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GLOBAL DYNAMICS OF A DELAYED DIFFUSIVE TWO–STRAIN DISEASE MODEL

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Abstract. The aim of this paper is to investigate the global dynamics of a delayed diffusive two-strain disease model. We first study the well-posedness of the model. And then, by selecting appropriate Lyapunov functionals, we demonstrate that the global stability of the model is fully determined by the basic reproduction number. Furthermore, using Schauder fixed point theorem and constructing a pair of upper-lower solutions, we show that the model admits a traveling wave solution connecting the disease free and co-existence equilibria.

1. Introduction

In [5], the author derived a mathematical model to describe the dynamics of a communicable disease through a vector population as follows

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \beta S(t)I(t-\tau), \\ \dot{I}(t) = \beta S(t)I(t-\tau) - (\mu+\gamma)I(t), \\ \dot{R}(t) = \gamma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$ and $R(t)$ represent the sub-populations of susceptible class, infective class and removed class at time t, respectively. Λ is the recruitment rate of the susceptible population, μ is the natural death rate of the population, β is the contact rate, γ is the recovery rate of the infective individuals, and $\tau \geqslant 0$ is a constant representing the length of the immunity period.

It is because that mutation of a pathogen is common and causes serious problems in treating the resulting disease that one often needs to deal with more than one strain, see [3, 7]. Thus, the study of disease dynamics with multiple strains is an important research topic and has attracted sustained attention of researchers in recent decades. Among those, in [1], by introducing another strain of the disease, the authors extended

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Cooke's idea, and proposed a two-strain disease model with latency and saturating incidence rate

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \frac{\beta_1 S(t) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - \frac{\beta_2 S(t) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)}, \\ \dot{I}_1(t) = \frac{\beta_1 S(t) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - (\mu_1+\gamma_1) I_1(t), \\ \dot{I}_2(t) = \frac{\beta_2 S(t) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - (\mu_2+\gamma_2) I_2(t), \\ \dot{R}(t) = \gamma_1 I_1(t) + \gamma_2 I_2(t) - \mu R(t), \end{cases} \tag{1.2}$$

where $I_1(t)$ and $I_2(t)$ represent the sub-populations of infective class with strain 1 and strain 2, respectively. As model (1.1), $S(t)$ and $R(t)$ still denote the population of susceptible and removed class, respectively. Λ denotes the recruitment of individuals, $1/\mu$ is the lift expectancy, β_1 (β_2) represents the transmission coefficient of susceptible individuals to strain 1 (strain 2), $1/\gamma_1$ $(1/\gamma_2)$ denotes the average infected period of strain 1 (strain 2), μ_1 (μ_2) is the combination of infection induced death rate and natural death rate of strain 1 (strain 2), $\tau_i \geqslant 0$, $i=1,2$, is a constant representing the length of the immunity period, and $\alpha_i \geqslant 0$, $i=1,2$, denotes the saturation level when the infection population is large. The readers are referred to [1] for the precise interpretation of the biological implication of (1.2).

Since R is decoupled in (1.2), the authors [1] analyzed the global dynamics of the following reduced dimensional system

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \frac{\beta_1 S(t) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - \frac{\beta_2 S(t) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)}, \\ \dot{I}_1(t) = \frac{\beta_1 S(t) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - (\mu_1+\gamma_1) I_1(t), \\ \dot{I}_2(t) = \frac{\beta_2 S(t) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - (\mu_2+\gamma_2) I_2(t). \end{cases} \tag{1.3}$$

More precisely, it was proved that if the basic reproduction number is less than one, then disease dies out, but if the number is large than one, then one or two of the strains become endemic.

Clearly, model (1.2) is of ODE type, which could only reflect the epidemiological and demographic process as the time changes. We note that the spatial content of the environment has been ignored in the model (1.2). To closely match the reality, considering an diffusive epidemic model of PDE type is natural and reasonable [17]. Inspired from [1], we here propose the following delayed diffusive two-strain disease model

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = d_s \Delta S(t,x) + \Lambda - \mu S(t,x) - \frac{\beta_1 S(t,x) I_1(t-\tau_1,x)}{1+\alpha_1 I_1(t-\tau_1,x)} - \frac{\beta_2 S(t,x) I_2(t-\tau_2,x)}{1+\alpha_2 I_2(t-\tau_2,x)}, \\ \frac{\partial I_1(t,x)}{\partial t} = d_1 \Delta I_1(t,x) + \frac{\beta_1 S(t,x) I_1(t-\tau_1,x)}{1+\alpha_1 I_1(t-\tau_1,x)} - (\mu_1+\gamma_1) I_1(t,x), \\ \frac{\partial I_2(t,x)}{\partial t} = d_2 \Delta I_2(t,x) + \frac{\beta_2 S(t,x) I_2(t-\tau_2,x)}{1+\alpha_2 I_2(t-\tau_2,x)} - (\mu_2+\gamma_2) I_2(t,x), \\ \frac{\partial R(t,x)}{\partial t} = d_R \Delta R(t,x) + \gamma_1 I_1(t,x) + \gamma_2 I_2(t,x) - \mu R(t,x), \end{cases} \tag{1.4}$$

in which $S(t,x)$, $I_1(t,x)$, $I_2(t,x)$ and $R(t,x)$ represent the sub-populations of susceptible class, infective class with strain 1 and strain 2, and removed class at time $t>0$ and position $x \in \Omega$, respectively. $d_s, d_1, d_2, d_R > 0$ are the diffusion rates, Δ is the Laplacian operator and Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The parameters $\Lambda, \mu, \beta_1, \beta_2, \gamma_1, \gamma_2, \mu_1, \mu_2, \alpha_1, \alpha_2$ are positive constants as in model (1.2).

In the biological context, one of the fundamental problems is to study the stability of the steady states, since this characterizes whether a disease will become endemic and this is a major concern for public health offices. For the model under consideration, on the other hand, the traveling wave describes the disease population into the susceptible population from an initial disease-free equilibrium to the endemic equilibrium. Biologically speaking, existence of an epidemic wave implies that the disease can invade successfully and an epidemics arises [17].

For model (1.4), in the absence of the strain 2, that is, $I_2(t,x) \equiv 0$, one gets the following delayed diffusive epidemic model

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = d_s \Delta S(t,x) + \Lambda - \mu S(t,x) - \frac{\beta_1 S(t,x) I_1(t-\tau_1,x)}{1+\alpha_1 I_1(t-\tau_1,x)}, \\ \frac{\partial I_1(t,x)}{\partial t} = d_1 \Delta I_1(t,x) + \frac{\beta_1 S(t,x) I_1(t-\tau_1,x)}{1+\alpha_1 I_1(t-\tau_1,x)} - (\mu_1+\gamma_1) I_1(t,x). \end{cases} \tag{1.5}$$

Yang et al [24] studied the existence of the traveling waves of (1.5). Very recently, Li et al [12] has further investigated the existence of the traveling waves and established the critical waves and the minimal speed of the model (1.5). The main purpose of this paper is to study the dynamical behaviors of this model. We focus on the global stability of the equilibria as well as the existence of traveling wave solutions of the model (1.4).

This paper is organized as follows. In section 2, we study the well-posedness for system (2.1)-(2.3). In section 3, by means of appropriate Lyapunov functionals and LaSalle's invariance principle, we investigate the global dynamics of the four equilibria, respectively. In section 4, we construct a pair of upper-lower solutions and employ the Schauder fixed point theorem to prove the existence of traveling wave solutions for system (2.1). Finally, a brief discussion is given in section 5.

2. The well-posedness

For simplicity, let

$$\bar{S} = \frac{\mu}{\Lambda} S,\ \bar{I}_i = \frac{\mu}{\Lambda} I_i,\ \bar{R} = \frac{\mu}{\Lambda} R,\ \bar{\alpha}_i = \frac{\alpha_i}{\mu}\Lambda,\ \bar{\beta}_i = \frac{\beta_i}{\mu}\Lambda,\ I_{\tau_i} = I(t-\tau_i,x),\ i=1,2.$$

Then, dropping the bars on S, I_i, R, we obtain

$$\begin{cases} \frac{\partial S}{\partial t} = d_s \Delta S + \mu(1-S) - \beta_1 S f_1(I_{\tau_1}) - \beta_2 S f_2(I_{\tau_2}), \\ \frac{\partial I_1}{\partial t} = d_1 \Delta I_1 + \beta_1 S f_1(I_{\tau_1}) - (\mu_1+\gamma_1) I_1, \\ \frac{\partial I_2}{\partial t} = d_2 \Delta I_2 + \beta_2 S f_2(I_{\tau_2}) - (\mu_2+\gamma_2) I_2, \\ \frac{\partial R}{\partial t} = d_R \Delta R + \gamma_1 I_1 + \gamma_2 I_2 - \mu R, \end{cases} \tag{2.1}$$

where $f_i(x) = \frac{x}{1+\alpha_i x}$. Accompanied with (2.1), we consider the initial conditions

$$S(t,x) = \varphi_1(t,x),\ I_1(t,x) = \varphi_2(t,x),\ I_2(t,x) = \varphi_3(t,x),\ R(t,x) = \varphi_4(t,x), \tag{2.2}$$

for all $(t,x) \in [-\tau,0] \times \overline{\Omega}$, and the Neumann boundary conditions

$$\frac{\partial S}{\partial n} = \frac{\partial I_1}{\partial n} = \frac{\partial I_2}{\partial n} = \frac{\partial R}{\partial n} = 0, \tag{2.3}$$

for all $(t,x) \in (0,+\infty) \times \partial\Omega$, where $\tau := \max\{\tau_1, \tau_2\}$, $\varphi_i(t,x)$ $(i = 1,2,3,4)$ are nonnegative and Hölder continuous in $[-\tau,\infty) \times \overline{\Omega}$, and $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$. The Neumann boundary conditions (2.3) imply that the two diseases do not move across the boundary $\partial\Omega$.

Define two threshold values as $\mathscr{R}_i = \frac{\beta_i}{\mu_i+\gamma_i}$, $i = 1,2$. The basic reproduction number of (2.1) is given by $\mathscr{R}_0 = \max\{\mathscr{R}_1, \mathscr{R}_2\}$.

By a direct computation, we get the following conclusion.

LEMMA 2.1. (1) *System (2.1) always has a disease-free equilibrium* $E_0 = (1,0,0,0)$.

(2) *If $\mathscr{R}_1 > 1$ and $\mathscr{R}_2 \leqslant 1$, then system (2.1) has the single-infection equilibrium $E_1 = (\bar{S}, \bar{I}_1, 0, \bar{R})$, where*

$$\bar{S} = \frac{1}{\mathscr{R}_1}(1+\alpha_1 \bar{I}_1),\ \bar{I}_1 = \frac{\mu}{\alpha_1\mu+\beta_1}(\mathscr{R}_1 - 1),\ \bar{R} = \frac{\gamma_1}{\mu}\bar{I}_1.$$

(3) If $\mathscr{R}_2 > 1$ and $\mathscr{R}_1 \leqslant 1$, then system (2.1) has the single-infection equilibrium $E_2 = (\widehat{S}, 0, \widehat{I}_2, \widehat{R})$, where

$$\widehat{S} = \frac{1}{\mathscr{R}_2}(1+\alpha_2 \widehat{I}_2),\ \widehat{I}_2 = \frac{\mu}{\alpha_2\mu+\beta_2}(\mathscr{R}_2 - 1),\ \widehat{R} = \frac{\gamma_2}{\mu}\widehat{I}_2.$$

(4) If $\overline{\mathscr{R}}_0 := \min\{\mathscr{R}_1 S^*, \mathscr{R}_2 S^*\} > 1$, then system (2.1) has the co-existence equilibrium (all components are positive) $E^* = (S^*, I_1^*, I_2^*, R^*)$, where

$$S^* = \frac{\alpha_1\alpha_2\mu + \alpha_1(\mu_2+\gamma_2) + \alpha_2(\mu_1+\gamma_1)}{\alpha_1\alpha_2\mu + \alpha_1\beta_2 + \alpha_2\beta_1},\ R^* = \frac{1}{\mu}(\gamma_1 I_1^* + \gamma_2 I_2^*),$$

and $I_i^* = \frac{1}{\alpha_i}(\mathscr{R}_i S^* - 1)$, $i = 1,2$.

