

A SURVEY ON THE EXISTENCE, UNIQUENESS AND REGULARITY QUESTIONS TO FULLY NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

J. TYAGI AND R.B. VERMA

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Abstract. The aim of this survey paper is to provide the recent developments on the existence, uniqueness and regularity results to fully nonlinear elliptic equations of the form

$$\begin{cases} F(x, u, Du, D^2u) = f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^n .

1. Introduction

Let us consider

$$F(x, u, Du, D^2u) = f(x) \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, unless we say explicitly otherwise, $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \rightarrow \mathbb{R}$, $S(n)$ is the set of all $n \times n$ real symmetric matrices equipped with usual ordering and $f : \bar{\Omega} \rightarrow \mathbb{R}$. Equations of the form (1.1) are called fully nonlinear second order partial differential equations (in short, PDEs) when F is not affine in D^2u . Hamilton-Jacobi-Bellman (in short, HJB) equations are examples of (1.1), which characterize the value functions of stochastic control problems. Hamilton-Jacobi-Isaac equations, which are the fundamental of the differential game theory, are also example of fully nonlinear elliptic equation. These problems arise from applications in engineering, physics, economics, and finance, see [116, 79, 110, 122, 123]. k -Hessian equations are another kind of examples of the fully nonlinear elliptic equations. These problems arise in geometry. If we take the extreme values of k [38], i.e, if $k = 1$, we get Laplace equation and if $k = n$, we get Monge-Ampère equation. Monge-Ampère equation is an interesting fully nonlinear elliptic PDEs that frequently arises in differential geometry, for example, in the Weyl and Minkowski problems in differential geometry of surfaces.

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It also has some geometric applications which can be found in [160]. For the existence and regularity results for Monge-Ampère equations, we refer to [33, 42, 43, 42]. Monge-Ampère equations have the divergence form while Bellman and Isaac's equations are of nondivergence form. Due to the nondivergence structure of Bellman and Isaac's equations, we can not apply the theory of the weak solutions that comes from the integration by parts. At the same time some fully nonlinear equations of the nondivergence as well as divergence form shares maximum principle with it which helps us to define another notion of the weak (viscosity) solution which is the most appropriate for the study of the nondivergence form equations.

Let us briefly present the developments on the theory of nonlinear PDEs. In the beginning, the general theory of some classes of nonlinear equations was not developed, however, most of the nonlinear problems were studied by the experts in those fields of mathematics, where these problems arose. The first nonlinear elliptic equation subjected to intensive investigations was the Monge-Ampère equation. This equation arises in the theory of convex surfaces. Due to the lack of general theory of Monge-Ampère equation, it was studied with the help of convex surfaces theory. In 1958, Aleksandrov introduced the notion of generalised solution of Monge-Ampère equation in [2]. This notion of solution is closely related to the normal mapping, see Chapter 1 [87]. Up to 1971, the smoothness of its generalized solutions was proved only for the case of two variables, see [12]. In 1971, Pogorelov proved the interior regularity for the multidimensional case, see [144]. The smoothness of generalised solution up to the boundary for the multidimensional Monge-Ampère equation was proved in 1982 after the general theory of nonlinear equations was developed see, [111, 161]. N.V Krylov in [108, 109] using probabilistic methods in 1972 showed the solvability of general degenerate Bellman equations in the whole space in the class of functions with bounded derivatives. These results seem to be very important because a large class of nonlinear equations are treated there, in fact, the Monge-Ampère equations are also a partial case of Bellman equations. Thus it appeared that there is a possibility of constructing a general theory of nonlinear equations including the Monge-Ampère equations. Till 1979, the probabilistic methods played the main role in the theory of the Bellman equations. In 1979, Brezis and Evans [27] considered the case of the Bellman equations with two elliptic operators and proved its solvability in $C^{2,\alpha}$. In [64], L.C. Evans obtained the local $C^{2,\alpha}$ estimates for elliptic Bellman equations with constant coefficients. In addition to $C^{2,\alpha}$ estimates, L.C. Evans also obtained the existence of solution by the method of continuity in [64]. Independently, in the same year, N.V Krylov [111, 112] obtained the same results for elliptic and parabolic Bellman equations with variable coefficients and also proved the $C^{2,\alpha}$ regularity up to the boundary for the elliptic case. The basis of the works [64, 111, 112] are the results of Krylov and Safonov [113, 114] on the Hölder estimates for solutions of linear equations with measurable coefficients. There are also some existence results for the weak solutions of Bellman equations, see [18]. The methods mentioned above for the study of Bellman equations had used the convexity(concavity) of F in D^2u . But there are also fully nonlinear elliptic equations which are not convex in D^2u , for instance, Isaac's equations. These equations arise from stochastic differential game theory, for the details, we refer to [110]. In order to take care of these equations, various methods developed for the existence of solutions to

the Dirichlet problems of the form (1.1), (1.2). These methods comprise the following:

- (i) the classical method of continuity which can be applied, when the function F is concave (or convex) in D^2u variable, see Chapter 17 [85], [108], Section 3 [37].
- (ii) utilization of the stochastic game or control theory representation, see [110, 123].
- (iii) approximation by nonlinear Poisson equations based on [66].
- (iv) the Perron's scheme developed in [94, 95], see also [47].
- (v) a discretization method which was developed by N. S. Trudinger and H. Kuo [115].

In this article, we are interested in (iv) i.e., viscosity solutions of the second order fully nonlinear elliptic PDEs. This notion of the solution first of all appears in [52] in the context of the first order Hamilton Jacobi equations. The idea to define the viscosity solutions has been taken from the early appeared articles by L. C. Evans [64, 66] concerning “weak passages to the limit” in equations satisfying certain maximum principle. In these developments, roadmaps were indicated by the aspects of nonlinear functional analysis and nonlinear semigroup theory. In [52], M. G. Crandall, P. L. Lions have defined various equivalent definitions of the viscosity solution.

Further, the results of [52] was simplified in [46] by using another equivalent definition given in [52]. This notion has been quite successful in the study of existence and uniqueness theory of solutions of Hamilton-Jacobi equations, see for example, G. Barles [14, 16], M. G. Crandall, H. Ishii, and P. L. Lions [48], M. G. Crandall, and P. L. Lions [53, 126], H. Ishii [95, 96, 97]. The notion of the viscosity solution in the context of the second order fully nonlinear elliptic equations was appeared in [17, 123]. In [123], P. L. Lions obtained the general uniqueness theorem by identifying the viscosity solution as the value function of the associated optimal control problem, see also [130, 128, 124]. In [123, 129], P. L. Lions made the assumption that F is convex or concave in (D^2u, Du, u) and that F grows linearly in (D^2u, Du, u) and not only this much but also, that F is uniformly decreasing in u . The main question concerning the uniqueness was that how to extend the result of P.L. Lions to more general F (like, nonconvex or nonconcave), and also the proof of P. L. Lions depends fully on the theory of the stochastic optimal control problems [124]. In order to solve this problem, the first step was taken by R. Jensen, using purely analytic techniques in [99]. He considered the operator F , which was independent of x , the solutions were in $W^{1,\infty}$ (i.e., Lipschitz continuous), and the domain was bounded. A close observation of the proof in [99] shows that the assumption of the independence of x was not necessary. The Lipschitz continuity and independence of x were relaxed in [100]. H. Ishii [94] refined the Jensen's results, and also he obtained the uniqueness results that cover the results of P.L. Lions [123]. In the sequel of Ishii's work, P. L. Lions and H. Ishii obtained very general existence and uniqueness results concerning various boundary conditions like, for instance Dirichlet and Neumann conditions. They also applied the method and results obtained there to quasilinear Monge-Ampère equations and obtained some regularity results like, Hölder continuity of the solutions and concavity of the solutions. One of the best references for theory of the viscosity solutions for the second order fully nonlinear elliptic PDEs is [47]. [102] is also a very good reference for the beginners. In [102], N. Katzourakis has taken the p -Laplacian and ∞ -Laplacian as the particular examples to compare the viscosity solution with variational method in L^∞ . In [47], M. G.

