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EXPONENTIAL DICHOTOMY OF LINEAR AUTONOMOUS SYSTEMS OVER TIME SCALES

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Abstract. In this paper we study the exponential stability and exponential dichotomy of the first order linear dynamic equation $z^{\Delta}(s) = Mz(s)$ in terms of the boundedness of solutions of the following Cauchy problems:

$$\begin{cases} z^{\Delta}(s) = Mz(s) + f(s)Qb, \ 0 \leqslant s \in \mathbb{T}, \\ z(0) = 0 \end{cases}$$

and

$$\begin{cases} w^{\Delta}(s) = -Mw^{\sigma} + f(s)(I-Q)b, \\ w(0) = 0, \end{cases}$$

where $\mathbb{T}$ is a time scale, M is a regressive matrix, b is a non-zero vector in $\mathbb{C}^m$, $f(s)$ is a bounded and right-dense continuous function on $\mathbb{T}$, and Q is a projection on $\mathbb{C}^m$.

1. Introduction

Let $\mathscr{A}$ be a bounded linear operator acting on a complex Banach space X. A well-known theorem of Daletckii and Krein [12] and Krein [21] says that the system $\dot{x}(t) = \mathscr{A}x(t)$ is uniformly exponentially stable if and only if for each $\mu \in \mathbb{R}$ and each $b \in X$ the solution of the Cauchy problem

$$\begin{cases} \dot{\mathscr{V}}(t) = \mathscr{A}\mathscr{V}(t) + e^{i\mu t}b, \\ \mathscr{V}(0) = 0 \end{cases}$$

is bounded. The proof of this classical result can be found in [2]. This result can also be extended for strongly continuous bounded semigroups; see [7, 8, 23].

Under a slightly different assumption the result on stability is also preserved for any strongly continuous semigroups acting on complex Hilbert spaces; see, for example, [24, 27] and the references cited therein. See also, [6, 16], for counter-examples. In discrete case, similar results can be found in [1, 29, 30].

In [9, 28] the same results were extended to dichotomy for square size matrices in both continuous and discrete cases. The main purpose of this article is to present the results of [9, 28] in a unified way, i.e., on time scales.

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2. Basic Notation and Preliminaries

In 1988, the theory of dynamic equations on time scales was introduced by Hilger [18, 19], with the motivation of providing a unification to continuous and discrete calculus. Since then, this theory has been developing rapidly and has received a lot of attention in recent years. The basic theory of time scales and dynamic equations on time scales can be found in the monographes by Bohner and Peterson [3, 4] and the references contained therein.

Recently, many researchers paid attention to the study of the different types of stabilities of dynamic equations on time scales, with different approaches; see, e.g., [5, 10, 11, 14, 15, 17, 20, 22, 25, 26, 31].

By a time scale $\mathbb{T}$, we mean a nonempty closed subset of real numbers. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess function $\mu : \mathbb{T} \to \mathbb{R}$ are respectively defined as

$$\sigma(s) = \inf\{n \in \mathbb{T} : n > s\},\ \rho(s) = \sup\{n \in \mathbb{T} : n < s\},\ \mu(s) = \sigma(s) - s.$$

A point $s \in \mathbb{T}$ will be called left-scattered and left-dense if $s > \rho(s)$ and $\rho(s) = s$, respectively. If $s < \sigma(s)$ and $\sigma(s) = s$, then s will be called right-scattered and right-dense, respectively. We also need the set $\mathbb{T}^{z}$ which is called the derived form of time scale $\mathbb{T}$ and is defined as follows: if $\mathbb{T}$ has a left-scattered maximum n, then $\mathbb{T}^{z} = \mathbb{T}\backslash\{n\}$. Otherwise, $\mathbb{T}^{z} = \mathbb{T}$. Shortly,

$$\mathbb{T}^{z} = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \text{if } \sup\mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup\mathbb{T} = \infty. \end{cases}$$

A function $h : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The set of all right-dense continuous functions is denoted by C_{rd}. If h is continuous at each right-dense and left-dense point, then h is said to be a continuous function on time scale $\mathbb{T}$.