Here, it is easy to see that the co-existence equilibrium is biologically meaningful if and only if $\mathscr{R}_i S^* > 1$, $i = 1,2$.

Next, we consider the positive invariance and uniform boundedness of solutions for the initial-boundary-value problem of system (2.1)-(2.3).

Let $\mathbb{X} := C(\overline{\Omega}, \mathbb{R}^4)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Clearly, any vector in $\mathbb{R}^4$ can be regarded as an element in $\mathbb{X}$. For $u = (u_1,u_2,u_3,u_4)$, $v = (v_1,v_2,v_3,v_4) \in \mathbb{X}$, we write $u \geqslant v$ $(u \leqslant v)$ provided $u_i(x) \geqslant v_i(x)$ $(u_i(x) \leqslant v_i(x))$, $i = 1,2,3,4$, $x \in \overline{\Omega}$. For $\tau := \max\{\tau_1,\tau_2\}$, define $\mathbb{C} = C([-\tau,0],\mathbb{X})$ with the norm $\|\phi\| = \max_{\theta\in[-\tau,0]} \|\phi\|_{\mathbb{X}}$ for $\phi \in \mathbb{C}$. Then $\mathbb{C}$ is a Banach space. Define $\mathbb{X}_+ := C(\overline{\Omega}, \mathbb{R}^4_+)$ and $\mathbb{C}_+ = C([-\tau,0],\mathbb{X}_+)$. Then both $(\mathbb{X},\mathbb{X}_+)$ and $(\mathbb{C},\mathbb{C}_+)$ are strongly ordered Banach spaces. As usual, we identify an element $\varphi \in \mathbb{C}$ as a function from $[-\tau,0]\times\mathbb{R}^n$ into $\mathbb{R}^4$ by $\varphi(s,x) = \varphi(s)(x)$. Given a function $u : [-\tau,b) \to \mathbb{X}$ for $b > 0$, define $u_t \in \mathbb{C}$ by $u_t(s) = u(t+s)$, $s \in [-\tau,0]$. Let $\mathbf{D} = (d_s,d_1,d_2,d_R)^T$. It follows from [6, Theorem 1.5] that $\mathbb{X}-$realization of $\mathbf{D}\Delta$ generates an analytic semi-group $\mathscr{T}(t)$ on $\mathbb{X}$.

For any $\varphi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in \mathbb{C}_+$ and $x \in \overline{\Omega}$, we define $F = (F_1,F_2,F_3,F_4) : \mathbb{C}_+ \to \mathbb{X}$ by

$$F_1(\varphi)(x) := \mu(1-\varphi_1(0,x)) - \beta_1\varphi_1(0,x)f_1(\varphi_2(-\tau,x))$$

$$
\begin{aligned}
&\quad -\beta_2\varphi_1(0,x)f_2(\varphi_3(-\tau,x)),\\
F_2(\varphi)(x) &:= \beta_1\varphi_1(0,x)f_1(\varphi_2(-\tau,x)) - (\mu_1+\gamma_1)\varphi_2(0,x),\\
F_3(\varphi)(x) &:= \beta_2\varphi_1(0,x)f_2(\varphi_3(-\tau,x)) - (\mu_2+\gamma_2)\varphi_3(0,x),\\
F_4(\varphi)(x) &:= \gamma_1\varphi_2(0,x)+\gamma_2\varphi_3(0,x)-\mu\varphi_4(0,x).
\end{aligned}
$$

Then F is Lipschitz continuous in any bounded subset of $\mathbb{C}_+$. Rewriting (2.1)-(2.3) as the following abstract functional differential equation

$$
\begin{cases}
\dfrac{\mathrm{du}}{\mathrm{dt}} = \mathbf{A}\mathrm{u}+\mathbf{F}(\mathrm{u_t}),\ t\geqslant 0,\ \mathrm{u_t}\in\mathbb{C},\\
\mathrm{u_0}=\varphi\in\mathbb{C}_+,
\end{cases}
$$

where $\mathrm{u}=(S,I_1,I_2,R)$, $\mathbf{A}\mathrm{u} := (d_s\Delta S, d_1\Delta I_1, d_2\Delta I_2, d_R\Delta R)^T$ and $\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$. Define

$$
[\mathbf{0},\mathbf{M}]_{\mathbb{C}} := \left\{\varphi\in\mathbb{C} : \mathbf{0}\leqslant\varphi(\theta,x)\leqslant\mathbf{M},\ \forall\theta\in[-\tau,0], x\in\overline{\Omega}\right\}
$$

with $\mathbf{0}:=(0,0,0,0)$, and

$$
\mathbf{M} := \left(1, \frac{\beta_1}{\alpha_1(\mu_1+\gamma_1)}, \frac{\beta_2}{\alpha_2(\mu_2+\gamma_2)}, \frac{1}{\mu}\left(\frac{\gamma_1\beta_1}{\alpha_1(\mu_1+\gamma_1)}+\frac{\gamma_2\beta_2}{\alpha_2(\mu_2+\gamma_2)}\right)\right).
$$

THEOREM 2.1. *For any given initial data $\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)\in[\mathbf{0},\mathbf{M}]_{\mathbb{C}}$, there exists a unique non-negative solution $u(t,x;\varphi)$ of (2.1)-(2.3) defined on $[0,\infty)$ and $u_t\in[\mathbf{0},\mathbf{M}]_{\mathbb{C}}$ for $t\geqslant 0$.*

Proof. For any $\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)\in[\mathbf{0},\mathbf{M}]_{\mathbb{C}}$ and any $\kappa\geqslant 0$, we have

$$
\begin{aligned}
&\varphi(0,x)+\kappa F(\varphi)(x)\\
&\quad=\begin{pmatrix}
\varphi_1(0,x)+\kappa(\mu(1-\varphi_1(0,x))-\beta_1\varphi_1(0,x)f_1(\varphi_2(-\tau,x))\\
-\beta_2\varphi_1(0,x)f_2(\varphi_3(-\tau,x)))\\
\varphi_2(0,x)+\kappa(\beta_1\varphi_1(0,x)f_1(\varphi_2(-\tau,x))-(\mu_1+\gamma_1)\varphi_2(0,x))\\
\varphi_3(0,x)+\kappa(\beta_2\varphi_1(0,x)f_2(\varphi_3(-\tau,x))-(\mu_2+\gamma_2)\varphi_3(0,x))\\
\varphi_4(0,x)+\kappa(\gamma_1\varphi_2(0,x)+\gamma_2\varphi_3(0,x)-\mu\varphi_4(0,x))
\end{pmatrix}.
\end{aligned}
$$

Note that $f_i(x)$ $(i=1,2)$ is increasing in $x>0$, and for any sufficiently small $\kappa>0$,

$$
\varphi(0,x)+\kappa F(\varphi)(x)\geqslant
\begin{pmatrix}
(1-\kappa(\mu+\beta_1+\beta_2))\varphi_1(0,x)\\
(1-\kappa(\mu_1+\gamma_1))\varphi_2(0,x)\\
(1-\kappa(\mu_2+\gamma_2))\varphi_3(0,x)\\
(1-\kappa\mu)\varphi_4(0,x)
\end{pmatrix}\geqslant\mathbf{0}.
$$

On the other hand, note that, for any sufficiently small $\kappa>0$, the functions $u_1+\kappa(\mu(1-u_1)-\beta_1u_1f(u_2)-\beta_2u_1f_2(u_3))$ is increasing in $u_1>0$ for any fixed $u_2>0$ and $u_3>0$; $u_2+\kappa(\beta_1u_1f_1(u_2)-(\mu_1+\gamma_1)u_2)$ is increasing in u_2 for any fixed $u_1>$

0, $u_3+\kappa(\beta_2 u_1 f_2(u_3)-(\mu_2+\gamma_2)u_3)$ is increasing in u_3 for any fixed $u_1>0$, and $u_4+\kappa(\gamma_1 u_2+\gamma_2 u_3-\mu u_4)$ is increasing in u_4 for any fixed $u_2>0$ and $u_3>0$. Then

$$\varphi(0,x)+\kappa F(\varphi)(x)\leqslant\begin{pmatrix}1\\ \frac{\beta_1}{\alpha_1(\mu_1+\gamma_1)}+\kappa\left(\frac{\beta_1^2}{\alpha_1(\beta_1+\mu_1+\gamma_1)}-\frac{\beta_1}{\alpha_1}\right)\\ \frac{\beta_2}{\alpha_2(\mu_2+\gamma_2)}+\kappa\left(\frac{\beta_2^2}{\alpha_2(\beta_2+\mu_2+\gamma_2)}-\frac{\beta_2}{\alpha_2}\right)\\ \varphi_4(0,x)\end{pmatrix}\leqslant\mathbf{M}.$$

Thus, $\varphi(0)+\kappa F(\varphi)\in[\mathbf{0},\mathbf{M}]_{\mathbb{C}}$. This implies that

$$\lim_{\kappa\to 0^+}\frac{1}{\kappa}\operatorname{dist}\big(\varphi(0)+\kappa F(\varphi),[\mathbf{0},\mathbf{M}]_{\mathbb{C}}\big)=0,\ \forall\varphi\in[\mathbf{0},\mathbf{M}]_{\mathbb{C}}.$$

Let $K=[\mathbf{0},\mathbf{M}]_{\mathbb{C}},S(t,x)=\mathscr{T}(t-x)$ and $B(t,\varphi)=F(\varphi)$. Then it follows from [16, Corollary 4], that (2.1) -(2.3) admit a unique non-continuable mild solution $\mathrm{u}(t,\varphi)\in[\mathbf{0},\mathbf{M}]_{\mathbb{C}}$ for $t\in[0,\infty)$. Furthermore, since the semigroup $\mathscr{T}(t)$ is analytic, the mild solution $\mathrm{u}(t,\varphi)$ of (2.1)-(2.3) is classic for $t\geqslant\tau$ (see [21, Corollary 2.2.5]). $\square$

3. Global Stability of the equilibria

In this section, we consider the global stability of the equilibria. Generally speaking, it is difficult for us to obtain the global properties of the diffusive model with nonlinear functional responses. Recently, there has been some works on the global asymptotic stability of the constant equilibria of reaction-diffusion models by constructing Lyapunov functionals and using LaSalle's invariant principle, see [10, 15, 23, 28], for example. Here, motivated by the works of [1], we investigate the global dynamics of system (2.1)-(2.3). Such Lyapunov functional takes advantages of the properties of the function

$$g(x):=x-1-\ln x,\ x\in(0,+\infty),$$

with $g(x)\geqslant 0$ for all $x\in(0,+\infty)$ and $\min_{0<x<+\infty}g(x)=g(1)=0$.

