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PRACTICAL STABILITY OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS BY LYAPUNOV FUNCTIONS

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Abstract. The practical stability of a nonlinear nonautonomous Caputo fractional differential equation is studied using Lyapunov like functions. The novelty of this paper is based on the new definition of the derivative of a Lyapunov like function along the given fractional differential equation. Comparison results using this definition for scalar fractional differential equations are presented. Several sufficient conditions for practical stability, practical quasi stability, strongly practical stability of the zero solution and the corresponding uniform types of practical stability are established.

1. Introduction

An important property in the qualitative theory of differential equations is stability of solutions. In [20] the authors pointed out that the stability and even asymptotic stability themselves are neither necessary nor sufficient to ensure practical stability. The desired state of a system may be mathematically unstable but however the system may oscillate sufficiently close to the desired state, and its performance is deemed acceptable. Practical stability is neither weaker nor stronger than the usual stability; an equilibrium can be stable in the usual sense, but not practically stable, and vice versa. Practical stability is studied for various types of differential equations (see, for example, [3], [5], [10], [11], [12], [13], [15], [17], [19], [21], [22], [28]).

The stability of fractional order systems is quite recent. There are several approaches in the literature to study stability and one is the Lyapunov approach. Results on stability in the literature via Lyapunov functions could be divided into two main groups:

- continuously differentiable Lyapunov functions (see, for example, the papers [2], [9], [14], [24]). Different types of stability are discussed using the Caputo derivative of Lyapunov functions which depends significantly of the unknown solution of the fractional equation.

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- continuous Lyapunov functions (see, for example, the papers [7], [16], [18]) in which the authors use the derivative of a Lyapunov function similar to the Dini derivative of Lyapunov functions.

In this paper the practical stability of nonlinear nonautonomous Caputo fractional differential equations is defined and studied using Lyapunov functions. The Caputo fractional Dini derivative of a Lyapunov function is defined in an appropriate way. Note this type of derivative is introduced in [1] and is used to study the stability and asymptotic stability of Caputo fractional equations. Comparison results using this new definition and scalar fractional differential equations are presented and sufficient conditions for practical stability, uniform practical stability, quasi practical stability, uniform quasi practical stability, strong practical stability and uniformly strongly practically stability of nonlinear nonautonomous Caputo fractional differential equations are obtained.

2. Notes on fractional calculus

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [16, 25, 26] and there are several definitions of fractional derivatives and fractional integrals. In engineering, the fractional order q is often less than 1, so we restrict our attention to $q \in (0,1)$.

1: The Riemann–Liouville (RL) fractional derivative of order $q \in (0,1)$ of $m(t)$ is given by (see, for example, Section 1.4.1.1 [6], or [25])

$$ {}_{t_0}^{RL}D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \qquad t \geqslant t_0, $$

where $\Gamma(.)$ denotes the Gamma function.

2: The Caputo fractional derivative of order $q \in (0,1)$ is defined by (see, for example, Section 1.4.1.3 [6])

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \qquad t \geqslant t_0. \tag{2.1} $$

The Caputo and Riemann-Liouville formulations coincide when $m(t_0) = 0$. The properties of the Caputo derivative are quite similar to those of ordinary derivatives. Also, the initial conditions of fractional differential equations with the Caputo derivative has a clear physical meaning and as a result the Caputo derivative is usually used in real applications.

3: The Grunwald – Letnikov fractional derivative is given by (see, for example, Section 1.4.1.2 [6])

$$ {}_{t_0}^{GL}D^q m(t) = \lim_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \quad t \geqslant t_0, $$

and the Grunwald – Letnikov fractional Dini derivative by

$$ {}_{t_0}^{GL}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \binom{q}{r} \quad t \geqslant t_0, \tag{2.2} $$

where $\binom{q}{r}$ are the Binomial coefficients and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

EXAMPLE 1. Note ${}_0^{GL}D^q \sin(t) = t^{1-q}E_{2,2-q}(-t^2)$ and note for $q = 0.5$ the derivative is not periodic but it converges to the periodic function $\sin(t+q\frac{\pi}{2})$. □

PROPOSITION 1. (Theorem 2.25 [8]). *Let $m \in C^1[t_0,b]$. Then, for $t \in (t_0,b]$, ${}_{t_0}^{GL}D^q m(t) = {}_{t_0}^{RL} D^q m(t)$.*

Also, according to Lemma 3.4 ([8]) the following equality

$$ {}_{t_0}^{c}D_t^q m(t) = {}_{t_0}^{RL} D_t^q m(t) - m(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)} $$

holds.

From the relation between the Caputo fractional derivative and the Grunwald – Letnikov fractional derivative using (2.2) we define the Caputo fractional Dini derivative as

$$ {}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{GL}D_+^q [m(t) - m(t_0)], $$

i.e.

$$ {}_{t_0}^{c}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q}\Big[m(t) - m(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\Big(m(t-rh) - m(t_0)\Big)\Big]. \tag{2.3} $$

DEFINITION 1. ([7]) We say $m \in C^q([t_0,T],\mathbb{R}^n)$ if $m(t)$ is differentiable (i.e. $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{c}D^q m(t)$ exists and satisfies (2.1) for $t \in [t_0,T]$.

REMARK 1. If $m \in C^q([t_0,T],\mathbb{R}^n)$ then ${}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{c}D^q m(t)$.

In this paper we will use the following result:

LEMMA 1. *([2]). Let $x \in C^q([t_0,\infty),\mathbb{R}^n)$. Then for any $t \geqslant t_0$ the inequality*

$$ {}_{t_0}^{c}D^q\Big(x^T(t)x(t)\Big) \leqslant 2\, x^T(t)\, {}_{t_0}^{c}D^q x(t) $$

holds.