For $h : \mathbb{T} \to \mathbb{R}$ and $s \in \mathbb{T}^{z}$, we define the delta derivative of $h(s)$ denoted by $h^{\Delta}(s)$, to be the number (if it exists) with the property that for any $\varepsilon > 0$, there exists a neighborhood $\mathbb{N}$ of s such that

$$|[h(\sigma(s)) - h(t)] - h^{\Delta}(s)[\sigma(s) - t]| \leqslant \varepsilon|\sigma(s) - t|,\ \forall\, t \in \mathbb{N}.$$

Also a function $h : \mathbb{T} \to \mathbb{R}$ is called regulated if its left-sided limits exist at all left-dense points and right-sided limits exist at all right-dense points. A function $H : \mathbb{T} \to \mathbb{R}$ is called an anti-derivative of $h : \mathbb{T} \to \mathbb{R}$ provided $H^{\Delta}(s) = h(s),\ \ \forall\, s \in \mathbb{T}^{z}$. Note that every right-dense continuous function has an anti-derivative. The indefinite integral of $h : \mathbb{T} \to \mathbb{R}$ is given by

$$\int h(s)\Delta s = H(s) + C$$

and the Cauchy integral is given by

$$\int_{r}^{s} h(t)\Delta t = H(s) - H(r).$$

A function $q:\mathbb{T}\to\mathbb{R}$ is called regressive provided $1+\mu(s)q(s)\neq 0$ for all $s\in\mathbb{T}^z$ and is called positively regressive if $1+\mu(s)q(s)>0$. The set of all regressive and right-dense continuous functions will be denoted by $\mathrm{Reg}(\mathbb{T})$ and the set of all right-dense and positively regressive functions will be denoted by $\mathrm{Reg}(\mathbb{T})^+$.

If $q\in\mathrm{Reg}(\mathbb{T})$, then the first order linear dynamic equation $z^\Delta(s)=q(s)z$ is called regressive and $e_q(\cdot,s_0)$ is the solution of initial value problem $z^\Delta(s)=q(s)z,\ z(s_0)=1$.

Finally, if $w:\mathbb{T}\to\mathbb{R}$ is a function, then $w^\sigma:\mathbb{T}\to\mathbb{R}$ is defined by

$$w^\sigma(s)=w(\sigma(s)) \text{ for all } s\in\mathbb{T}.$$

DEFINITION 1. The function $\lambda\in\mathrm{Reg}(\mathbb{T}^z,\mathbb{C})$ is said to be uniformly regressive if there exists an $\alpha>0$ such that it satisfies the following condition

$$\alpha^{-1}\leqslant|\,1+\mu(t)\lambda(t)\,| \ \text{ for } t\in\mathbb{T}^z. \tag{2.1}$$

DEFINITION 2. If $q\in\mathrm{Reg}(\mathbb{T})$, then the generalized exponential function $e_q(x,y)$ on time scale $\mathbb{T}$ is defined by

$$e_q(x,y)=\exp\left(\int_x^y \xi_{\mu(s)}q(s)\Delta s\right) \text{ for all } x,y\in\mathbb{T},$$

with the cylindrical transformation given by

$$\xi_g(y)=\begin{cases}\frac{\mathrm{Log}(1+gy)}{h}, & \text{if } h\neq 0,\\ y, & \text{if } h=0.\end{cases}$$

DEFINITION 3. Let $g,h:\mathbb{T}\to\mathbb{R}$ be two regressive functions. The operations $\oplus$ and $\ominus$ can be define as

$$g\oplus h=g+h+\mu gh,\ \ominus g=-\frac{g}{1+\mu g},\ \text{ and } \ g\ominus h=g\oplus(\ominus h).$$

Here we recall few results from [3], without proofs, which will be useful later on.