Noting that $R(t,x)$ does not appear in the first three equations of (2.1), we first consider the following subsystem

$$\begin{cases}\frac{\partial S}{\partial t}=d_s\Delta S+\mu(1-S)-\beta_1 Sf_1(I_{\tau_1})-\beta_2 Sf_2(I_{\tau_2}),\\ \frac{\partial I_1}{\partial t}=d_1\Delta I_1+\beta_1 Sf_1(I_{\tau_1})-(\mu_1+\gamma_1)I_1,\\ \frac{\partial I_2}{\partial t}=d_2\Delta I_2+\beta_2 Sf_2(I_{\tau_2})-(\mu_2+\gamma_2)I_2,\end{cases}\tag{3.1}$$

with the initial conditions

$$S(t,x)=\varphi_1(t,x),\ I_1(t,x)=\varphi_2(t,x),I_2(t,x)=\varphi_3(t,x),\ \forall(t,x)\in[-\tau,0]\times\overline{\Omega},\tag{3.2}$$

and the Neumann boundary conditions

$$\frac{\partial S}{\partial n}=\frac{\partial I_1}{\partial n}=\frac{\partial I_2}{\partial n}=0,\ t>0,\ x\in\partial\Omega,\tag{3.3}$$

where $\varphi_i(t,x) \geqslant 0$ and $\varphi_i(t,x) \neq 0$ for any $(t,x) \in [-\tau,0] \times \overline{\Omega}$. It follows from the maximum principle [25] that $S(t,x) > 0$, $I_1(t,x) > 0$ and $I_2(t,x) > 0$ for any $(t,x) \in [0,\infty) \times \Omega$.

THEOREM 3.1. *If $\mathscr{R}_0 \leqslant 1$, then $E_0(1,0,0)$ of system (3.1)-(3.3) is globally attractive.*

Proof. We consider the following Lyapunov functional

$$L(t) = \int_\Omega (L_1(t,x) + L_2(t,x))\mathrm{d}x,$$

where $L_1(t,x) = g(S(t,x))$, and

$$L_2(t,x) = I_1(t,x) + I_2(t,x) + \beta_1 \int_{t-\tau_1}^{t} I_1(\theta,x)\mathrm{d}\theta + \beta_2 \int_{t-\tau_2}^{t} I_2(\theta,x)\mathrm{d}\theta.$$

Obviously, $L(t)$ is non-negative in $\mathbb{R}^+$ and attains zero at E_0. Next, we calculate the time derivative of $L(t)$ along the solution of system (3.1)-(3.3).

Note that

$$\frac{\partial L_1}{\partial t} = \frac{1}{S}(S-1)d_s\Delta S - \frac{\mu}{S}(S-1)^2 - (S-1)(\beta_1 f_1(I_{\tau_1}) + \beta_2 f_2(I_{\tau_2})), \tag{3.4}$$

and

$$\begin{aligned}\frac{\partial L_2}{\partial t} &= d_1\Delta I_1 + d_2\Delta I_2 + \beta_1 S f_1(I_{\tau_1}) + \beta_2 S f_2(I_{\tau_2}) \\ &\quad + \beta_1 I_1 (1 - \frac{1}{\mathscr{R}_1}) + \beta_2 I_2 (1 - \frac{1}{\mathscr{R}_2}) - \beta_1 I_{\tau_1} - \beta_2 I_{\tau_2}.\end{aligned} \tag{3.5}$$

Consequently, by (3.4) and (3.5), we obtain

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = \int_\Omega \Big(\frac{1}{S}(S-1)d_s\Delta S + d_1\Delta I_1 + d_2\Delta\, I_2 - \frac{\mu}{S}(S-1)^2 + C_1(t,x)\Big)\mathrm{d}x,$$

where

$$C_1(t,x) = -\alpha_1\beta_1 f_1(I_{\tau_1})I_{\tau_1} - \alpha_2\beta_2 f_2(I_{\tau_2})I_{\tau_2} + \beta_1 I_1\Big(1 - \frac{1}{\mathscr{R}_1}\Big) + \beta_2 I_2\Big(1 - \frac{1}{\mathscr{R}_2}\Big),$$

which follows that $C_1(t,x) \leqslant 0$ for $\mathscr{R}_0 \leqslant 1$.

By the Divergence Theorem and (3.3), we get

$$\int_\Omega \Delta S\mathrm{d}x = 0,\ \int_\Omega \Delta I_1\mathrm{d}x = 0,\ \int_\Omega \Delta I_2\mathrm{d}x = 0, \tag{3.6}$$

and

$$\int_\Omega \frac{\Delta S}{S}\mathrm{d}x = \int_\Omega \frac{\|\nabla S\|^2}{S^2}\mathrm{d}x. \tag{3.7}$$

It is easy to see that

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = -d_s\int_\Omega \frac{1}{S^2}||\nabla S||^2\mathrm{d}x - \mu\int_\Omega \frac{1}{S}(S-1)^2\mathrm{d}x + \int_\Omega C_1(t,x)\mathrm{d}x.$$

Hence, $\mathscr{R}_0 \leqslant 1$ ensures $\frac{\mathrm{d}L(t)}{\mathrm{d}t} \leqslant 0$ on $\mathbb{R}^+$. And also, for $\mathscr{R}_0 = 1$, $\frac{\mathrm{d}L(t)}{\mathrm{d}t} = 0$ if and only if $S=1, I_1=0, I_2=0$. The largest compact invariant set in $\{(S,I_1,I_2)\in\mathbb{R}^3_+ : \frac{\mathrm{d}L(t)}{\mathrm{d}t}=0\}$ is the singleton E_0. By LaSalle's invariant principle [11, Theorem 4.3.4], E_0 is globally attractive when $\mathscr{R}_0 \leqslant 1$. $\square$

Next, we show that the single-infective equilibria $E_1(\bar{S},\bar{I}_1,0)$ and $E_2(\widehat{S},0,\widehat{I}_2)$ are globally attractive.

THEOREM 3.2. *If $E_1(\bar{S},\bar{I}_1,0)$ exists (i.e., $\mathscr{R}_1>1$), but $E_2(\widehat{S},0,\widehat{I}_2)$ does not exist (i.e., $\mathscr{R}_2\leqslant 1$), then E_1 of system (3.1)-(3.3) is globally attractive.*

Proof. We construct the following Lyapunov functional

$$V(t) = \frac{1}{\beta_1 f_1(\bar{I}_1)}\int_\Omega\Big(V_S + \frac{\bar{I}_1}{\bar{S}}V_{I_1} + \beta_1 f_1(\bar{I}_1)\overline{V}_{I_1} + \frac{1}{\bar{S}}I_2 + \beta_2\int_{t-\tau_2}^{t} I_2(\theta,x)\mathrm{d}\theta\Big)\mathrm{d}x,$$

where

$$V_S = g\big(\frac{S}{\bar{S}}\big),\ V_{I_1} = g\big(\frac{I_1}{\bar{I}_1}\big),\ \overline{V}_{I_1} = \int_{t-\tau_1}^{t} g\Big(\frac{1}{\bar{I}_1}I_1(s,x)\Big)\mathrm{d}s.$$

By the properties of the function $g(x)$, it is easy to see that the Lyapunov functional V is non-negative and attains to zero at E_1. That is to say, V is positive define. Next, we show that $\frac{\mathrm{d}V(t)}{\mathrm{d}t}\leqslant 0$ along the solution of system (3.1)-(3.3).

For the simplicity of notation, let

$$u=\frac{S}{\bar{S}},\ v=\frac{I_1}{\bar{I}_1},\ w=\frac{I_{\tau_1}}{\bar{I}_1},\ F_1(w)=\frac{f_1(\bar{I}_1 w)}{f_1(\bar{I}_1)}=\frac{f_1(I_{\tau_1})}{f_1(\bar{I}_1)}.$$

A short calculation gives

$$\begin{aligned}\frac{\partial V_S}{\partial t} &= \Big(\frac{1}{\bar{S}}-\frac{1}{S}\Big)d_s\Delta S - \frac{\mu}{\bar{S}S}(S-\bar{S})^2\\ &\quad + \beta_1 f_1(\bar{I}_1)\Big(1-\frac{1}{u}-uF_1(w)+F_1(w)\Big) - u\beta_2\Big(1-\frac{1}{u}\Big)f_2(I_{\tau_2}),\\ \frac{\partial V_{I_1}}{\partial t} &= \Big(\frac{1}{\bar{I}_1}-\frac{1}{I_1}\Big)d_1\Delta I_1 + \frac{1}{\bar{I}_1}\beta_1\bar{S}f_1(\bar{I}_1)\Big(uF_1(w)-v-\frac{u}{v}F_1(w)+1\Big),\end{aligned}$$

and

$$\frac{\partial \overline{V}_{I_1}}{\partial t} = g(v)-g(w) = v-w+\ln w-\ln v.$$

Thus,

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \frac{1}{\beta_1 f_1(\bar{I}_1)} \int_\Omega \Big(\Big(\frac{1}{\bar{S}_1} - \frac{1}{S} \Big) d_s \Delta S + \frac{\bar{I}_1}{\bar{S}} \Big(\frac{1}{\bar{I}_1} - \frac{1}{I_1} \Big) d_1 \Delta I_1 + \frac{1}{\bar{S}} d_2 \Delta I_2 \Big) \mathrm{d}x$$
$$+ \int_\Omega \Big(- \frac{\mu}{\beta_1 f_1(\bar{I}_1)} \frac{(S-\bar{S}))^2}{\bar{S}S} - g\Big(\frac{1}{u}\Big) - g\Big(\frac{uF_1(w)}{v}\Big)\Big) \mathrm{d}x$$
$$+ \int_\Omega (F_1(w) - w + \ln w - \ln F_1(w) + C_2(t,x)) \mathrm{d}x,$$

where

$$C_2(t,x) = -\frac{\beta_2}{\beta_1} u \Big(1 - \frac{1}{u}\Big) \frac{f_2(I_{\tau_2})}{f_1(\bar{I}_1)} + \frac{1}{\beta_1 \bar{S} f_1(\bar{I}_1)} (\beta_2 S f_2(I_{\tau_2}) - (\mu_2 + \gamma_2) I_2)$$
$$+ \frac{\beta_2}{\beta_1 f_1(\bar{I}_1)} (I_2 - I_{\tau_2}).$$

Recall that (3.6) and (3.7), we see the first integration of the above is non-positive. By the same arguments as the proof of [1, Theorem 4.2], we get

$$F_1(w) - w + \ln w - \ln F_1(w) < 0 \text{ for all } w > 0 \text{ except at } w = 1 \text{ where it vanishes,}$$

and $C_2(t,x) \leqslant 0$ for $(t,x) \in (0,+\infty) \times \Omega$. Hence, $\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant 0$ on $\mathbb{R}^+$ with equality holding only at E_1. The largest compact invariant set in $\{(S, I_1, I_2) \in \mathbb{R}^3_+ : \frac{\mathrm{d}V(t)}{\mathrm{d}t} = 0\}$ is the singleton E_1. Hence, it follows from LaSalle's invariant principle [11, Theorem 4.3.4] that E_1 is globally attractive. □

By symmetry, we can prove the following theorem parallel to Theorem 3.2 in a similar fashion.

THEOREM 3.3. *If $E_2(\widehat{S}, 0, \widehat{I}_2)$ exists (i.e. $\mathscr{R}_2 > 1$), but $E_1(\bar{S}, \bar{I}_1, 0)$ does not exist (i.e. $\mathscr{R}_1 \leqslant 1$), then $E_2(\widehat{S}, 0, \widehat{I}_2)$ of system (3.1)-(3.3) is globally attractive.*

Proof. Define the following Lyapunov functional

$$V(t) = \frac{1}{\beta_2 f_2(\widehat{I}_2)} \int_\Omega \Big(V_S + \frac{\widehat{I}_2}{\widehat{S}} V_{I_2} + \beta_2 f_2(\widehat{I}_2) \widehat{V}_{I_2} + \frac{1}{\widehat{S}} I_1 + \beta_1 \int_{t-\tau_1}^t I_1(\theta,x) \mathrm{d}\theta \Big) \mathrm{d}x,$$

where

$$V_S = g\Big(\frac{S}{\widehat{S}}\Big), \; V_{I_2} = g\Big(\frac{I_2}{\widehat{I}_2}\Big), \; \widehat{V}_{I_2} = \int_{t-\tau_2}^t g\Big(\frac{1}{\widehat{I}_2} I_2(s,x)\Big) \mathrm{d}s.$$

The rest of the proof is similar to that of Theorem 3.2, so we omit it here. □

In the following, we study the global stability of $E^*(S^*, I_1^*, I_2^*)$.