Crandall, H. Ishii, P. L. Lions have established the existence of the viscosity solutions by Perron's method. They have given the full exposure to the comparison principle which was needed for applying the Perron's method of sub and super solutions. Regularity of viscosity solutions to fully nonlinear elliptic PDEs are given in [36]. Up to now, in order to define the viscosity solutions of F , we need that it is continuous in the x variable. L. Caffarelli [34, 35] and N. Trudinger [158] started the study of viscosity solutions of fully nonlinear second order uniformly elliptic equations from the L^p point of view. In particular, $W^{2,p}$ estimates have been obtained in [35] and later, by using the results of [69], were generalised by L. Escauriza [63]. The similar results have been extended to the parabolic case by L. Wang [162, 163]. Later in 1995, L. Caffarelli [50] defined the precise notion of the L^p -viscosity solution. In order to define the notion of L^p -viscosity solution, we do not need the continuity of F in the x variable.

The aim of this survey is to provide the recent results on the existence, uniqueness and regularity questions to fully nonlinear elliptic equations. More specifically, we discuss more on these questions to equations involving Pucci's operator and also focus more on eigenvalue problems for singular fully nonlinear second order elliptic operators.

The organization of this papers is as follows. We present basic definitions and examples in Section 2, which are needed for our presentation. In Section 3, we present the comparison principle and the existence results for proper operator by Perron's method. In order to deal with existence results for nonproper operator, we need Liouville type theorem that is presented in Section 4. This section also contains some existence results. Section 5 deal with the eigenvalue and eigenfunctions for Pucci's operator and more general operators. It contains several other results related to eigenvalue, like, isolation as well as the simplicity of the eigenfunction and bifurcation results. There are many existence and nonexistence results which are also characterized in terms of the conditions on nonlinear terms. In Section 6, we present the comparison principle, Liouville type theorem and some existence and regularity results related to singular fully nonlinear elliptic equation and finally Section 7 deals with some regularity results to the solutions of fully nonlinear elliptic equations.

2. Basic Definitions and Examples

Let us first recall some basic definitions for second order fully nonlinear elliptic PDEs, see [47] for details.

DEFINITION 1. [47] An equation of the form (1.1) is said to be degenerate elliptic if it satisfies the following monotonicity condition:

$$F(x, r, p, X) \leq F(x, r, p, Y) \text{ whenever } Y \leq X, \text{ for all } (x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \quad (2.1)$$

further, we say that F is proper if it also satisfies following monotonicity condition

$$F(x, r, p, X) \leq F(x, s, p, X) \text{ whenever } r \leq s, \text{ for all } (x, p, X) \in \Omega \times \mathbb{R}^n \times S(n). \quad (2.2)$$

The operator F is said be uniformly elliptic if there exist $0 < \lambda \leq \Lambda$ such that

$$\lambda \text{trace}(N) \leq F(x, r, p, M - N) - F(x, r, p, M) \leq \Lambda \text{trace}(N), \tag{2.3}$$

or equivalently

$$\lambda \text{trace}(N) \leq F(x, r, p, M) - F(x, r, p, M + N) \leq \Lambda \text{trace}(N), \tag{2.4}$$

for every nonnegative definite matrix $0 \leq N$. In this case, we say that the numbers λ and Λ are ellipticity constants for F , see example 1.12 [47]. Here and below, for any two symmetric matrix X, Y we will write $X \leq Y$ whenever $Y - X$ is a nonnegative semidefinite matrix.

EXAMPLE 1. ([47]) Let us start from the simple partial differential equation

$$F(x, u, Du, D^2u) = L(u) = - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \tag{2.5}$$

where $A(x) = [a_{i,j}(x)]$ is an $n \times n$ real symmetric matrix. Clearly (2.5) is degenerate elliptic if $A(x) \geq 0$ and it is proper if $c(x) \geq 0$. Further, it is also uniformly elliptic if there also exist constants $0 < \lambda \leq \Lambda$, such that $\lambda |\xi|^2 \leq A(x) \xi_i \xi_j \leq \Lambda |\xi|^2$ for all $0 \neq \xi \in \mathbb{R}^n$.

Let us take $A(x) = I, b(x) = 0$ and $c(x) = 0$, we get the following equation

$$F(x, u, Du, D^2u) = -\text{trace}(D^2u) = -\Delta u = 0. \tag{2.6}$$

Clearly, (2.6) is uniformly elliptic with ellipticity constants $\lambda = \Lambda = 1$.

EXAMPLE 2. ([47]) Let us consider following second order quasilinear equation

$$F(x, u, Du, D^2u) = - \sum_{i,j=1}^n a_{ij}(x, Du) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(x, u, Du) = 0, \tag{2.7}$$

where $A(x, p) = [a_{i,j}]$ is an $n \times n$ symmetric matrix with real entries. The above equation can also be written as follows

$$F(x, u, Du, D^2u) = -\text{trace}(A(x, Du)D^2u) + b(x, u, Du) = 0. \tag{2.8}$$

Equation (2.8) is degenerate elliptic if $A \geq 0$, and it is proper if b is also nondecreasing with respect to u .

EXAMPLE 3. ([47]) Let us consider quasilinear equation in the divergent form

$$F(x, u, Du, D^2u) = - \sum_{i=1}^n \frac{\partial}{\partial x_i} (a_i(x, Du)) + b(x, u, Du) = 0. \tag{2.9}$$

Equation (2.9) can be put in the form (2.7) (if enough regularity is available) as follows

$$F(x, u, Du, D^2u) = - \sum_{i,j=1}^n \frac{\partial a_i}{\partial p_j}(x, Du) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(x, u, Du) - \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}(x, Du) = 0, \tag{2.10}$$

and so it takes the form

$$F(x, u, Du, D^2u) = -\text{trace}(D_p a(x, Du) D^2u) + b(x, u, Du) - \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}(x, Du) = 0. \tag{2.11}$$

Thus (2.9) is degenerate elliptic if $a(x, p)$ is nondecreasing in second variable and further it is also proper if b is nondecreasing in second variable. For more about the quasilinear equations, see [85].