LEMMA 1. (see [3, Theorem 2.36]) *If* $g,h\in\mathrm{Reg}(\mathbb{T})$ *and* $c,\ x,\ y,\ z\in\mathbb{T}$, *then*

(i) $e_0(x,y)=1$ *and* $e_g(x,x)=1$;

(ii) $e_g(\sigma(x),y)=(1+\mu(s)g(s))e_g(x,y)$;

(iii) $e_g(x,y)=\frac{1}{e_g(y,x)}=e_{\ominus g}(x,y)$;

(iv) $e_g(x,y)e_g(y,z)=e_g(x,z)$;

(v) $(e_{\ominus g}(x,y))^\Delta=(\ominus g)(s)e_{\ominus g}(x,y)$;

(vi) $\int_x^y g(s)e_g(cz,\sigma(s))\Delta s=e_g(x,y)-e_g(y,z)$.

Consider the first order linear dynamic system

$$z^{\Delta}(s) = Mz(s);\ z(s_0) = z_0,\ s \in \mathbb{T}, \tag{M}$$

where M is a square matrix of order m.

Concerning the exponential stability of system (M), in [17] the following result was obtained.

LEMMA 2. (see [17, Theorem 6.3]) *System (M) is exponentially stable if and only if there exists a $\gamma > 0$ with $-\gamma \in \mathrm{Reg}(\mathbb{T})^+$ such that for any $s_0 \in \mathbb{T}$, there exists an $\alpha = \alpha(s_0) \geqslant 1$ such that for any solution $\psi(s)$ of (M), we have*

$$||\psi(s)|| \leqslant \alpha \|z_0\| e_{-\gamma}(s, s_0),\ s \geqslant s_0,\ s \in \mathbb{T}.$$

LEMMA 3. (see [13, Lemma 3.1]) *Let $\mathbb{T}$ be a time scale and $\beta > 0$ be a positive number such that $\beta \in \mathrm{Reg}(\mathbb{T})^+$. Then for the corresponding scalar system $z^{\Delta} = \beta z$ the following inequality holds*

$$e_{\beta}(u, v) \leqslant e^{\beta(u-v)} \text{ for all } u \geqslant v.$$

In [27], a result concerning the system (M) in scalar case was obtained as follows.

PROPOSITION 1. (see [27, Proposition 6]) *Let $\mathbb{T}$ be an unbounded time scale and let $\lambda \in \mathbb{C}$. The scalar system $z^{\Delta} = \lambda z$, $z \in \mathbb{C}$ is exponentially stable if and only if one of the following conditions is satisfied for arbitrary $s_0 \in \mathbb{T}$.*

(i)

$$\omega(\lambda) = \limsup_{S \to \infty} \frac{1}{S - s_0} \int_{s_0}^{S} \lim_{u \to \mu(s)} \frac{\log|1 + u\lambda|}{u} \Delta s < 0.$$

(ii) $\forall\, S \in \mathbb{T} : \exists s \in \mathbb{T}$ *with* $s > S$ *such that* $1 + \mu(s)\lambda = 0$.

DEFINITION 4. Let $\mathbb{T}$ be an unbounded time scale and we define for arbitrary $s_0 \in \mathbb{T}$, the sets

$$E_{\mathbb{C}}(\mathbb{T}) := \left\{ \lambda \in \mathbb{C} : \limsup_{S \to \infty} \frac{1}{S - s_0} \int_{s_0}^{S} \lim_{u \to \mu(s)} \frac{\log|1 + u\lambda|}{u} \Delta s < 0 \right\}$$

and

$$E_{\mathbb{R}}(\mathbb{T}) := \{\lambda \in \mathbb{R} | \forall\, S \in \mathbb{T} : \exists\, s \in \mathbb{T} \text{ with } s > S \text{ such that } 1 + \mu(s)\lambda = 0\}.$$

On time scale $\mathbb{T}$ there is a set of exponential stability defined by

$$E(\mathbb{T}) = E_{\mathbb{C}}(\mathbb{T}) \cup E_{\mathbb{R}}(\mathbb{T}).$$

REMARK 1. For any time scale $\mathbb{T}$ we have $E_{\mathbb{C}}(\mathbb{T}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$, because if $\mathrm{Re}(\lambda) \geqslant 0$, then $|1 + u\lambda| \geqslant 1$ for all non-negative $u \in \mathbb{R}$.

Let $\mathscr{S}(M)$ denote the spectrum of the matrix M.