THEOREM 3.4. *If $E^*(S^*, I_1^*, I_2^*)$ exists (i.e., $\overline{\mathscr{R}}_0 > 1$), then $E^*(S^*, I_1^*, I_2^*)$ of system (3.1)-(3.3) is globally attractive.*

Proof. We construct the following Lyapunov functional

$$W(t)=\frac{1}{\beta_1 f_1(I_1^*)}\int_\Omega\Big(W_S+\frac{I_1^*}{S^*}W_{I_1}+\frac{I_2^*}{S^*}W_{I_2}+\beta_1 f_1(I_1^*)W_{I_1}^*+\beta_2 f_2(I_2^*)W_{I_2}^*\Big)\mathrm{d}x,$$

where

$$W_S=g\Big(\frac{S}{S^*}\Big),\ W_{I_i}=g\Big(\frac{I_i}{I_i^*}\Big),\ W_{I_i}^*=\int_{t-\tau_i}^t g\Big(\frac{1}{I_i^*}I_i(s,x)\Big)\mathrm{d}s,\ i=1,2.$$

Obviously, $W(t)$ is non-negative in $\mathbb{R}^+$ and attains zero at E^*. Following the approach in Theorem 3.2 with the following modification on function F_1 as

$$u=\frac{S}{S^*},\ v_i=\frac{I_i}{I_i^*},\ w_i=\frac{I_{\tau_i}}{I_i^*},\ F_i(w_i)=\frac{f_i(I_i^*w_i)}{f_i(I_i^*)}=\frac{f_i(I_{\tau_i})}{f_i(I_i^*)},\ i=1,2,$$

we can find the derivative of $W(t)$ along the solution of (3.1), and obtain

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t}=&-\frac{1}{\beta_1 f_1(I_1^*)}\Big(\int_\Omega\Big(\Big(\frac{1}{S^*}-\frac{1}{S}\Big)d_s\Delta S+\frac{I_1^*}{S^*}\Big(\frac{1}{I_1^*}-\frac{1}{I_1}\Big)d_1\Delta I_1\\&+\frac{I_2^*}{S^*}\Big(\frac{1}{I_2^*}-\frac{1}{I_2}\Big)d_2\Delta I_2\Big)\mathrm{d}x-\frac{\mu}{S^*}\int_\Omega\frac{1}{S}(S-S^*)^2\mathrm{d}x\Big)\\&+\int_\Omega\Big(2-\frac{1}{u}+F_1(w_1)-\frac{u}{v_1}F_1(w_1)-w_1+\ln w_1-\ln v_1\Big)\mathrm{d}x\\&+\frac{\beta_2 f_2(I_2^*)}{\beta_1 f_1(I_1^*)}\int_\Omega\Big(2-\frac{1}{u}+F_2(w_2)-\frac{u}{v_2}F_2(w_2)-w_2+\ln w_2-\ln v_2\Big)\mathrm{d}x.\end{aligned}$$

Recall that (3.6) and (3.7), we get the first term of the above is non-negative, and

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t}\leqslant&\int_\Omega\Big(2-\frac{1}{u}+F_1(w_1)-\frac{u}{v_1}F_1(w_1)-w_1+\ln w_1-\ln v_1\Big)\mathrm{d}x\\&+\frac{\beta_2 f_2(I_2^*)}{\beta_1 f_1(I_1^*)}\int_\Omega\Big(2-\frac{1}{u}+F_2(w_2)-\frac{u}{v_2}F_2(w_2)-w_2+\ln w_2-\ln v_2\Big)\mathrm{d}x\\=&\int_\Omega\Big(-g\Big(\frac{1}{u}\Big)-g\Big(\frac{u}{v_1}F_1(w_1)\Big)+(F_1(w_1)-w_1+\ln w_1-\ln F_1(w_1))\Big)\mathrm{d}x\\&+\frac{\beta_2 f_2(I_2^*)}{\beta_1 f_1(I_1^*)}\int_\Omega\Big(-g\Big(\frac{1}{u}\Big)-g\Big(\frac{u}{v_2}F_2(w_2)\Big)\\&+(F_2(w_2)-w_2+\ln w_2-\ln F_2(w_2))\Big)\mathrm{d}x.\end{aligned}$$

Similarly, we can see that $\frac{\mathrm{d}W(t)}{\mathrm{d}t}\leqslant 0$ in $\mathbb{R}^+$ with the equality holding only at E^*. The largest compact invariant set in $\{(S^*,I_1^*,I_2^*)\in\mathbb{R}_+^3:\frac{\mathrm{d}W(t)}{\mathrm{d}t}=0\}$ is the singleton E^*. Hence, it follows from LaSalle's invariant principle [11, Theorem 4.3.4] that the co-endemic equilibrium E^* is globally attractive. □

In what follows, we study the fourth equation of system (2.1). The equation reads as

$$\frac{\partial R(t,x)}{\partial t} = d_R \Delta R(t,x) + \gamma_1 I_1(t,x) + \gamma_2 I_2(t,x) - \mu R(t,x). \tag{3.8}$$

Then we have the following result.

LEMMA 3.1. *(i) If* $\mathscr{R}_0 \leqslant 1$*, then* $\lim_{t\to\infty} R(t,x) = 0$*;*
(ii) If $\mathscr{R}_1 > 1$ *but* $\mathscr{R}_2 \leqslant 1$*, then* $\lim_{t\to\infty} R(t,x) = \bar{R}$*;*
(iii) If $\mathscr{R}_2 > 1$ *but* $\mathscr{R}_1 \leqslant 1$*, then* $\lim_{t\to\infty} R(t,x) = \widehat{R}$*;*
(iv) If $\overline{\mathscr{R}_0} > 1$*, then* $\lim_{t\to\infty} R(t,x) = R^*$*.*

Proof. Using the Green's function Γ associated with $d_R\Delta$ and the Neumann boundary condition and integrating (3.8) yields

$$\begin{aligned} R(t,x) = {} & e^{-\mu t}\int_\Omega \Gamma(d_R t,x,y)\varphi_4(0,y)\mathrm{d}y \\ & + \int_0^t e^{-\mu s}\int_\Omega \Gamma(d_R s,x,y)(\gamma_1 I_1(t-s,y) + \gamma_2 I_2(t-s,y))\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.9}$$

Due to $E_0(1,0,0)$ of system (3.1) is globally attractive by Theorem 3.1, we then have

$$\lim_{t\to\infty} S(t,x) = 1, \quad \lim_{t\to\infty} I_1(t,x) = 0, \quad \lim_{t\to\infty} I_2(t,x) = 0.$$

Hence, noting that $\int_\Omega \Gamma(d_R s,x,y)\mathrm{d}y = 1$ and using the Lebesgue dominated convergence theorem to (3.9), we get $\lim_{t\to\infty} R(t,x) = 0$; i.e., the conclusion (i) is true. By the same way, we can proof the remaining conclusions, and omit it here. □

Now, summing up Theorems 3.1-4 and Lemma 3.1, we can obtain the following conclusion.

THEOREM 3.5. *(i) If* $\mathscr{R}_0 \leqslant 1$*, then* $E_0(1,0,0,0)$ *of system (2.1)-(2.3) is globally attractive;*

(ii) If $\mathscr{R}_1 > 1$ *but* $\mathscr{R}_2 \leqslant 1$*, then* $E_1(\bar{S},\bar{I}_1,0,\bar{R})$ *of system (2.1)-(2.3) is globally attractive;*

(iii) If $\mathscr{R}_2 > 1$ *but* $\mathscr{R}_1 \leqslant 1$*, then* $E_2(\widehat{S},0,\widehat{I}_2,\widehat{R})$ *of system (2.1)-(2.3) is globally attractive;*

(iv) If $\overline{\mathscr{R}_0} > 1$*, then* $E^*(S^*,I_1^*,I_2^*,R^*)$ *of system (2.1)-(2.3) is globally attractive.*

4. Existence of traveling wave solutions

According to the stable analysis of system (3.1)-(3.3) at the equilibrium E_0, E_1, E_2, and E^*, let $\Omega = \mathbb{R}^n$, we will first establish the existence of traveling waves to system (3.1) connecting the equilibria E_0 and E^*. More precisely, we look for a special translation invariant solution of the form $(S(x\cdot e+ct), I_1(x\cdot e+ct), I_2(x\cdot e+ct))$ of (3.1), where $c>0$ is the wave speed and e is a unit vector in $\mathbb{R}^n$. Without loss generality, we consider $n=1$. Letting $x+ct$ by t, then $S(t),I_1(t),I_2(t)$ satisfy the following system

$$\begin{cases} cS'(t)=d_sS''(t)+\mu(1-S(t))-\beta_1S(t)f_1(I_1(t-c\tau_1)) \\ \qquad\qquad -\beta_2S(t)f_2(I_2(t-c\tau_2)), \\ cI_1'(t)=d_1I_1''(t)+\beta_1S(t)f_1(I_1(t-c\tau_1))-(\mu_1+\gamma_1)I_1(t), \\ cI_2'(t)=d_2I_2''(t)+\beta_2S(t)f_2(I_2(t-c\tau_2))-(\mu_2+\gamma_2)I_2(t), \end{cases} \tag{4.1}$$

and the boundary condition

$$S(-\infty)=1,\ S(+\infty)=S^*,\ I_i(-\infty)=0,\ I_i(+\infty)=I_i^*,\ i=1,2. \tag{4.2}$$

The characteristic equations associated with the linearized equations of the second and third equation of (4.1) at $E_0(1,0,0)$ are

$$\Delta_i(\lambda,c)=d_i\lambda^2-c\lambda+\beta_ie^{-\lambda c\tau_i}-(\mu_i+\gamma_i)=0,\ i=1,2.$$

It is easy to show the following lemma [22, lemma 3.1], see also, [12, lemma 2.1] or [18, lemma 4.4].

LEMMA 4.1. *Assume that $\mathscr{R}_i>1$. Then there exist $c_i^*>0$ and $\lambda_i^*>0$ such that*

$$\frac{\partial\Delta_i(\lambda,c)}{\partial\lambda}\Big|_{(\lambda_i^*,c_i^*)}=0,\ \Delta_i(\lambda_i^*,c_i^*)=0,\ i=1,2.$$

Furthermore, $\Delta_i(\lambda,c)$ also satisfies

(i) if $c<c_i^*$, then $\Delta_i(\lambda,c)>0$ for any $\lambda\geqslant 0$.