EXAMPLE 4. ([30]) Let us consider the minimal surface equation

$$F(x, u, Du, D^2u) = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial u}{\partial x_i}}{\sqrt{1 + |Du|^2}} \right), \tag{2.12}$$

which is the Euler-Lagrange equation of the area functional

$$\int_{\Omega} \sqrt{1 + |Du|^2} dx, \quad \Omega \subset \mathbb{R}^n.$$

An easy differentiation in (2.12) yields the following equation

$$F(x, u, Du, D^2u) = -\text{trace}(A(x, Du) D^2u) = 0, \tag{2.13}$$

where

$$A(x, Du) = [a_{i,j}] = \left[\delta_{i,j} - \frac{\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}}{1 + |Du|^2} \right].$$

A simple computation shows that

$$a_{i,j} \xi_i \xi_j \geq |\xi|^2 \left(1 - \frac{|Du|^2}{1 + |Du|^2} \right),$$

since $\frac{|Du|^2}{1 + |Du|^2} < 1$, thus (2.12) is degenerate elliptic but not uniformly elliptic because as $|Du| \rightarrow \infty$, implies that $\frac{|Du|^2}{1 + |Du|^2} \rightarrow 1$. For more details about Example (4), see [30].

EXAMPLE 5. ([30]) Finally, let us consider some examples of fully nonlinear partial differential equation. Let $z = u(x, y)$ be surface in \mathbb{R}^3 . Its Gaussian curvature $K(x, y)$ is given by the formula

$$K(x, y) = \frac{1}{(1 + |Du|^2)^2} \left| \begin{array}{cc} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y \partial x} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{array} \right|. \tag{2.14}$$

One question that naturally comes into mind that given a smooth function $K(x, y)$, can we find a function having the graph with Gaussian curvature $K(x, y)$? In the case $n = 2$, the answer of this question can be given by the solution of following problem

$$F(x, Du, D^2u) = -\text{Det}(D^2u) + K(x, y)(1 + |Du|^2)^2, \tag{2.15}$$

while for any n , the answer of the above question can be given by solution of the following equation

$$F(x, Du, D^2u) = -\text{Det}(D^2u) + K(x)(1 + |Du|^2)^{\frac{n+2}{2}}. \tag{2.16}$$

Equations (2.15), (2.16) are called Monge-Ampère equation. For the details, we refer to [30]. The next two examples on the Monge-Ampère and Hamilton Jacobi Isaac’s equations are taken from the celebrated papers of Crandall, Ishii and Lions [47].

EXAMPLE 6. ([47]) General Monge-Ampère equation, which contains (2.15) and (2.16) as a particular case, is

$$F(x, u, Du, D^2u) = -\text{Det}(D^2u) + f(x, u, Du) = 0, \tag{2.17}$$

where u is convex and $f(x, r, p) \geq 0$. Observe that (2.17) is degenerate elliptic and proper if f is nondecreasing in u . It has many applications in geometry like affine geometry, see[160]. For more information on the Monge-Ampère equations, see [87, 60].

EXAMPLE 7. ([47]) Let us recall two fully nonlinear second order PDEs called HJB equation and Hamilton-Jacobi-Isaac’s equations. These are the fundamental equations of the stochastic control problem and differential game theory, respectively. We will also present some examples based on these equations. For this, let us consider a family of linear elliptic operators

$$L^\alpha u = - \sum_{i,j=1}^n a_{ij}^\alpha(x) \frac{\partial^2 u}{\partial x_i \partial x_j} u + \sum_{i=1}^n b_i^\alpha(x) \frac{\partial u}{\partial x_i} + c^\alpha(x)u,$$

and

$$L^{\alpha\beta} u = - \sum_{i,j=1}^n a_{ij}^{\alpha\beta}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} u + \sum_{i=1}^n b_i^{\alpha\beta}(x) \frac{\partial u}{\partial x_i} + c^{\alpha\beta}(x)u.$$

Let us define

$$F(x, u, Du, D^2u) = \sup_{\alpha \in \mathcal{A}} (L^\alpha u - f^\alpha(x)), \tag{2.18}$$

and

$$F(x, u, Du, D^2u) = \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} (L^{\alpha\beta} u - f^{\alpha\beta}(x)). \tag{2.19}$$

The statements (2.18) and (2.19) are called HJB equation and Hamilton-Jacobi-Isaac equation, respectively.

Let us refer to the work of J. Kovats [106, 107] for the series of the following examples.

EXAMPLE 8. ([106]) Let us consider following second order PDE

$$F(x, u, Du, D^2u) = -\Delta u + \left(\frac{\partial^2 u}{\partial x_1^2}\right)^- = 0, \tag{2.20}$$

where $(t)^+ = \max\{t, 0\}$ and $(t)^- = \max\{-t, 0\}$. (2.20) is clearly fully nonlinear second order (possibly simplest one) PDE. Let us present the PDE (2.20) in the form of (2.18) by introducing the matrix

$$A^\alpha = \begin{bmatrix} \alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{2.21}$$

$$F(x, u, Du, D^2u) = \max_{\alpha \in [1,2]} [-\text{trace}(A^\alpha D^2u)] = 0. \tag{2.22}$$

EXAMPLE 9. ([106]) Let us consider one more example in this sequel

$$F(x, u, Du, D^2u) = -\Delta u - \left(\frac{\partial^2 u}{\partial x_1^2}\right)^+ + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^- = 0. \tag{2.23}$$

Further, this equation can be written in the form (2.16) with the help of the following matrix

$$A^{\alpha\beta} = \begin{bmatrix} \alpha & & & \\ & \beta & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}. \tag{2.24}$$

$$F(x, u, Du, D^2u) = \max_{\alpha \in [1,2]} \min_{\beta \in [1,2]} [-\text{trace}(A^{\alpha\beta} D^2u)] = 0. \tag{2.25}$$

EXAMPLE 10. ([106]) Let us consider the perturbed equation for $\varepsilon \in [0, 1]$,

$$F(x, u, Du, D^2u) = \Delta u + \varepsilon \left(\frac{\partial^2 u}{\partial x_1^2}\right)^+ - (1 - \varepsilon) \left(\frac{\partial^2 u}{\partial x_2^2}\right)^- = 0, \tag{2.26}$$

which can also be put in the form (2.28). For this, we will introduce the following equation matrix

$$A^{\alpha\beta}(x) = \begin{bmatrix} 1 + \varepsilon\alpha & & & \\ & 1 + (1 - \varepsilon)\beta & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}. \tag{2.27}$$

With the help of this matrix, (2.27), (2.26) can be written as follows:

$$F(x, u, Du, D^2u) = \max_{\alpha \in [1,2]} \min_{\beta \in [1,2]} [-\text{trace}(A^{\alpha\beta}(x)D^2u)] = 0. \tag{2.28}$$

EXAMPLE 11. ([107]) Let us consider the following equation

$$F(x, u, Du, D^2u) = \max \left\{ -\Delta u + \left(\frac{\partial^2 u}{\partial x_2^2} - \frac{1}{2} \frac{\partial^2 u}{\partial x_1^2} \right)^+, -\Delta u + \left(\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{2} \frac{\partial^2 u}{\partial x_2^2} \right)^+ \right\} \tag{2.29}$$

$$A^{\alpha\beta} = \begin{bmatrix} \frac{\alpha}{\beta} & & & & \\ & \frac{\beta}{\alpha} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}. \tag{2.30}$$

With the help of this matrix (2.30) can be written as

$$F(x, u, Du, D^2u) = \max_{1 \leq \alpha < 2} \min_{1 \leq \beta < 2} [-\text{trace}[A^{\alpha\beta}D^2u]] = 0. \tag{2.31}$$

All the equations (2.25), (2.28), (2.31) are uniformly elliptic PDEs. HJB equations defined by (2.18) arise in the optimal control problem of stochastic process, see [68, 92, 93]. If in a stochastic control problem, the diffusion coefficient is a control variable, we get a very important class of second order fully nonlinear PDEs called Pucci’s extremal operators.