3. Statement of the problem

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}_{t_0}^{c}D^q x = f(t,x), \quad x(t_0) = x_0 \tag{3.1} $$

where $x, x_0 \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0) \equiv 0$, $t_0 \geqslant 0$.

We will assume in this paper that the function $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is such that for any initial data $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the IVP for FrDE (3.1) has a solution $x(t;t_0,x_0) \in C^q([t_0,\infty),\mathbb{R}^n)$. Note some sufficient conditions for global existence of solutions of IVP for FrDE (3.1) are given in [4], [8], [16].

The goal of our paper is to study various types of practical stability of the zero solution of the IVP for FrDE (3.1). We now present some types of practical stability of the zero solution of fractional differential equations. In the definition below we assume $x(t;t_0,x_0)$ is any solution of the FrDE (3.1).

DEFINITION 2. Let positive constants λ, A, $\lambda < A$ be given. The zero solution of the system of FrDE (3.1) is said to be

(S1) *practically stable with respect to (λ, A)* if there exists $t_0 \geqslant 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $||x_0|| < \lambda$ implies $||x(t;t_0,x_0)|| < A$ for $t \geqslant t_0$;

(S2) *uniformly practically stable with respect to (λ, A)* if (S1) holds for all $t_0 \in \mathbb{R}_+$;

(S3) *practically quasi stable with respect to (λ, A, T)* if there exists $t_0 \geqslant 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $||x_0|| < \lambda$ implies $||x(t;t_0,x_0)|| < A$ for $t \geqslant t_0 + T$, where the positive constant T is given;

(S4) *uniformly practically quasi stable with respect to (λ, A, T)* if (S3) holds for all $t_0 \in \mathbb{R}_+$;

(S5) *strongly practically stable with respect to (λ, A, K, T)* if there exists $t_0 \geqslant 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $||x_0|| < \lambda$ implies $||x(t;t_0,x_0)|| < A$ for $t \geqslant t_0$ and $||x(t;t_0,x_0)|| < K$ for $t \geqslant t_0 + T$, where the positive constants λ, A, K, T, $K < \lambda < A$ are given;

(S6) *uniformly strongly practically stable with respect to (λ, A, B, T)* if (S5) holds for all $t_0 \in \mathbb{R}_+$.

EXAMPLE 2. Consider the scalar FrDE

$$ {}_{t_0}^{c}D^q x = Cx, \quad x(t_0) = x_0, \tag{3.2} $$

where $x \in \mathbb{R}$, C is a constant. Its solution is $x(t;t_0,x_0) = x_0\, E_q(C(t-t_0)^q)$ for $t \geqslant t_0$, where E_q denotes the one parametric Mittag-Leffler function. For $C \leqslant 0$ the zero solution is uniformly stable. Also, for any given couple (λ, A), $0 < \lambda < A$, the zero solution is uniformly practically stable w.r.t. to (λ, A) (see Figure 1 for $C = -1$ and $q = 0.2$). $\square$

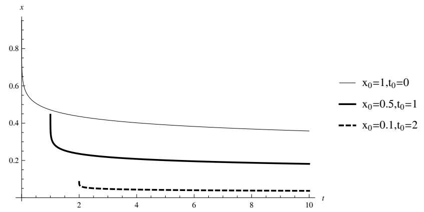

Figure 1. Example 2: Graphs of solution of (3.2) for $q = 0.2$, $C = -1$ and various initial points.

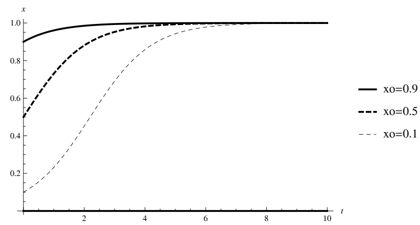

Figure 2. Example 3: Graphs of solutions with various initial points x_0.

REMARK 2. In the case $q = 1$ Definition 2 reduces to a definition for the corresponding types of practical stability of zero solution of ordinary differential equations, given in the books [17], [22].

REMARK 3. The practical stability of a nonzero solution $x^*(t) = x(t;t_0,x_0)$ of the system FrDE (3.1) could be reduced to studying the practical stability of the zero solution of an appropriate FrDE. Indeed, define $z(t) = y(t;t_0,y_0) - x^*(t)$, where $y(t;t_0,y_0)$ is any solution of (3.1). Then $z(t)$ is a solution of ${}_{t_0}^{c}D^q z(t) = F(t,z)$ for $t \geqslant t_0$, $z(t_0) = y_0 - x_0$, where $F(t,z) = f(t,z+x^*(t)) - f(t,x^*(t))$ and $F(t,0) = 0$.

EXAMPLE 3. Consider the ordinary differential equation

$$x' = (1-x)x \tag{3.3}$$

which is similar to logistic model in population dynamic. The solution of that equation with an initial value condition $x(t_0) = x_0, x_0 > 0$ is $x(t) = \frac{x_0 e^t}{(1-x_0)e^{t_0}+x_0 e^t}$. The zero solution is not stable, but it is uniformly practically stable w.r.t. $(\lambda,1)$, $0 < \lambda < 1$. Zero solution of (3.3) is also uniformly practically stable w.r.t. (λ,A), $0 < \lambda < A$, $A \geqslant 1$, but it is not practically stable w.r.t. (λ,A), $0 < \lambda < A < 1$, (see the graphs of solutions for $t_0 = 0$, and various x_0 on Figure 2).