(ii) if $c>c_i^*$, then $\Delta_i(\lambda,c)=0$ has two different positive solutions $\lambda_{i1}(c)<\lambda_{i2}(c)$ with

$$\Delta_i(\lambda,c)\begin{cases} >0, & \lambda\in(0,\lambda_{i1}(c))\cup(\lambda_{i2}(c),+\infty), \\ <0, & \lambda\in(\lambda_{i1}(c),\lambda_{i2}(c)). \end{cases}$$

4.1. Construction of the upper-lower solutions

In this subsection, we assume that $\mathscr{R}_0>1$ and $\overline{\mathscr{R}}_0>1$. In addition, we fix $c>c^*:=\max\{c_1^*,c_2^*\}$ and always denote $\lambda_{i1}(c)$ by λ_{i1}, $i=1,2$. Now, by the ideas [2, 12, 19, 20, 22, 27], we define six continuous functions as follows, for $t\in\mathbb{R}$,

$$\overline{S}(t)=1,\quad \underline{S}(t)=\max\left\{1-\frac{1}{\sigma}e^{\sigma t},\frac{\mu\alpha_1\alpha_2}{\mu\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1}\right\},$$

$$\overline{I}_i(t)=\min\left\{e^{\lambda_{i1}t},\frac{1}{\alpha_i}(\mathscr{R}_i-1)\right\},\quad \underline{I}_i(t)=\max\left\{e^{\lambda_{i1}t}(1-M_ie^{\varepsilon_it}),0\right\},\ i=1,2,$$

where σ,M_i,ε_i $(i=1,2)$ are positive constants to determine in the following lemmas.

LEMMA 4.2. *The functions $\overline{S}(t)$ and $\overline{I}_1(t)$ satisfy the inequality*

$$d_1\overline{I}_1''(t)-c\overline{I}_1'(t)+\beta_1\overline{S}(t)f_1(\overline{I}_1(t-c\tau_1))-(\mu_1+\gamma_1)\overline{I}_1(t)\leqslant 0, \tag{4.3}$$

for all $t\neq t_1:=\frac{1}{\lambda_{11}}\ln\left(\frac{1}{\alpha_1}(\mathscr{R}_1-1)\right)$.

Proof. When $t < t_1$, $\overline{I}_1(t) = e^{\lambda_{11} t}$. Note that $\overline{S}(t) = 1$ and $\overline{I}_1(t) \leqslant e^{\lambda_{11} t}$ for all $t \in \mathbb{R}$. Then

$$\begin{aligned}
& d_1 \overline{I}_1''(t) - c\overline{I}_1'(t) + \beta_1 \overline{S}(t) f_1(\overline{I}_1(t - c\tau_1)) - (\mu_1 + \gamma_1)\overline{I}_1(t) \\
& \quad \leqslant d_1 \overline{I}_1''(t) - c\overline{I}_1'(t) + \beta_1 \overline{S}(t)\overline{I}_1(t - c\tau_1) - (\mu_1 + \gamma_1)\overline{I}_1(t) \\
& \quad \leqslant e^{\lambda_{11} t}\big(d_1 \lambda_{11}^2 - c\lambda_{11} + \beta_1 e^{-\lambda_{11} c\tau_1} - (\mu_1 + \gamma_1)\big) \\
& \quad = 0.
\end{aligned}$$

When $t \geqslant t_1$, $\overline{I}_1(t) = \frac{1}{\alpha_1}(\mathscr{R}_1 - 1)$. It follows from the fact $\overline{I}_1(t) \leqslant \frac{1}{\alpha_1}(\mathscr{R}_1 - 1)$ for all $t \in \mathbb{R}$ that

$$\begin{aligned}
& d_1 \overline{I}_1''(t) - c\overline{I}_1'(t) + \beta_1 \overline{S}(t) f_1(\overline{I}_1(t - c\tau_1)) - (\mu_1 + \gamma_1)\overline{I}_1(t) \\
& \quad \leqslant d_1 \overline{I}_1''(t) - c\overline{I}_1'(t) + \beta_1 f_1(\overline{I}_1(t - c\tau_1)) - (\mu_1 + \gamma_1)\overline{I}_1(t) \\
& \quad = 0.
\end{aligned}$$

This completes the proof. $\square$

Similarly, we can get the following lemma.

LEMMA 4.3. *The functions $\overline{S}(t)$ and $\overline{I}_2(t)$ satisfy the inequality*

$$d_2 \overline{I}_2''(t) - c\overline{I}_2'(t) + \beta_2 \overline{S}(t) f_2(\overline{I}_2(t - c\tau_2)) - (\mu_2 + \gamma_2)\overline{I}_2(t) \leqslant 0, \tag{4.4}$$

for all $t \neq t_2 := \frac{1}{\lambda_{21}} \ln\big(\frac{1}{\alpha_2}(\mathscr{R}_2 - 1)\big)$.

LEMMA 4.4. *The functions $\overline{S}(t)$, $\underline{I}_1(t)$ and $\underline{I}_2(t)$ satisfy the following inequality*

$$d_s \overline{S}''(t) - c\overline{S}'(t) + \mu(1 - \overline{S}(t)) - \beta_1 \overline{S}(t) f_1(\underline{I}_1(t - c\tau_1)) - \beta_2 \overline{S}(t) f_2(\underline{I}_2(t - c\tau_2)) \leqslant 0. \tag{4.5}$$

The proof is trivial and we omit it.

LEMMA 4.5. *Let $\sigma \in (0, \min\{\lambda_{11}, \lambda_{21}\})$ be sufficiently small. Then the functions $\underline{S}(t)$, $\overline{I}_1(t)$ and $\overline{I}_2(t)$ satisfy the inequality*

$$d_s \underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \beta_1 \underline{S}(t) f_1(\overline{I}_1(t - c\tau_1)) - \beta_2 \underline{S}(t) f_2(\overline{I}_2(t - c\tau_2)) \geqslant 0, \tag{4.6}$$

for all $t \neq t_3 := \frac{1}{\sigma} \ln \frac{\sigma(\alpha_1\beta_2 + \alpha_2\beta_1)}{\mu\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1} < 0$.

Proof. If $t \geqslant t_3$, then $\underline{S}(t) = \frac{\mu\alpha_1\alpha_2}{\mu\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1}$. Hence,

$$\begin{aligned}
& d_s \underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \beta_1 \underline{S}(t) f_1(\overline{I}_1(t - c\tau_1)) - \beta_2 \underline{S}(t) f_2(\overline{I}_2(t - c\tau_2)) \\
& \quad \geqslant d_s \underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \frac{\beta_1}{\alpha_1}\underline{S}(t) - \frac{\beta_2}{\alpha_2}\underline{S}(t)
\end{aligned}$$

$$
\begin{aligned}
&= \mu(1-\underline{S}(t)) - \frac{\beta_1}{\alpha_1}\underline{S}(t) - \frac{\beta_2}{\alpha_2}\underline{S}(t) \\
&= 0.
\end{aligned}
$$

If $t < t_3$, then $\underline{S}(t) = 1 - \frac{1}{\sigma}e^{\sigma t}$. Note that the function $f_i(x) \leqslant x\ (i=1,2)$ for all $x \geqslant 0$ and $\overline{I}_i(t) \leqslant e^{\lambda_{i1}t}$, $i=1,2$, for all $t \in \mathbb{R}$. One gets

$$
\begin{aligned}
& d_s\underline{S}''(t) - c\underline{S}'(t) + \mu(1-\underline{S}(t)) - \beta_1\underline{S}(t)f_1(\overline{I}_1(t-c\tau_1)) - \beta_2\underline{S}(t)f_2(\overline{I}_2(t-c\tau_2)) \\
&\quad \geqslant d_s\underline{S}''(t) - c\underline{S}'(t) + \mu(1-\underline{S}(t)) - \beta_1\underline{S}(t)\overline{I}_1(t-c\tau_1) - \beta_2\underline{S}(t)\overline{I}_2(t-c\tau_2) \\
&\quad \geqslant \Big(-d_s\sigma + c + \frac{\mu}{\sigma}\Big)e^{\sigma t} - \beta_1 e^{\lambda_{11}(t-c\tau_1)} - \beta_2 e^{\lambda_{21}(t-c\tau_2)} \\
&\quad \geqslant \Big(-d_s\sigma + c + \frac{\mu}{\sigma} - \beta_1 e^{-\lambda_{11}c\tau_1} - \beta_2 e^{-\lambda_{21}c\tau_2}\Big)e^{\sigma t}.
\end{aligned}
$$

Then, for sufficiently small $\sigma > 0$,

$$
-d_s\sigma + c + \frac{\mu}{\sigma} - \beta_1 e^{-\lambda_{11}c\tau_1} - \beta_2 e^{-\lambda_{21}c\tau_2} > 0,
$$

which gives that (4.6) holds for $t < t_3$, which completes the proof. □

LEMMA 4.6. *Let $0 < \varepsilon_1 < \min\{\sigma, \lambda_{11}, \lambda_{12} - \lambda_{11}\}$. Then, for $M_1 > 1$ sufficiently large, the functions $\underline{S}(t)$ and $\underline{I}_1(t)$ satisfy the inequality*

$$
d_1\underline{I}_1''(t) - c\underline{I}_1'(t) + \beta_1\underline{S}(t)f_1(\underline{I}_1(t-c\tau_1)) - (\mu_1+\gamma_1)\underline{I}_1(t) \geqslant 0, \tag{4.7}
$$

for all $t \neq t_4 := \frac{1}{\varepsilon_1}\ln\frac{1}{M_1}$.

Proof. For $t \geqslant t_4$, then the inequality (4.7) holds immediately since $\underline{I}(t) = 0$ on $[t_4, \infty)$. For $t < t_4$, then $\underline{I}_1(t) = e^{\lambda_{11}t}\left(1 - M_1 e^{\varepsilon_1 t}\right)$. In view of the fact $f_1(x) \geqslant x(1-\alpha_1 x)$ for all $x \geqslant 0$, and

$$
e^{\lambda_{11}t}\left(1 - M_1 e^{\varepsilon_1 t}\right) \leqslant \underline{I}_1(t) \leqslant e^{\lambda_{11}t}, \quad 1 - \frac{1}{\sigma}e^{\sigma t} \leqslant \underline{S}(t) \leqslant 1,\ \forall t \in \mathbb{R},
$$

we get

$$
\begin{aligned}
& d_1\underline{I}_1''(t) - c\underline{I}_1'(t) + \beta_1\underline{S}(t)f_1(\underline{I}_1(t-c\tau_1)) - (\mu_1+\gamma_1)\underline{I}_1(t) \\
&\quad \geqslant d_1\underline{I}_1''(t) - c\underline{I}_1'(t) + \beta_1\underline{S}(t)\underline{I}_1(t-c\tau_1)(1-\alpha_1\underline{I}_1(t-c\tau_1)) - (\mu_1+\gamma_1)\underline{I}_1(t) \\
&\quad \geqslant -M_1\Delta_1(\lambda_{11}+\varepsilon_1, c)e^{(\lambda_{11}+\varepsilon_1)t} - \beta_1\alpha_1 e^{2\lambda_{11}t} - \frac{\beta_1}{\sigma}e^{(\lambda_{11}+\sigma)t} \\
&\quad = e^{(\lambda_{11}+\varepsilon_1)t}\Big(-M_1\Delta_1(\lambda_{11}+\varepsilon_1, c) - \beta_1\alpha_1 e^{(\lambda_{11}-\varepsilon_1)t} - \frac{\beta_1}{\sigma}e^{(\sigma-\varepsilon_1)t}\Big).
\end{aligned}
$$

Note that $e^{(\sigma-\varepsilon_1)t} < 1$ and $e^{(\lambda_{11}-\varepsilon_1)t} < 1$ since $\sigma - \varepsilon_1 > 0$, $\lambda_{11} - \varepsilon_1 > 0$, and $t < t_4 < 0$. Therefore

$$
-M_1\Delta_1(\lambda_{11}+\varepsilon_1, c) - \beta_1\alpha_1 e^{(\lambda_{11}-\varepsilon_1)t} - \frac{\beta_1}{\sigma}e^{(\sigma-\varepsilon_1)t}
$$

$$> -M_1\Delta_1(\lambda_{11}+\varepsilon_1,c)-\beta_1\alpha_1-\frac{\beta_1}{\sigma}.$$

Consequently, we need only to choose

$$M_1 \geqslant \max\left\{1, -\frac{\sigma\beta_1\alpha_1+\beta_1}{\sigma\Delta_1(\lambda_{11}+\varepsilon_1,c)}\right\},$$

then (4.7) holds. □

Similar to the proof of Lemma 4.6, we can get

LEMMA 4.7. *Let* $0<\varepsilon_2<\min\{\sigma,\lambda_{21},\lambda_{22}-\lambda_{21}\}$. *Then, for* $M_2>1$ *sufficiently large, the functions* $\underline{S}(t)$ *and* $\underline{I}_2(t)$ *satisfy the inequality*

$$d_2\underline{I}_2''(t)-c\underline{I}_2'(t)+\beta_2\underline{S}(t)f_2(\underline{I}_2(t-c\tau_2))-(\mu_2+\gamma_2)\underline{I}_2(t)\geqslant 0, \tag{4.8}$$

for all $t\neq t_5:=\frac{1}{\varepsilon_2}\ln\frac{1}{M_2}$.