DEFINITION 2. (Pucci’s Operator [146, 147]) Let us define, for fixed $0 < \lambda \leq \Lambda$, the Pucci’s extremal operators

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^-(M) = \sup_{A \in \mathcal{A}[\lambda, \Lambda]} \{-\text{trace}(AM)\} \\ \mathcal{P}_{\lambda, \Lambda}^+(M) = \inf_{A \in \mathcal{A}[\lambda, \Lambda]} \{-\text{trace}(AM)\}, \end{cases} \tag{2.32}$$

where $\mathcal{A}[\lambda, \Lambda]$ denotes the set of real symmetric matrices having the eigenvalues in $[\lambda, \Lambda]$. Another equivalent definition of the Pucci’s extremal operators are given below:

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^-(M) = -\Lambda \sum_{e_i > 0} e_i - \lambda \sum_{e_i < 0} e_i, \\ \mathcal{P}_{\lambda, \Lambda}^+(M) = -\lambda \sum_{e_i > 0} e_i - \Lambda \sum_{e_i < 0} e_i, \end{cases} \tag{2.33}$$

where e_i are the eigenvalues of $M \in S(n)$.

For more properties of the Pucci’s operator, see Lemma 2.13 [45]. Note that this definition is more suitable from the computation point of view, and also when we take

$\lambda = \Lambda = 1$, these operators coincides with $-\Delta u$. At the same time some authors use another definition of the Pucci’s operator

$$\mathcal{M}_{\lambda,\Lambda}^+(M) = -\mathcal{P}_{\lambda,\Lambda}^-(M) \text{ and } \mathcal{M}_{\lambda,\Lambda}^-(M) = -\mathcal{P}_{\lambda,\Lambda}^+(M).$$

In this case if we take $\lambda = \Lambda = 1$, then it coincides with Δu . For more properties of $\mathcal{M}_{\lambda,\Lambda}^\pm$, see Lemma 2.10 [36]. These operators play almost same role in the study of the regularity theory of fully nonlinear elliptic PDEs as Laplace equation in the regularity theory for linear second order elliptic PDEs.

LEMMA 1. (Equivalent forms of uniform ellipticity conditions [82]) For $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \longrightarrow \mathbb{R}$, following conditions are equivalent

(i) F is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$

(ii)

$$F(x, r, p, M) - F(x, r, p, M + N) \leq \Lambda \text{trace}(N^+) - \lambda \text{trace}(N^-) \tag{2.34}$$

(iii)

$$\mathcal{P}_{\lambda,\Lambda}^-(M - N) \leq F(x, r, p, M) - F(x, r, p, N) \leq \mathcal{P}_{\lambda,\Lambda}^+(M - N), \tag{2.35}$$

where $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $M, N \in S(n)$.

For the proof, we refer to Proposition 1.1.2 [82].

REMARK 1. We remark that in proving (2.35), it is easy to see that every uniformly elliptic equation can be written as Isaac’s equation.

Recently, P. Felmer and H. Chen in [70] introduced another operator resembling with Pucci’s extremal operators and defined as follows:

$$\begin{cases} \mathcal{P}^-(r, \lambda, \Lambda)(M) = -\Lambda(r) \sum_{e_i > 0} e_i - \lambda(r) \sum_{e_i < 0} e_i, \\ \mathcal{P}^+(r, \lambda, \Lambda)(M) = -\lambda(r) \sum_{e_i > 0} e_i - \Lambda(r) \sum_{e_i < 0} e_i, \end{cases} \tag{2.36}$$

where $\lambda, \Lambda : (0, \infty) \longrightarrow \mathbb{R}$, are continuous functions. Let us recall the basic definitions of classical and viscosity solutions to fully nonlinear elliptic PDEs.

DEFINITION 3. A function $u \in C^2(\Omega)$ is called classical solution of (1.1) in Ω if it satisfies the equation point-wise. Further, it is called the classical solution of the Dirichlet problem (1.1), (1.2) if $u \in C^2(\bar{\Omega})$, classical solution of (1.1) and $u = 0$ on $\partial\Omega$.

DEFINITION 4. (Sub and Superjet [47]) Let Ω be a bounded open subset of \mathbb{R}^n and $u : \Omega \longrightarrow \mathbb{R}$ be an upper semi continuous function at \hat{x} . Then the second order superjet of u at \hat{x} is denoted by $J_{\Omega}^{2,+} u(\hat{x})$ and is given by

$$\left\{ (p, X) \in \mathbb{R}^n \times S(N) \mid u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2) \text{ and } x \in \Omega \text{ as } x \rightarrow \hat{x} \right\} \tag{2.37}$$

Thus the superjet of an upper semi continuous function in Ω is a map

$$J_{\Omega}^{2,+}u : \Omega \longrightarrow P(\mathbb{R}^N \times S(N)),$$

and its value at \hat{x} is given by (2.37). Similarly, for a lower semi continuous function $u : \Omega \longrightarrow \mathbb{R}$ the second order subjet of u at the point \hat{x} is denoted by $J_{\Omega}^{2,-}u(\hat{x})$ and its value at the point \hat{x} is given by

$$\left\{ \begin{aligned} \{ (p, X) \in \mathbb{R}^n \times S(N) \mid u(x) \geq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle \\ + o(|x - \hat{x}|^2) \text{ and } x \in \Omega \text{ as } x \rightarrow \hat{x} \}. \end{aligned} \right. \quad (2.38)$$

DEFINITION 5. (*C*-viscosity solution [47]) An upper (resp., lower) semi continuous function $u : \Omega \longrightarrow \mathbb{R}$, is called a viscosity sub (resp., super) solution of (1.1) at \hat{x} if for any $\phi \in C^2(\Omega)$, such that $u - \phi$ has local maximum (resp., minimum) at \hat{x} then

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq (\text{resp.}, \geq) f(\hat{x}). \quad (2.39)$$

If u is a viscosity sub and super solution at \hat{x} then u is called viscosity solution of (1.1) at \hat{x} . Further, if u is viscosity (resp., sub, super) solution of (1.1) at each point of Ω then u is called viscosity (resp., sub, super) solution of (1.1) in Ω . Sometimes we will also say *C*-viscosity solution to separate it from L^p -viscosity solution, which be will introduced below.