Consider the Lyapunov function $V(t,x) = e^{-2t}x^2$. Then for $0 \leqslant x \leqslant 1$, $t_0 \geqslant 0$ we obtain $D_{(3.3)}V(t,x) = 2e^{-2t}x^2(1-x) - 2e^{-2t}x^2 = -2x^3 \leqslant 0$. □

In this paper we will use the followings sets:

$$\mathscr{K} = \{a \in C[\mathbb{R}_+,\mathbb{R}_+] : a \text{ is strictly increasing and } a(0) = 0\},$$
$$B(\lambda) = \{x \in \mathbb{R}^n : \|x\| \leqslant \lambda\}, \quad \lambda = const > 0.$$

In our results we will use the initial value problem for scalar fractional differential equations of the form

$${}_{t_0}^{c}D^q u = g(t,u), \quad t \geqslant t_0, \quad u(t_0) = u_0 \tag{3.4}$$

where $u, u_0 \in \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $g(t,0) \equiv 0$, $t_0 \geqslant 0$.We will assume in the paper that the function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is such that for any initial data $(t_0,u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the IVP for the scalar FrDE (3.4) with $u(t_0)=u_0$ has a solution $u(t;t_0,u_0)\in C^q([t_0,\infty),\mathbb{R})$ (we will assume the existence of a maximal solution in Section 5). Note some existence results are given in [4], [8], [16].

In this paper we will study the connection between the practical stability of the system FrDE (3.1) and the practical stability of the scalar FrDE (3.4).

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the practical stability of the system FrDE (3.1).

DEFINITION 3. Let $J\subset \mathbb{R}_+$ and $\Delta\subset \mathbb{R}^n$, $0\in\Delta$. We will say that the function $V(t,x): J\times\Delta\to\mathbb{R}_+$ belongs to the class $\Lambda(J,\Delta)$ if $V(t,x)\in C(J\times\Delta,\mathbb{R}_+)$ is locally Lipschitzian with respect to its second argument and $V(t,0)\equiv 0$ for $t\in J$.

Lyapunov like functions used to discuss stability for differential equations require an appropriate definition of the derivative of the Lyapunov function along the studied differential equations. Note for the ordinary differential equation $x'=f(t,x)$ the following derivative of the Lyapunov function $V(t,x)$ along the ordinary differential equation is

$$DV(t,x)=\limsup_{h\to 0}\frac{1}{h}\Big[V(t,x)-V(t-h,x-hf(t,x)\Big]. \tag{3.5}$$

In some papers, for example [16], [18], the derivative of the Lyapunov function from the class Λ along the FrDE(3.1) is introduced as a natural generalization of (3.5) as

$$D^qV(t,x)=\limsup_{h\to 0}\frac{1}{h^q}\Big[V(t,x)-V(t-h,x-h^qf(t,x)\Big]. \tag{3.6}$$

EXAMPLE 4. Consider the quadratic Lyapunov function $V(x)=x^2$, $x\in\mathbb{R}$. Then using (3.5) and (3.6) we have

$$DV(x)=2xf(t,x)$$

and

$$D^qV(x)=\limsup_{h\to 0}\frac{1}{h^q}\Big[x^2-(x-h^qf(t,x))^2\Big]=2xf(t,x).$$

Both derivatives are of the same type, i.e. (3.6) could be considered as a generalization of (3.5).

In some papers, for example [24], the fractional derivative of the Lyapunov function with the unknown solution $x(t)$ of the FrDE (3.1) is applied. In the case of quadratic Lyapunov functions according to Lemma 1 we get

$${}_{t_0}^{c}D^qV(x(t))={}_{t_0}^{c}D^q\Big(x^2(t)\Big)\leqslant 2\,x(t)\,{}_{t_0}^{c}D^qx(t)=2x(t)f(t,x(t))$$

or

$${}_{t_0}^{c}D^q\Big(x^2(t)\Big)\leqslant D^qV(x(t)).$$

Therefore, in the case of a quadratic Lyapunov function one could use the derivative (3.6) instead of the Caputo fractional derivative ${}_{t_0}^{c}D^qV(x(t))$. □

The formula (3.6) for the derivative of a Lyapunov function could lead to problems when the Lyapunov function depends on t.

EXAMPLE 5. Let $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = \sin^2 t\, x^2$. It is locally Lipshitz with respect to its second argument x.

Apply formula (3.6) to obtain the derivative of V, namely

$$\begin{aligned} D^q V(t,x) &= \limsup_{h\to 0} \frac{1}{h^q}\Big[\sin^2 t\, x^2 - \sin^2(t-h)(x-h^q f(t,x))^2\Big] \\ &= \limsup_{h\to 0} \frac{1}{h^q}\Big\{\Big(\sin^2 t - \sin^2(t-h)\Big)x^2 + \sin^2(t-h)h^q f(t,x)(2x-h^q f(t,x))\Big\} \\ &= x^2 \limsup_{h\to 0} \frac{\sin^2 t - \sin^2(t-h)}{h^q} + \limsup_{h\to 0} \sin^2(t-h) f(t,x)(2x-h^q f(t,x)) \\ &= 2x\, \sin^2(t) f(t,x). \end{aligned} \tag{3.7}$$

Let $f(t,x) \equiv 0$. Then the solution of (3.1) for $n = 1$ and $t_0 = 0$ is $x(t) \equiv x_0$, $t \geqslant 0$ and $V(t,x(t)) = x_0^2 \sin^2 t$. All the conditions of Corollary 2.2 [18] are satisfied so the inequality $V(t,x(t)) \leqslant V(t_0,x_0)$, $t \geqslant t_0$ has to be hold. However in this case the inequality $x_0^2 \sin^2 t \leqslant x_0^2\ \sin^2 0 = 0$ is not satisfied for all $t \geqslant t_0$. □

The formula (3.6) in the case of Lyapunov function depending on t gives a quite different result.

