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EXISTENCE OF POSITIVE SOLUTION AND NEW OSCILLATION CRITERIA FOR NONLINEAR FIRST–ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

SHYAM SUNDAR SANTRA

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Abstract. In this work, oscillatory behaviour of the solutions of a class of nonlinear first-order neutral delay differential equations of the form

$$(E_1) \qquad (x(t)+p(t)x(t-\tau))' + q(t)H(x(t-\sigma)) = f(t)$$

and

$$(E_2) \qquad (x(t)+p(t)x(t-\tau))' + q(t)H(x(t-\sigma)) = 0$$

are studied under various ranges of $p(t)$. Sufficient conditions are obtained for existence of bounded positive solutions of (E_1).

1. Introduction

For the last decade, the study of the asymptotic and oscillatory behavior of solutions of neutral differential equations is a concerned of major area of research. This is because of the development in science and technology and the challenges that the new classes of such equations provide in these application areas. Equations involving delay, and those involving advance and a combination of both arise in the models on lossless transmission lines in high speed computers which are used to interconnect switching circuits. The construction of these models using delays is complemented by the mathematical investigation of nonlinear equations. Moreover, the delay differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. There has been many investigations into the oscillation and nonoscillation of first order nonlinear neutral delay differential equations (See for example, [1]-[5], [7], [9], [11]-[25]). However, the study of oscillatory behaviour of solutions of (E_1) has received much less attention, which is due to mainly to the technical difficulties arising in its analysis.

Mathematics subject classification (2010): 34C10, 34C15, 34K40.

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In [1], Ahmed et al. have studied the oscillation properties of a linear differential equations of the form

$$(E_3) \qquad (r(t)(x(t)+p(t)x(t-\tau))'+q(t)x(t-\sigma)=0,$$

for the cases $p(t) \leqslant -1$, $-1 \leqslant p(t) < 0$ and $p(t) \equiv p \neq \pm 1$ and established sufficient conditions so that every solution of (E_3) is oscillates. Their method has made the proof unnecessarily complicated and applicable to only homogeneous equations. In an another paper [3], Ahmed et al. considered the first order nonlinear neutral delay differential equations with variable coefficients of the form

$$(E_4) \qquad [r(t)(a(t)x(t)+p(t)x(t-\tau))]'+q(t)f(x(t-\sigma))=0,$$

and obtained some new sufficient conditions for the oscillation of all solutions of (E_4) by employing the Riccati transformation. In [9], Graef et al. considered (E_4) when $a(t)=1=r(t)$ and developed some sufficient conditions for the oscillation of all solutions of (E_4). In [7], Elabbasy et al. have studied first-order nonlinear neutral delay differential equation of the form

$$(E_5) \qquad (x(t)-q(t)x(t-\tau))]'+f(t,x(\tau(t)))=0,$$

and established oscillation criteria for all solutions of (E_5) for $q(t) \neq 1$.

In [5], Das and Misra have made an attempt to study the oscillation properties of a nonlinear differential equations of type

$$(E_6) \qquad (x(t)-px(t-\tau))'+q(t)H(x(t-\sigma))=f(t),$$

where $0 \leqslant p < 1$, $f(t) > 0$, and H satisfies the generalized sublinear condition

$$\int_0^{\pm k} \frac{dt}{H(t)} < \infty,$$

for every positive constant k, and established necessary and sufficient conditions so that every solution of (E_6) either oscillates or tends to zero. Their method has made the proof unnecessarily complicated and does not allow $f \equiv 0$ and H to be superlinear. Thus their result is applicable to only strictly nonhomogeneous equations. In [18], Parhi and Rath considered (E_6) for $p = \pm 1$ and established sufficient conditions under which every solution of (E_6) either oscillates or tends to zero or $\pm\infty$ as $t \to \infty$.

Hence in this work, the author have made an attempt to establish the sufficient condition for oscillation (every solution oscillates) of a class of nonlinear neutral delay differential equation

$$(x(t)+p(t)x(t-\tau))'+q(t)H(x(t-\sigma))=f(t), \tag{1.1}$$

where

$$\tau,\sigma \in \mathbb{R}_+ = (0,+\infty),\ p \in C([0,\infty),\mathbb{R}),\ q \in (\mathbb{R}_+,\mathbb{R}_+),\ f \in C(\mathbb{R},\mathbb{R}),$$

and H satisfies

$$H \in C(\mathbb{R},\mathbb{R}) \text{ with } uH(u) > 0 \ for \ u \neq 0.$$

The objective of this work to establish the sufficient conditions for oscillation of solutions of (1.1) under various ranges of $p(t)$. Its associated homogenous equation

$$(x(t)+p(t)x(t-\tau))' + q(t)H(x(t-\sigma)) = 0, \tag{1.2}$$

is also considered. Unlike the work in [1], [3], [5], [7], [9] and [18] an attempt is made here to establish sufficient conditions under which every solution or every bounded solution of (1.1) and (1.2) oscillates. Of course, the impact of forcing term is considered. keeping in view of the influence of forcing function, this work is separated for forced and unforced equations.

By a solution of (1.1) / (1.2) we understand a function $x \in C([-\rho,\infty),\mathbb{R})$ such that $x(t)+p(t)x(t-\tau)$ is once continuously differentiable and (1.1) or (1.2) is satisfied for $t \geqslant 0$, where $\rho = \max\{\tau,\sigma\}$ and $\sup\{|x(t)| : t \geqslant t_0\} > 0$ for every $t_0 \geqslant 0$. A solution of (1.1) / (1.2) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

2. Oscillation properties of Eq. (1.1)

In this section, sufficient conditions are obtained for oscillation of solutions of the equation (1.1). We need the following conditions for this work in the sequel.

(A_1) *there exists* $\lambda > 0$ *such that* $H(u)+H(v) \geqslant \lambda H(u+v)$, $for \ u,v > 0$;

(A_2) $H(uv) = H(u)H(v)$, $for\, u,v \in \mathbb{R}$;

(A_3) $H(-u) = -H(u)$, $for\, u \in \mathbb{R}$;

(A_4) *there exists* $F \in C(\mathbb{R},\mathbb{R})$ *such that* $F(t)$ *changes sign with*

$$-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty \ and \ F'(t) = f(t);$$

(A_5) $F^+(t) = \max\{F(t),0\}$, $F^-(t) = \max\{-F(t),0\}$;

(A_6) *there exists* $F \in C(\mathbb{R},\mathbb{R})$ *such that* $F(t)$ *changes sign with*

$$\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = +\infty \ and \ F'(t) = f(t).$$

REMARK 1. Assumption (A_2) implies that (A_3) Indeed, $H(1)H(1) = H(1)$ and $H(1) > 0$ imply that $H(1) = 1$. Further,

$$H(-1)H(-1) = H(1) = 1$$

implies that $(H(1))^2 = 1$. Since $H(-1) < 0$, we conclude that $H(-1) = -1$. Hence,

$$H(-u) = H(-1)H(-u) = -H(u).$$

On the other hand, $H(uv) = H(u)H(v)$ for $u > 0$ and $v > 0$ and $H(-u) = -H(u)$ imply that $H(xy) = H(x)H(y)$ for every $x,y \in \mathbb{R}$.