4.2. The verification of Schauder fixed point theorem

In this subsection, we shall use the usual Banach space $\mathbb{B}:=C(\mathbb{R},\mathbb{R}^3)$ of bounded continuous functions endowed with the maximum norm

$$\|\Phi\|=\sup_{t\in\mathbb{R}}(|\phi_1(t)|+|\phi_2(t)|+|\phi_3(t)|) \text{ for } \Phi=(\phi_1,\phi_2,\phi_3)\in\mathbb{B},$$

see [14, 26]. Next, for any $c>c^*$, we construct the profile set as follows

$$\Gamma_c=\left\{(S,I_1,I_2)\in\mathbb{B}:(\underline{S},\underline{I}_1,\underline{I}_2)(t)\leqslant(S,I_1,I_2)(t)\leqslant(\overline{S},\overline{I}_1,\overline{I}_2)(t)\right\}.$$

Clearly, Γ_c is a bounded nonempty closed convex subset of $\mathbb{B}$.

Define $\rho_{11}<0<\rho_{12}$ satisfying $d_s\rho^2-c\rho-\gamma=0$, $\rho_{21}<0<\rho_{22}$ satisfying $d_1\rho^2-c\rho-\gamma=0$ and $\rho_{31}<0<\rho_{32}$ satisfying $d_2\rho^2-c\rho-\gamma=0$, where $\gamma> max\{\mu_1+\gamma_1,\mu_2+\gamma_2\}$ is a constant such that

$$\gamma S(t)+\mu(1-S(t))-\beta_1S(t)f_1(I_1(t-c\tau_1))-\beta_2S(t)f_2(I_2(t-c\tau_2))$$

is monotone increasing in $S>0$. For $(S,I_1,I_2)\in\Gamma_c$, we denote

$$\begin{aligned}
H_1(S,I_1,I_2)(t)&=\gamma S(t)+\mu(1-S(t))-\beta_1S(t)f_1(I_1(t-c\tau_1))\\
&\quad-\beta_2S(t)f_2(I_2(t-c\tau_2)),\\
H_2(S,I_1,I_2)(t)&=\gamma I_1(t)+\beta_1S(t)f_1(I_1(t-c\tau_1))-(\mu_1+\gamma_1)I_1(t),\\
H_3(S,I_1,I_2)(t)&=\gamma I_2(t)+\beta_2S(t)f_2(I_2(t-c\tau_2))-(\mu_2+\gamma_2)I_2(t).
\end{aligned}$$

By the definition of γ, we see that H_1 is monotone increasing in S and monotone decreasing in I_1 and I_2; H_2 is monotone increasing in both S and I_1; and H_3 is monotone increasing in both S and I_2.

Then we define an operator $F=(F_1,F_2,F_3):\Gamma_c\to\mathbb{B}$ as follows

$$\begin{aligned}
F_1(S,I_1,I_2)(t)&=\frac{1}{\Lambda_1}\Big(\int_{-\infty}^{t}e^{\rho_{11}(t-s)}+\int_{t}^{+\infty}e^{\rho_{12}(t-s)}\Big)H_1(S,I_1,I_2)(s)\mathrm{d}s,\\
F_2(S,I_1,I_2)(t)&=\frac{1}{\Lambda_2}\Big(\int_{-\infty}^{t}e^{\rho_{21}(t-s)}+\int_{t}^{+\infty}e^{\rho_{22}(t-s)}\Big)H_2(S,I_1,I_2)(s)\mathrm{d}s,\\
F_3(S,I_1,I_2)(t)&=\frac{1}{\Lambda_3}\Big(\int_{-\infty}^{t}e^{\rho_{31}(t-s)}+\int_{t}^{+\infty}e^{\rho_{32}(t-s)}\Big)H_3(S,I_1,I_2)(s)\mathrm{d}s,
\end{aligned}$$

where $\Lambda_1:=d_s(\rho_{12}-\rho_{11}),\ \Lambda_2:=d_1(\rho_{22}-\rho_{21}),\ \Lambda_3:=d_2(\rho_{32}-\rho_{31})$.

LEMMA 4.8. *The map $F:\Gamma_c\to\Gamma_c$.*

Proof. For $(S,I_1,I_2)\in\Gamma_c$, we only need to prove that for all $t\in\mathbb{R}$,

$$\begin{aligned}
&\underline{S}(t)\leqslant F_1(S,I_1,I_2)(t)\leqslant 1,\quad \underline{I}_1(t)\leqslant F_2(S,I_1,I_2)(t)\leqslant \overline{I}_1(t),\\
&\underline{I}_2(t)\leqslant F_3(S,I_1,I_2)(t)\leqslant \overline{I}_2(t).
\end{aligned}$$

Here, we only prove the first inequality of the above holds since the proofs of the others are similar to that of the first one of the above. Indeed, according to the monotonicity of H_1 with respect to S, I_1 and I_2, we have

$$F_1(\underline{S},\overline{I}_1,\overline{I}_2)(t)\leqslant F_1(S,I_1,I_2)(t)\leqslant F_1(\overline{S},\underline{I}_1,\underline{I}_2)(t),\ t\in\mathbb{R}.$$

Thus it is sufficient to verify

$$\underline{S}(t)\leqslant F_1(\underline{S},\overline{I}_1,\overline{I}_2)(t)\leqslant F_1(\overline{S},\underline{I}_1,\underline{I}_2)(t)\leqslant 1. \tag{4.9}$$

For $t\neq t_3$, by (4.6), we have

$$\begin{aligned}
F_1(\underline{S},\overline{I}_1,\overline{I}_2)(t)&=\frac{1}{\Lambda_1}\Big(\int_{-\infty}^{t}e^{\rho_{11}(t-s)}+\int_{t}^{+\infty}e^{\rho_{12}(t-s)}\Big)H_1(\underline{S},\overline{I}_1,\overline{I}_2)(s)\mathrm{d}s,\\
&\geqslant\frac{1}{\Lambda_1}\Big(\int_{-\infty}^{t}e^{\rho_{11}(t-s)}+\int_{t}^{+\infty}e^{\rho_{12}(t-s)}\Big)(\gamma\underline{S}(s)+c\underline{S}'(s)-d_s\underline{S}''(s))\mathrm{d}s.
\end{aligned}$$

When $t>t_3$, since $\underline{S}'(t_3-)\leqslant 0$ and $\rho_{12}>0>\rho_{11}$, it follows that

$$\begin{aligned}
F_1(\underline{S},\overline{I}_1,\overline{I}_2)(t)&\geqslant\frac{1}{\Lambda_1}\Big(\int_{-\infty}^{t_3}+\int_{t_3}^{t}\Big)e^{\rho_{11}(t-s)}(\gamma\underline{S}(s)+c\underline{S}'(s)-d_s\underline{S}''(s))\mathrm{d}s\\
&\quad+\frac{1}{\Lambda_1}\int_{t}^{+\infty}e^{\rho_{12}(t-s)}(\gamma\underline{S}(s)+c\underline{S}'(s)-d_s\underline{S}''(s))\mathrm{d}s\\
&=\frac{1}{\Lambda_1}(\frac{\gamma}{\rho_{12}}-\frac{\gamma}{\rho_{11}})\underline{S}(t)-\frac{d_s}{\Lambda_1}e^{\rho_{11}(t-t_3)}\underline{S}'(t_3-)\\
&=\underline{S}(t)-\frac{d_s}{\Lambda_1}e^{\rho_{11}(t-t_3)}\underline{S}'(t_3-)\\
&\geqslant\underline{S}(t).
\end{aligned}$$

Similarly, when $t < t_3$, we also have $F_1(\underline{S},\overline{I}_1,\overline{I}_2)(t) \geqslant \underline{S}(t)$. By the continuity of both $\underline{S}(t)$ and $F_1(\underline{S},\overline{I}_1,\overline{I}_2)(t)$, we obtain $F_1(\underline{S},\overline{I}_1,\overline{I}_2)(t) \geqslant \underline{S}(t)$ for all $t \in \mathbb{R}$.

On the other hand, for any $t \in \mathbb{R}$, it follows from (4.5) that

$$\begin{aligned}
F_1(\overline{S},\underline{I}_1,\underline{I}_2)(t) &= \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{t} e^{\rho_{11}(t-s)} + \int_{t}^{+\infty} e^{\rho_{12}(t-s)}\Big)H_1(\overline{S},\underline{I}_1,\underline{I}_2)(s)\mathrm{d}s, \\
&\leqslant \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{t} e^{\rho_{11}(t-s)} + \int_{t}^{+\infty} e^{\rho_{12}(t-s)}\Big)\big(\gamma\overline{S}(s) + c\overline{S}'(s) - d_s\overline{S}''(s)\big)\mathrm{d}s \\
&= \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{t} e^{\rho_{11}(t-s)} + \int_{t}^{+\infty} e^{\rho_{12}(t-s)}\Big)(\gamma\overline{S}(s))\mathrm{d}s \\
&= 1.
\end{aligned}$$

This completes the proof of (4.9). $\square$

LEMMA 4.9. *The map $F : \Gamma_c \to \Gamma_c$ is complete continuous with respect to $||\cdot||$ in $\mathbb{B}$.*

Proof. We first show that the operator F_1 is continuous. Note that, for any $\Phi_1 = (S_1,I_{11},I_{21}), \Phi_2 = (S_2,I_{12},I_{22}) \in \mathbb{B}$, it is easy to see that there exists $L > 0$ such that, for all $t \in \mathbb{R}$,

$$\begin{aligned}
|H_1(S_1,I_{11},I_{21})(t) - H_1(S_2,I_{12},I_{22})(t)| &\leqslant L(|I_{11}(t-c\tau_1) - I_{12}(t-c\tau_1)| \\
&\quad + |I_{21}(t-c\tau_2) - I_{22}(t-c\tau_2)| + |S_1(t) - S_2(t)|) \\
&\leqslant L\|\Phi_1 - \Phi_2\|.
\end{aligned}$$

Then

$$\begin{aligned}
&|F_1(S_1,I_{11},I_{21})(t) - F_1(S_2,I_{12},I_{22})(t)| \\
&\quad\leqslant \frac{L}{\Lambda_1}\Big(\int_{-\infty}^{t} e^{\rho_{11}(t-s)} + \int_{t}^{+\infty} e^{\rho_{12}(t-s)}\Big)\|\Phi_1 - \Phi_2\| \\
&\quad\leqslant \frac{L}{\gamma}\|\Phi_1 - \Phi_2\|.
\end{aligned}$$

Then, $F_1 : \Gamma_c \to \Gamma_c$ is continuous with respect to the norm $\|\cdot\|$.