EXAMPLE 6. Let $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = x^2 m(t)$, where $m \in C^1(\mathbb{R}_+,\mathbb{R}_+)$. Then from (3.5) we obtain

$$DV(t,x) = 2xm(t)f(t,x) + x^2\big(m(t)\big)' \tag{3.8}$$

and from (3.6) we get

$$\begin{aligned} D^q V(t,x) &= \limsup_{h\to 0} \frac{1}{h^q}\Big[x^2 m(t) - m(t-h)\big(x-h^q f(t,x)\big)^2\Big] \\ &= \limsup_{h\to 0} \frac{1}{h^q}\Big[x^2\big(m(t)-m(t-h)\big) - m(t-h)\Big(\big(x-h^q f(t,x)\big)^2 - x^2\Big)\Big] \\ &= x^2 \limsup_{h\to 0} \frac{m(t)-m(t-h)}{h^q} + f(t,x)\limsup_{h\to 0} m(t-h)(\big(2x-h^q f(t,x)\big) \\ &= 2xm(t)f(t,x). \end{aligned} \tag{3.9}$$

Formula (3.9) gives only one term for the derivative of Lyapunov function, which differs to the classical derivative (3.8) of Lyapunov function for ordinary differential equations.

Formula (3.9) is independent on the order q of the fractional differential equation. However the behavior of solutions of fractional differential equations depends significantly on the order q. For example, consider the IVP for FrDE ${}_0^cD^q x = 1-x$, $x(0) = 0$. Note $x(t) = t^q E_{q,1+q}(-t^q)$. Now $\lim_{t\to\infty} x(t)$ varies for different q (see the graphs of solutions for different q in Figure 3). □

Now we define in a new way a derivative of Lyapunov function among a given fractional differential equation. The formula will differ from the derivative of Lyapunov functions (3.6).

To define the derivative of the Lyapunov function we will use the Caputo fractional Dini derivative of a function $m(t)$ given in (2.3). We define the generalized *Caputo fractional Dini derivative* of the function $V(t,x) \in \Lambda([t_0,T),\Delta)$ along trajectories of solutions of the system FrDE (3.1) as follows::

$$\begin{aligned}
&{}^c_+D^q_{(3.1)}V(t,x;t_0,x_0) \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{V(t,x)-V(t_0,x_0) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big[V(t-rh,x-h^q f(t,x))-V(t_0,x_0)\Big]\Big\}, \ \text{for } t\geqslant t_0,
\end{aligned} \tag{3.10}$$

where $t\in(t_0,T)$, $x,x_0\in\Delta$, and there exists $h_1>0$ such that $t-h\in[t_0,T)$, $x-h^q f(t,x)\in\Delta$ for $0<h\leqslant h_1$.

Note the definition (3.10) has been introduced and used in [1] for studying the stability properties of zero solution of FrDE (3.1).

REMARK 4. Let $q=1$ in formula (3.10). Then using $\binom{q}{r}=0$ for $n<r$, $n,r>0$ integers, we obtain for any $t\geqslant 0$ the formula

$$D_+V(t,x) = \limsup_{h\to 0^+}\frac{1}{h}\Big\{V(t,x)-V(t-h,x-hf(t,x))\Big\}$$

which is used in the literature.

We will give an example to illustrate the application of the introduced Caputo fractional Dini derivative of the function $V(t,x)\in\Lambda([t_0,T),\Delta)$ along trajectories of solutions of the initial value problem for the system FrDE (3.1) and we willl make comparisons wih other derivatives of Lyapunov functios in the literature.

EXAMPLE 7. Let $V:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ be given by $V(t,x)=x^2m(t)$, where $m\in C^1(\mathbb{R}_+,\mathbb{R}_+)$. Then the Caputo fractional Dini derivative along trajectories of solutions of the initial value problem for the system FrDE (3.1) given by formula (3.10) is reduced to

$$\begin{aligned}
&{}^c_+D^q_{(3.1)}V(t,x;t_0,x_0) \\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{x^2m(t)-x_0^2m(t_0) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big[\Big(x-h^q f(t,x)\Big)^2 m(t-rh)-x_0^2m(t_0)\Big]\Big\}
\end{aligned}$$

$$= x^2 \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh) - x_0^2 m(t_0) \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r}$$

$$- f(t,x) \limsup_{h\to 0^+} \Big(2x - h^q f(t,x)\Big) \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh).$$

Using $\lim_{N\to\infty} \sum_{r=0}^{N} (-1)^r \binom{q}{r} = 0$, where N is a natural number, and the limit $lim_{h\to 0^+} [\frac{t-t_0}{h}] = \infty$ we obtain

$$\lim_{h\to 0^+} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} = -1 \tag{3.11}$$

and from the definition of Grunwald – Letnikov fractional derivative we obtain

$${}_{+}^{c}D^q_{(3.1)} V(t,x;t_0,x_0) = x^2 \, {}_{t_0}^{GL}D^q m(t) - x_0^2 \, {}_{t_0}^{GL}D^q m(t_0) + 2xf(t,x)m(t). \tag{3.12}$$

According to Proposition 1 we obtain

$$\begin{aligned} {}_{+}^{c}D^q_{(3.1)} V(t,x;t_0,x_0) &= x^2 \, {}_{t_0}^{RL}D^q m(t) - x_0^2 \, {}_{t_0}^{RL}D^q m(t_0) + 2xf(t,x)m(t) \\ &= x^2 \, {}_{t_0}^{RL}D^q m(t) - \frac{x_0^2 m(t_0)}{(t-t_0)^q \Gamma(1-q)} + 2xf(t,x)m(t). \end{aligned} \tag{3.13}$$

Consider the Lyapunov function in the case $t_0 = 0$, $x_0 = 0$. Then the Caputo fractional Dini derivative of V from (3.13) is given by

$${}_{+}^{c}D^q_{(3.1)} V(t,x;0,0) = x^2 \, {}_{0}^{RL}D^q m(t) + 2xf(t,x)m(t). \tag{3.14}$$

The derivative (3.14) obtained from the formula (3.10) is similar to the classical derivative $DV(t,x) = x^2\big(m(t)\big)' + 2xf(t,x)m(t)$ used for studying zero solution of ordinary differential equation (compare with Example 6). □

REMARK 5. Note in some papers, for example [24], the fractional derivative of the Lyapunov function with the unknown solution $x(t)$ of the FrDE (3.1) is applied. In the case of the Lyapunov function $V(t,x)$ depending directly on t the Caputo fractional derivative ${}_{t_0}^{c}D^q_{(3.1)} V(t,x(t))$ is very difficult to obtain. For example, if $V(t,x) = m(t)x^2$ as in Example 7, then the derivative ${}_{t_0}^{c}D^q\Big(m(t)\big(x(t)\big)^2\Big)$ is very complicated to apply.