REMARK 2. We may note that if $x(t)$ is a solution of (1.1), then $y(t) = -x(t)$ is also a solution of (1.1) provided that H satisfies (A_2) or (A_3).

THEOREM 1. *Let* $p(t) \geqslant 0$, $t \in \mathbb{R}_+$. *If* (A_2) *and* (A_6) *hold, then every solution of the equation* (1.1) *is oscillatory.*

Proof. Suppose for contrary that $x(t)$ is a nonoscillatory solution of equation (1.1). Then there exists $t_0 \geqslant \rho$ such that $x(t) > 0$ or $x(t) < 0$, for $t \geqslant t_0$. Assume that $x(t) > 0$ for $t \geqslant t_0$. Setting

$$z(t) = x(t) + p(t)x(t-\tau), \tag{2.1}$$

and

$$w(t) = z(t) - F(t), \tag{2.2}$$

it follows from (1.1) that

$$w'(t) = -q(t)H(x(t-\sigma)) \leqslant 0 \tag{2.3}$$

for $t \geqslant t_1 > t_0 + \sigma$. Consequently, $w(t)$ is nonincreasing on $[t_2, \infty)$, $t_2 > t_1$. Hence we have $w(t) < 0$ or $w(t) > 0$ for $t \geqslant t_2$. Since $z(t) > 0$, then $w(t) < 0$, for $t \geqslant t_2$ implies that $\liminf_{t\to\infty} F(t) \geqslant 0$, for $t \geqslant t_2$, a contradiction to (A_6). Hence, $w(t) > 0$ for $t \geqslant t_2$, then $\lim_{t\to\infty} w(t)$ exists.Writing

$$z(t) = w(t) + F(t),$$

we notice that

$$\begin{aligned}
0 \leqslant \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty} (w(t) + F(t)) \\
&\leqslant \limsup_{t\to\infty} w(t) + \liminf_{t\to\infty} F(t) \\
&= \lim_{t\to\infty} w(t) + \liminf_{t\to\infty} F(t) \\
&= -\infty,
\end{aligned}$$

a contradiction due to (A_6).

If $x(t) < 0$, for $t \geqslant t_0$, then we set $y(t) = -x(t)$, for $t \geqslant t_0$ in (1.1) and we find

$$(y(t) + p(t)y(t-\tau))' + q(t)H(y(t-\sigma)) = \widetilde{f}(t), \tag{2.4}$$

where $\widetilde{f}(t) = -f(t)$ due to (A_2). Let $\widetilde{F}(t) = -F(t)$. Then

$$-\infty < \liminf_{t\to\infty} \widetilde{F}(t) < 0 < \limsup_{t\to\infty} \widetilde{F}(t) < \infty$$

and $\widetilde{F}'(t) = \widetilde{f}(t)$ hold. Hence proceeding as above, we find a contradiction to (A_6). This completes the proof of the theorem.

THEOREM 2. *Let $0 \leqslant p(t) \leqslant p < \infty$, $t \in \mathbb{R}_+$. Assume that (A_1), (A_2), (A_4) and (A_5) hold. Furthermore, assume that*

(A_7) $\int_T^\infty Q(t)H(F^+(t-\sigma))dt = \infty,\ T > 0$

and

(A_8) $\int_T^\infty Q(t)H(F^-(t-\sigma))dt = \infty,\ T > 0$

hold, then conclusion of the Theorem 1 is true, where for $t > \tau$,

$$Q(t) = min\{q(t), q(t-\tau)\}.$$

Proof. On the contrary, we proceed as in the proof of the Theorem 1 to obtain that $w(t)$ is monotonic on $[t_2,\infty)$. Since $z(t) > 0$, then $w(t) < 0$, for $t \geqslant t_2$ implies that $F(t) > 0$, for $t \geqslant t_2$, a contradiction to (A_4). Hence, $w(t) > 0$ for $t \geqslant t_2$. Ultimately, $z(t) > F(t)$ and hence $z(t) > \max\{0, F(t)\} = F^+(t)$, for $t \geqslant t_2$. Note that $\lim_{t\to\infty} w(t)$ exists. Due to (2.2), (1.1) becomes

$$0 = w'(t) + q(t)H(x(t-\sigma)) + H(p)[w'(t-\tau) + q(t-\tau)H(x(t-\tau-\sigma))]$$

for $t \geqslant t_2$ and because of (A_1) and (A_2), we find that

$$\begin{aligned} 0 &\geqslant w'(t) + H(p)w'(t-\tau) + Q(t)[H(x(t-\sigma)) + H(p\,x(t-\tau-\sigma))] \\ &\geqslant w'(t) + H(p)w'(t-\tau) + \lambda Q(t)H(z(t-\sigma)) \\ &\geqslant w'(t) + H(p)w'(t-\tau) + \lambda Q(t)H(F^+(t-\sigma)), \end{aligned} \tag{2.5}$$

for $t \geqslant t_3 > t_2 + \sigma$. Integrating (2.5) from t_3 to $t(> t_3)$, we obtain

$$\lambda \int_{t_3}^{t} Q(s)H(F^+(t-\sigma))ds \leqslant -[w(s) + H(p)w(s-\tau)]_{t_3}^{t} < \infty,\ as\, t \to \infty,$$

a contradiction to (A_7).

If $x(t) < 0$, for $t \geqslant t_0$, then we set $y(t) = -x(t)$ to obtain $y(t) > 0$ for $t \geqslant t_0$ and hence using equation (2.4), we obtain a contradiction due to (A_8). This completes the proof of the theorem.

