'(t)) \mid < 2\sqrt{2}$ for all t.

Choose $u \in K \cap \partial\Omega_1$ and $w \in Au$ with its integrable selection f. Then we have:

$$
\begin{aligned}
w'(t) &= \lambda \int_0^t \left(\int_r^1 \left[\int_s^1 f(x)dx \right] ds \right) dr \\
&\leqslant \lambda \int_0^t \left(\int_r^1 \left[\int_s^1 (h_{2\sqrt{2}}(x)+M)dx \right] ds \right) dr \\
&\leqslant M_1 \lambda \int_0^1 \left(\int_r^1 ds \right) dr = \frac{\lambda M_1}{2} < 2
\end{aligned}
$$

since $\lambda < \frac{4}{M_1}$. Thus we know that $\sup_{w \in Au} \parallel w \parallel \leqslant \parallel u \parallel$ for $u \in K \cap \partial\Omega_1$.

Choose $N > 0$ so that

$$
\frac{N\alpha}{2}\left[\frac{\beta^2}{2}(\beta-\alpha) - \frac{\beta}{2}(\beta^2-\alpha^2) + \frac{1}{6}(\beta^3-\alpha^3)\right] \geqslant 1.
$$

Note that the quantity inside the brackets is positive for $0 < \alpha < \beta < 1$.

Choose $\overline{R} > 2$ such that $1 - \frac{\lambda M}{2\overline{R}} \geqslant \frac{1}{2}$ and such that if $h \geqslant \frac{1}{2}\ \overline{R}\alpha$, then $\frac{y}{h} \geqslant N$ for all $y \in G(t,x,h)$ with $t \in F_1$ and $x \in (-\infty,+\infty)$.

This is possible because of property (2) in the statement of theorem 1.

Now let $\Omega_2 = \{u \in C_0^1[0,1], \parallel u \parallel < \overline{R}\}$ and let $u \in K \cap \partial\Omega_2$. Using lemma 3 we find that

$$
z'(s) = \lambda M w_0'(s) \leqslant \lambda M \frac{1}{2} s \leqslant \frac{1}{2}\lambda M \frac{u'(s)}{\parallel u \parallel} = \frac{\lambda M}{2\overline{R}} u'(s).
$$

Thus following [9], we can see that for $s \in [\alpha,\beta]$,

$$
u'(s) - z'(s) \geqslant (1 - \frac{\lambda M}{2\overline{R}})u'(s) \geqslant \frac{1}{2}u'(s) \geqslant \frac{1}{2} \parallel u \parallel s \geqslant \frac{1}{2}\overline{R}\alpha
$$

since $s \geqslant \alpha$ and $\parallel u \parallel = \overline{R}$.

Suppose $s \in F_1$. Then for all $y \in G(s, u(s)-z(s), u'(s)-z'(s))$ we have

$$
y \geqslant N(u'(s)-z'(s)) \geqslant N\frac{\overline{R}\alpha}{2},
$$

where we are letting $h = u'(s) - z'(s)$.

Since F_1 is of full measure in $[\alpha,\beta]$ the previous statement holds almost everywhere in this interval.

Let $w \in Au$. Then

$$w'(1) = \lambda \int_0^1 \left[\int_r^1 \left(\int_s^1 f(x)dx \right) ds \right] dr,$$

where $f(x)$ is an integrable nonnegative selection of $G(x, u(x) - z(x), u'(x) - z'(x))$ as before.

Note that $\| w \| \geqslant w'(1) \geqslant 0$. Thus

$$\begin{aligned} w'(1) &\geqslant \lambda \int_\alpha^\beta \left[\int_r^1 \left(\int_s^1 f(x)dx \right) ds \right] dr \\ &\geqslant \lambda \int_\alpha^\beta \left[\int_r^\beta \left(\int_s^\beta f(x)dx \right) ds \right] dr, \end{aligned}$$

where all integrals are again Lebesque integrals. Note that s, x and r will all be in $[\alpha, \beta]$ in this integral.