THEOREM 1. (see [27, Theorem 21]) *Let the time scale $\mathbb{T}$ be unbounded from above and let M be a regressive matrix. Then the following statements are true.*
(i) If the system $z^\Delta = Mz$ is exponentially stable, then $\mathscr{S}(M) \subset E_{\mathbb{C}}(\mathbb{T})$.
(ii) If (2.1) is true for all eigenvalues λ of M and if $\mathscr{S}(M) \subset E_{\mathbb{C}}(\mathbb{T})$, then the system $z^\Delta = Mz$ is exponentially stable.

3. Spectral Decomposition Theorem on Time Scales

In [9, 28], the authors used the idea of spectral decomposition theorems in discrete and continuous cases, respectively. Here, first we recall the spectral decomposition theorems in continuous and discrete cases and then we prove them over time scales, which is the main tool for proving our main results.

Let M be a square matrix of order m and q_M be the characteristic polynomial associated with M and let $\mathscr{S}(M) := \{\lambda_1, \lambda_2, \dots, \lambda_k\}$, $k \leqslant m$ be its spectrum. It is clear that there exist integer numbers $m_1, m_2, \dots, m_k \geqslant 1$ such that

$$q_M(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_k)^{m_k}, \quad m_1 + m_2 + \dots + m_k = m.$$

Let $j \in \{1, 2, \dots, k\}$ and $\mathscr{Z}_j := \ker(M - \lambda_j I)^{m_j}$. It is easy to check that $\mathscr{Z}_j$ is an e^{tM}-invariant subspace of $\mathbb{C}^m$ and $\dim(\mathscr{Z}_j) \geqslant 1$.

In [28], the spectral decomposition theorem in continuous case was stated as follows.

THEOREM 2. (see [28, Theorem 1.1]) *For each $z \in \mathbb{C}^m$ there exist $r_j \in \mathscr{Z}_j$ $(j \in \{1, 2, \dots, k\})$ such that*

$$e^{sM} z = e^{sM} r_1 + e^{sM} r_2 + \dots + e^{sM} r_k, \quad s \in \mathbb{R}.$$

Moreover, if $r_j(s) := e^{sM} r_j$, then $r_j(s) \in \mathscr{Z}_j$ for all $s \in \mathbb{R}$ and there exist $\mathbb{C}^m$-valued polynomials $t_j(s)$ with $\deg(t_j) \leqslant m_j - 1$ such that

$$r_j(s) = e^{\lambda_j s} t_j(s), \quad s \in \mathbb{R},\ j \in \{1, 2, \dots, k\}.$$

Similarly, for discrete case the similar spectral decomposition theorem was proved in [9] and is stated as follows.

THEOREM 3. (see [9, Theorem 1]) *For each $z \in \mathbb{C}^m$ there exist $y_j \in Y_j$, $(j \in \{1, 2, \dots, k\})$ such that*

$$A^s z = A^s y_1 + A^s y_2 + \dots + A^s y_k, \text{ for any } s \in \mathbb{Z}_+.$$

Moreover, if $y_j(s) := A^s y_j$, then $y_j(s) \in Y_j$ for all $s \in \mathbb{Z}_+$ and there exist $\mathbb{C}^m$-valued polynomials $t_j(s)$ with $\deg(t_j) \leqslant m_j - 1$ such that

$$y_j(s) = \lambda_j^s t_j(s), \quad s \in \mathbb{Z}_+,\ j \in \{1, 2, \dots, k\}.$$

Consider the system $z^{\Delta}(s) = Mz(s),\ s \in \mathbb{T}$. We know that the solution of the Cauchy problem

$$z^{\Delta}(s) = Mz(s),\ z(0) = z_0$$

is given by $z(s) = e_M(s,0)z_0$. And the solution of the Cauchy problem

$$\begin{cases} z^{\Delta}(s) = Mz(s) + f(s), \\ z(0) = 0 \end{cases}$$

is given by

$$z(s) = \int_0^s e_M(s,\sigma(t))f(t)\Delta t.$$

We are in the position to state and prove the spectral decomposition theorem on time scales.