4. Fractional differential inequalities and comparison results

Again in this section we assume $0 < q < 1$. Now we will give some comparison results. Note similar results were obtained by the authors in paper ([1]).

LEMMA 2. *([1]). Let $m \in C([t_0,T],\mathbb{R})$ and suppose that there exists $t^* \in (t_0,T]$, such that $m(t^*) = 0$ and $m(t) < 0$ for $t_0 \leqslant t < t^*$. Then if the Caputo fractional Dini derivative (2.3) of m exists at t^* then the inequality ${}^{c}D^q_{+} m(t^*) > 0$ holds.*

LEMMA 3. (Comparison result) [1]. *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;t_0,x_0) \in C^q([t_0,T],\Delta)$ *is a solution of the FrDE (3.1) where* $\Delta \subset \mathbb{R}^n$, t_0, $T \in \mathbb{R}_+$, $t_0 < T$ *are given constants,* $x_0 \in \Delta$.

2. *The function* $g \in C([t_0,T] \times \mathbb{R}, \mathbb{R})$.

3. *The function* $V \in \Lambda([t_0,T],\Delta)$ *and for any point* $t \in [t_0,T]$ *the inequality*

$$ {}_{+}^{c}D^q_{(3.1)} V(t,x(t;t_0,x_0)) \leqslant g(t,V(t,x(t;t_0,x_0))) $$

holds.

4. *The function* $u^*(t) = u(t;t_0,u_0) \in C^q([t_0,T],\mathbb{R})$ *is the maximal solution of the initial value problem (3.4).*

Then the inequality $V(t_0,x_0) \leqslant u_0$ *implies* $V(t,x^*(t)) \leqslant u^*(t)$ *for* $t \in [t_0,T]$.

If $g(t,x) \equiv 0$ in Lemma 3 we obtain the following result:

COROLLARY 1. *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;t_0,x_0) \in C^q([t_0,T],\Delta)$ *is a solution of the FrDE (3.1) where* $\Delta \subset \mathbb{R}^n$.

2. *The function* $V \in \Lambda([t_0,T],\Delta)$ *and the inequality*

$$ {}_{+}^{c}D^q_{(3.1)} V(t,x(t;t_0,x_0) \leqslant 0, \quad t \in [t_0,T] $$

holds.

Then for $t \in [t_0,T]$ *the inequality* $V(t,x^*(t)) \leqslant V(t_0,x_0)$ *holds.*

EXAMPLE 8. Let $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = \sin^2 t\, x^2$ and $t_0 = 0$. From (3.13) we obtain the Caputo fractional Dini derivative of V, namely

$$ {}_{+}^{c}D^q_{(3.1)} V(t,x;0,x_0) = x^2\, {}_{0}^{RL}D^q (\sin(t))^2 + 2xf(t,x)(\sin(t))^2. $$

Use $(\sin(t))^2 = 0.5 - 0.5\cos(2t)$ and ${}_{0}^{RL}D^q \cos(2t) = 2^q \cos(2t + \frac{q\pi}{2})$ and obtain

$$ {}_{+}^{c}D^q_{(3.1)} V(t,x;0,x_0) = x^2\Big(0.5\frac{t^{-q}}{\Gamma(1-q)} + 2^{q-1}\cos(2t + \frac{q\pi}{2})\Big) + 2xf(t,x)(\sin(t))^2. $$

Let $f(t,x) \equiv 0$. The solution of (3.1) for $n = 1$ is $x(t) \equiv x_0$, $t \geqslant 0$ and the Caputo fractional Dini derivative

$$ {}_{+}^{c}D^q_{(3.1)} V(t,x;0,x_0) = x^2\Big(0.5\frac{t^{-q}}{\Gamma(1-q)} + 2^{q-1}\cos(2t + \frac{q\pi}{2})\Big). $$

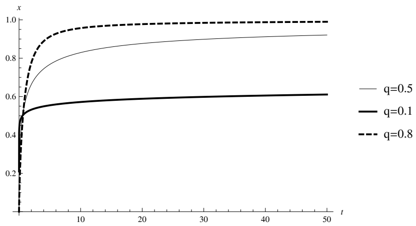

Figure 3. Example 6: $q = 0.1, 0.5$ *and* 0.8.

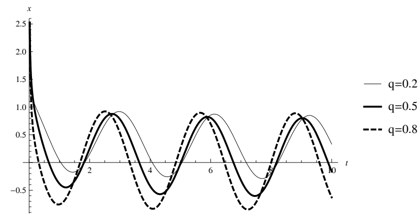

Figure 4. Example 8: $q = 0.2, 0.5$ *and* 0.8.

The sign of the Caputo fractional Dini derivative of V changes (see Figure 4 for the graph of $p(t) = 0.5\frac{t^{-q}}{\Gamma(1-q)} + 2^{q-1}\cos(2t + \frac{q\pi}{2})$, $q = 0.2, 0.5, 0.8$). Therefore, the conditions of Corollary 1 are not satisfied (compare with Example 4 for the derivative (3.6)).