THEOREM 3. *Let $-1 \leqslant p(t) \leqslant 0$, $t \in \mathbb{R}_+$. Suppose that (A_2), (A_4) and (A_5) hold. If any one of the following conditions:*

(A_9) $\int_T^\infty q(t)H(F^+(t-\sigma))dt = \infty,\ T > 0,$

(A_{10}) $\int_T^\infty q(t)H(F^-(t-\sigma))dt = \infty,\ T > 0,$

(A_{11}) $\int_T^\infty q(t)H(F^+(t+\tau-\sigma))dt = \infty,\ T > 0,$

and

(A_{12}) $\int_T^\infty q(t)H(F^-(t+\tau-\sigma))dt = \infty,\ T > 0,$

holds, then conclusion of the Theorem 1 is true.

Proof. On the contrary, we proceed as in the proof of the Theorem 1 to obtain that $w(t)$ is monotonic on $[t_2,\infty)$. If $w(t) < 0$ for $t \geqslant t_2$, then $z(t) < F(t)$ is a contradiction due to (A_4) when $z(t) > 0$. Ultimately, $z(t) < 0$ and $z(t) < F(t)$ for $t \geqslant t_3 > t_2$. Using the fact $z(t) < 0$ for $t \geqslant t_3$, it follows that

$$x(t) < -p(t)x(t-\tau) \leqslant x(t-\tau) \leqslant x(t-2\tau) \leqslant x(t-3\tau) \leqslant \dots \leqslant x(t_3),$$

that is, $x(t)$ is bounded on $[t_3, \infty)$. Consequently, $\lim_{t\to\infty} w(t)$ exists. Clearly,

$$-z(t) > -F(t) \text{ implies that } -z(t) > \max\{0, -F(t)\} = F^-(t).$$

Therefore, for $t \geqslant t_3$

$$-x(t-\tau) \leqslant p(t)x(t-\tau) \leqslant z(t) < -F^-(t)$$

gives rise to $x(t-\sigma) > F^-(t+\tau-\sigma)$, $t \geqslant t_4 > t_3$ and hence (2.3) reduced to

$$w'(t) + q(t)H(F^-(t+\tau-\sigma)) \leqslant 0,$$

for $t \geqslant t_4$. Integrating the last inequality from t_4 to $t(> t_4)$, we obtain

$$\int_{t_4}^{t} q(s)H(F^-(s+\tau-\sigma))ds \leqslant -[w(s)]_{t_4}^{t} < \infty, \, as\, t \to \infty,$$

which contradicts (A_{12}). Hence $w(t) > 0$, for $t \geqslant t_2$. We note that $z(t) > F(t)$ and $z(t) < 0$ is not possible due to (A_4). Therefore $z(t) > 0$ and $z(t) \leqslant x(t)$, for $t \geqslant t_3 > t_2$. In this case, $\lim_{t\to\infty} w(t)$ exists. Because, it happens that $z(t) > F^+(t)$ for $t \geqslant t_3$, then (2.3) can be viewed as

$$w'(t) + q(t)H(F^+(t-\sigma)) \leqslant 0.$$

Integrating the last inequality from t_3 to $t(> t_3)$, we obtain

$$\int_{t_3}^{t} q(s)H(F^+(s-\sigma))ds \leqslant -[w(s)]_{t_3}^{t} < \infty, \, as\, t \to \infty,$$

a contradiction to (A_9). The case $x(t) < 0$, for $t \geqslant t_0$ is similar. Hence, the theorem is proved.

THEOREM 4. *Let $-\infty < -p \leqslant p(t) \leqslant -1$, $t \in \mathbb{R}_+$ and $p > 0$. If all conditions of Theorem 3 are satisfied, then every bounded solution of (1.1) oscillates.*

Proof. The proof of the theorem can be followed from the proof of the Theorem 3. Hence the details are omitted.

REMARK 3. In Theorem 2-4, H could be linear, sublinear or superlinear.

THEOREM 5. *Let $-\infty < -p \leqslant p(t) \leqslant -1$, $t \in \mathbb{R}_+$, $p > 0$ and $\tau \geqslant \sigma$. Assume that (A_2), (A_4), (A_5), (A_9) and (A_{12}) hold. Furthermore, assume that*

(A_{13}) $\frac{H(u)}{u^\beta} \geqslant \frac{H(v)}{v^\beta}$, $u \geqslant v$, $\beta > 1$,

(A_{14}) $\int_T^\infty \frac{q(t)H(F^+(t+\tau-\sigma))}{[F^+(t+\tau-\sigma)]^\beta}dt = \infty$, $T > 0$,
and
(A_{15}) $\int_T^\infty \frac{q(t)H(F^-(t+\tau-\sigma))}{[F^-(t+\tau-\sigma)]^\beta}dt = \infty$, $T > 0$,
hold. Then conclusion of the Theorem 1 is true.

Proof. The proof of the theorem follows from the proof of Theorem 3 except the case when $w(t) < 0$, $z(t) < 0$, for $t \geqslant t_3$. Since $z(t) \geqslant p(t)x(t-\tau)$, then

$$w(t) = z(t) - F(t) \geqslant p(t)x(t-\tau) - F(t), \ \ t \geqslant t_3$$

implies that $w(t) - p(t)x(t-\tau) \geqslant -F(t)$, for $t \geqslant t_3$. Clearly, $w(t) - p(t)x(t-\tau) < 0$ is not possible due to (A_4) and the fact that $w(t) - p(t)x(t-\tau) = x(t) - F(t) \geqslant -F(t)$ if and only if $x(t) > 0$, for $t \geqslant t_3$. Ultimately, $w(t) - p(t)x(t-\tau) > 0$ and hence

$$w(t) - p(t)x(t-\tau) \geqslant \max\{0, -F(t)\} = F^-(t),$$

that is,

$$w(t) \geqslant p(t)x(t-\tau) + F^-(t) \geqslant -px(t-\tau) + F^-(t) > -px(t-\tau) \tag{2.6}$$

for $t \geqslant t_4 > t_3$. Since $w(t)$ is decreasing and $\tau \geqslant \sigma$, then it follows that

$$-w(t) \leqslant -w(t+\tau-\sigma) < px(t-\sigma), \ \ t \geqslant t_4.$$

Therefore,

$$\frac{H(x(t-\sigma))}{[-w(t)]^\beta} \geqslant \frac{H(x(t-\sigma))}{p^\beta x^\beta(t-\sigma)}, \ \ t \geqslant t_4. \tag{2.7}$$