THEOREM 4. *Let M be a regressive matrix of order m. For each $w \in \mathbb{C}^m$ there exist $z_j \in \mathscr{Z}_j$ $(j \in \{1,2,\dots,k\})$ such that*

$$e_M(s,0)w = e_M(s,0)z_1 + e_M(s,0)z_2 + \ldots + e_M(s,0)z_k, \quad s \in \mathbb{T}.$$

Moreover, if $z_j(s) := e_M(s,0)z_j$, then $z_j(s) \in \mathscr{Z}_j$ for all $s \in \mathbb{T}$ and there exist $\mathbb{C}^m$-valued polynomials $t_j(s)$ with $\deg(t_j) \leqslant m_j - 1$ such that

$$z_j(s) = e_{\lambda_j}(s,0)t_j(s), \quad s \in \mathbb{T},\ j \in \{1,2,\dots,k\}.$$

From Cayley–Hamilton theorem and the fact that

$$\ker[gh(M)] = \ker[g(M)] \oplus \ker[h(M)],$$

whenever the complex valued polynomials g and h are relatively prime, it follows that

$$\mathbb{C}^m = \mathscr{Z}_1 \oplus \mathscr{Z}_2 \oplus \ldots \oplus \mathscr{Z}_k. \tag{3.1}$$

Let $w \in \mathbb{C}^m$. For each $j \in \{1,2,\dots,k\}$ there exist unique $z_j \in \mathscr{Z}_j$ such that

$$w = z_1 + z_2 + \ldots + z_k.$$

Thus

$$e_M(s,0)w = e_M(s,0)z_1 + e_M(s,0)z_2 + \ldots + e_M(s,0)z_k, \quad s \in \mathbb{T}.$$

Let $t_j(s) := e_{\ominus\lambda_j}(s,0)z_j(s)$. A simple calculation shows that

$$t_j^{\Delta^{m_j}}(s) = \frac{e_{\ominus\lambda}(s,0)(M-\lambda_j I)^{m_j} z_j e_M(s,0)}{(1+\mu\lambda_j)^{m_j}} = 0.$$

The last equality follows because $z_j(s)$ belongs to $\mathscr{Z}_j$ for each $s \in \mathbb{T}$. Thus t_j is a $\mathbb{C}^m$-valued polynomial having degree less than m_j.

4. Exponential Dichotomy in terms of Boundedness of Solution of a Cauchy Problem

Using the idea of [27], here first we decompose $\mathbb{C}$ into three parts and then with the help of this decomposition we decompose $\mathbb{C}^m$ into three spectral subspaces which will help us to use the idea of dichotomy.

Let us divide $\mathbb{C}$ into three sets as follows.

$$E_{\mathbb{C}}(\mathbb{T}) = \left\{\lambda \in \mathbb{C} : \limsup_{S\to\infty} \frac{1}{S-s_0}\int_{s_0}^{S} \lim_{u\to\mu(s)} \frac{\log|1+u\lambda|}{s}\Delta s < 0\right\},$$

$$E^{+}_{\mathbb{C}}(\mathbb{T}) = \left\{\lambda \in \mathbb{C} : \limsup_{S\to\infty} \frac{1}{S-s_0}\int_{s_0}^{S} \lim_{u\to\mu(s)} \frac{\log|1+u\lambda|}{s}\Delta s > 0\right\}$$

and

$$E^{0}_{\mathbb{C}}(\mathbb{T}) = \left\{\lambda \in \mathbb{C} : \limsup_{S\to\infty} \frac{1}{S-s_0}\int_{s_0}^{S} \lim_{u\to\mu(s)} \frac{\log|1+u\lambda|}{s}\Delta s = 0\right\}.$$

Clearly, $\mathbb{C} = E_{\mathbb{C}}(\mathbb{T}) \cup E^{+}_{\mathbb{C}}(\mathbb{T}) \cup E^{0}_{\mathbb{C}}(\mathbb{T})$. By using this decomposition of $\mathbb{C}$, we can state the following definition.