Now let $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = x^2$. According to Eq. (3.13) for $m(t) \equiv 1$ we obtain the equality

$$ {}_{+}^{c}D^{q}_{(3.1)}V(t,x;0,x_0) = \frac{x^2 - x_0^2}{t^q\Gamma(1-q)} + 2xf(t,x). $$

Let $f(t,x) = -\frac{x}{t^q\Gamma(1-q)}$. Then ${}_{+}^{c}D^{q}_{(3.1)}V(t,x;0,x_0) \leqslant 0$ and according to Corollary 1 the inequality $|x(t:t_0,x_0)| \leqslant |x_0|$, $t \geqslant 0$, holds for any solution $x(t;t_0,x_0)$ of (3.1). □

REMARK 6. Corollary 1 is similar to Corollary 2.2 [18] where instead of derivative (3.10) is used (3.6).

The result of Lemma 3 is also true on the half line.

COROLLARY 2. [1]. *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;t_0,x_0) \in C^q([t_0,\infty),\Delta)$ *is a solution of the FrDE (3.1) where* $\Delta \subset \mathbb{R}^n$.

2. *The function* $g \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R})$.

3. *The function* $V \in \Lambda([t_0,\infty),\Delta)$ *and the inequality*

$$ {}_{+}^{c}D^{q}_{(3.1)}V(t,x(t;t_0,x_0)) \leqslant g(t,V(t,x(t;t_0,x_0))), \quad t \geqslant t_0 $$

holds.

4. *The function* $u^*(t) = u(t;t_0,u_0) \in C^q([t_0,\infty),\mathbb{R})$ *is the maximal solution of the initial value problem (3.4).*

Then the inequality $V(t_0,x_0) \leqslant u_0$ *implies* $V(t,x^*(t)) \leqslant u^*(t)$ *for* $t \geqslant t_0$.

5. Main results

We obtain sufficient conditions for practical stability of the zero solution of the system FrDE (3.1). Further we assume $0 < q < 1$.

THEOREM 1. *Suppose the following conditions hold:*

1. The function $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $g(t,0) \equiv 0$.

2. There exists $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ *such that for any* $t_0 \in \mathbb{R}_+$ *and* $x_0 \in \Delta$ *the FrDE (3.1) has a solution* $x(t;t_0,x_0) \in C^q([t_0,\infty),\Delta)$.

3. There exists a function $V \in \Lambda(\mathbb{R}_+,\Delta)$ *such that*

(i) the inequality

$$ {}_{+}^{c}D^q_{(3.1)}V(t,x;t_0,x_0) \leqslant g(t,V(t,x)) \tag{5.1} $$

holds for any $t_0,t \in \mathbb{R}_+$, $t \geqslant t_0$ *and* $x,x_0 \in B(A)$ *where* A *is a given constant such that* $B(A) \subset \Delta$;

(ii) $b(||x||) \leqslant V(t,x) \leqslant a(||x||)$ *for* $t \in \mathbb{R}_+$, $x \in B(A)$, *where* $a,b \in \mathscr{K}$.

4. The zero solution of the scalar FrDE (3.4) is practically stable (uniformly practically stable) w.r.t. $(a(\lambda),b(A))$ *where the constant* $\lambda > 0$ *is given such that* $\lambda < A$, $a(\lambda) < b(A)$.

Then the zero solution of the system of FrDE (3.1) is practically stable (uniformly practically stable) w.r.t. (λ,A).

REMARK 7. Note in the conditions of Theorem 1 we could have $\Delta \equiv \mathbb{R}^n$.

Proof. Let the zero solution of the scalar FrDE (3.4) be practically stable w.r.t. the couple $(a(\lambda),b(A))$. Thus, there exists a point $t_0 \geqslant 0$ such that $|\overline{u}_0| < a(\lambda)$ implies

$$ |\overline{u}(t;t_0,\overline{u}_0)| < b(A) \quad \text{for } t \geqslant t_0, \tag{5.2} $$

where $\overline{u}(t;t_0,\overline{u}_0)$ is a solution of (3.4) (with initial point $(t_0,\overline{u}_0)$ i.e. $\overline{u}(t_0) = \overline{u}_0$).

Choose a point $x_0 \in B(\lambda)$ and let $x(t;t_0,x_0) \in \Delta$, $t \geqslant t_0$, be a solution of the IVP for the FrDE (3.1) for the chosen x_0 and the above t_0. Assume the inequality

$$ ||x(t;t_0,x_0)|| < A \ \text{ for } t \geqslant t_0 \tag{5.3} $$

is not true. Then there exists a point $t^* > t_0$ such that

$$ ||x(t;t_0,x_0)|| < A \ \text{ for } t \in [t_0,t^*) \quad \text{and} \quad ||x(t^*;t_0,x_0)|| = A. \tag{5.4} $$

Let $u_0 = V(t_0,x_0)$. According to condition 3(ii) and the choice of x_0 we obtain $u_0 < a(\lambda)$. From Lemma 3 for the interval $[t_0,t^*]$ we obtain

$$ V(t,x(t;t_0,x_0)) \leqslant u^*(t;t_0,u_0) \quad \text{for } t \in [t_0,t^*]; \tag{5.5} $$

here $u^*(t;t_0,u_0)$ is the maximal solution of (3.4). From inequality (5.5) and condition 3(*ii*) we get

$$b(A) = b(||x(t^*;t_0,x_0)||) \leqslant V(t^*,x(t^*;t_0,x_0)) \leqslant u^*(t^*;t_0,u_0) < b(A). \tag{5.6}$$

The obtained contradiction proves inequality (5.3) is true. Thus the zero solution of FrDE (3.1) is practically stable w.r.t. (λ,A).