Consequently,

$$\begin{aligned}
-\frac{d}{dt}[-w(t)]^{1-\beta} &= -(1-\beta)[-w(t)]^{-\beta}\left[-w('t)\right] \\
&= (\beta-1)[-w(t)]^{-\beta} q(t)H(x(t-\sigma)) \\
&\geqslant (\beta-1)q(t)\frac{H(x(t-\sigma))}{p^\beta x^\beta(t-\sigma)}, \ \ t \geqslant t_4
\end{aligned}$$

due to (2.3) and (2.7). We may note from (2.6) that $0 > w(t) > -px(t-\tau) + F^-(t)$ implies that $x(t-\sigma) > p^{-1}F^-(t+\tau-\sigma)$ and hence

$$-\frac{d}{dt}[-w(t)]^{1-\beta} \geqslant (\beta-1)\, q(t)\, \frac{H(p^{-1}F^-(t+\tau-\sigma))}{p^\beta[p^{-1}F^-(t+\tau-\sigma)]^\beta}, \tag{2.8}$$

for $t \geqslant t_4$ due to (A_{13}). Integrating (2.8) from t_4 to $t(> t_4)$, we get

$$(\beta-1)H(p^{-1})\int_{t_4}^t q(s)\, \frac{H(F^-(s+\tau-\sigma))}{[F^-(s+\tau-\sigma)]^\beta}ds \leqslant -\left[-w(s)^{1-\beta}\right]_{t_4}^t < \infty, \text{ as } t \to \infty,$$

due to (A_2), a contradiction to (A_{14}).

The case $x(t) < 0$ for $t \geqslant t_0$ can similarly be dealt with. Hence the theorem is proved.

REMARK 4. It seems that the solution in Theorem 4 is bounded which makes equation (1.1) oscillatory. However, Theorem 5 holds for any solution. The conditions (A_9) - (A_{12}) and (A_{14}), (A_{15}) are not comparable and hence Theorem 4 and Theorem 5 are different.

EXAMPLE 1. Consider

$$(x(t)+x(t-\frac{\pi}{2}))'+x(t-\frac{\pi}{2}) = \sin t. \tag{2.9}$$

Here $p(t)=1,\ Q(t)\equiv 1,\ f(t)=\sin t$. If we set $F(t)=-\cos t$, then $F'(t)=f(t)$,

$$F^+(t)=\begin{cases} -\cos t, & 2n\pi+\frac{\pi}{2}\leqslant t\leqslant 2n\pi+\frac{3\pi}{2} \\ 0, & \text{otherwise,}\end{cases}$$

and

$$F^-(t)=\begin{cases} \cos t, & 2n\pi+\frac{3\pi}{2}\leqslant t\leqslant 2n\pi+\frac{5\pi}{2} \\ 0, & \text{otherwise.}\end{cases}$$

Therefore

$$F^+(t-\frac{\pi}{2})=\begin{cases} -\sin t, & 2n\pi+\pi\leqslant t\leqslant 2n\pi+2\,\pi \\ 0, & \text{otherwise,}\end{cases}$$

and

$$F^-(t-\frac{\pi}{2})=\begin{cases} \sin t, & 2n\pi+2\,\pi\leqslant t\leqslant 2n\pi+3\,\pi \\ 0, & \text{otherwise.}\end{cases}$$

For $n=0,1,2...$, we get

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\infty} Q(t)F^+(t-\frac{\pi}{2})dt &= \sum_{n=0}^{\infty}\int_{2n\pi+\pi}^{2n\pi+2\,\pi}[-\sin t]dt \\ &= \sum_{n=0}^{\infty}[\cos t]_{2n\pi+\pi}^{2n\pi+2\,\pi} = +\infty.\end{aligned}$$

Clearly, (A_1), (A_2), (A_4), (A_5), and (A_8) are satisfied. Hence, by Theorem 2, every solution of (2.9) is oscillatory. Thus, in particular, $x(t)=\sin t$ is an oscillatory solution of the equation (2.9).

EXAMPLE 2. Consider

$$(x(t)-x(t-\pi))'+x(t-\frac{\pi}{2}) = \cos t. \tag{2.10}$$

Here $p(t)=-1,\ q(t)\equiv 1,\ f(t)=\cos t$. If we set $F(t)=\sin t$, then $F'(t)=f(t)$,

$$F^+(t)=\begin{cases} \sin t, & 2n\pi\leqslant t\leqslant 2n\pi+\pi \\ 0, & \text{otherwise,}\end{cases}$$

and

$$F^{-}(t)=\begin{cases}-\sin t, & 2n\pi+\pi\leqslant t\leqslant 2n\pi+2\pi\\ 0, & \text{otherwise.}\end{cases}$$

Therefore

$$F^{+}(t-\frac{\pi}{2})=\begin{cases}-\cos t, & 2n\pi+\frac{\pi}{2}\leqslant t\leqslant 2n\pi+\frac{3\pi}{2}\\ 0, & \text{otherwise,}\end{cases}$$

and

$$F^{-}(t-\frac{\pi}{2})=\begin{cases}\cos t, & 2n\pi+\frac{3\pi}{2}\leqslant t\leqslant 2n\pi+\frac{5\pi}{2}\\ 0, & \text{otherwise.}\end{cases}$$

Also

$$F^{+}(t+\frac{\pi}{2})=\begin{cases}\cos t, & 2n\pi-\frac{\pi}{2}\leqslant t\leqslant 2n\pi+\frac{\pi}{2}\\ 0, & \text{otherwise,}\end{cases}$$

and

$$F^{-}(t+\frac{\pi}{2})=\begin{cases}-\cos t, & 2n\pi+\frac{\pi}{2}\leqslant t\leqslant 2n\pi+\frac{3\pi}{2}\\ 0, & \text{otherwise.}\end{cases}$$

For $n=0,1,2...$, we get

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\infty}q(t)F^{+}(t-\frac{\pi}{2})dt&=\sum_{n=0}^{\infty}\int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}}[-\cos t]dt\\&=-\sum_{n=0}^{\infty}[\sin t]_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}}=+\infty.\end{aligned}$$

and

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\infty}q(t)F^{+}(t+\frac{\pi}{2})dt&=\sum_{n=0}^{\infty}\int_{2n\pi-\frac{\pi}{2}}^{2n\pi+\frac{\pi}{2}}[\cos t]dt\\&=\sum_{n=0}^{\infty}[\sin t]_{2n\pi-\frac{\pi}{2}}^{2n\pi+\frac{\pi}{2}}=+\infty.\end{aligned}$$

Clearly, (A_2), (A_4), (A_5), (A_{10}) and (A_{12}) are hold true. Hence, by Theorem 3, every solution of (2.10) is oscillatory. Thus, in particular, $x(t)=\sin t$ is an oscillatory solution of the equation (2.10).