DEFINITION 5. Let $\mathscr{S}(M)$ denote the spectrum of the regressive matrix M. The system

$$z^{\Delta}(s) = Mz(s);\ z(s_0) = z_0 \tag{M}$$

is said to be exponentially stable if all eigenvalues of matrix M are uniformly regressive and $\mathscr{S}(M) \subset E_{\mathbb{C}}(\mathbb{T})$ and it is said to be expansive if $\mathscr{S}(M) \subset E^{+}_{\mathbb{C}}(\mathbb{T})$. If $\mathscr{S}(M) \cap E^{0}_{\mathbb{C}}(\mathbb{T}) = \phi$, then the system is said to be dichotomic.

REMARK 2. By the decomposition of $\mathbb{C}^m$ in (3.1), let us consider

$$\mathbb{C}^m = \mathscr{W}_s(M) \oplus \mathscr{W}_0(M) \oplus \mathscr{W}_u(M),$$

where

$$\mathscr{W}_s(M) = \bigoplus_{j=1,\lambda_j\in E_{\mathbb{C}}(\mathbb{T})}^{k} \ker(M-\lambda_j I)^{n_j},$$

$$\mathscr{W}_0(M) = \bigoplus_{j=1,\lambda_j\in E^{0}_{\mathbb{C}}(\mathbb{T})}^{k} \ker(M-\lambda_j I)^{n_j},$$

and

$$\mathscr{W}_u(M) = \bigoplus_{j=1,\lambda_j\in E^{+}_{\mathbb{C}}(\mathbb{T})}^{k} \ker(M-\lambda_j I)^{n_j}.$$

Now if M is a dichotomic matrix, then $\mathscr{W}_0(M) = \{0\}$, and so $\mathbb{C}^m = \mathscr{W}_s(M) \oplus \mathscr{W}_u(M)$.

Our result concerning the exponential stability of (M) is stated as follows.

THEOREM 5. *The system $z^{\Delta}(s) = Mz(s);\ z(s_0) = z_0$ is exponentially stable if and only if for each $b \in \mathbb{C}^m$ and each bounded function f the unique solution of the Cauchy problem*

$$\begin{cases} z^{\Delta}(s) = Mz(s) + f(s)b, \\ z(0) = 0 \end{cases} \qquad (M,b,z_0)$$

is bounded.

Proof. **Necessity:** Let $z^{\Delta}(s) = Mz(s);\ z(s_0) = z_0$ be exponentially stable. Then by Lemma 2,

$$||\psi(s)|| \leqslant \alpha \|z_0\| e_{-\gamma}(s,s_0),\ s,s_0 \in \mathbb{T}.$$

We need to prove that for each $b \in \mathbb{C}^m$ and each bounded function f the unique solution of the Cauchy problem (M,b,z_0) is bounded. Taking norm of the solution of (M,b,z_0), we have

$$\begin{aligned} ||z(s)|| &= \sup_{s\in\mathbb{T}} \left| \int_0^s e_M(s,\sigma(t)) f(t) b \Delta t \right| \\ &\leqslant \sup_{s\in\mathbb{T}} \left| \int_0^s e_M(s,\sigma(t)) \Delta t \right| ||f(t)b|| \\ &= \sup_{s\in\mathbb{T}} \left| M^{-1} \int_0^s M e_M(s,\sigma(t)) \Delta t \right| C \\ &= CM^{-1} \sup_{s\in\mathbb{T}} |e_M(s,0) - e_M(s,s)| \\ &= CM^{-1} ||e_M(s,0) - 1|| \\ &\leqslant CM^{-1} ||\psi(s)|| + CM^{-1} \\ &\leqslant CM^{-1} \gamma \|z_0\| e_{-\gamma}(s,s_0) + CM^{-1} \\ &\leqslant CM^{-1} \gamma \|z_0\| e^{-\gamma(s-s_0)} + CM^{-1}. \end{aligned}$$

Hence, the unique solution of (M,b,z_0) is bounded.