We say that $F(x, u, Du, D^2u) \leq (\text{resp.}, \geq, =) f$ in Ω in the viscosity sense when $u \in C(\Omega)$ is a viscosity subsolution (resp., supersolution, solution) of $F = f$ in Ω .

PROPOSITION 1. (Sub and Superjet in terms of test functions [103]) Let $u : \Omega \rightarrow \mathbb{R}$ be an upper (resp., lower) semi continuous function. Then

$$\begin{aligned} J_{\Omega}^{2,+}u(\hat{x}) &= \{ (D\phi(\hat{x}), D^2\phi(\hat{x})) \in \mathbb{R}^N \times S(N) \mid \text{there exists } \phi \in C^2(\Omega) \text{ such that} \\ &\quad u - \phi \text{ attains its maximum at } \hat{x} \}. \\ J_{\Omega}^{2,-}u(\hat{x}) &= \{ (D\phi(\hat{x}), D^2\phi(\hat{x})) \in \mathbb{R}^N \times S(N) \mid \text{there exists } \phi \in C^2(\Omega) \text{ such that} \\ &\quad u - \phi \text{ attains its minimum at } \hat{x} \}. \end{aligned}$$

For the proof, we refer to Proposition 2.6 [103].

DEFINITION 6. (*C*-Viscosity solution [47]) An upper (resp., lower) semi continuous function $u : \Omega \longrightarrow \mathbb{R}$ is said to be viscosity sub (resp., super) solution of (1.1) at \hat{x} if,

$$\begin{aligned} F(\hat{x}, u(\hat{x}), p, X) \leq 0 \text{ (resp., } F(\hat{x}, u(\hat{x}), q, Y) \geq 0) \text{ for } (p, X) \in J_{\Omega}^{2,+}u(\hat{x}) \\ \text{(resp., } (q, Y) \in J_{\Omega}^{2,-}u(\hat{x}) \text{)}. \end{aligned}$$

If u is a viscosity sub (resp., supersolution) at each point of Ω , then we say that u is a viscosity sub (resp., supersolution) of (1.1) in Ω . Further, if u is a viscosity sub and supersolution of (1.1) at each point of Ω then it called viscosity solution of (1.1) in Ω .

REMARK 2. We have taken the above definition of C -viscosity solution from [47]. But in [94], the author has considered more general definition of viscosity solutions.

REMARK 3. Both Definitions (5) and (6) of viscosity solution are equivalent in view of Proposition 1.

Now we recall the following relation between classical and viscosity solution for degenerate elliptic PDEs by the following theorem.

THEOREM 1. Assume that F is degenerate elliptic. Then a function $u : \Omega \rightarrow \mathbb{R}$ is a classical solution of (1.1) in Ω if and only if u is a viscosity solution of (1.1) in Ω and $u \in C^2(\Omega)$.

For the proof of Theorem 1, we refer to Proposition 2.3 [103]. In the above definition of viscosity solution of (1.1), we assume that F, f are continuous function of the x variable. In order to define the notion of L^p -viscosity solution, we do not need the continuity of F in the x variable. We just only need the measurability of F in the x variable. The following definition of the L^p -viscosity solution, we have taken from [45].

DEFINITION 7. (L^p -viscosity solution) Let F be proper, $n < 2p$ and $f \in L^p_{loc}(\Omega)$. A function $u \in C(\Omega)$ is an L^p -viscosity sub solution (resp., supersolution) of (1.1) in Ω if for all $\phi \in W^{2,p}_{loc}(\Omega)$, whenever $\varepsilon > 0, \mathcal{O} \subset \Omega$ is open and

$$F(x, u(x), D\phi(x), D^2\phi(x)) - f(x) \geq +\varepsilon \text{ a.e. in } \mathcal{O} \tag{2.40}$$

$$(\text{resp.}, F(x, u(x), D\phi(x), D^2\phi(x)) - f(x) \leq -\varepsilon \text{ a.e. in } \mathcal{O}),$$

then $u - \phi$ cannot have a local maximum (resp., minimum) in \mathcal{O} .

Equivalently, u is an L^p -viscosity subsolution (resp., supersolution) if for all $\phi \in W^{2,p}_{loc}(\Omega)$, and point $\hat{x} \in \Omega$ at which $u - \phi$ has a local maximum (resp., minimum) one has

$$\begin{cases} \text{ess lim inf}_{x \rightarrow \hat{x}} (F(x, u(x), D\phi(x), D^2\phi(x)) - f(x)) \leq 0 \\ \text{(ess lim sup}_{x \rightarrow \hat{x}} (F(x, u(x), D\phi(x), D^2\phi(x)) - f(x)) \geq 0). \end{cases} \tag{2.41}$$

Moreover, u is an L^p -viscosity solution of (1.1) in Ω if it is both an L^p -viscosity subsolution and L^p -viscosity supersolution.

REMARK 4. The restriction $n < 2p$, guarantees that the test function ϕ is continuous. Further, particular case of the classical result of Calderón and Zygmund stating that if $n < 2p$, then the functions in $W^{2,p}_{loc}$ are pointwise twice differentiable (in the sense of possessing second order Taylor expansions) a.e. Thus the derivatives of ϕ appearing in the definition have a pointwise as well as a distributional sense.

DEFINITION 8. (L^p -strong solution) A function $u \in W_{loc}^{2,p}(\Omega)$ is called L^p -strong subsolution (resp., strong supersolution) of $F = 0$ in Ω if

$$F(x, u, Du, D^2u) \leq 0 \text{ (resp., } F(x, u, Du, D^2u) \geq 0) \text{ a.e in } \Omega.$$

Further, if u is a L^p -strong sub and supersolution of $F = 0$ in Ω it is called L^p -strong solution of the same equation.

Following theorem gives the conditions under which different type of solutions are equivalent.

THEOREM 2. Let F satisfies the structure (SC), $n \leq p < \infty$ and $f \in L_{loc}^p(\Omega)$.

(i) If u is an L^p -strong solution of $F \leq f$ in Ω , then u is an L^p -viscosity solution of $F \leq f$ in Ω .

(ii) If F, f are continuous in x , then u is a C -viscosity solution of $F \leq f$ in ω if and only if u is an L^p -viscosity solution of $F \leq f$ in Ω .

(iii) If $n \leq q < p$, then u is an L^q -viscosity solution of $F \leq f$ in Ω if and only if u is an L^p -viscosity solution of $F \leq f$ in Ω .

All the statements (i)-(iii) are also true for $F \geq f$ and $F = f$.

For the proof of Theorem 2, we refer to Theorem 2.1 [51]. Structural condition (SC) appeared in the statement of the theorem is given below.