EXAMPLE 3. Consider

$$(x(t)-2x(t-\pi))'+x(t-\frac{\pi}{2})=-2\sin t. \tag{2.11}$$

Here $p(t)=-2,\ q(t)\equiv 1,\ f(t)=-2\sin t$. If we set $F(t)=2\cos t$, then $F'(t)=f(t)$,

$$F^{-}(t)=\begin{cases}-2\cos t, & 2n\pi+\frac{\pi}{2}\leqslant t\leqslant 2n\pi+\frac{3\pi}{2}\\ 0, & \text{otherwise.}\end{cases}$$

and

$$F^{+}(t)=\begin{cases}2\cos t, & 2n\pi+\frac{3\pi}{2}\leqslant t\leqslant 2n\pi+\frac{5\pi}{2}\\ 0, & \text{otherwise,}\end{cases}$$

Clearly, all the assumptions of Theorem 3 are satisfied. Hence, by Theorem 4, every bounded solution of (2.11) is oscillatory. Thus, in particular, $x(t)=\cos t$ is an bounded oscillatory solution of the equation (2.11).

EXAMPLE 4. Consider

$$(x(t)+p(t)x(t-\pi))'+q(t)x^{3}(t-\frac{\pi}{2})=\cos t, \tag{2.12}$$

where $p(t)=-\frac{3}{2}$, $q(t)=\frac{5}{2}$, $H(x)=x^3$ and $f(t)=\cos t$. If we set $F(t)=\sin t$, then $F'(t)=f(t)$. Clearly all the assumptions of the Theorem 5 are hold true for $\beta=2$. Hence by Theorem 5, every solution of the equation (2.12) oscillates.

3. Oscillation properties of Eq. (1.2)

This section deals with the oscillatory behaviour of solutions of equation (1.2).

THEOREM 6. *Let $0\leqslant p(t)\leqslant p<\infty$, $t\in\mathbb{R}_+$ and $\tau\leqslant\sigma$. Assume that (A_1) and (A_2) hold. Furthermore, assume that*

(A_{16}) *H is sublinear and* $\int\limits_0^{\pm c}\frac{dt}{H(t)}<\infty,\ \ c>0$

and

(A_{17}) $\int\limits_0^{\infty}Q(t)dt=\infty$

hold, where $Q(t)$ is defined as in Theorem 2. Then every solution of the equation (1.2) is oscillatory.

Proof. Let $x(t)$ be nonoscillatory solution of equation (1.2) such that $x(t)>0$ for $t\geqslant t_0$. Setting as in (2.1) then (1.2) can be written as

$$z'(t)=-q(t)H(x(t-\sigma))\leqslant 0 \tag{3.1}$$

for $t\geqslant t_1>t_0+\sigma$. Consequently, $z(t)$ is nonincreasing on $[t_2,\infty)$, $t_2>t_1$. Since $z(t)>0$ for $t_2>t_1$. Due to (3.1), (1.2) becomes

$$0=z'(t)+q(t)H(x(t-\sigma))+H(p)[z'(t-\tau)+q(t-\tau)H(x(t-\tau-\sigma))]$$

for $t\geqslant t_3=\max\{t_2,\tau+\sigma\}$ and because of (A_1) and (A_2), we find that

$$\begin{aligned}0&\geqslant z'(t)+H(p)z'(t-\tau)+Q(t)[H(x(t-\sigma))+H(p\,x(t-\tau-\sigma))]\\&\geqslant z'(t)+H(p)z'(t-\tau)+\lambda Q(t)H(z(t-\sigma)).\end{aligned}$$

Consequently, there exists $t_4 > t_3$ such that

$$\frac{z'(t)}{H(z(t-\sigma))} + H(p)\frac{z'(t-\tau)}{H(z(t-\sigma))} + \lambda Q(t) < 0, \tag{3.2}$$

Because of $z(t)$ is non-increasing on $[t_4,\infty)$ and $\tau \leqslant \sigma$, the inequalities in (3.2) become

$$\frac{z'(t)}{H(z(t))} + H(p)\frac{z'(t-\tau)}{H(z(t-\tau))} + \lambda Q(t) < 0.$$

Note that $\lim_{t\to\infty} z(t)$ exists. Integrating the last inequality from t_4 to t, we get

$$\int_{t_4}^{t} \frac{z'(s)}{H(z(s))} ds + H(p)\int_{t_4}^{t} \frac{z'(s-\tau)}{H(z(s-\tau))} ds + \lambda \int_{t_4}^{t} Q(s) ds < 0,$$

that is

$$\lambda \int_{t_4}^{t} Q(s)ds < -\left[\int_{z(t_4)}^{z(t)} \frac{dy}{H(y)} + H(p)\int_{z(t_4-\tau)}^{z(t-\tau)} \frac{dy}{H(y)}\right] < \infty, \, as\, t \to \infty,$$

due to (A_{16}), a contradiction to (A_{17}).

If $x(t) < 0$, for $t \geqslant t_0$, then we set $y(t) = -x(t)$, for $t \geqslant t_0$ in (1.1) and we find

$$(y(t) + p(t)y(t-\tau))' + q(t)H(y(t-\sigma)) = 0,$$

then proceeding as above, we find a same contradiction. This completes the proof of the theorem.

THEOREM 7. *Let $-\infty < -p \leqslant p(t) \leqslant -1$, $t \in \mathbb{R}_+$, $p > 0$ and $\tau > \sigma$. Assume that (A_2) hold. If*

(A_{18}) *H is superlinear and* $\int_0^{\pm\infty} \frac{dt}{H(t)} < \infty$

and

(A_{19}) $\int_0^{\infty} q(t)dt = \infty$

hold, then also conclusion of the Theorem 6 is true.

Proof. On the contrary, we proceed as in the proof of the Theorem 6 to obtain $z(t)$ is monotonic on $[t_2,\infty)$. We claim that $z(t) < 0$, for $t \geqslant t_2$. If not, let $z(t) \geqslant 0$, for $t \geqslant t_2 > t_1$. Consequently,

$$x(t) \geqslant -p(t)x(t-\tau) \geqslant x(t-\tau) \geqslant x(t-2\tau) \geqslant x(t-3\tau) \geqslant \ldots \geqslant x(t_2),$$

implies that, $x(t)$ is bounded from below by $m > 0$ for $t \geqslant t_2$. Integrating (3.1) from t_2 to $t(> t_2)$, we obtain

$$z(t) - z(t_2) + \int_{t_2}^{t} q(s)H(x(s-\sigma))ds = 0,$$

that is,

$$z(t)-z(t_2)+H(m)\int_{t_2}^{t} q(s)ds<0.$$

Therefore,

$$z(t)<z(t_2)-H(m)\int_{t_2}^{t} q(s)ds\to-\infty \ as \ t\to\infty,$$

a contradiction to the fact that $z(t)>0$ on $[t_2,\infty)$. So our claim holds. From (2.1), it follows that $z(t+\tau-\sigma)>p(t+\tau-\sigma)x(t-\sigma)$. Hence, (3.1) becomes