Since the above argument shows that $f \geqslant N\frac{\overline{R}\alpha}{2}$ a.e. on F_1 and F_1 is of full measure in $[\alpha, \beta]$ we therefore have that

$$\begin{aligned} w'(1) &\geqslant \frac{\lambda N \overline{R}\alpha}{2} \int_\alpha^\beta \left[\int_r^\beta \left(\int_s^\beta dx \right) ds \right] dr \\ &= \frac{\lambda N \overline{R}\alpha}{2} \left[\frac{\beta^2}{2}(\beta - \alpha) - \frac{\beta}{2}(\beta^2 - \alpha^2) + \frac{1}{6}(\beta^3 - \alpha^3) \right] \\ &\geqslant \overline{R} = \| u \| . \end{aligned}$$

Thus for $u \in K \cap \partial\Omega_2$ we have $\| w \| \geqslant \| u \|$.

Thus in order to apply theorem 2 to find a fixed point it remains to show that A is a completely continuous operator. To do so we will use theorem 3 which is our extension of the Ascoli theorem.

Let us first apply A to a bounded set in $C_0^1[0,1]$,

Consider the ball of radius ε centered about 0, $B_\varepsilon(0) \subset C_0^1[0,1]$ and let $v \in B_\varepsilon(0)$. Since $v(t) = \int_0^t v'(x)dx$ it is clear that $\| v \|_0 < \varepsilon$ and $\| v' \|_0 < \varepsilon$. Thus $|(v, v')| < \varepsilon\sqrt{2}$.

Let $w \in Av$. Then

$$w'(t) = \lambda \int_0^t \left[\int_r^1 \left(\int_s^1 f(x)dx \right) ds \right] dr,$$

where $f(x) \in G(x, v(x) - z(x), v'(x) - z'(x))$ *a.e.* as above.

Recall that

$$| z | = \lambda M \, | w_0 | \leqslant \lambda M \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{24} \right) = \frac{11}{24} \lambda M.$$

Similarly $| z' | \leqslant \frac{7}{6}\lambda M$. Thus we find that $|(v - w, v' - w')| \leqslant \sqrt{2}\varepsilon + \frac{13}{8}\lambda M$.

Let us call this last quantity N_0. By assumption 1) in theorem 1, the integral boundedness assumption, we have that $f(x) \leqslant h_{N_0}(x) + M$ a.e. .

We will let $N_1 = \int_0^1 (h_{N_0}(x)+M)dx$. Then it follows that:

$$w'(t) \leqslant \lambda \int_0^t \left[\int_r^1 \left(\int_s^1 h_{N_0}(x)dx\right)ds\right]dr \leqslant \lambda N_1 \int_0^t \left[\int_r^1 ds\right]dr \leqslant \lambda N_1.$$

Thus we have that $\{w' \mid w \in Av\}$ is uniformly bounded satisfying condition 1) of theorem 3.

Now consider $t, t_1 \in [0,1]$ with $t_1 < t$, then

$$\mid w'(t) - w'(t_1) \mid = \lambda \left| \int_{t_1}^t \left[\int_r^1 \left(\int_s^1 f(x)dx\right)ds\right]dr \right| \leqslant \lambda N_1 \mid t - t_1 \mid$$

which means that the equicontinuity condition of theorem 3 is satisfied.

Thus theorem 3 implies that $A(B_\varepsilon(0))$ has compact closure in $C_0^1[0,1]$ which means that A is completely continuous.

We can apply theorem 2 to show that A has a fixed point $\widehat{u} \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$ which implies that $2 \leqslant \| \widehat{u} \| \leqslant \overline{R}$.

Lemma 3 implies that $w_0'(t) \leqslant \frac{1}{2}t < t$. Using this and the facts that $\widehat{u} \in K$ and $\lambda < \frac{1}{M}$, it is clear that

$$\widehat{u}'(t) \geqslant \| \widehat{u} \| t \geqslant 2t > 2\lambda M t > 2\lambda M w_0'(t) = 2z'(t).$$

Thus letting $u(t) = \widehat{u}(t) - z(t)$ we find that $u'(t) = \widehat{u}'(t) - z'(t) \geqslant \frac{1}{2}\widehat{u}'(t) > 0$ for $t \in (0,1)$ since $\frac{\widehat{u}'(t)}{2} \geqslant z'(t)$.

Also

$$u(t) = \widehat{u}(t) - z(t) = \int_0^t (\widehat{u}'(x) - z'(x))dx \geqslant \frac{1}{2}\int_0^t \widehat{u}'(x)dx \geqslant \int_0^t z'(x)dx > 0$$

for $t \in (0,1)$ so our solution, u, to problem (1) is positive.

This could also be obtained by using lemma 4. □

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