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SOME GENERALIZATIONS OF A HYBRID FIXED POINT THEOREM IN PARTIALLY ORDERED METRIC SPACES AND NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS

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Abstract. In this paper, the author presents some generalizations of a measure theoretic hybrid fixed point theorem of Dhage for the monotone nondecreasing mappings in a partially ordered metric space and then applies to a nonlinear functional integral equation for proving the existence as well as local ultimate attractivity of the comparable solutions defined on a unbounded interval of real line. An algorithm is constructed and it is shown that the sequence of successive approximations of the considered integral equation converges monotonically to the solution under weak partial Lipschitz and partial compactness type conditions.

1. Introduction

It is well-known that the measure theoretic hybrid fixed point theorems of Dhage in a partially ordered metric space are very much useful in nonlinear analysis for proving the existence as well other numerical and qualitative aspects of the solutions of nonlinear differential and integral equations. See for example, Dhage [5, 6, 7, 8, 9], Dhage and Dhage [10, 11, 12, 13], Dhage *et.al.* [15, 14] and the references therein. The purpose of the present paper is to obtain some generalizations of above mentioned fixed point theorem of Dhage [5, 6] in a partially ordered metric space and discuss some of their applications to nonlinear functional integral equations. Before going to the main results, we give some notations, definitions and auxiliary facts which are needed in the subsequent development of the paper.

Unless otherwise mentioned, we assume that $(E,\preceq,d)$ is a partially ordered complete metric space. Two elements x and y in E are said to be **comparable** if either the relation $x\preceq y$ or $y\preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n\to x^*$ as $n\to\infty$, then $x_n\preceq x^*$ (resp. $x_n\succeq x^*$) for all $n\in\mathbb{N}$. If C is a chain in E, then C' denotes the set of all limit points of C in E. The symbol $\overline{C}$ stands for the closure of C in E defined by $\overline{C}=C\cup C'$. Then $\overline{C}$ is called a closed chain in E. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Clearly, $\inf C, \sup C\in\overline{C}$ provided $\inf C$ and $\sup C$ exist.

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The $\sup C$ is an element $z \in E$ such that for every $\varepsilon > 0$ there exists a $c \in C$ such that $d(c,z) < \varepsilon$ and $x \leqslant z$ for all $x \in C$. Similarly, $\inf C$ is defined in the same way. A few details of a partially ordered sets and chains appear in Hekillä and Lakshmikantham [17], Dhage [5] and the references therein.

In what follows, we denote by $\mathscr{P}_{cl}(E)$, $\mathscr{P}_{bd}(E)$, $\mathscr{P}_{rcp}(E)$, $\mathscr{P}_{ch}(E)$, $\mathscr{P}_{bd,ch}(E)$, $\mathscr{P}_{rcp,ch}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively. The following concept of a partially measure of noncompactness of sets in E has been introduced in Dhage [7, 8] on the lines of usual classical theory. See Banas and Goebel [1], Banas and Dhage [2], Dhage [3, 4] and the references therein.

DEFINITION 1.1. A mapping $\mu^p : \mathscr{P}_{bd,ch}(E) \to \mathbb{R}_+ = [0,\infty)$ is said to be a partial measure of noncompactness in E if it satisfies the following conditions:

1^o $\emptyset \neq (\mu^p)^{-1}(\{0\}) \subset \mathscr{P}_{rcp,ch}(E)$,

2^o $\mu^p(\overline{C}) = \mu^p(C)$,

3^o μ^p is nondecreasing, i.e., if $C_1 \subset C_2 \Rightarrow \mu^p(C_1) \leqslant \mu^p(C_2)$, and

4^o if $\{C_n\}$ is a sequence of closed chains from $\mathscr{P}_{bd,ch}(E)$ such that $C_{n+1} \subset C_n, (n = 1,2,\dots)$ and if $\lim\limits_{n\to\infty} \mu^p(C_n) = 0$, then the set $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty.

The family of sets described in condition 1^o is said to be the *kernel of the partial measure of noncompactness* μ^p and is defined as

$$\ker \mu^p = \big\{ C \in \mathscr{P}_{bd,ch}(E) \,\big|\, \mu^p(C) = 0 \big\}.$$

Clearly, $\ker \mu^p \subset \mathscr{P}_{rcp,ch}(E)$. Observe that the intersection set C_∞ from condition 4^o is a member of the family $\ker \mu^p$. In fact, since $\mu^p(C_\infty) \leqslant \mu^p(C_n)$ for any n, we infer that $\mu^p(C_\infty) = 0$. This yields that $C_\infty \in \ker \mu^p$. This simple observation will be essential in our further investigations.

If $(E, \preceq, \|\cdot\|)$ is a partially ordered Banach space, then the partially measure μ^p of noncompactness in E is called **sublinear** if it satisfies

5^o $\mu^p(C_1 + C_2) \leqslant \mu^p(C_1) + \mu^p(C_2)$ for all $C_1, C_2 \in \mathscr{P}_{bd,ch}(E)$, and

6^o $\mu^p(\lambda C) = |\lambda| \mu^p(C)$ for $\lambda \in \mathbb{R}$.

Now we state our basic hybrid fixed point theorem for its further generalizations. We need the following definitions in what follows.

DEFINITION 1.2. A mapping $\mathscr{T} : E \to E$ is called **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ for all $x, y \in E$. Similarly, $\mathscr{T} : E \to E$ is called **monotone nonincreasing** if it preserves the inverse order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \succeq \mathscr{T}y$ for all $x, y \in E$. A monotone mapping $\mathscr{T}$ is one which is either monotone nondecreasing or monotone nonincreasing on E into itself.

DEFINITION 1.3. (Dhage [5, 6]) A mapping $\mathscr{T}: E \to E$ is called **partially continuous** at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(\mathscr{T}x, \mathscr{T}a) < \varepsilon$ whenever x is comparable to a and $d(x,a) < \delta$. $\mathscr{T}$ called partially continuous on E if it is partially continuous at every point of it.

It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

DEFINITION 1.4. (Dhage [5, 6]) A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. An operator $\mathscr{T}$ on a partially normed linear space E is said to be **partially bounded** if $\mathscr{T}(C)$ is bounded for every chain C in E. The operator $\mathscr{T}$ is **uniformly partially bounded** if all chains $\mathscr{T}(C)$ in E are bounded by a unique constant.

DEFINITION 1.5. (Dhage [5]) A non-empty subset S of E is called a **partially compact** if every chain C in S is compact. An operator $\mathscr{T}$ on a partially ordered metric space E into itself is called **partially compact** if $\mathscr{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathscr{T}$ is called **uniformly partially compact** if $\mathscr{T}$ is uniformly bounded and partially compact. $\mathscr{T}$ is called **partially totally bounded** if for any totally ordered and bounded subset C of E, $\mathscr{T}(C)$ is a relatively compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

DEFINITION 1.6. (Dhage [5]) The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leqslant$ and the norm defined by the absolute value function has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

To state the measure theoretic hybrid fixed point theorems, we need the following notion of a $\mathscr{D}$-function which is useful in the generalizations of well-known Lipschitz condition for a mapping in metric spaces.