$$z'(t)+\frac{q(t)}{H(-p)}H(z(t+\tau-\sigma))\leqslant 0, \tag{3.3}$$

due to (A_2). Because z is decreasing on $[t_2,\infty)$, then

$$z'(t)+\frac{q(t)}{H(-p)}H(z(t))\leqslant 0.$$

Integrating the last inequality from t_2 to $t(>t_1)$, we get

$$\int_{t_2}^{t}\frac{z'(s)}{H(z(s))}ds+\frac{1}{H(-p)}\int_{t_2}^{t} q(s)ds\geqslant 0,$$

that is,

$$\int_{t_2}^{t} q(s)ds\leqslant -H(-p)\int_{z(t_2)}^{z(t)}\frac{dy}{H(y)}<\infty, \ as \ t\to\infty,$$

due to (A_{18}), a contradiction to (A_{19}). The case $x(t)<0$ is similar. Hence the theorem is proved.

THEOREM 8. *Let* $-\infty<-p\leqslant p(t)\leqslant -1$, $t\in\mathbb{R}_+$ *and* $p>0$. *Assume that* (A_2) *and* (A_{19}) *hold. Then every bounded solutions of* (1.2) *are oscillatory.*

Proof. Proceeding as in the proof of Theorem 7, we have that $z(t)<0$, for $t\geqslant t_2$. Hence the inequality (3.3) holds. Because z is decreasing, there exist $t_3>t_2$ and $k>0$ such that $z(t)\leqslant -k$, for $t\geqslant t_3$. Therefore, the inequality (3.3) can be viewed as

$$z'(t)+\frac{H(-k)}{H(-p)}q(t)<0, \tag{3.4}$$

for $t\geqslant t_3$. Integrating (3.4) from t_3 to $t(>t_3)$, we obtain

$$\frac{H(-k)}{H(-p)}\int_{t_3}^{t} q(s)ds<-\left[z(s)\right]_{t_3}^{t}$$

Since $x(t)$ is bounded, then $z(t)$ is bounded and hence for $t \to \infty$ the last inequality becomes

$$\frac{H(-k)}{H(-p)} \int_{t_3}^{\infty} q(s)ds < \infty,$$

a contradiction to (A_{19}). the case $x(t) < 0$ is similar dealt with. Hence the proof of the theorem is completed.

REMARK 5. Theorem 8 and Theorem 7 are different in their own rights, especially due to H. We note that Theorem 7 is restricted to superlinear H but in Theorem 8, H could be linear, sublinear or superlinear.

THEOREM 9. *Let* $-1 < -p \leqslant p(t) \leqslant 0$, $t \in \mathbb{R}_+$, $p > 0$ *and* $\tau > \sigma$. *If* (A_1), (A_{16}) *and* (A_{19}) *hold, then also conclusion of the Theorem 6 is true.*

Proof. Proceeding as in Theorem 6, we may note that $z(t)$ is monotonic on $[t_2, \infty)$. Hence there exists $t_3 > t_2$ such that $z(t) > 0$ or $z(t) < 0$. Let $z(t) > 0$ for $t_3 > t_2$. From (2.1), it follows that $z(t) \leqslant x(t)$ on $[t_3, \infty)$. Consequently, (3.1) becomes

$$z'(t) + q(t)H(z(t-\sigma)) < 0,$$

that is,

$$\frac{z'(t)}{H(z(t))} + q(t) < 0,$$

Note that $\lim_{t\to\infty} z(t)$ exists. Integrating the last inequality from t_3 to t, we get

$$\int_{t_3}^{t} q(s)ds < -\int_{z(t_3)}^{z(t)} \frac{dy}{H(y)} < \infty, \ \ as \ t \to \infty,$$

due to (A_{16}), a contradiction to (A_{19}). Hence $z(t) < 0$, for $t_3 > t_2$. Proceedings as above proof of the Theorem 3, we obtain $x(t)$ is bounded on $[t_3, \infty)$. The rest of the theorem follows from the Theorem 8. This completes the proof of the theorem.

4. Existence of positive solution

In this section, necessary conditions are obtained to show that equation (1.1) admits a positive bounded solution.

THEOREM 10. *Let* H *be Lipschitzian on the interval of the form* $[a, b]$, $0 < a < b < \infty$. *Suppose that* $f(t)$ *satisfies* (A_4). *If* $p(t)$ *is bounded and*

$$\int_0^{\infty} q(t)dt < \infty,$$

then the equation (1.1) *admits a positive bounded solution.*

Proof. The proof of the theorem is divided accordingly with respect to different ranges of $p(t)$.

(i) Let $0 \leqslant p(t) \leqslant p_1 < 1$. It is possible to find $t_1 > 0$ such that

$$\int_{t_1}^{\infty} q(s)ds < \frac{1-p_1}{5K},$$

where $K = \max\{K_1, H(1)\}$, K_1 is the Lipschitz constant on $[\frac{1-p_1}{10}, 1]$. Let F be such that $-\frac{1-p_1}{10} \leqslant F(t) \leqslant \frac{1-p_1}{10}$ for $t \geqslant t_2$. For $t_3 > \max\{t_1, t_2\}$, we set $Y = BC([t_3, \infty), \mathbb{R})$, the space of real valued bounded continuous functions on $[t_3, \infty)$. Clearly, Y is a Banach space with respect to supremum norm defined by

$$||y|| = \sup\{|y(t)| : t \geqslant t_3\}.$$

Let's define

$$S = \left\{u \in Y : \frac{1-p_1}{10} \leqslant u(t) \leqslant 1,\ t \geqslant t_3\right\}.$$

Clearly, S is a closed and convex subspace of Y. Let $T : S \to S$ be defined by

$$Tx(t) = \begin{cases} Tx(t_3+\rho), & t \in [t_3, t_3+\rho] \\ \\ -p(t)x(t-\tau) + \frac{1+4p_1}{5} + F(t) + \int_t^{\infty} q(s)H(x(s-\sigma))ds, & t \geqslant t_3+\rho. \end{cases}$$