3. Existence and uniqueness of solution

3.1. Perron’s Method

The first result concerning the existence of viscosity solution for the second order fully nonlinear elliptic PDEs was given by P. L. Lions [123]. It was the second in a series and the first was [122], in which he had given the conditions ensuring the continuity of the cost function of the optimal control problem. Later in [123], he considered the following equation

$$F(x, u, Du, D^2u) = \sup_{\alpha \in \mathcal{A}} (L^\alpha u(x) - f^\alpha(x)) = 0 \text{ in } \Omega, \tag{3.1}$$

and showed that every continuous optimal cost function is a viscosity solution of (3.1). The proof of P. L. Lions was based on stochastic control theory and also used the convexity of the function F in D^2u . From (3.1) it is clear that F is convex in (u, Du, D^2u) , however, we have seen several examples which are not convex in D^2u . Later in [95], H. Ishii extended the classical Perron’s method to the case of viscosity solutions. It covers the results of P.L Lions, moreover, it also covers a class of fully nonlinear operators which are not convex in D^2u . Below, we will present some existence results based on the Perron’s method, where F need not to be convex. However, the convexity of F in D^2u plays an important role in the uniqueness and regularity theory for the viscosity solutions to fully nonlinear elliptic PDEs. Before presenting the existence results, let us present some definitions and notations to make the statements clear.

DEFINITION 9. Given a function $u : \Omega \rightarrow \mathbb{R}$, we denote the upper (resp., lower) semi continuous envelope of u by u^* (resp., u_*) and is defined by

$$u^*(x) = \limsup_{\varepsilon \rightarrow 0} \{u(y) : 0 < |x - y| < \varepsilon, \text{ and } y \in \Omega\}$$

$$(\text{resp.}, u_*(x) = \liminf_{\varepsilon \rightarrow 0} \{u(y) : 0 < |x - y| < \varepsilon, \text{ and } y \in \Omega\}.$$

EXAMPLE 12. Consider a function $u : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$u(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

It is easy to see that the upper (resp., lower) semi continuous envelope u^* (resp. u_*) of u i.e, $u^* = 1$ (resp., $u_* = 0$).

For the properties of the upper and lower semicontinuous envelopes of a function, we refer to page 296 [13]. Below, we will denote $\Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n)$ and $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S(n)$ by Γ and $\bar{\Gamma}$, respectively. In [94], H. Ishii proved the following proposition.

PROPOSITION 2. Let F be a continuous function on Γ . Let S be a nonempty family of viscosity subsolutions of $F = 0$ in Ω . Define a function u on Ω by

$$u(x) = \sup\{w(x) \mid w \in S\} \text{ for } x \in \Omega.$$

Assume that $u^*(x) < \infty$ for $x \in \bar{\Omega}$, where u^* is an upper semicontinuous envelop of u . Then u is a viscosity subsolution of $F = 0$ in Ω .

For the proof of Proposition 2, we refer to Proposition 2.2 [94].

PROPOSITION 3. Let F be a continuous function on Γ . Suppose that there is a viscosity subsolution w and a viscosity supersolution v of $F = 0$ satisfying $w \leq v$ in Ω and $w, v \in C(\bar{\Omega})$. Then there is a viscosity solution u of $F = 0$ satisfying $w \leq u \leq v$ in Ω .

For the proof, we refer to Proposition 2.3 [94] and see also [95].

THEOREM 3. Let Ω be an open subset of \mathbb{R}^n and let $F : \mathbb{R} \times \mathbb{R}^n \times S(n) \rightarrow \mathbb{R}$ be continuous and degenerate elliptic. Assume the function $r \rightarrow F(r, p, X)$ is non-decreasing for $(p, X) \in \mathbb{R}^n \times S(n)$. Let w and v be respectively, viscosity sub and supersolutions of

$$u + F(u, p, X) = 0 \text{ in } \Omega. \tag{3.2}$$

Assume further that $w(x) \leq v(x)$ for $x \in \partial\Omega$. Then $w \leq v$ in Ω .

For the proof of Theorem 3, see Theorem 3.1 [94].

THEOREM 4. If the hypotheses of Theorem 3 are satisfied and there exist w and v , respectively continuous sub and supersolution of (3.2) satisfying $w \leq v$ in Ω and $w = v$ on $\partial\Omega$, then there exists a solution u of (3.2) satisfying $w \leq u \leq v$ in Ω .

For the proof of Theorem 4, we refer to Theorem 3.2 [94].

The next theorem gives the existence of the viscosity solution to $F = 0$ in Ω , where F is given by (2.19) under certain conditions on the coefficients of $L^{\alpha\beta}$. Let us consider the required hypotheses;

(A1) Suppose that matrix $A^{\alpha\beta} = (a_{ij}^{\alpha\beta}) = \Sigma^T \Sigma$ with $\Sigma : \Omega \times \mathcal{A} \times \mathcal{B} \longrightarrow M(m, n)$, where $M(m, n)$ is the set all $m \times n$ matrices.

(A2) The functions $A^{\alpha\beta}$, $b^{\alpha\beta}$, $c^{\alpha\beta}$ and $f^{\alpha\beta}$ are uniformly bounded in α, β .

(A3) The functions $\Sigma(\cdot, \alpha, \beta)$, $b^{\alpha\beta}$ are Lipschitz continuous in Ω for each α, β .

(A4) The functions $c^{\alpha\beta}$ and $f^{\alpha\beta}$ are continuous in x for each α, β . Further, suppose that

$$\inf\{c^{\alpha\beta} \mid \alpha \in \mathcal{A}, \beta \in \mathcal{B}\} > 0.$$

REMARK 5. Note that F in this case need not to be convex in D^2u .

THEOREM 5. Assume Ω is bounded and that (A1)-(A4) hold. Let w and v be respectively, viscosity sub and supersolutions of

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega.$$

Assume that $w^*(x) \leq v_*(x)$ for $x \in \partial\Omega$. Then $w^* \leq v_*$ in Ω .

THEOREM 6. Suppose the hypotheses of Theorem 5 hold. Suppose that there are $w, v \in C(\bar{\Omega})$ respectively, viscosity sub and supersolution of

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega. \tag{3.3}$$

Assume further $w \leq v$ in Ω and $w = v$ on $\partial\Omega$. Then there exists a viscosity solution u of (3.3) satisfying $w \leq u \leq v$.

For the proof of Theorems 5 and 6, we refer to Theorems 3.3 and 3.4 [94], respectively.

3.2. Comparison Principle

From Theorems 4 and 6, one question that naturally comes into the mind that under what conditions on F , the conclusion of Theorems 5, 3 holds. Because if it holds, then finding viscosity solutions is equivalent to find the viscosity sub and super-solutions of equation $F = f$, which is very easy in comparison to find the viscosity solution. Furthermore, it also gives the uniqueness of the viscosity solution. Whenever, under the assumption $w \leq v$ on $\partial\Omega$ the conclusion of the Theorems 5, 3 holds, we say that comparison principle holds for the equation under consideration. More precisely, comparison principle states that if $w \in USC(\bar{\Omega})$, $v \in LSC(\bar{\Omega})$ are respectively, sub and supersolution of

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega, \tag{3.4}$$

and satisfy $w \leq v$ on $\partial\Omega$. Then $w \leq v$ in $\bar{\Omega}$. From Theorems 3 and 5 it is clear that the structural conditions on F , which ensures to comparison principle holds vary. In fact,

these conditions vary with regularity of the viscosity solutions. One of the sufficient conditions on F ensuring the comparison principle to hold is given below.