DEFINITION 1.7. (Dhage [5]) A mapping $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, **$\mathscr{D}$-function** if it is an upper semi-continuous and monotonic nondecreasing function satisfying $\psi(r) = 0 \iff r = 0$.

REMARK 1.1. We mention there do exist $\mathscr{D}$-functions in the mathematical literature and commonly used $\mathscr{D}$-functions in the fixed point theory and applications are $\psi_1(r) = kr,\ k > 0,\ \ \psi_2(r) = \frac{Lr}{K+r},\ L > 0,\ K > 0,\ \ \psi_3(r) = \arctan r,\ \ \psi_4(r) = \log(1+r)$ and $\psi_5(r) = e^r - 1$. The first three of these $\mathscr{D}$-functions have been widely used in the existence theory of nonlinear differential and integral equations.

REMARK 1.2. If $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two $\mathscr{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\phi$, $\lambda > 0$, and iii) $\phi \circ \psi$ are also $\mathscr{D}$-functions on $\mathbb{R}_+$. The class of $\mathscr{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.

The following hybrid fixed point result is a slight improvement of the applicable hybrid fixed point theorems proved in Dhage [6, 7, 8] and Dhage and Dhage [10] in a partially ordered metric space.

THEOREM 1.1. *Let S be a non-empty closed subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the metric d are compatible in every compact chain C of S. Let $\mathscr{T} : S \to S$ be a nondecreasing, partially bounded and partially continuous mapping satisfying the inequality*

$$\mu^p(\mathscr{T}(C)) \leqslant \psi(\mu^p(C)) \tag{1.1}$$

for all bounded chains C in S, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{D}$-function such that $\psi(r) < r$ for $r > 0$. Suppose that there exists and element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$. Then $\mathscr{T}$ has at least one fixed point x^* in S and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.

We remark that Theorem 1.1 includes the fixed point theorems of Dhage [5, 6, 7, 8], Dhage *et.al* [15], Ran and Reurings [20] and Nieto an Lopez [19] in a partially ordered metric space as the special cases and has some interesting applications to nonlinear differential and integral equations. See Dhage [9], Dhage and Dhage [10, 11, 12, 13] and the references therein.

REMARK 1.3. The regularity of the partially ordered metric space E in above Theorem 1.1, may be replaced with a stronger continuity condition than partial continuity of the mappings $\mathscr{T}$ on E. Again, the compatibility of every chain w.r.t. the order relation $\preceq$ and the metric d holds if every partially compact subset of E possesses the compatibility property w.r.t. $\preceq$ and d. Furthermore, fixed point set $\mathscr{F}_{\mathscr{T}}$ of comparable elements of the mapping $\mathscr{T}$ in S is a member of $\ker \mu^p$.

2. Hybrid Fixed Point Theory

The following measure theoretic hybrid fixed point theorem is a generalization of Theorem 1.1 under weaker upper semi-continuity of the $\mathscr{D}$-function ψ on $\mathbb{R}_+$.

THEOREM 2.1. *Let S be a non-empty closed subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the metric d are* *compatible in every compact chain C of S. Let $\mathscr{T}: S \to S$ be a nondecreasing, partially bounded and partially continuous mapping satisfying the inequality*

$$\mu^p(\mathscr{T}(C)) \leqslant \psi(\mu^p(C)) \tag{2.1}$$

for all bounded chains C in S, where $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$. Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$. Then $\mathscr{T}$ has at least one fixed point x^* in S and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.

Proof. The proof is similar to that given in Dhage [7, 8], but for the sake of completeness we give the details of it. Define a sequence $\{x_n\}$ of points in S by

$$x_{n+1} = \mathscr{T}x_n, \quad n = 0,1,2,\dots. \tag{2.2}$$

Suppose that $x_0 \preceq \mathscr{T}x_0$. Since $\mathscr{T}$ is nondecreasing, we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{2.3}$$

Denote

$$C_n = \{x_n, x_{n+1}, \dots\}$$

for $n = 0,1,2,\dots$. By construction, each C_n is a bounded chain in S and

$$C_n = \mathscr{T}(C_{n-1}), \quad n = 0,1,2,\dots.$$

Moreover,

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots,$$

and so

$$\overline{C_0} \supset \overline{C_1} \supset \cdots \supset \overline{C_n} \supset \cdots. \tag{2.4}$$

Therefore, by nondecreasing nature of μ^p we obtain

$$\begin{aligned}
\mu^p(\overline{C_n}) &= \mu^p(C_n) \\
&= \mu^p(\mathscr{T}(C_{n-1})) \\
&\leqslant \psi(\mu^p(C_{n-1})) \\
&\leqslant \psi^2(\mu^p(C_{n-2})) \\
&\;\vdots \\
&\leqslant \psi^n(\mu^p(C_0)).
\end{aligned} \tag{2.5}$$

Taking the limit superior as $n \to \infty$ in the above equality (2.5), we obtain that

$$\lim_{n\to\infty} \mu^p(\overline{C_n}) = \lim_{n\to\infty} \mu^p(C_n) \leqslant \limsup_{n\to\infty} \psi^n(\mu^p(C_0)) = \lim_{n\to\infty} \psi^n(\mu^p(C_0)) = 0. \tag{2.6}$$

Hence, by condition (4^o) of μ^p,

$$\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n \neq \emptyset \quad \text{and} \quad C_\infty \in \mathscr{P}_{rcp,ch}(E).$$

From (2.5) it follows that for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\mu^p(C_n) < \varepsilon \quad \forall\, n \geqslant n_0.$$

This shows that $\overline{C}_{n_0}$ and consequently $\overline{C}_0$ is a compact chain in E. Hence, $\{x_n\}$ has a convergent subsequence. Further since the order relation $\preceq$ and the metric d are compatible in the compact chain in S, the original sequence $\{x_n\} = \{\mathscr{T}^n x_0\}$ is convergent and converges monotonically to a point, say $x^* \in \overline{C}_0$. Since the ordered metric space E is regular, we have that $x_n \preceq x^*$. Finally, from the partial continuity of $\mathscr{T}$, we get

$$\mathscr{T}x^* = \mathscr{T}\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \mathscr{T}x_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Similarly, if $x_0 \succeq \mathscr{T}x_0$, it is proved that $\mathscr{T}$ has a fixed point S. This completes the proof. □

COROLLARY 2.1. *Let S be a non-empty and closed subset of a regular partially ordered complete metric space $(E,\preceq,d)$ such that the order relation $\preceq$ and the metric d are compatible in every compact chain C of S. Let $\mathscr{T} : S \to S$ be a monotone nondecreasing, partially continuous and partially compact mapping. Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$. Then $\mathscr{T}$ has at least one fixed point x^* in S and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. Let C be arbitrary bounded chain in S. Then from partial compactness of $\mathscr{T}$ it follows that $\mathscr{T}(C)$ is a relatively compact chain in S. As a result, we have that $\mu^p(\mathscr{T}(C)) = 0 \leqslant \psi(\mu^p(C))$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$. Now the desired conclusion follows by an application of Theorem 2.1.