Sufficiency: Suppose to the contrary that system (M) is not exponentially stable. Then for $\nu \in \{1,2,\ldots,k\}$ there exists an eigenvalue λ_ν of M such that $\lambda_\nu \notin E_{\mathbb{C}}(\mathbb{T})$. By Remark 1, $E_{\mathbb{C}}(\mathbb{T}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$. Thus $\mathrm{Re}(\lambda_\nu) \geqslant 0$, which implies that $|1 + u\lambda_\nu| \geqslant 1$ for all non-negative $u \in \mathbb{R}$. Letting $f(t) = C$ and $b = 0 + 0 + \ldots + 0 + b_\nu + 0 + \ldots + 0$, then by spectral decomposition theorem in Theorem 4, $e_M(s,\sigma(t))b = e_{\lambda_\nu}(s,\sigma(t)) p_\nu(t)$, where $p_\nu(t)$ is a polynomial of degree less than or equal to $m_\nu - 1$. In this case the solution of (M,b,z_0) can be written as

$$\begin{aligned} z(s) &= \int_0^s e_M(s,\sigma(t)) f(t) b \Delta t \\ &= \int_0^s e_{\lambda_\nu}(s,\sigma(t)) p_\nu(t) C \Delta t \\ &= C \int_0^s \frac{e_{\lambda_\nu}(s,t)}{1 + \mu\lambda_\nu} p_\nu(t) \Delta t. \end{aligned}$$

Since $\mathrm{Re}(\lambda_\nu) \geqslant 0$, so $z(s) \to \infty$ as $s \to \infty$, i.e., the unique solution of (M,b,z_0) is unbounded, which is a contradiction. Hence, system (M) is exponentially stable. This completes the proof.

We know that the system $z^{\Delta}(s) = Mz^{\sigma}(s);\ z(s_0) = z_0$ is expansive if and only if $z^{\Delta}(s) = -Mz(s);\ z(s_0) = z_0$ is stable. Thus we can state the following corollary.

COROLLARY 1. *The system $z^{\Delta}(s) = Mz(s);\ z(s_0) = z_0$ is expansive if and only if for each $b \in \mathbb{C}^m$ and each bounded function f the unique solution of the Cauchy problem*

$$\begin{cases} z^{\Delta}(s) = -Mz^{\sigma} + f(s)b, \\ z(0) = 0 \end{cases}$$

is bounded.

In the next theorem we give our main result.

THEOREM 6. *System (M) is dichotomic if and only if there exists a projection Q having the property $e_M(s,0)Q = Qe_M(s,0)$ for all $s \geqslant 0$ such that for each non-zero vector b in $\mathbb{C}^m$ and each bounded function f such that $e_M(s,0)f(s) = f(s)e_M(s,0)$, the solutions of the following Cauchy problems*

$$\begin{cases} z^{\Delta}(s) = Mz(s) + f(s)Qb, \\ z(0) = 0 \end{cases} \tag{(M,Qb,z_0)}$$

and

$$\begin{cases} w^{\Delta}(s) = -Mw^{\sigma} + f(s)(I-Q)b, \\ w(0) = 0 \end{cases} \tag{$(-M,(I-Q)b,w_0)$}$$

are bounded.

Proof. **Necessity:** Suppose that system (M) is dichotomic. By Remark 2, $\mathbb{C}^m = \mathscr{W}_s(M) \oplus \mathscr{W}_u(M)$. Let us define $Q : \mathbb{C}^m \to \mathbb{C}^m$ by $Qw := w_s$, where $w = w_s + w_u$, $w_s \in \mathscr{W}_s$, and $w_u \in \mathscr{W}_u$. Clearly, Q is a projection. Moreover, for all $w \in \mathbb{C}^m$ and all $s \geqslant 0$, we have

$$\begin{aligned} Qe_M(s,0)w &= Qe_M(s,0)(w_s + w_u) \\ &= Q(e_M(s,0)w_s + e_M(s,0)w_u) \\ &= e_M(s,0)w_s \\ &= e_M(s,0)Qw, \end{aligned}$$

where the fact that $\mathscr{W}_s(M)$ is an $e_M(s,0)$-invariant subspace is used. Thus $Qe_M(s,0) = e_M(s,0)Q$. Now the solution of (M,Qb,z_0) is given by