1. Assume that there exists $\gamma > 0$ such that

$$\gamma(r - s) \leq F(x, r, p, X) - F(x, s, p, X) \text{ for } r \geq s, (x, p, X) \in \bar{\Omega} \times \mathbb{R}^n \times S(n). \quad (3.5)$$

2. There is a function $\omega : [0, \infty) \rightarrow [0, \infty)$ that satisfies $\omega(0+) = 0$ such that

$$F(y, r, \alpha(x - y), Y) - F(x, r, \alpha(x - y), X) \leq \omega(\alpha|x - y|^\tau + |x - y|) \quad (3.6)$$

whenever $x, y \in \Omega, r \in \mathbb{R}$ and $X, Y \in S(n)$ such that

$$-3\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \quad (3.7)$$

For the details, see Theorem 3.3 [47]. Further, one can relax the conditions if sub and supersolution are more regular. The right hand side of (3.6) is optimal in the sense that we can not replace the right hand side of (3.6) by $\omega(\alpha|x - y|^\tau + |x - y|)$ with $\tau < 2$ unless the solution is more regular. For example, if we remove the Lipschitz continuity in (A2) of Theorem 5, we may not get the conclusion. For instance, let us consider the following counter example.

EXAMPLE 13. ([94]) Let us consider the following linear second order equation

$$u(x) - \text{trace}A(x)D^2u(x) = 0 \text{ in } \mathbb{R}^n, \quad (3.8)$$

where

$$A(x) = \frac{|x|^{2-\alpha}}{(n-1)\alpha} \left(I - \frac{1}{|x|^\alpha} x \otimes x \right),$$

for some $\alpha \in (0, 2)$ and $x \otimes x$ denotes the matrix $[x_i x_j]_{1 \leq i, j \leq n}$.

If we define

$$\Sigma(x) = \frac{|x|^{1-\alpha/2}}{((n-1)\alpha)^{1/2}} \left(I - \frac{1}{|x|^\alpha} x \otimes x \right) \text{ for } x \in \mathbb{R}^n,$$

then $\Sigma(x) \in S(n), \Sigma^2(x) = A(x)$ for $x \in \mathbb{R}^n$, and $\Sigma \in C^{0,1-\alpha/2}(\mathbb{R}^{n^2})$. Define $u, v : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$u(x) = \exp|x|^\alpha \text{ and } v(x) = \begin{cases} \exp|x|^\alpha & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

If $x \neq 0$, then it satisfies

$$u(x) - \text{trace}A(x)D^2u(x) = v(x) - \text{trace}A(x)D^2v(x) = 0 \quad (3.9)$$

and at $x = 0, A(0) = 0, v$ is lower semicontinuous and $J_\Omega^{2,+}v(0) = \phi$. So it follows from definition that u and v both are viscosity solutions of (3.9) in \mathbb{R}^n . Let us consider a domain Ω with $0 \in \Omega$. It is easy to see that u and v are solutions of (3.9) with $u = v$ on $\partial\Omega$ but $u \not\leq v$ in Ω .

REMARK 6. If the viscosity solutions of

$$F(x, u, Du, D^2u) = f \text{ in } \Omega, \tag{3.10}$$

are Hölder continuous with exponent, $\theta > 0$, and if F satisfies the condition (3.6) with the right hand side $\omega(\alpha|x - y|^\tau + |x - y|)$ for some $\tau > 2 - \theta$ along with (3.5), then the comparison principle still holds. For different type of relaxation in the structural conditions on F with different type of regularity of the viscosity solution, see Section 5A [47].

Next, we want to present comparison theorem from [98] with a slight modification in the hypotheses made by H. Ishii and P.L Lions. These modifications were made by Ishii and Yoshimura which, we have taken the statement from [6]. So let us define the structure condition on F , required in the statement of the next comparison theorem. For each $R > 0$, there exists a modulus $\omega_R : [0, \infty) \rightarrow [0, \infty)$, such that

$$\sup_{t>0} \frac{\omega_R(t)}{(t+1)} < \infty$$

and a positive constant $\frac{1}{2} < \tau \leq 1$, depending on R such that

$$|F(x, r, p, X) - F(x, s, p, X)| \leq \omega_R((|X| + 1)|r - s|^\tau), \tag{3.11}$$

THEOREM 7. Assume that F satisfies (3.11), and a structure condition (SCI), (which is defined next) and for some $\sigma > 0$,

$$r \rightarrow F(x, r, p, X) - \sigma r \text{ is nondecreasing} \tag{3.12}$$

for every $X \in S(n)$, $p \in \mathbb{R}^n$, and $x \in \Omega$. Suppose that $w, v \in C(\bar{\Omega})$ are respectively, a sub and supersolution of

$$F(x, r, p, X) = 0 \text{ in } \Omega,$$

and that $w \leq v$ on $\partial\Omega$. Then $w \leq v$ in Ω .

REMARK 7. The condition (3.12) in the Theorem 7 can be further weakened to the properness of the function F , if either sub or supersolution is strict in the following sense.

We say here w is a strict subsolution of $F = 0$ in Ω if for some $\varepsilon > 0$, $w \in C(\bar{\Omega})$ satisfies

$$F(x, w, Dw, D^2w) \leq -\varepsilon \text{ in } \Omega$$

in the viscosity sense. Similarly, we say that $v \in C(\bar{\Omega})$ is a strict supersolution of $F = 0$ in Ω if for some $\varepsilon > 0$, v satisfies

$$F(x, v, Dv, D^2v) \geq \varepsilon \text{ in } \Omega,$$

in the viscosity sense.

For the proof of Theorem 7 and Remark 7, see Theorem 2.12 and Proposition 2.13 [6], respectively, see also Section 5A of [47]. One consequence of the comparison principle is the following existence and uniqueness result. With the help of the comparison principle we can also obtain estimates for the viscosity solutions of the equations under consideration, see Section 5B [47].

THEOREM 8. *Let us suppose that the comparison principle holds for (3.13). Suppose also that there is a subsolution w and a supersolution v of*

$$\begin{cases} F(x, u(x), Du, D^2u) = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.13}$$

and satisfies

$$w_*(x) = v^*(x) = 0 \text{ for } x \in \partial\Omega.$$

Then $u(x) = \sup\{\omega(x) : w \leq \omega(x) \leq v \text{ and } \omega \text{ is subsolution of (3.13)}\}$, is a viscosity solution of the (3.13).

The proof of above theorem is an application of Perron’s method, see [94, 95, 47]. For the proof of Theorem 8, we refer to Theorem 4.1 [47].