For every $x \in S$,

$$\begin{aligned} Tx(t) &\leqslant \frac{1-p_1}{10} + \frac{1+4p_1}{5} + H(1)\left[\int_t^{\infty} q(s)ds\right] \\ &< \frac{1-p_1}{10} + \frac{1+4p_1}{5} + \frac{1-p_1}{5} = \frac{1+p_1}{2} < 1 \end{aligned}$$

and

$$Tx(t) \geqslant -p(t)x(t-\tau) + \frac{1+4p_1}{5} + F(t) \geqslant -p_1 + \frac{1+4p_1}{5} - \frac{1-p_1}{10} = \frac{1-p_1}{10}$$

implies that $Tx \in S$. Now, for $y_1, y_2 \in S$

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leqslant |p(t)||y_1(t-\tau) - y_2(t-\tau)| \\ &\quad + K_1 \int_t^{\infty} q(s)|y_1(s-\sigma) - y_2(s-\sigma)|ds \\ &\leqslant p_1||y_1 - y_2|| + K_1||y_1 - y_2||\left[\int_t^{\infty} q(s)ds\right] \\ &< \left(p_1 + \frac{1-p_1}{5}\right)||y_1 - y_2|| \end{aligned}$$

implies that

$$||Ty_1 - Ty_2|| \leqslant \mu||y_1 - y_2||,$$

that is, T is a contraction mapping, where $\mu = \frac{1+4p_1}{5} < 1$. Since S is complete and T is a contraction on S, then by the Banach's fixed point theorem T has a unique fixed point on $\left[\frac{1-p_1}{10}, 1\right]$. Hence $Tx = x$ and

$$x(t) = \begin{cases} x(t_3+\rho), & t \in [t_3, t_3+\rho] \\ \\ -p(t)x(t-\tau) + \frac{1+4p_1}{5} + F(t) + \int_t^\infty q(s)H(x(s-\sigma))ds & t \geqslant t_3+\rho \end{cases}$$

is a bounded positive solution of the equation (1.1) on $\left[\frac{1-p_1}{10}, 1\right]$.

(ii) Let $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ and $p_2^2 > p_3$. It is possible to find a $t_1 > 0$ such that

$$\int_{t_1}^\infty q(t)dt < \frac{p_2-1}{2K},$$

where $K = \max\{K_1, K_2\}$, K_1 is the Lipschitz constant of H on $[\alpha, \beta]$ and $K_2 = H(\beta)$ such that

$$\alpha = \frac{2\gamma(p_2{}^2 - p_3) - p_3(p_2 + p_2{}^2 - 2)}{2p_2{}^2 p_3}$$

$$\beta = \frac{p_2 - 1 + \gamma}{p_2}, \qquad \gamma > \frac{p_3(p_2 + p_2{}^2 - 2)}{2(p_2{}^2 - p_3)} > 0.$$

Let $F(t)$ be such that $-\frac{1}{2}(p_2-1) \leqslant F(t) \leqslant \frac{1}{2}(p_2-1)$, for $t \geqslant t_2 > t_1$. Let $Y = BC([t_2,\infty),\mathbb{R})$ be the space of real valued bounded continuous functions on $[t_2,\infty)$. Clearly, Y is a Banach space with respect to supremum norm defined by

$$||y|| = \sup\{|y(t)| : t \geqslant t_2\}.$$

Define

$$S = \{u \in Y : \ \alpha \leqslant u(t) \leqslant \beta, \, t \geqslant t_2\}.$$

It is easy to verify that S is a closed convex subspace of Y. Let $T : S \to S$ be such that

$$Tx(t) = \begin{cases} Tx(t_2+\rho), & t \in [t_2, t_2+\rho] \\ \\ -\frac{x(t+\tau)}{p(t+\tau)} + \frac{F(t+\tau)}{p(t+\tau)} + \frac{\gamma}{p(t+\tau)} + \frac{1}{p(t+\tau)}\left[\int_{t+\tau}^\infty q(s)H(x(s-\sigma))ds\right], & t \geqslant t_2+\rho. \end{cases}$$

For every $x \in S$,

$$\begin{aligned} Tx(t) &\leqslant \frac{H(\beta)}{p(t+\tau)}\left[\int_{t+\tau}^\infty q(s)ds\right] + \frac{p_2-1}{2p(t+\tau)} + \frac{\gamma}{p(t+\tau)} \\ &\leqslant \frac{1}{p_2}\left[\frac{2(p_2-1)}{2} + \gamma\right] = \beta \end{aligned}$$

and

$$Tx(t) \geqslant -\frac{x(t+\tau)}{p(t+\tau)} + \frac{F(t+\tau)}{p(t+\tau)} + \frac{\gamma}{p(t+\tau)}$$

$$
\begin{aligned}
&> -\frac{\beta}{p_2} - \frac{p_2-1}{2\,p_2} + \frac{\gamma}{p_3}\\
&= -\frac{p_2-1+\gamma}{{p_2}^2} - \frac{p_2-1}{2p_2} + \frac{\gamma}{p_3}\\
&= \frac{2\gamma({p_2}^2-p_3)-p_3(p_2-2+{p_2}^2)}{2{p_2}^2p_3} = \alpha
\end{aligned}
$$

implies that $Tx \in S$. For $y_1, y_2 \in S$

$$
\begin{aligned}
|Ty_1(t)-Ty_2(t)| &\leqslant \frac{1}{|p(t+\tau)|}|y_1(t+\tau)-y_2(t+\tau)|\\
&+ \frac{H(\beta)}{|p(t+\tau)|}\left[\int_{t+\tau}^{\infty} q(s)|y_1(s-\sigma)-y_2(s-\sigma)|ds\right]\\
&\leqslant \frac{1}{p_2}||y_1-y_2|| + \frac{H(\beta)}{p_2}||y_1-y_2||\left[\int_{t+\tau}^{\infty} q(s)ds\right]\\
&< \left(\frac{1}{p_2}+\frac{p_2-1}{2p_2}\right)||y_1-y_2||,
\end{aligned}
$$

that is,

$$
||Ty_1-Ty_2|| \leqslant \mu||y_1-y_2||
$$

implies that T is a contraction, where $\mu = \frac{p_2+1}{2p_2} < 1$. Hence by the Banach's fixed point theorem T has a unique fixed point $x(t)$ in the interval $[\alpha,\beta]$. In fact, $x(t)$ is the positive bounded solution of equation (1.1).