COROLLARY 2.2. *Let S be a non-empty closed subset of a regular partially ordered complete metric space $(E,\preceq,d)$ such that $\preceq$ and d are compatible in every compact chain C of S. Let $\mathscr{T} : S \to S$ be a nondecreasing, partially bounded and partially continuous mapping satisfying the inequality*

$$d(\mathscr{T}x, \mathscr{T}y) \leqslant \psi(d(x,y)) \tag{2.7}$$

for all $x,y \in S$, $x \geqslant y$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$. Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$. Then $\mathscr{T}$ has at least one fixed point x^* in S and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.

Proof. Let $\mu^p : \mathscr{P}_{bd,ch}(E) \to \mathbb{R}_+$ be a set-function defined by

$$\mu^p(C) = \operatorname{diam}(C) \tag{2.8}$$

where, $\operatorname{diam}(C) = \sup\{d(x,y) : x,y \in C\}$ is the diameter of the chain C in E. It is easily verified that μ^p is a partially measure of noncompactness in E. Since the function ψ is nondecreasing, in view of the inequality (2.7), we obtain

$$\sup_{x,y\in C} d(\mathscr{T}x, \mathscr{T}y) \leqslant \sup_{x,y\in C} \psi(d(x,y)) \leqslant \psi\Big(\sup_{x,y\in C} d(x,y)\Big).$$

This implies that

$$\mu^p(\mathscr{T}(C)) \leqslant \psi(\mu^p(C)).$$

Now the required result follows by a direct application Theorem 2.1. This completes the proof. □

Note that we do not assume the upper semi-continuity of the function ψ in above measure theoretic hybrid fixed point theorems for nonlinear set-contractions in ordered metric spaces. However, if ψ is upper semi-continuous, then the following result follows immediately.

LEMMA 2.1. (Dhage [3]) *If $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{D}$-function, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in \mathbb{R}_+$ if and only if $\psi(t) < t$ for each $t > 0$.*

THEOREM 2.2. *Let S be a non-empty closed subset of a regular partially ordered metric space $(E, \preceq, d)$ such that $\preceq$ and d are compatible in every compact chain C of S. Let $\mathscr{T} : S \to S$ be a nondecreasing, partially bounded and partially continuous mapping satisfying the inequality*

$$\mu^p(\mathscr{T}(C)) \leqslant \mu^p(C) - \phi(\mu^p(C)) \tag{2.9}$$

for all bounded chains C in S, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing lower semi-continuous function such that $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$. Then $\mathscr{T}$ has at least one fixed point x^* in S and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.

Proof. Define a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi(t) = t - \phi(t)$. Then the inequality (2.9) takes the form (1.1). We show that ψ is a $\mathscr{D}$-function satisfying $\psi(t) < t$ for $t > 0$. Clearly, $\psi(0) = 0$. Since ϕ is lower semi-continuous, the function $-\phi$ is an upper semi-continuous on $\mathbb{R}_+$. Next, we show that ψ is nondecreasing on $\mathbb{R}_+$. Let $t_1, t_2 \in \mathbb{R}_+$ be arbitrary with $t_1 \leqslant t_2$. Then, we have $\psi(t_1) = t_1 - \phi(t_1) \leqslant t_2 - \phi(t_2) = \psi(t_2)$, and so, ψ is nondecreasing on $\mathbb{R}_+$. Finally, if $t > 0$, then $\phi(t) > 0$ and by construction, $\psi(t) = t - \phi(t) < t$. Now the desired conclusion follows by an application of Theorem 2.1.

REMARK 2.1. The regularity of the partially ordered metric space E in above Theorems 2.1 and 2.2 may be replaced with a stronger continuity condition than partial continuity of the mappings $\mathscr{T}$ on S. Furthermore, the fixed point set $\mathscr{F}_{\mathscr{T}}$ of comparable elements of the mapping $\mathscr{T}$ in S is a member of $\ker \mu^p$. This simple observation is used for proving the attractivity aspect of the comparable solutions of the nonlinear integral equation considered in the following section.

EXAMPLE 2.1. Let $E = \mathbb{R}$ and define the usual standard order relation $\leqslant$ and the metric d in $\mathbb{R}$ defined by $x \leqslant y$ if and only if $y - x \geqslant 0$ and $d(x,y) = |x-y|$. Then $(\mathbb{R}, \leqslant, d)$ is a regular partially ordered metric space in which $\leqslant$ and d are compatible. Take a subset S of $\mathbb{R}$ defined by $S = \{0, \frac{1}{2^n} : n \in \mathbb{N}_0\}$. Clearly, S is a non-empty, partially bounded and closed subset of $\mathbb{R}$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\psi(r) = \frac{3r}{4}$. Clearly, ψ is nondecreasing function satisfying $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$. Let $\mathscr{T} : S \to S$ be a mapping defined by $\mathscr{T}x = \dfrac{x}{2}$. The $\mathscr{T}$ is partially continuous and satisfies the condition (2.1) on S. Let C be any chain in S and define a partial measure of noncompactness in $\mathbb{R}$ by $\mu^p(C) = \operatorname{diam}(C) = \delta$. By definition of the μ^p, we obtain

$$\mu^p(\mathscr{T}(C)) = \frac{\delta}{2} \leqslant \psi(\mu^p(C)).$$

Moreover, there is an element $x_0 = 1 \in S$ such that $x_0 \geqslant \mathscr{T}x_0$. Thus all the conditions of Theorem 2.1 are satisfied and $\mathscr{T}$ has a fixed point. Here, $\mathscr{T}$ has only one fixed point $x^* = 0$ in S and the sequence $\{\mathscr{T}^n(x_0)\}$ of successive iterations converges monotonically to $x^* = 0$.

Finally we state a couple of hybrid fixed point theorems concerning the sum of two operators in a partially ordered complete normed linear space and the sum-product of three operators in a partially ordered complete normed linear algebra which are useful in the applications to perturbed nonlinear differential or integral equations.

THEOREM 2.3. *Let S be a non-empty closed subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathscr{A}, \mathscr{B} : S \to E$ be two nondecreasing mappings satisfying the following conditions:*

(a) $\mathscr{A}$ is partially bounded and partially continuous mapping satisfying $\|\mathscr{A}x - \mathscr{A}y\| \leqslant \psi(\|x-y\|)$ for all $x, y \in S$, $x \geqslant y$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$,

(b) $\mathscr{B}$ is partially completely continuous, and

(c) $\mathscr{A}x + \mathscr{B}x \in S$ for all $x \in S$.

Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{A}x_0 + \mathscr{B}x_0$ or $x_0 \succeq \mathscr{A}x_0 + \mathscr{B}x_0$. Then the operator equation $\mathscr{A}x + \mathscr{B}x = x$ has at least one solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_n = \mathscr{A}x_{n-1} + \mathscr{B}x_{n-1}$ converges monotonically to x^*.

Next, let $(E,\preceq,\|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$\mathscr{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u,v \in E^+\}. \tag{2.10}$$

The elements of the set $\mathscr{K}$ are called the positive vectors in E. The following lemma follows immediately from the definition of the set $\mathscr{K}$ which is often times used in the hybrid fixed point theory in Banach algebras and applications to nonlinear differential and integral equations.