$$z(s) = \int_0^s e_M(s,\sigma(t))f(t)Qb\Delta t.$$

Applying Theorem 4, we have

$$\begin{aligned}
e_M(s,\sigma(t))f(t)Qb &= f(t)e_M(s,\sigma(t))Qb \\
&= f(t)Q(e_{\lambda_1}(s,\sigma(t))p_1(t)+e_{\lambda_2}(s,\sigma(t))p_2(t)+\ldots \\
&\quad +e_{\lambda_\nu}(s,\sigma(t))p_\nu(t)+\ldots+e_{\lambda_k}(s,\sigma(t))p_k(t)) \\
&= f(t)(e_{\lambda_1}(s,\sigma(t))p_1(t)+\ldots+e_{\lambda_\nu}(s,\sigma(t))p_\nu(t)),
\end{aligned}$$

where $\lambda_i \in \mathscr{W}_s(M)$ for $i \in \{1,\ldots,\nu\}$ and $\lambda_j \in \mathscr{W}_u(M)$ for $j \in \{\nu+1,\ldots,k\}$. It is clear that the solution of (M,Qb,z_0) is bounded. Similarly, we can show that the solution of $(-M,(I-Q)b,w_0)$ is bounded.

Sufficiency: Suppose to the contrary that system (M) is not dichotomic. Then $\mathscr{W}_0(M) \neq 0$ and so there exists an eigenvalue λ_l of M such that $\lambda_l \in E^0_{\mathbb{C}}(\mathbb{T})$. Since $b \in \mathbb{C}^m$, so let us choose b such that

$$b = 0+0+\ldots+0+b_l+0+\ldots+0.$$

Then either $b_l \in \mathscr{W}_s(M)$ or $b_l \in \mathscr{W}_u(M)$. If $l \leqslant \nu$, then $b_l \in \mathscr{W}_s(M)$ and if $l > \nu$, then $b_l \in \mathscr{W}_u(M)$.

Case-1: Assume $b_l \in \mathscr{W}_s(M)$. Then by Theorem 4,

$$e_M(s,\sigma(t))f(t)Qb = e_M(s,\sigma(t))f(t)Qb_l = f(t)e_{\lambda_l}(s,\sigma(t))p_l(t),\ l \in \{1,2,\ldots,\nu\}.$$

So the solution of (M,Qb,z_0) becomes

$$z(s) = \int_0^s f(t)e_{\lambda_l}(s,\sigma(t))p_l(t)\Delta t,$$

where $p_l(t)$ is a polynomial of degree less than or equal to m_l-1. Let $f(t)=C$. Then

$$z(s) = \int_0^s Ce_{\lambda_l}(s,\sigma(t))p_l(t)\Delta t = C\int_0^s e_{\lambda_l}(s,\sigma(t))p_l(t)\Delta t,$$

which is clearly unbounded. Hence, we arrived at a contradiction.

Case-2: Suppose $b_l \in \mathscr{W}_u(M)$. The solution of $(-M,(I-Q)b,w_0)$ is given by

$$w(s) = \int_0^s e_{\ominus M}(s,t)f(t)(I-Q)b\Delta t.$$

Now again by Theorem 4,

$$\begin{aligned}
e_{\ominus M}(s,\sigma(t))f(t)(I-Q)b &= e_{\ominus M}(s,\sigma(t))f(t)(I-Q)b_l \\
&= f(t)e_{\ominus M}(s,\sigma(t))(I-Q)b_l \\
&= f(t)e_{\ominus\lambda_l}(s,\sigma(t))p_l(t),
\end{aligned}$$

where $p_l(t)$ is a polynomial of degree less than or equal to m_l-1. So the solution of $(-M,(I-Q)b,w_0)$ becomes

$$w(s) = \int_0^s f(t)e_{\ominus\lambda_l}(s,\sigma(t))p_l(t)\Delta t.$$

Letting $f(t) = C$, then

$$w(s) = C\int_0^s e_{\ominus\lambda_l}(s,\sigma(t))p_l(t)\Delta t.$$

Clearly, $e_{\ominus\lambda_l}(s,\sigma(t)) \to 1$ as $s \to \infty$ and so $w(s) \to \infty$ as $s \to \infty$, which is again unbounded. Hence, in this case we also arrived at a contradiction. As in both cases we have contradictions, and thus we accept that system (M) is dichotomic. The proof is complete.

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