(iii) Let $-1 < -p_4 \leqslant p(t) \leqslant 0$, $p_4 > 0$. Then there exist $t_1, t_2 > 0$ such that

$$
\int_{t_1}^{\infty} q(t)dt < \frac{1-p_4}{10H(1)},\ \ t \geqslant t_1
$$

and $-\frac{1-p_4}{20} \leqslant F(t) < \frac{1-p_4}{20}$, for $t \geqslant t_2$. For $t_3 > \max\{t_1,t_2\}$, we let $Y = BC([t_3,\infty),\mathbb{R})$, be the space of all real valued bounded continuous functions defined on $[t_3,\infty)$. Clearly, Y is a Banach space with respect to supremum norm defined by

$$
||y|| = \sup\{|y(t)| : t \geqslant t_3\}.
$$

Let $L = \{y \in Y : y(t) \geqslant 0,\ t \geqslant t_3\}$. Then, Y is a partially ordered Banach pace (see for e.g [6],p.30). For $u, v \in Y$, we define $u \leqslant v$ if and only if $u - v \in L$. Let

$$
S = \left\{x \in Y :\ \frac{1-p_4}{20} \leqslant x(t) \leqslant 1,\ t \geqslant t_3\right\}.
$$

If $x_0(t) = \frac{1-p_4}{20}$, then $x_0 \in S$ and $x_0 = glb\,S$. Further, if $\Phi \subset S^* \subset S$, then

$$
S^* = \left\{x \in Y :\ l_1 \leqslant x(t) \leqslant l_2,\ \frac{1-p_4}{20} \leqslant l_1,\ l_2 \leqslant 1\right\}.
$$

Let $v_0(t) = l_2'$, $t \geqslant t_3$, where $l_2' = \sup\{l_2 : \frac{1-p_4}{20} \leqslant l_2 \leqslant 1\}$. Then $v_0 \in S$ and $v_0 = lub\, S^*$. For $t_4 = t_3 + \rho$, define $T : S \to S$ by

$$Tx(t) = \begin{cases} Tx(t_4), & t \in [t_3, t_4] \\ \\ -p(t)x(t-\tau) + \frac{1-p_4}{10} + F(t) + \int_t^\infty q(s)H(x(s-\sigma))ds, & t \geqslant t_4. \end{cases}$$

For every $x \in S$,

$$Tx(t) \leqslant p_4 + H(1)\left[\int_t^\infty q(s)ds\right] + \frac{1-p_4}{20} + \frac{1-p_4}{10} < \frac{1+3p_4}{4} < 1$$

and

$$Tx(t) \geqslant \frac{1-p_4}{10} + F(t) > \frac{1-p_4}{10} - \frac{1-p_4}{20} = \frac{1-p_4}{20}$$

implies that $Tx \in S$. Now, for $x_1, x_2 \in S$, it is easy to verify that $x_1 \leqslant x_2$ implies that $Tx_1 \leqslant Tx_2$. Hence by Knaster-Tarski fixed point theorem (see for e.g [8], Theorem 1.7.3), T has a unique fixed point $x(t)$ in the interval $[\frac{1-p_4}{20}, 1]$. In fact, $x(t)$ is a positive bounded solution of the equation (1.1).

In the other ranges of $p(t)$, the above procedure is same except the procedure of (iii). Hence without details, the necessary informations are given below :

(iv) Let $-\infty < -p_5 \leqslant p(t) \leqslant -p_6 < -1$, $p_5, p_6 > 0$. Choose $t_1 > 0$ sufficiently large such that

$$\int_{t_1}^\infty q(t)dt < \frac{p_6 - 1}{2K}, \ -\frac{1}{2}(p_6 - 1) \leqslant F(t) \leqslant \frac{1}{2}(p_6 - 1),$$

where $K = \max\{K_1, K_2\}$, K_1 is the Lipschitz constant of H on $[\alpha, \beta]$ and $K_2 = H(\beta)$ such that

$$\alpha = \frac{\gamma p_6 - p_5(p_6 - 1)}{p_5 p_6}, \ \beta = \frac{1}{2} + \frac{\gamma}{p_6 - 1},$$

for

$$\gamma > \frac{p_5(p_6 - 1)}{p_6} > 0.$$

We set

$$S = \{x \in Y : \ \alpha \leqslant x(t) \leqslant \beta, \, t \geqslant t_0\}$$

and

$$Tx(t) = \begin{cases} Tx(t_2 + \rho), & t \in [t_2, t_2 + \rho] \\ \\ -\frac{x(t+\tau)}{p(t+\tau)} + \frac{F(t+\tau)}{p(t+\tau)} - \frac{\gamma}{p(t+\tau)} + \frac{1}{p(t+\tau)}\left[\int_{t+\tau}^\infty q(s)H(x(s-\sigma))ds\right], & t \geqslant t_2 + \rho. \end{cases}$$

Therefore T is a contraction with a contraction constant $\frac{1+p_6}{2p_6}$.

(v) Let $p(t) \equiv +1$. Let $0 < p_7 < 1$ be such that $p_7 \neq \frac{1}{2}$. Choose $t_0 > 0$ sufficiently large such that

$$\int_{t_1}^{\infty} q(t)dt < \frac{1-2p_7}{20K} \text{ and } -\frac{1-2p_7}{40} \leqslant F(t) \leqslant \frac{1-2p_7}{20},$$

where $K = \max\{K_1, H(p_7)\}$, K_1 is the Lipschitz constant of H on $\left[\frac{1-42p_7}{40}, p_7\right]$. We set

$$S = \left\{x \in Y : \frac{7-42p_7}{40} \leqslant x(t) \leqslant p_7,\, t \geqslant t_0\right\}$$

and

$$Tx(t) = \begin{cases} Tx(t_0+\rho), & t \in [t_0, t_0+\rho] \\ -x(t-\tau) + \frac{2-p_7}{10} + F(t) + \int_t^{\infty} q(s)H(x(s-\sigma))ds, & t \geqslant t_4. \end{cases}$$

Therefore T is a contraction with a contraction constant $\frac{21-2p_7}{20}$.

(vi) When $p(t) \equiv -1$ for all t. Let $-1 < p_7 < 0$ be such that $p_7 \neq -\frac{1}{2}$. If we take $\int_{t_1}^{\infty} q(t)dt < \frac{1+2p_7}{10K}$ and replace $-p_7$ in the place of p_7, in the earlier settings in (v), then we have T is a contraction with a contraction constant $\frac{11+2\,p_7}{10}$

This completes the proof of the theorem.

5. Summary

It is worth observation that both unforced and forced equations (1.1) and (1.2) are studied keeping in view of assumptions (A_1) - (A_{19}). The results concerning equations (1.1) and (1.2) are completely oscillatory due to the analysis corporated here. Of course the forcing term can be considered to (1.1).

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Shyam Sundar Santra
Department of Mathematics
Sambalpur University
Sambalpur - 768019
INDIA
e-mail: shyam01.math@gmail.com