LEMMA 2.2. (Dhage [3]) *If $u_1,u_2,v_1,v_2 \in \mathscr{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

As an application of Lemma 2.2, we obtain the following interesting hybrid fixed point theorem in a partially ordered Banach algebra which is useful in the theory of nonlinear quadratic differential and integral equations.

THEOREM 2.4. *Let S be a non-empty closed subset of a regular partially ordered complete normed linear algebra $(E,\preceq,\|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathscr{A},\mathscr{C} : S \to \mathscr{K}$ and $\mathscr{B} : S \to E$ be three nondecreasing mappings satisfying the following conditions:*

(a) $\mathscr{A}$ and $\mathscr{C}$ are partially bounded and there exist $\mathscr{D}$-functions ψ_1 and ψ_2 such that $\|\mathscr{A}x - \mathscr{A}y\| \leqslant \psi_1(\|x-y\|)$ and $\|\mathscr{C}x - \mathscr{C}y\| \leqslant \psi_2(\|x-y\|)$ for all $x,y \in S$, $x \geqslant y$,

(b) $\mathscr{B}$ is partially completely continuous, and

(c) $\mathscr{A}x\mathscr{B}x + \mathscr{C}x \in S$ for all $x \in S$, and

(d) $M\psi_1(r) + \psi_2(r) < r$ for $r > 0$, where $M = \sup\{\|\mathscr{B}(C)\| : C \text{ is a chain in } S\}$.

Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{A}x_0\mathscr{B}x_0 + \mathscr{C}x_0$ or $x_0 \succeq \mathscr{A}x_0\mathscr{B}x_0 + \mathscr{C}x_0$. Then the operator equation $\mathscr{A}x\mathscr{B}x + \mathscr{C}x = x$ has at least one solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_n = \mathscr{A}x_{n-1}\mathscr{B}x_{n-1} + \mathscr{C}x_{n-1}$ converges monotonically to x^*.

COROLLARY 2.3. *Let S be a non-empty, closed subset of a regular partially ordered complete normed linear algebra $(E,\preceq,\|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathscr{A},\mathscr{B} : S \to \mathscr{K}$ be two nondecreasing mappings satisfying the following conditions:*

(a) $\mathscr{A}$ is partially bounded and there exists a $\mathscr{D}$-function ψ_1 such that $\|\mathscr{A}x - \mathscr{A}y\| \leqslant \psi_1(\|x-y\|)$ for all $x,y \in S$, $x \geqslant y$,

(b) $\mathscr{B}$ is partially completely continuous, and

(c) $\mathscr{A}x\mathscr{B}x \in S$ *for all* $x \in S$, *and*

(d) $M\psi_1(r) < r$ *for* $r > 0$, *where* $M = \sup\{\|\mathscr{B}(C)\| : C \text{ is a chain in } S\}$.

Suppose that there exists an element $x_0 \in S$ *such that* $x_0 \preceq \mathscr{A}x_0\mathscr{B}x_0$ *or* $x_0 \succeq \mathscr{A}x_0\mathscr{B}x_0$. *Then the operator equation* $\mathscr{A}x\mathscr{B}x = x$ *has at least one positive solution* x^* *in* S *and the sequence* $\{x_n\}$ *of successive iterations defined by* $x_n = \mathscr{A}x_{n-1}\mathscr{B}x_{n-1}$ *converges monotonically to* x^*.

The proofs of Theorems 2.3 and 2.4 are similar to that given in Dhage [7, 8] and now the desired result follows by an application of Theorem 2.1. Hence we omit the details. Furthermore, Corollary 2.3 is used in Dhage and Dhage [12, 13] in the study of nonlinear quadratic differential equations for proving the existence and approximation theorems under mixed partial Lipschitz and partial compactness type conditions.

REMARK 2.2. We note that the partial boundedness of the mappings in all hybrid fixed point theorems of this section may be replaced with the partial boundedness of their domain S which is a slightly stronger condition and follows if S is a bounded subset of E. This simple fact is often times used in applications of the hybrid fixed point theorems to nonlinear hybrid equations.

We mention that the common existence principle hidden or involved in all classical or measure theoretic hybrid fixed point theorems of this section in a partially ordered metric or Banach space is called "**Dhage iteration principle**" in the subject of mathematical analysis and it may be described as "**the sequence of successive approximations of a nonlinear equation beginning with a lower or an upper solution as its first or initial approximation converges monotonically to the solution.**" The aforesaid convergence principle is interesting and very much useful tool in the theory of nonlinear differential and integral equations for proving the existence as well as approximation of the solutions under some mixed hybrid conditions of nonlinearities and it is known as "**Dhage iteration method**" for nonlinear equations. See Dhage [5, 6, 7, 8, 9], Dhage and Dhage [10, 11], Dhage *et al.* [14, 15] and the references therein for details.

In the following section we apply the measure theoretic hybrid fixed point theorem to nonlinear functional integral equations for proving the existence and algorithms for the local attractivity of comparable solutions.

3. Nonlinear Functional Integral Equations

Consider the following nonlinear hybrid functional integral equation (in short HFIE),

$$x(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))\, ds \tag{3.1}$$

for all $t, s \in \mathbb{R}_+$, $t \geqslant s$, where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

DEFINITION 3.1. By a **solution** of HFIE (3.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (3.1) on $\mathbb{R}_+$, where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions defined on $\mathbb{R}_+$. We say that solutions of the equation HFIE (3.1) are locally ultimately attractive if there exists a closed ball $B[x_0, r_0]$ in the space $C(\mathbb{R}_+, \mathbb{R})$ for some $x_0 \in C(\mathbb{R}_+, \mathbb{R})$ such that, for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation HFIE (3.1) belonging to $B[x_0, r_0]$, we have

$$\lim_{t \to \infty} (x(t) - y(t)) = 0. \tag{3.2}$$

In case the limit (3.2) is uniform with respect to the set $B[x_0, r_0]$, i.e., for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leqslant \varepsilon \tag{3.3}$$

for all solutions $x, y \in B[x_0, r_0]$ of the HFIE (3.1) and for $t \geqslant T$, we will then say that solutions of equation (3.1) are uniformly locally ultimately attractive on $\mathbb{R}^+$.

The HFIE (3.1) has already been discussed in Dhage [7] for global existence and attractivity results of comparable solutions under a growth condition on the nonlinearity f in terms of a special $\mathscr{D}$-function ψ_2 given in Remark 1.2. In the following we prove the local existence and attractivity of comparable solutions under full generality of $\mathscr{D}$-function ψ on f.

We place the HFIE (3.1) in the space $E = BC(\mathbb{R}_+, \mathbb{R})$ of continuous and bounded real-valued functions defined on $\mathbb{R}_+$. Define a norm $\|\cdot\|$ and the order relation $\leqslant$ in E by

$$\|x\| = \sup\{|x(t)| : t \geqslant 0\}. \tag{3.4}$$

and

$$x \leqslant y \iff x(t) \leqslant y(t) \;\forall\, t \in \mathbb{R}_+. \tag{3.5}$$

Clearly, E is a partially ordered Banach space with respect to the above norm $\|\cdot\|$ and the order relation $\leqslant$. The following lemma follows immediately by an application of Arzelá-Ascoli theorem.