If the zero solution of the scalar FrDE (3.4) is uniformly practically stable, then the proof above with an arbitrary t_0 shows the uniform practical stability w.r.t. (λ,A) of the zero solution of FrDE (3.1). □

THEOREM 2. *Suppose the following conditions hold:*

1. The conditions 1 and 2 of Theorem 1 are fulfilled.

2. There exists a function $V \in \Lambda(\mathbb{R}_+,\Delta)$ such that

(i) the inequality

$$ {}_{+}^{c}D^{q}_{(3.1)}V(t,x;t_0,x_0) \leqslant g(t,V(t,x)) \tag{5.7}$$

holds for any $t_0,t \in \mathbb{R}_+$, $t \geqslant t_0$ and $x,x_0 \in \Delta$;

(ii) $b(||x||) \leqslant V(t,x) \leqslant a(||x||)$ for $t \in \mathbb{R}_+$, $x \in \Delta$, where $a,b \in \mathscr{K}$.

3. The zero solution of the scalar FrDE (3.4) is practically quasi stable (uniformly practically quasi stable) w.r.t. $(a(\lambda),b(A),T)$ where the positive constants T,λ,A are given such that $\lambda < A$, $a(\lambda) < b(A)$, $B(A) \subset \Delta$.

Then the zero solution of the system of FrDE (3.1) is practically quasi stable (uniformly quasi practically stable) w.r.t. (λ,A,T).

REMARK 8. Note in the conditions of Theorem 2 we could have $\Delta \equiv \mathbb{R}^n$.

Proof. Let the zero solution of the scalar FrDE (3.4) be practically quasi stable w.r.t. $(a(\lambda),b(A),T)$. Thus there exists a point $t_0 \geqslant 0$ such that $|\overline{u}_0| < a(\lambda)$ implies

$$|\overline{u}(t;t_0,\overline{u}_0)| < b(A) \quad \text{for } t \geqslant t_0 + T, \tag{5.8}$$

where $\overline{u}(t;t_0,\overline{u}_0)$ is a solution of (3.4) (with $\overline{u}(t_0) = \overline{u}_0$)..

Choose a point $x_0 \in B(\lambda)$ and let $x(t;t_0,x_0) \in \Delta$ be a solution of the IVP for the FrDE (3.1) for the chosen x_0 and the above t_0. Assume the inequality

$$||x(t;t_0,x_0)|| < A \ \text{ for } t \geqslant t_0 + T \tag{5.9}$$

is not true. Then there exists a point $t^* \geqslant t_0 + T$ such that $||x(t^*;t_0,x_0)|| \geqslant A$.

Let $u_0 = V(t_0,x_0)$. According to condition 2(*ii*) and the choice of x_0 we obtain $u_0 < a(\lambda)$ and from Corollary 2 we obtain

$$V(t,x(t;t_0,x_0)) \leqslant u^*(t;t_0,u_0) \quad \text{for } t \geqslant t_0; \tag{5.10}$$

here $u^*(t;t_0,u_0)$ is the maximal solution of (3.4).

From inequality (5.10) and condition 2(*ii*) we get

$$b(A) \leqslant b(||x(t^*;t_0,x_0)||) \leqslant V(t^*,x(t^*;t_0,x_0)) \leqslant u^*(t^*;t_0,u_0) < b(A). \tag{5.11}$$

The obtained contradiction proves inequality (5.9) is true. Therefore, the zero solution of FrDE (3.1) is practically quasi stable w.r.t. (λ,A,T).

Let the zero solution of the scalar FrDE (3.4) be uniformly practically quasi stable w.r.t. $(a(\lambda),b(A),T)$. Then the proof above with an arbitrary initial time t_0 shows the uniform practical quasi stability w.r.t. (λ,A,T) of the zero solution of FrDE (3.1). □

THEOREM 3. *Suppose the following conditions hold:*

1. The conditions 1 and 2 of Theorem 1 are fulfilled.

2. There exists a function $V \in \Lambda(\mathbb{R}_+,\Delta)$ such that

(i) the inequality

$$ {}_{+}^{c}D^q_{(3.1)}V(t,x;t_0,x_0) \leqslant g(t,V(t,x)) \tag{5.12}$$

holds for any $t_0,t \in \mathbb{R}_+,\ t \geqslant t_0$ and $x,x_0 \in \Delta$;

(ii) $b(||x||) \leqslant V(t,x) \leqslant a(||x||)$ for $t \in \mathbb{R}_+,\ x \in \Delta$, where $a,b \in \mathscr{K}$.

3. The zero solution of scalar FrDE (3.4) is strongly practically stable (uniformly strongly practically stable) w.r.t. $(a(\lambda),b(A),b(K),T)$ where the positive constants T,λ,A,K are given such that $K < \lambda < A,\ b(K) < a(\lambda) < b(A),\ B(A) \subset \Delta$.

Then the zero solution of the system of FrDE (3.1) is strongly practically stable (uniformly strongly practically stable) w.r.t. (λ,A,K,T).

REMARK 9. Note in the conditions of Theorem 3 we could have $\Delta \equiv \mathbb{R}^n$.

The proof of Theorem 3 is similar to that in Theorem 1 and Theorem 2 so we omit it.

EXAMPLE 9. Consider the scalar FrDE

$$ {}_{0}^{c}D^q x(t) = (1-x)x \tag{5.13}$$

which is the fractional generalization of the logistic model in population dynamic (see Example 3).

Note, that if $x_0 > 0$ then $x(t) > 0,\ t \geqslant 0$. Indeed, let $h(t) = -x(t)$. Then $h(0) = -x_0 < 0$. If we assume there exists a point $t^* > 0$ such that $h(t) < 0$ for $t \in [0,t^*)$ and $h(t^*) = 0$ then according to Lemma 2 ${}_{0}^{c}D^q h(t^*) > 0$. Also, from (5.13) we obtain ${}_{0}^{c}D^q h(t^*) = 0$. The obtained contradiction proves that the solution of (5.13) is positive for positive initial values.

Let $\Delta = (0,\infty)$. Let the function $m(t) \in C^1(\mathbb{R}_+,\Delta), m(0) = C > 0$ be such that ${}_{0}^{RL}D^q m(t) = -2m(t)$. Note that $m(t) < m(0)$ for $t \geqslant 0$ ([7]).