Next, we prove that F_1 is compact. Let

$$M_1 = \sup_{(S,I_1,I_2)\in\Gamma_c, \forall t\in\mathbb{R}} |H_1(S(t),I_1(t),I_2(t))|.$$

For any given $\Delta t > 0$ and $(S,I_1,I_2) \in \Gamma_c$, keeping in mind that $\rho_{11} < 0 < \rho_{12}$, it follows from the definition of F_1 that

$$\begin{aligned}
&|F_1(S,I_1,I_2)(t+\Delta t) - F_1(S,I_1,I_2)(t)| \\
&\quad= \frac{1}{\Lambda_1}\Big|\Big(\int_{-\infty}^{t+\Delta t} e^{\rho_{11}(t+\Delta t-s)} - \int_{-\infty}^{t} e^{\rho_{11}(t-s)}\Big)H_1(S,I_1,I_2)(s)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\int_{t+\Delta t}^{\infty} e^{\rho_{12}(t+\Delta t-s)}-\int_{t}^{\infty} e^{\rho_{12}(t-s)}\Big)H_1(S,I_1,I_2)(s)\mathrm{d}s\Big| \\
\leqslant\ & \frac{1}{\Lambda_1}\Big(|e^{\rho_{11}\Delta t}-1|\int_{-\infty}^{t} e^{\rho_{11}(t-s)}|H_1(S,I_1,I_2)(s)|\mathrm{d}s \\
&+\int_{t}^{t+\Delta t} e^{\rho_{11}(t+\Delta t-s)}|H_1(S,I_1,I_2)(s)|\mathrm{d}s \\
&+(e^{\rho_{12}\Delta t}-1)\int_{t}^{\infty} e^{\rho_{12}(t-s)}|H_1(S,I_1,I_2)(s)|\mathrm{d}s \\
&+\int_{t}^{t+\Delta t} e^{\rho_{12}(t-s)}|H_1(S,I_1,I_2)(s)|\mathrm{d}s\Big) \\
\leqslant\ & \frac{M_1}{\Lambda_1}\Big(e^{\rho_{11}t}(1-e^{\rho_{11}\Delta t})\int_{-\infty}^{t} e^{-\rho_{11}s}\mathrm{d}s+e^{\rho_{11}(t+\Delta t)}\int_{t}^{t+\Delta t} e^{-\rho_{11}s}\mathrm{d}s \\
&+e^{\rho_{12}t}(e^{\rho_{12}\Delta t}-1)\int_{t+\Delta t}^{\infty} e^{-\rho_{12}s}\mathrm{d}s+e^{\rho_{12}t}\int_{t}^{t+\Delta t} e^{-\rho_{12}s}\mathrm{d}s\Big) \\
=\ & \frac{M_1}{\Lambda_1}\Big(\frac{2}{\rho_{11}}(e^{\rho_{11}\Delta t}-1)+\frac{2}{\rho_{12}}(1-e^{-\rho_{12}\Delta t})\Big) \\
\leqslant\ & \frac{4M_1}{\Lambda_1}\Delta t.
\end{aligned}
$$

Here, we use the inequality $e^x \geqslant 1+x$ for $x\in\mathbb{R}$. It follows that $\{F_1(S,I_1.I_2)(t) : (S,I_1.I_2)\in\Gamma_c\}$ is a family of equicontinuous functions.

Similarly, one also can show $\{F_i(S,I_1,I_2)(t) : (S,I_1,I_2)\in\Gamma_c\}$, $i=2,3$, is a family of equicontinuous functions. Thus, $\{F(S,I_1,I_2)(t) : (S,I_1,I_2)\in\Gamma_c\}$ represents a family of equicontinuous functions. Then the Arzelà-Ascoli theorem implies that F takes the bounded convex subset of Γ_c into a compact subset of Γ_c. The proof is completed. □

Now we are in a position to state and show our main results as follows.

THEOREM 4.1. *Assume that $\mathscr{R}_0>1$ and $\overline{\mathscr{R}_0}>1$. Then for every $c>c^*$, model (4.1) admits a nontrivial traveling wave solution $\big(S(x+ct),I_1(x+ct),I_2(x+ct)\big)$ satisfying the asymptotic boundary condition (4.2), and*

$$\lim_{t\to-\infty} e^{-\lambda_{11}t}I_1(t)=1,\ \lim_{t\to-\infty} e^{-\lambda_{21}t}I_2(t)=1.$$

Proof. By Lemmas 4.8 and 4.9 and using Schauder fixed point theorem, we can conclude that there exists a pair of $(S(t),I_1(t),I_2(t))\in\Gamma_c$, which is a fixed point of the operator F, consequently, (S,I_1,I_2) is a solution of (4.1). Also, in view of Lemmas 4.2-4.7, it is easy to see that $S(-\infty)=1$, $I_1(-\infty)=0$, $I_2(-\infty)=0$. Moreover, we see that $\lim_{t\to-\infty} e^{-\lambda_{11}t}I_1(t)=1$, $\lim_{t\to-\infty} e^{-\lambda_{21}t}I_2(t)=1$.

On the other hand, for any $t\in\mathbb{R}$, observing that $(S(t),I_1(t),I_2(t))\in\Gamma_c$, we then get $S(t)>0$, $I_1(t)\geqslant 0$ and $I_2(t)\geqslant 0$. Further, we claim that $I_1(t)>0$ for any $t\in\mathbb{R}$. Indeed, if there exists $t_0\in\mathbb{R}$ such that $I_1(t_0)=0$, then there exist constants $a,b\in\mathbb{R}$ such that $a<\frac{1}{\varepsilon_1}\ln\frac{1}{M_1}\leqslant b$ and $t_0\in(a,b)$. It implies $I_1(t)$ attains its minimum in (a,b)

for any $t \in [a,b]$. From the second equation of (4.1), I_1 satisfies

$$-d_1 I_1''(t) + cI_1'(t) + (\mu_1+\gamma_1)I_1(t) \geqslant 0,\ t \in [a,b].$$

By the elliptic strong maximum principle (see, [25, Lemma 2.1.2]), it follows that $I_1(t) \equiv 0$ for $t \in [a,b]$. On the other hand, by Lemma 4.6, we have $I_1(t) > 0$ for $t \in [a, \frac{1}{\varepsilon_1}\ln\frac{1}{M_1})$. This is a contradiction. Similarly, we can proof that $I_2(t) > 0$ for any $t \in \mathbb{R}$.

We next show that $\frac{S'(t)}{S(t)}$, $\frac{I_1'(t)}{I_1(t)}$ and $\frac{I_2'(t)}{I_2(t)}$ are bounded for any $t \in \mathbb{R}$ by similar arguments as those in [8, Theorem 4.2(ii)]. Indeed, system (4.1) can be rewritten as

$$-\begin{pmatrix} d_s \\ d_1 \\ d_2 \end{pmatrix}\begin{pmatrix} S \\ I_1 \\ I_2 \end{pmatrix}'' + c\begin{pmatrix} S \\ I_1 \\ I_2 \end{pmatrix}' + \begin{pmatrix} q_{11}(t) & 0 & 0 \\ q_{21}(t) & -(\mu_1+\gamma_1) & 0 \\ q_{31}(t) & 0 & -(\mu_2+\gamma_2) \end{pmatrix}\begin{pmatrix} S \\ I_1 \\ I_2 \end{pmatrix} = 0,$$

where

$$q_{11}(t) = \frac{\mu}{S(t)} - \mu - \beta_1 f_1(I_1(t-c\tau_1)) - \beta_2 f_2(I_2(t-c\tau_2))$$

and

$$q_{21}(t) = \beta_1 f_1(I_1(t-c\tau_1)), \quad q_{31}(t) = \beta_2 f_2(I_2(t-c\tau_2)).$$

Note that $(S(t), I_1(t), I_2(t)) \in \Gamma_c$, we see the functions $q_{i1}(t)$, $i = 1,2,3$, is bounded. Apply Harnack inequality (see [4, Theorem 1.1]), it follows that there exists $C > 0$ such that for any $t \in \mathbb{R}$,

$$\max_{[t-1,t+1]} S(\xi) \leqslant C \min_{[t-1,t+1]} S(\xi), \quad \max_{[t-1,t+1]} I_i(\xi) \leqslant C \min_{[t-1,t+1]} I_i(\xi),\ i = 1,2.$$

Therefore, there exists $\overline{C} > 0$ such that

$$\left|\frac{S'(t)}{S(t)}\right| + \left|\frac{I_1'(t)}{I_1(t)}\right| + \left|\frac{I_2'(t)}{I_2(t)}\right| \leqslant \overline{C}, \quad t \in \mathbb{R}. \tag{4.10}$$

Next, motivated by [8, 9, 12], we use the Laypunov method to show $S(+\infty) = S^*$, $I_1(+\infty) = I_1^*$, $I_2(+\infty) = I_2^*$. To this end, we consider the following Lyapunov functional

$$\begin{aligned} V(t) =& c(V_S(t) + V_{I_1}(t) + V_{I_2}(t)) + (\mu_1+\gamma_1)I_1^* V_1(t) + (\mu_2+\gamma_2)I_2^* V_2(t) \\ &+ d_s S'\Big(\frac{S^*}{S(t)} - 1\Big) + d_1 I_1'\Big(\frac{f_1(I_1^*)}{f_1(I_1(t))} - 1\Big) + d_2 I_2'\Big(\frac{f_2(I_2^*)}{f_2(I_2(t))} - 1\Big), \end{aligned}$$

where

$$V_S(t) = S - S^* - S^*\ln\frac{S}{S^*},\ V_{I_i}(t) = I_i(t) - I_i^* - \int_{I_i^*}^{I_i(t)} \frac{f_i(I_i^*)}{f_i(\eta)}\mathrm{d}\eta,\ i = 1,2,$$

and

$$V_i(t) = \int_{t-c\tau_i}^{t} \Big(\frac{f_i(I_i(s))}{f_i(I_i^*)} - 1 - \ln\frac{f_i(I_i(s))}{f_i(I_i^*)}\Big)\mathrm{d}s,\ i = 1,2.$$

By (4.10), we know the function V is defined and bounded from below.

A direct calculation shows that

$$
\begin{aligned}
c\frac{\mathrm{d}V_S}{\mathrm{d}t} &= \frac{S-S^*}{S}(cS') = \frac{S-S^*}{S}\big(d_sS'' + \mu(S^*-S) + \beta_1 S^* f_1(I_1^*) + \beta_2 S^* f_2(I_2^*)\\
&\quad - \beta_1 S f_1(I_1(t-c\tau_1)) - \beta_2 S f_2(I_2(t-c\tau_2))\big),\\
c\frac{\mathrm{d}V_{I_i}}{\mathrm{d}t} &= \frac{f_i(I_i)-f_i(I_i^*)}{f_i(I_i)}(cI_i')\\
&= \frac{f_i(I_i)-f_i(I_i^*)}{f_i(I_i)}\big(d_iI_i'' + \beta_i S f_i(I_i(t-c\tau_i)) - (\mu_i+\gamma_i)I_i\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(S'\Big(\frac{S^*}{S}-1\Big)\Big) &= \frac{S^*-S}{S}S'' - \frac{S^*(S')^2}{S^2},\\
\frac{\mathrm{d}}{\mathrm{d}t}\Big(I_i'\Big(\frac{f_i(I_i^*)}{f_i(I_i)}-1\Big) &= I_i''\Big(\frac{f_i(I_i^*)}{f_i(I_i)}-1\Big) - \frac{f_i(I_i^*)}{f_i(I_i)}f_i'(I_i)(I_i')^2,
\end{aligned}
$$

and

$$
\frac{\mathrm{d}V_i}{\mathrm{d}t} = -\frac{f_i(I_i(t-c\tau_i))}{f_i(I_i^*)} + \frac{f_i(I_i)}{f_i(I_i^*)} + \ln\frac{f_i(I_i(t-c\tau_i))}{f_i(I_i)},\ i=1,2.
$$