LEMMA 3.1. *Let $\big(BC(\mathbb{R}_+, \mathbb{R}), \leqslant, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leqslant$ defined by (3.4) and (3.5) respectively. Then, every partially compact subset S of $BC(\mathbb{R}_+, \mathbb{R})$ is Janhavi, i.e., $\|\cdot\|$ and $\leqslant$ are compatible in in every compact chain C of S.*

Proof. Let S be a partially compact subset of $BC(\mathbb{R}_+, \mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leqslant x_2(t) \leqslant \cdots \leqslant x_n(t) \leqslant \cdots \tag{$*$}$$

for each $t \in \mathbb{R}_+$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the original sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in \mathbb{R}_+$. This shows that the sequence $\{x_n(t)\}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leqslant$ are compatible in S and so S is a Janhavi set in E. This completes the proof. □

We will employ a handy tool of the partial ball measure of noncompactness in the partially ordered Banach space $BC(\mathbb{R}_+,\mathbb{R})$ which relates the ultimate attractivity of the considered integral equation. Let us fix a bounded chain A of E and a positive real number T. For any $x \in A$ and $\varepsilon \geqslant 0$, denote by $\omega^T(x,\varepsilon)$, the modulus of continuity of x on the interval $[0,T]$ defined by

$$\omega^T(x,\varepsilon) = \sup\{|x(t)-x(s)| : t,s \in [0,T], |t-s| \leqslant \varepsilon\}.$$

Moreover, let

$$\omega^T(A,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in A\},$$

$$\omega_0^T(A) = \lim_{\varepsilon \longrightarrow 0} \omega^T(A,\varepsilon),$$

$$\omega_0(A) = \lim_{T \longrightarrow \infty} \omega_0^T(A).$$

By $A(t)$ we mean a set in $\mathbb{R}$ defined by $A(t) = \{x(t) | x \in A\}$ which is again a China in $\mathbb{R}$ for each $t \in \mathbb{R}^+$. We denote

$$\operatorname{diam}(A(t)) = \sup\{|x(t)-y(t)| : x,y \in A\}$$

and

$$\delta_c^T(A) = \sup_{t \geqslant T} \operatorname{diam}(A(t)).$$

Next, we let

$$\delta_c(A) = \lim_{T \to \infty} \delta_c^T(A) = \limsup_{t \longrightarrow \infty} \operatorname{diam}(A(t)).$$

Finally, we define a function μ^p on $\mathscr{P}_{bd}(E)$ by the formula

$$\mu^p(A) = \omega_0(A) + \delta_c(A). \tag{3.6}$$

It has been shown as in Banas and Goebel [1] that μ^p is a sublinear partially measure of noncompactness in E. From the definition of the measure μ^p, it is clear that the thickness of the bundle of functions from $A(t)$ tends to zero as t tends to ∞. This particular characteristic of μ^p has been utilized in formulating the main existence and attractiivity result for the integral equation (3.1) on $\mathbb{R}_+$.

Let Ψ be the class of all $\mathscr{D}$-functions ψ on $\mathbb{R}_+$ into itself satisfying the limit condition that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$.

We consider the following set of hypotheses in what follows.

(H_1) There exists a function $\psi \in \Psi$ such that

$$0 \leqslant f(t,x) - f(t,y) \leqslant \psi(x-y),$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ with $x \geqslant y$. Moreover, ψ is superadditive, i.e., $\psi(t+s) \geqslant \psi(t) + \psi(s)$ for all $t, s \in \mathbb{R}_+$.

(H_2) The function $t \mapsto f(t,0)$ is a member of $BC(\mathbb{R}_+,\mathbb{R})$ with $F_0 = \sup_{t\geqslant 0} |f(t,0)|$.

(H_3) The function $x \mapsto g(t,s,x)$ are nondecreasing in $\mathbb{R}$ for all $t, s \in \mathbb{R}_+$ with $t \geqslant s$.

(H_4) There exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,s,x)| \leqslant a(t)b(s)$$

for all $t, s \in \mathbb{R}_+$ such that $t \geqslant s$ and $x \in \mathbb{R}$. Moreover,

$$\lim_{t\to\infty} a(t) \int_0^t b(s)\,ds = 0,$$

and $V = \sup_{t\geqslant 0} a(t) \int_0^t b(s)\,ds$.

(H_5) There exists a lower solution $u \in E$ of the HFIE (3.1), i.e., it satisfies the integral inequality

$$u(t) \leqslant f(t,u(t)) + \int_0^t g(t,s,u(s))\,ds,\ t \in \mathbb{R}_+.$$

(H_6) There exists a positive solution r_0 of the inequality

$$\psi(r) + F_0 + V \leqslant r.$$

REMARK 3.1. Notice that the hypothesis (H_6) makes sense because $\psi(r) < r$, $r > 0$ in view of Lemma 2.1.

THEOREM 3.1. *Assume that the hypotheses (H_1) through (H_6) hold. Then the functional integral equation (3.1) has at least one solution x^* in the space $BC(\mathbb{R}_+,\mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_n(t) = f(t,x_{n-1}(t)) + \int_0^t g(t,s,x_{n-1}((s)))\,ds,\ t \in \mathbb{R}_+, \tag{3.7}$$

for each $n \in \mathbb{N}$ with $x_0 = u$, converges monotonically to x^*. Moreover, the comparable solutions of the HFIE (3.1) are uniformly locally ultimately attractive on $\mathbb{R}_+$.

Proof. Set $E = BC(\mathbb{R}_+,\mathbb{R})$. Then, in view of Lemma 3.1, every partially compact subset S of E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leqslant$ in E. Hence in view of Remark 1.2, every compact chain C possesses the compatibility property w. r. t. $\leqslant$ and $\|\cdot\|$ and so is Janhavi in E.

Define the operator Q defined on the space E by the formula

$$Qx(t) = f(t,x((t)))) + \int_0^t g(t,s,x(s))\,ds,\ t \in \mathbb{R}_+. \tag{3.8}$$

Observe that in view of our assumptions, for any function $x \in E$ the function Qx is continuous on $\mathbb{R}_+$. As a result, Q defines a mapping $Q:\ E \to E$. Now, for $x_0 = u \in E$, we define an open ball $\mathscr{B}(x_0,r)$ in E, where $r = \|x_0\| + r_0$ and r_0 is a positive real number given in hypothesis (H$_6$). We show that Q satisfies all the conditions of Theorem 2.3 on $S = \overline{\mathscr{B}}(x_0,r)$. This will be achieved in a series of following steps:

Step I: *Q is a nondecreasing operator on S.*

Let $x, y \in S$ be such that $x \leqslant y$. Then by hypotheses (H$_1$) and (H$_3$), we obtain

$$Qx(t) = f(t,x(t)) + \int_0^t g(t,s,x(s))\,ds \leqslant f(t,y(t)) + \int_0^t g(t,s,y(s))\,ds = Qy(t)$$

for all $t \in \mathbb{R}_+$. This shows that Q is a nondecreasing operator on S.