Now consider the Lyapunov function $V(t,x) = m(t)x^2$. Then according to Example 7 for $(t,x) \in \mathbb{R}_+ \times \Delta$ we obtain

$$\begin{aligned} {}_{+}^{c}D^q_{(5.13)}V(t,x;0,x_0) &= -2x^2m(t) - \frac{x_0^2 m(0)}{t^q\Gamma(1-q)} + 2xm(t)x(1-x) \\ &\leqslant -2x^2m(t) - \frac{x_0^2 m(0)}{t^q\Gamma(1-q)} + 2x^2m(t) - 2m(t)x^3 \leqslant 0. \end{aligned}$$

According to Theorem 1 the zero solution of the FrDE (5.13) is practically stable w.r.t. (λ, A), $0 < C < \lambda < A$.

Note that in the case of ordinary derivatives $q = 1$, the Riemann-Louiville fractional equation ${}_{0}^{RL}D^q m(t) = -2m(t)$ for $m(t)$ reduces to $m'(t) = -2m(t)$, $m(0) = 1$, whose solution is e^{-2t} and the Lyapunov function is the same as in Example 3. $\Box$

REFERENCES

[1] R. Agarwal, D. O'Regan and S. Hristova, Stability of Caputo fractional differential equations by Lyapunov functions, *Appl. Math.* (accepted).

[2] N. Aguila-Camacho, M. A. Duarte-Mermoudand J. A. Gallegos, Lyapunov functions for fractional order systems, *Comm. Nonlinear Sci. Numer. Simul.*, **19** (2014), 2951–2957.

[3] D. Bainov, A. Dishliev and I. Stamova, Practical stability of the solutions of impulsive sytems of differential difference equations via the method of comparison and some applications to population dynamics, *ANZIAM J.*, **43**, (2002), 525–539.

[4] D. Baleanu and O. G. Mustafa , On the global existence of solutions to a class of fractional differential equations, *Comput. Math. Appl.* **59**, (2010) ,1835–1841.

[5] S. R. Bernfeld. and V. Lakshmikantham, Practical stability and Lyapunov functions, *Tohoku Math. J.*, **32**, (1980), 607–613.

[6] Sh. Das, *Functional Fractional Calculus*, Springer-Verlag Berlin Heidelberg, 2011.

[7] J. V. Devi , F.A. Mc Rae and Z. Drici, Variational Lyapunov method for fractional differential equations, *Comput. Math. Appl.*, **64** (2012), 2982–2989.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag Berlin Heidelberg, 2010.

[9] M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos and R. Castro-Linares, Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems, *Comm. Nonlinear Sci. Numer. Simul.*, **22** (2015), 650–659.

[10] J. Henderson and S. Hristova, Eventual practical stability and cone valued Lyapunov functions for differential equations with "Maxima", *Commun. Appl. Anal.*, **14**, 4, (2010), 515–524.

[11] S. G. Hristova, Practical stability and cone valued Lyapunov functions for differential equations with "maxima", *Int. J. Pure Appl. Math*, **57**, 3, (2009), 313–323.

[12] S. G. Hristova and K. Stefanova, Practical stability of impulsive differential equations with "supremum" by integral inequalities, *Eur. J. Pure Appl. Math.*, **5**, 1 (2012), 30–44.

[13] S. Hristova, Generalization of practical stability for delay differential equations with respect to initial time difference, *AIP Conf. Proc.* **1570**, 313 (2013).

[14] J. B. Hu, G.P. Lu, S.B. Zhang and L.-D, Zhao, Lyapunov stability theorem about fractional system without and with delay, *Commun. Nonlinear Sci. Numer. Simulat.*, **20** (2015), 905–913.

[15] T. Kaczorek, Practical stability and asymptotic stability of positive fractional 2D linear systems, *Asian J. Control*, **12**, 2, (2010) 200–207.

[16] V. Lakshmikantham, S. Leela and J.V. Devi, *Theory of Fractional Dynamical Systems*, Cambridge Scientific Publishers, 2009.

[17] V. Lakshmikantham, S. Leela and A. A. Martynyuk, *Practical Stability of Nonlinear Systems*, World Scientific, Singapore, 1990.

[18] V. LAKSHMIKANTHAM, S. LEELA AND M. SAMBANDHAM, Lyapunov theory for fractional differential equations, *Commun. Appl. Anal.* **12**, 4, (2008), 365–376.
[19] V. LAKSHMIKANTHAM AND Y. ZHANG, Strict practical stability of delay differential equation, *Appl. Math. Comput.* **118**, 2–3, (2001) 275–285.
[20] J. LA SALLE AND S. LEFSCHETZ, *Stability by Lyapunov's Direct Method and Applications*, Academic Press, New York, 1961.
[21] A. A. MARTYNYUK, On practical stability of hybrid systems, *Appl. Mech.* **25**, (1989), 101–107. (in Russian).
[22] A. A. MARTYNYUK, Practical stability of motion, Naukova Dumka, Kiev, 1983. (in Russian)
[23] C. LI, D. QIAN AND Y. CHEN, On Riemann-Liouville and Caputo Derivatives, *Discrete Dynamics in Nature and Society*, Vol. 2011, Article ID 562494, 15 pages.
[24] Y. LI, Y. CHEN AND I. PODLUBNY, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.* **59**, (2010), 1810–1821.
[25] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[26] G. SAMKO, A.A. KILBAS AND O.I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, 1993.
[27] J.C. TRIGEASSOU, N. MAAMRI, J. SABATIER AND A. OUSTALOUP, A Lyapunov approach to the stability of fractional differential equations, *Signal Processing*, **91**, (2011), 437–445.
[28] P. WANG AND W. SUN, Practical Stability in terms of Two Measures for Set Differential Equations on Time Scales, *The Scientific World Journal* **2014** (2014), Article ID 241034, 7 pages.

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