Thus,

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= -\frac{d_sS^*(S')^2}{S^2} - \frac{d_1f_1(I_1^*)f_1'(I_1)(I_1')^2}{f_1^2(I_1)} - \frac{d_2f_2(I_2^*)f_2'(I_2)(I_2')^2}{f_2^2(I_2)} - \frac{\mu(S-S^*)^2}{S}\\
&\quad + \beta_1S^*f_1(I_1^*) + \beta_2S^*f_2(I_2^*) - \frac{\beta_1f_1(I_1^*)(S^*)^2}{S} - \frac{\beta_2f_2(I_2^*)(S^*)^2}{S}\\
&\quad + \beta_1S^*f_1(I_1(t-c\tau_1)) - (\mu_1+\gamma_1)I_1 + \beta_2S^*f_2(I_2(t-c\tau_2)) - (\mu_2+\gamma_2)I_2\\
&\quad - \beta_1Sf_1(I_1(t-c\tau_1))\frac{f_1(I_1^*)}{f_1(I_1)} + (\mu_1+\gamma_1)I_1\frac{f_1(I_1^*)}{f_1(I_1)}\\
&\quad - \beta_2Sf_2(I_2(t-c\tau_2))\frac{f_2(I_2^*)}{f_2(I_2)} + (\mu_2+\gamma_2)I_2\frac{f_2(I_2^*)}{f_2(I_2)}\\
&\quad - (\mu_1+\gamma_1)I_1^*\frac{f_1(I_1(t-c\tau_1))}{f_1(I_1^*)} + (\mu_1+\gamma_1)I_1^*\frac{f_1(I_1)}{f_1(I_1^*)}\\
&\quad + (\mu_1+\gamma_1)I_1^*\ln\frac{f_1(I_1(t-c\tau_1))}{f_1(I_1)} - (\mu_2+\gamma_2)I_2^*\frac{f_2(I_2(t-c\tau_2))}{f_2(I_2^*)}\\
&\quad + (\mu_2+\gamma_2)I_2^*\frac{f_2(I_2)}{f_2(I_2^*)} + (\mu_2+\gamma_2)I_2^*\ln\frac{f_2(I_2(t-c\tau_2))}{f_2(I_2)}.
\end{aligned}
$$

Note that

$$
\ln\frac{f_i(I_i(t-c\tau_i))}{f_i(I_i)} = \ln\frac{S^*}{S} + \ln\frac{Sf_i(I_i(t-c\tau_i))}{S^*f_i(I_i)},\ (\mu_i+\gamma_i)I_i^* = \beta_iS^*f_i(I_i^*),\ i=1,2,
$$

it follows that

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} =& -\frac{d_s S^*(S')^2}{S^2} - \frac{d_1 f_1(I_1^*) f_1'(I_1)(I_1')^2}{f_1^2(I_1)} - \frac{d_2 f_2(I_2^*) f_2'(I_2)(I_2')^2}{f_2^2(I_2)} - \frac{\mu(S-S^*)^2}{S} \\
& - (\mu_1+\gamma_1) I_1^* \Big(\frac{S^*}{S} - 1 - \ln\frac{S^*}{S}\Big) - (\mu_2+\gamma_2) I_2^* \Big(\frac{S^*}{S} - 1 - \ln\frac{S^*}{S}\Big) \\
& - (\mu_1+\gamma_1) I_1^* \Big(\frac{S f_1(I_1(t-c\tau_1))}{S^* f_1(I_1)} - 1 - \ln\frac{S f_1(I_1(t-c\tau_1))}{S^* f_1(I_1)}\Big) \\
& - (\mu_2+\gamma_2) I_2^* \Big(\frac{S f_2(I_2(t-c\tau_2))}{S^* f_2(I_2)} - 1 - \ln\frac{S f_2(I_2(t-c\tau_2))}{S^* f_2(I_2)}\Big) \\
& + (\mu_1+\gamma_1) I_1^* \Big(-\frac{I_1}{I_1^*} + \frac{I_1 f_1(I_1^*)}{I_1^* f_1(I_1)} + \frac{f_1(I_1)}{f_1(I_1^*)} - 1\Big) \\
& + (\mu_2+\gamma_2) I_2^* \Big(-\frac{I_2}{I_2^*} + \frac{I_2 f_2(I_2^*)}{I_2^* f_2(I_2)} + \frac{f_2(I_2)}{f_2(I_2^*)} - 1\Big) \\
=& -\frac{d_s S^*(S')^2}{S^2} - \frac{d_1 f_1(I_1^*) f_1'(I_1)(I_1')^2}{f_1^2(I_1)} - \frac{d_2 f_2(I_2^*) f_2'(I_2)(I_2')^2}{f_2^2(I_2)} - \frac{\mu(S-S^*)^2}{S} \\
& - (\mu_1+\gamma_1) I_1^* \Big(\frac{S^*}{S} - 1 - \ln\frac{S^*}{S}\Big) - (\mu_2+\gamma_2) I_2^* \Big(\frac{S^*}{S} - 1 - \ln\frac{S^*}{S}\Big) \\
& - (\mu_1+\gamma_1) I_1^* \Big(\frac{S f_1(I_1(t-c\tau_1))}{S^* f_1(I_1)} - 1 - \ln\frac{S f_1(I_1(t-c\tau_1))}{S^* f_1(I_1)}\Big) \\
& - (\mu_2+\gamma_2) I_2^* \Big(\frac{S f_2(I_2(t-c\tau_2))}{S^* f_2(I_2)} - 1 - \ln\frac{S f_2(I_2(t-c\tau_2))}{S^* f_2(I_2)}\Big) \\
& + (\mu_1+\gamma_1) I_1^* \Big(\frac{I_1}{I_1^*} - \frac{f_1(I_1)}{f_1(I_1^*)}\Big)\Big(\frac{f_1(I_1^*)}{f_1(I_1)} - 1\Big) \\
& + (\mu_2+\gamma_2) I_2^* \Big(\frac{I_2}{I_2^*} - \frac{f_2(I_2)}{f_2(I_2^*)}\Big)\Big(\frac{f_2(I_2^*)}{f_2(I_2)} - 1\Big).
\end{aligned}$$

In views of $f_i(I_i) = \frac{I_i}{1+\alpha_i I_i}$, $i = 1,2$, the last two term reduces to

$$-\frac{\alpha_1(\mu_1+\gamma_1)(I_1-I_1^*)^2}{(1+\alpha_1 I_1)(1+\alpha_1 I_1^*)} - \frac{\alpha_2(\mu_2+\gamma_2)(I_2-I_2^*)^2}{(1+\alpha_2 I_2)(1+\alpha_2 I_2^*)} \leqslant 0.$$

In addition, note that $f_i'(I_i) > 0$ for $I_i > 0$, $i = 1,2$, and

$$\frac{S^*}{S} - 1 - \ln\frac{S^*}{S} \geqslant 0 \quad \text{for } S > 0,$$

and

$$\frac{S f_i(I_i(t-c\tau_i))}{S^* f_i(I_i)} - 1 - \ln\frac{S f_i(I_i(t-c\tau_i))}{S^* f_i(I_i)} \geqslant 0 \text{ for } I_i(t-c\tau_1) > 0,\ S > 0,\ i = 1,2.$$

Consequently, V is decreasing and bounded from below on $\mathbb{R}^+$. Note that $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ if and only if $S \equiv S^*$, $I_1 \equiv I_1^*$, $I_2 \equiv I_2^*$, $S' = 0$, $I_1' = 0$ and $I_2' = 0$. Thus, this shows the boundary condition (4.2) holds, which completes the proof. $\square$

In what follows, we turn to study the traveling wave solution of the fourth equation of (2.1). Letting $x+ct$ by t, then the equation reads

$$d_R R''(t) - cR'(t) + \gamma_1 I_1(t) + \gamma_2 I_2(t) - \mu R(t) = 0. \tag{4.11}$$

Then we have the following conclusion.

LEMMA 4.10. *Assume that $\mathscr{R}_0 > 1$ and $\overline{\mathscr{R}}_0 > 1$. Then for every $c > c^*$, Eq.(4.11) has a traveling wave solution $R(t)$ with $\lim\limits_{t\to-\infty} R(t) = 0$ and $\lim\limits_{t\to\infty} R(t) = R^*$.*

Proof. Note that equation (4.11) has a bounded solution given by

$$R(t) = \frac{1}{\Lambda_4}\Big(\int_{-\infty}^{t} e^{\rho_{41}(t-s)} + \int_{t}^{\infty} e^{\rho_{42}(t-s)}\Big)(\gamma_1 I_1(s) + \gamma_2 I_2(s))\mathrm{d}s,$$

where

$$\rho_{41} = \frac{c - \sqrt{c^2 + 4d_R\mu}}{2d_R} < 0,\ \rho_{42} = \frac{c + \sqrt{c^2 + 4d_R\mu}}{2d_R} > 0,\ \Lambda_4 = d_R(\rho_{42} - \rho_{41}).$$

By using the L'Hôspital rule, we get

$$\lim_{t\to-\infty} R(t) = \frac{1}{\Lambda_4}\Big(\frac{1}{\rho_{42}} - \frac{1}{\rho_{41}}\Big)\lim_{t\to-\infty}(\gamma_1 I_1(t) + \gamma_2 I_2(t)) = 0,$$

and

$$\lim_{t\to\infty} R(t) = \frac{1}{\mu}\lim_{t\to\infty}(\gamma_1 I_1(t) + \gamma_2 I_2(t)) = \frac{\gamma_1}{\mu} I_1^* + \frac{\gamma_2}{\mu} I_2^* = R^*.$$

This completes the proof. $\square$

Finally, summing up Theorem 4.1 and Lemma 4.10, we obtain the main result in this section.

THEOREM 4.2. *Assume that $\mathscr{R}_0 > 1$ and $\overline{\mathscr{R}}_0 > 1$. Then for every $c > c^*$, system (2.1) admits a traveling wave solution with speed c which connects the equilibria $E_0(1,0,0,0)$ and $E^*(S^*, I_1^*, I_2^*, R^*)$.*

5. Conclusion and discussion

In this paper, starting from the works [1, 5], we presented a mathematical model to describe the spatial dynamics of a diffusive disease model with two-strains and latency delays. For this model, by an abstract treatment, we studied the existence, uniqueness and positive of the solution to the initial-boundary-value problem associated to system (2.1).

For the model under consideration, the global stability of the equilibria is completely determined by selecting suitable Lyapunov functionals. More precisely, the disease dies out from the population if the basic reproduction $\mathscr{R}_0 \leqslant 1$; but, if $\mathscr{R}_0 > 1$, then the disease will persist and one or both of the strains become endemic: depending the model parameter values, either one or both of the two boundary (one-strain) equilibrium or the co-persistence equilibrium is globally attractive.

As we know, the traveling wave solutions describes the disease propagation into the susceptible individuals from an initial disease-free equilibrium to the disease equilibrium. Here, we proved that existence such a traveling wave solution connecting the disease free and endemic equilibria is totally determined by the threshold values $\mathscr{R}_0$ and $\overline{\mathscr{R}}_0$.

As a final remark, it is worth to mention, in this paper, we do not show whether c^* is the minimal wave speed $c_{\min}$, that is, there does not exist traveling wave solution connecting the two steady states for $0 < c < c_{\min}$. Recently, Liang and Zhao [13] have proved that for a class of system with certain monotonicity, these two speeds indeed coincide. Clearly, the results in [13] can not apply to the model directly, since system (2.1) is non-monotone. Hence, it is of interest to study the relation of the minimal wave speed and the asymptotic speeds of spread for system (2.1). This problem will be left for our further investigation.

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