Step II: *Q maps a closed and partially bounded set S into itself.*

Let X be a chain in S and let $x \in X$. Since the function $v : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v(t) = a(t)\int_0^t b(s)\,ds \tag{3.9}$$

is continuous and in view of hypothesis (H$_4$), the number $V = \sup_{t \geqslant 0} v(t)$ exists. Moreover if $x \geqslant 0$, then for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain:

$$\begin{aligned} |Qx(t)| &\leqslant |f(t,x(t))| + \int_0^t |g(t,s,x(s))|\,ds \\ &\leqslant |f(t,x(t)) - f(t,0)| + |f(t,0)| + a(t)\int_0^t b(s)\,ds \\ &\leqslant \psi(|x(t)|) + F_0 + V \end{aligned} \tag{3.10}$$

Similarly, if $x \leqslant 0$, then it can be shown that $|Qx(t)| \leqslant \psi(|x(t)|) + F_0 + V$ for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain $\|Qx\| \leqslant \psi(\|x\|) + F_0 + V \leqslant r_0$ for all $x \in X$ with $\|x\| \leqslant r_0$. This means that the operator Q transforms any chain X into a bounded chain in E. Moreover, we have

$$\|x_0 - Qx\| \leqslant \|x_0\| + \|Qx\| \leqslant \|x_0\| + \psi(\|x\|) + F_0 + V \leqslant \|x_0\| + r_0$$

for all $x \in X$, $\|x\| \leqslant r_0$. More precisely, we infer that the operator Q transforms every bounded chain X in $\overline{\mathscr{B}}(x_0,r)$ into the chain $Q(X)$ contained in the ball $\overline{\mathscr{B}}(x_0,r)$, where $r = \|x_0\| + r_0$. As a result, Q defines a mapping $Q : \mathscr{P}_{ch}(\overline{\mathscr{B}}(x_0,r)) \to \mathscr{P}_{ch}(\overline{\mathscr{B}}(x_0,r))$ and so Q maps a closed and partially bounded set $S = \overline{\mathscr{B}}(x_0,r)$ into itself. Moreover, Q is a partially bounded operator on S in view of Remark 2.2. Furthermore, in view of Lemma 3.1, every compact chain in S possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leqslant$ in E.

Step III: *Q is a partially continuous operator on S.*

Now we show that the operator Q is partially continuous on S. To do this, let X be a chain in S and let us fix arbitrarily $\varepsilon > 0$ and take $x, y \in X$ such that $x \geqslant y$ and $||x-y|| \leqslant \varepsilon$. Then we get:

$$\begin{aligned}|Qx(t)-Qy(t)| &\leqslant \big|f(t,x(t))-f(t,y(t))\big| + \left|\int_0^t g(t,s,x(s))\,ds - \int_0^t g(t,s,y(s))\,ds\right| \\ &\leqslant \psi(|x(t)-y(t)|) + 2a(t)\int_0^t b(s)\,ds \\ &\leqslant \psi(\|x-y\|) + 2v(t) \\ &< \varepsilon + 2v(t).\end{aligned}$$

Hence, in virtue of hypothesis (H_4) we infer that there exists $T > 0$ such that $v(t) \leqslant \frac{\varepsilon}{2}$ for $t \geqslant T$. Thus, for $t \geqslant T$ we derive that

$$|Qx(t)-Qy(t)| < 2\varepsilon\ . \tag{3.11}$$

Further, let us assume that $t \in [0,T]$. Then, evaluating similarly as above we get:

$$\begin{aligned}|Qx(t)-Qy(t)| &\leqslant \psi(|x(t)-y(t)|) + \int_0^t |g(t,s,x(s)-g(t,s,y(s)|\,ds \\ &< \varepsilon + T\,\omega_r^T(g,\varepsilon)\ ,\end{aligned} \tag{3.12}$$

where we have denoted

$$\omega_r^T(g,\varepsilon) = \sup\big\{|g(t,s,x)-g(t,s,y)| : \ t,s \in [0,T],\ x,y \in [-r,r],\ |x-y| \leqslant \varepsilon\big\}\ .$$

From the uniform continuity of the function $g(t,s,x)$ on the set $[0,T]\times[0,T]\times[-r,r]$ we derive that $\omega_r^T(g,\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, linking (3.11), (3.12) and the above established facts we conclude that the operator Q maps partially continuously the closed ball $\overline{\mathscr{B}}(x_0,r)$ into itself.

Step IV: *Q is a nonlinear $\mathscr{D}$-set-contraction w.r.t. characteristic value ω_0.*

Further on let us take a bounded chain X in S with bound $r_0 > 0$, i.e., the chain X belonging to the ball $\overline{\mathscr{B}}(x_0,r)$. Next, fix arbitrarily $T > 0$ and $\varepsilon > 0$. Let us choose $x \in X$ and $t_1,t_2 \in [0,T]$ with $|t_2-t_1| \leqslant \varepsilon$. Without loss of generality we may assume that $x(t_1) \geqslant x(t_2)$. Then, by our assumptions, we get:

$$\begin{aligned}|Qx(t_1)-Qx(t_2)| &\leqslant \big|f(t_1,x(t_1))-f(t_2,x(t_2))\big| \\ &\qquad + \left|\int_0^{t_1} g(t_1,s,x(s)) - \int_0^{t_2} g(t_2,s,x(s))\,ds\right| \\ &\leqslant \big|f(t_1,x(t_1))-f(t_2,x(t_2))\big| \\ &\qquad + \left|\int_0^{t_1} g(t_1,s,x(s)) - \int_0^{t_2} g(t_2,s,x(s))\,ds\right| \\ &\qquad + \left|\int_0^{t_1} g(t_1,s,x(s)) - \int_0^{t_2} g(t_2,s,x(s))\,ds\right|\end{aligned}$$

$$\leqslant \left|f(t_1,x(t_1))-f(t_2,x(t_2))\right| + \int_0^{t_1}|g(t_1,s,x(s))-g(t_2,s,x(s))|\,ds + \left|\int_{t_2}^{t_1}|g(t_1,s,x(s))|\,ds\right|$$
$$\leqslant \left|f(t_1,x(t_1))-f(t_2,x(t_2))\right| + \int_0^{T}|g(t_1,s,x(s))-g(t_2,s,x(s))|\,ds + G_T^r\,|t_1-t_2|, \tag{3.13}$$

where

$$G_T^r = \sup\{|g(t,s,x)| : t\in[0,T],\ s\in[0,T],\ x\in[-r,r]\}$$

which does exist in view of continuity of the function g on compact $[0,T]\times[0,T]\times[-r,r]$.

Now, from (3.13), we obtain

$$\begin{aligned}|Qx(t_2)-Qx(t_1)| &\leqslant \left|f(t_1,x(t_1))-f(t_2,x(t_1))\right| + \psi(|x(t_1)-x(t_2)|)\\ &\quad + \int_0^T|g(t_1,s,x(s))-g(t_2,s,x(s)|\,ds + G_T^r\,|t_1-t_2|\\ &\leqslant \psi(\omega^T(x,\varepsilon)) + \omega_r^T(f,\varepsilon) + \int_0^T \omega_r^T(g,\varepsilon)\,ds + G_T^r\,\omega^T\varepsilon,\end{aligned} \tag{3.14}$$

where we have denoted

$$\omega_r^T(f,\varepsilon) = \sup\{|f(t_2,x)-f(t_1,x)| : t_1,t_2\in[0,T],\ |t_2-t_1|\leqslant\varepsilon,\ x\in[-r,r]\},$$

and

$$\omega_r^T(g,\varepsilon) = \sup\left\{|g(t_2,s,x)-g(t_1,s,x)| : t_1,t_2,s\in[0,T],\ |t_2-t_1|\leqslant\varepsilon,\ x\in[-r,r]\right\}.$$

From the above estimate we derive the following one:

$$\omega^T(Q(X),\varepsilon) \leqslant \psi\left(\omega^T(x,\varepsilon)\right) + \omega_r^T(f,\varepsilon) + \int_0^T \omega_r^T(g,\varepsilon)\,ds + G_T^r\,\varepsilon. \tag{3.15}$$

Observe that $\omega_r^T(f,\varepsilon)\to 0$ and $\omega_r^T(g,\varepsilon)\to 0$ as $\varepsilon\to 0$, which is a simple consequence of the uniform continuity of the functions f and g on the sets $[0,T]\times[-r,r]$ and $[0,T]\times[0,T]\times[-r,r]$ respectively. Thus, linking the established facts with the estimate (3.15) we get $\omega_0^T(Q(X)) \leqslant \psi\left(\omega_0^T(X)\right)$. Consequently, we obtain

$$\omega_0(Q(X)) \leqslant \psi(\omega_0(X)). \tag{3.16}$$

Step V: *Q is a nonlinear $\mathscr{D}$-set-contraction w.r.t. characteristic value δ_c.*

Now, taking into account our assumptions, for arbitrarily fixed $t\in\mathbb{R}_+$ and for $x,y\in X$ with $x\geqslant y$, we deduce the following estimate (cf. the estimate (3.5)):

$$|(Qx)(t)-(Qy)(t)| \leqslant |f(t,x(t))-f(t,y(t))| + 2\left(a(t)\int_0^t b(s)\,ds\right)$$

$$\leqslant \psi(|x(t)-y(t)|)+2v(t).$$

From the above inequality it follows that

$$\text{diam}\,(QX(t)) \leqslant \psi(\,\text{diam}\,(X(t)))+v(t)$$

for each $t \in \mathbb{R}_+$. Therefore, taking limit superior over $t \to \infty$, we obtain

$$\delta_c(QX) = \limsup_{t\to\infty} \text{diam}\,(Q(X(t))) \leqslant \limsup_{t\to\infty} \psi(\,\text{diam}\,(X(t))) = \psi(\delta_c(X)). \quad (3.17)$$

Step VI: *Q is a partially nonlinear $\mathscr{D}$-set-contraction on S.*

Further, using the measure of noncompactness μ^p defined by the formula (3.2) and keeping in mind the estimates (3.16) and (3.17), we obtain

$$\mu^p(QX) = \omega_0(QX)+\delta_c(QX) \leqslant \psi(\omega_0(X)+\delta_c(X)) = \psi(\mu^p(X))$$

for all chains X in S. Hence, the operator Q is a partially nonlinear $\mathscr{D}$-set-contraction on S Again, by hypothesis (H_5), there exists an element $u \in S$ such that $u \leqslant Qu$, that is, u is a lower solution of the HFIE (3.1) defined on $\mathbb{R}_+$.

Thus Q satisfies all the conditions of Theorem 2.1 on S. Hence we apply it to the operator equation $Qx = x$ and deduce that the operator Q has a fixed point x^* in S. Obviously x^* is a solution of the functional integral equation (3.1) and the sequence $\{x_n\}$ of successive approximations defined by (3.7) converges monotonically to x^*. Moreover, taking into account that the image of every chain X under the operator Q is again a chain $Q(X)$ contained in the ball $\overline{\mathscr{B}}(x_0,r)$ we infer that the set $\mathscr{F}(Q)$ of all fixed points of Q is contained in $\overline{\mathscr{B}}(x_0,r)$. If the set $\mathscr{F}(Q)$ contains all comparable solutions of the equation (3.1), then we conclude from Remark 1.2 that the set $\mathscr{F}(Q)$ belongs to the family ker μ_c^p. Now, taking into account the description of sets belonging to ker μ_c^p (given in section 2) we deduce that all comparable solutions of the equation (3.1) are uniformly locally ultimately attractive on $\mathbb{R}_+$. This completes the proof. □

REMARK 3.2. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (H_5) with the following one.

(H'_5) There exists an upper solution $v \in E$ of the HFIE (3.1), i.e., it satisfies the functional integral inequality

$$v(t) \geqslant f(t,v(t)) + \int_0^t g(t,s,v(s))\,ds,$$

for all $t \in \mathbb{R}_+$.

REMARK 3.3. The superadditivity of the function ψ given in the hypothesis (H_3) of Theorem 3.1 may be relaxed if we define the partial measure μ_c^p of noncompactness in the partially ordered Banach space $(BC(\mathbb{R}_+,\mathbb{R}),\leqslant,\|\cdot\|)$ by the formula

$$\mu_c^p(X) = \max\{\omega_0(X),\, \delta_c(X)\} \quad (3.18)$$

for any bounded chain X in $BC(\mathbb{R}_+,\mathbb{R})$.

REMARK 3.4. If we consider the partial measure of noncompactness μ_c^p given by (3.18) for the application in Theorem 3.1, then the inequality in hypothesis (H_1) may be replaced with the inequality,

$$0 \leqslant f(t,x) - f(t,y) \leqslant \frac{L(x-y)}{K+(x-y)}$$

for all $t \in J$ and $x,y \in \mathbb{R}$ with $x \geqslant y$, where $L > 0$ and $K > 0$ are constants satisfying $L \leqslant K$. Here, the growth or $\mathscr{D}$-function ψ is defined by $\psi(r) = \frac{Lr}{K+r}$.

In the following we give a numerical example to illustrate the abstract theory developed in this paper.

EXAMPLE 3.1. Consider the linearly perturbed nonlinear hybrid integral equation,

$$x(t) = f_1(t,x(t)) + \int_0^t \frac{1}{5(t^2+1)}\, g_1(s,x(s))\, ds \tag{3.19}$$

for all $t \in \mathbb{R}_+$, where $f_1, g_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions defined by

$$f_1(t,x) = \begin{cases} \dfrac{1}{5}, & \text{if} \quad x \leqslant 0, \\ \frac{1}{5} + \log\left(1 + \dfrac{x}{5}\right), & \text{if} \quad x > 0. \end{cases}$$

and

$$g_1(t,x) = \begin{cases} 1, & \text{if} \quad x \leqslant 0, \\ 1 + \dfrac{x}{x+1}, & \text{if} \quad x > 0. \end{cases}$$

We shall show that all the hypotheses of Theorem 3.1 are satisfied by the functions involved in HFIE (3.19). Here, $f(t,x) = f_1(t,x)$ so that f is nondecreasing in x for each $t \in \mathbb{R}_+$ and continuous on $\mathbb{R}_+ \times \mathbb{R}$.

Now, we show that f_1 is partially nonlinear $\mathscr{D}$-contraction on $\mathbb{R}_+ \times \mathbb{R}$. Let $x,y \in \mathbb{R}$ be arbitrary elements with $0 \geqslant x \geqslant y$. Then,

$$0 \leqslant f_1(t,x) - f_1(t,y) = 0 \leqslant \log\left(1 + \frac{1}{5}(x-y)\right).$$

Similarly, if $x \geqslant y > 0$, then

$$\begin{aligned} 0 \leqslant f_1(t,x) - f_1(t,y) &= \log\left(1 + \frac{x}{5}\right) - \log\left(1 + \frac{y}{5}\right) \\ &= \log\left(\frac{1 + \frac{x}{5}}{1 + \frac{y}{5}}\right) \\ &= \log\left(1 + \frac{\frac{1}{5}(x-y)}{1 + \frac{y}{5}}\right) \\ &\leqslant \log\left(1 + \frac{1}{5}(x-y)\right). \end{aligned}$$

Therefore, hypothesis (H_3) is satisfied with the $\mathscr{D}$-function $\psi(r) = \log\left(1 + \frac{1}{5}r\right) < r$ for $r > 0$. Note that ψ is not supperadditive on $\mathbb{R}_+$. Moreover, $f_1(t,0) = \frac{1}{5}$ so that the hypotheses (H_1) and (H_2) are satisfied.

The function $g(t,s,x)$ is given by $g(t,s,x) = \dfrac{1}{5(t^2+1)} g_1(s,x)$. Obviously g is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$. Next, $g(t,s,x)$ is nondecreasing function in x for each $t,s \in \mathbb{R}_+$ and so (H_3) holds. Furthermore, $|g_1(t,x)| \leqslant 2$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Thus, we have

$$v(t) = \int_0^t \frac{1}{5(t^2+1)} \cdot 2\,ds = \frac{2t}{5(t^2+1)}.$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{2t}{5(t^2+1)} = 0.$$

and so hypothesis (H_4) holds.

Finally, it is easy to prove that $u(t) = \dfrac{t}{5(t^2+1)}$ is a lower solution of the HFIE (3.19) defined on $\mathbb{R}_+$ and hence the hypothesis (H_5) is satisfied. Moreover, hypothesis (H_6) holds with $r_0 = 1$. Thus in view of Remark 3.3, all the conditions of Theorem 3.1 are satisfied and by a direct application, we conclude that the HFIE (3.19) has a solution x^* in $\overline{\mathscr{B}}(x_0, 1)$ and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = f_1(t, x_n(t)) + \int_0^t \frac{1}{5(t^2+1)}\, g_1(s, x_n(s))\,ds,\ t \in \mathbb{R}_+,$$

converges monotonically to x^*, where $x_0 = u$. Moreover, the comparable solutions of the HFIE (3.19) are uniformly locally ultimately attractive and stable to 0 defined on $\mathbb{R}_+$.

EXAMPLE 3.2. Consider the linearly perturbed nonlinear hybrid integral equation,

$$x(t) = \frac{1}{3}\arctan x(t) + \int_0^t \frac{1}{5(t^2+1)} \tanh x(s)\,ds \tag{3.20}$$

for all $t \in \mathbb{R}_+$.

Here, $f(t,x) = \dfrac{1}{3}\arctan x$, and $g(t,s,x) = \dfrac{1}{5(t^2+1)} \tanh x$ for all $t,s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Then proceeding with the arguments that given in Example 3.1, it can be shown that the functions f and g satisfy all hypotheses (H_1) through (H_4). Finally, it is easy to prove that $u(t) = -\dfrac{2}{3} - \dfrac{t}{5(t^2+1)}\,ds$ is a lower solution of the HFIE (3.20) defined on $\mathbb{R}_+$ and hence the hypothesis (H_5) is satisfied. Furthermore, hypothesis (H_6) holds with $r_0 = 1$. Thus in view of Remark 3.2, all the conditions of Theorem 3.1 are satisfied and by a direct application, we conclude that the HFIE (3.20) has a solution x^* in $\overline{\mathscr{B}}(x_0, 1)$ and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \frac{1}{3}\arctan x_n(t) + \int_0^t \frac{1}{5(t^2+1)} \tanh x_n(s)\,ds,\ t \in \mathbb{R}_+,$$

converges monotonically to x^*, where $x_0 = u$. Moreover, the comparable solutions of the HFIE (3.20) are uniformly locally ultimately attractive and stable to 0 defined on $\mathbb{R}_+$.

REMARK 3.5. In this paper, we have considered a very simple nonlinear functional integral equation for proving the existence as well as algorithm for the local attractive comparable solutions. However, a similar study may also be made for the more general nonlinear functional integral equation of the type, viz.,

$$x(t) = F\left(t, f(t,x(t)), \int_0^t g(t,s,x(s))\,ds\right), \; t \in \mathbb{R}_+, \tag{3.21}$$

where $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The HFIE (3.21) is studied in Dhage and Lakshmikantam [16] via classical measure theoretic fixed point theorem of Dhage [4] for attractivity of solutions under a strong Lipschitz condition. Therefore, the obtained existence and approximation results would be under weaker conditions.

4. Notes and Comments

Finally, while concluding this paper we mention that local or global existence and attractivity results for the HFIE (3.1) can also be obtained under some mixed stronger hypotheses of usual Lipschitz and compactness type conditions using the classical measure of noncompatness. See Banas and Dhage [2] and the references therein. But nevertheless, we do not get the algorithm or constructive method for approximating the solutions of the integral equation (3.1) on $\mathbb{R}_+$. The novelty of the present approach lies in the fact that we get an algorithm for the local existence and attractive comparable solutions of the HFIE (3.1) under a condition involving general $\mathscr{D}$-function ψ on f. Again, if the hypothesis (H_5) holds, then in view of hypotheses (H_1), (H_2) and (H_4), the HFIE (3.1) has another lower solution u_1 such that $u_1(t) \leqslant u(t)$ for all $t \in \mathbb{R}_+$. Now, by Theorem 3.1, the sequences $\{Q^n u_1\}$ and $\{Q^n u\}$ converge respectively to the comparable solutions x_1^* and x^* satisfying the inequality $x_1^*(t) \leqslant x^*(t)$ for all $t \in \mathbb{R}_+$, where the operator Q is defined by (3.8). Therefore the HFIE (3.1) has more than one comparable solutions and so the conclusion of Theorem 3.1 is meaningful. A similar conclusion also remains true for the HFIE (3.1) if the hypothesis (H_5) in Theorem 3.1 is replaced with (H'_5). Furthermore we conjecture that similar results are also true for the HFIE (3.21) or its generalizations under some suitable hybrid conditions. Again the partial measure of noncompactness may have numerous applications to other related areas of nonlinear analysis which are yet to be investigated. Some of the results along these lines is our future plan and will be reported elsewhere.

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