Let us consider:

$$\text{(SC)} \begin{cases} \mathcal{P}_{\lambda, \Lambda}^-(X - Y) - \gamma|p - q| \leq F(x, r, p, X) - F(x, r, q, Y) \\ \hspace{10em} \leq \mathcal{P}_{\lambda, \Lambda}^+(X - Y) + \gamma|p - q| \\ \text{for } x \in \bar{\Omega}, r \in \mathbb{R}, p, q \in \mathbb{R}^n, X, Y \in S(n), \text{ and} \\ F(x, r, p, X) \text{ is nondecreasing in } r. \end{cases} \tag{3.14}$$

REMARK 8. F is uniformly elliptic if and only if

$$\mathcal{P}_{\lambda, \Lambda}^-(X - Y) \leq F(x, r, p, X) - F(x, r, q, Y) \leq \mathcal{P}_{\lambda, \Lambda}^+(X - Y).$$

In literature, some authors says that F is uniformly elliptic and Lipschitz continuous in p instead of saying that F satisfies (SC) defined above.

If F satisfies condition given by (3.14), then we say that F satisfies structure condition (SC). By using approximation of F to obtain the conditions ensuring the comparison principle to hold, we get the following theorem.

THEOREM 9. *Let Ω satisfy a uniform exterior cone condition. Suppose F is continuous, satisfies (SC) and $\phi \in C(\partial\Omega)$ then there are viscosity solutions $\bar{u}, \underline{u} \in C(\bar{\Omega})$ of*

$$\begin{cases} F(x, u, Du, D^2u) = 0 \text{ in } \Omega, \\ u = \phi \text{ on } \partial\Omega, \end{cases} \tag{3.15}$$

such that any other viscosity solution of (3.15) satisfies $\underline{u} \leq u \leq \bar{u}$.

For the complete proof of Theorem 9, we refer to Theorem 1.1 of [49]. Next, we will consider the equations of the form

$$\begin{cases} F(x, u, Du, D^2u) = f(x) \text{ in } \Omega, \\ u = \phi \text{ on } \partial\Omega, \end{cases} \tag{3.16}$$

where $F(x, 0, 0, 0) \equiv 0$.

REMARK 9. All equations $F(x, u, Du, D^2u) = g(x)$ can be put in the form of (3.16) with right hand side $f(x) = g(x) - F(x, 0, 0, 0)$.

Suppose that F is given by (3.16), satisfies

$$(SC1) \begin{cases} \text{(i)} \begin{cases} \mathcal{P}_{\lambda, \Lambda}^-(X - Y) - \gamma|p - q| - \delta|r - s| \leq F(x, r, p, X) - F(x, s, q, Y) \\ \leq \mathcal{P}_{\lambda, \Lambda}^+(X - Y) + \gamma|p - q| + \delta|r - s|, \end{cases} \\ \text{(ii) and the function } r \rightarrow F(x, r, p, X) \text{ is nondecreasing for all } (x, p, X) \in \\ \Omega \times \mathbb{R}^n \times S(n). \end{cases} \tag{3.17}$$

where $\gamma, \delta \geq 0$. If F satisfies only the condition (i) in (3.17), then we say that F satisfies (SC1)(i), similarly if F satisfies (ii) in (3.17), we say that it satisfies (SC1)(ii). Further, if it satisfies both the conditions (i) and (ii) in (3.17), then we say that F satisfies (SC1). Note that if F satisfies (SC1)(i), then $F + \delta u$ is proper. Next, we want to present two existence theorems of [165]. The proofs of these theorems are based on the Perron’s method.

PROPOSITION 4. Suppose $\Omega \subset\subset \mathbb{R}^n$ and satisfy a uniform exterior cone condition. Suppose that $f \in C(\Omega)$ is bounded, $\phi \in C(\partial\Omega)$ and F is Lipschitz continuous and satisfies (SC1), $F(x, 0, 0, 0) = 0$ and

$$d(r - s) \leq F(x, r, p, X) - F(x, s, p, X) \text{ for all } (x, p, X) \in \Omega \times \mathbb{R}^n \times S(n)$$

and $r, s \in \mathbb{R}, r \geq s$. Then there exists a C -viscosity solution u of the Dirichlet problem (3.16).

For the proof of Theorem 4, we refer to Proposition 1.6 [165]. With the help of the convolution and Jensen’s regularization, introduced by Jensen in [99], N. Winter also proved the following theorem.

THEOREM 10. Suppose $\Omega \subset\subset \mathbb{R}^n$ and satisfy a uniform exterior cone condition. Suppose further that F is continuous on Γ and satisfies (SC1). Then for $f \in C(\Omega)$, bounded, and $\psi \in C(\partial\Omega)$, there exists at least one viscosity solution of the Dirichlet problem (3.16).

In [165], N. Winter proved Theorem 10 by using Perron’s method but this method requires comparison principle to hold for which sufficient conditions are given by (3.5)

and (3.6). In general F , given in statement of Theorem 10, does not satisfy (3.5). In order to tackle this, N. Winter approximated given F by $F_\eta(x, r, p, X) = F(x, r, p, X) + \eta r$ for appropriate η so that (3.5) holds, and then regularized F_η by sub-convolution of F_η introduced by R. Jensen in [99], so resulting function F_η^ε becomes Lipschitz continuous, and therefore it will satisfy both (3.5) and (3.6) and comparison principle holds for F_η^ε . With comparison principle, it also satisfies the assumptions of Proposition 4 and therefore has a solution. Finally, N. Winter used the stability result for the viscosity solutions to obtain the viscosity solution of the original Equation (3.16). For the details, we refer to Proposition 1.11 [165].

REMARK 10. In [47], M. G. Crandall, M. Kocan, P. L. Lions and A. Świech relaxed the continuity of F in x and also (SC1) condition by using the notion of L^p -viscosity solution. More precisely, authors assumed that F is measurable in x and also satisfies following condition in r .

For each $R > 0$, there exists $\omega_R : [0, \infty) \rightarrow [0, \infty)$, such that $\omega_R(0+) = 0$ and

$$|F(x, r, p, X) - F(x, s, p, X)| \leq \omega(|r - s|), \tag{3.18}$$

holds for almost all $x \in \Omega$ and $|r| + |s| + |p| + \|X\| \leq R$.

THEOREM 11. Let Ω satisfy a uniform exterior cone condition and F satisfies (SC) a.e. in Ω and (3.18). Let $F(x, 0, 0, 0) \equiv 0$, $f \in L^n(\Omega)$ and $\phi \in C(\partial\Omega)$ then (3.16) has an L^n -viscosity solution.

In order to prove Theorem 11 authors mollify F in x as follows:

$$F_\varepsilon(x, r, p, X) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{\varepsilon}\right) F(y, r, p, X) dy,$$

where $\eta \in C_o^\infty(\mathbb{R}^n)$ satisfies $\eta \geq 0$, $\int_{\mathbb{R}^n} \eta(x) dx = 1$, and used the fact that this F_ε also satisfies (3.18) and $F_\varepsilon(x, 0, 0, 0) \equiv 0$. Using (SC), (3.18) and $|\mathcal{P}^\pm| \leq \Lambda \|X\|$, it is easy to see that F_ε bounded and measurable in x for fixed $(r, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S(n)$ and further, they also used the boundedness of F to get the Lipschitz continuity of F_ε . Further, by approximating $f \in L^n$ by a continuous function in $\bar{\Omega}$, the authors applied Theorem 9 to get a viscosity solution of approximated problem. Further, using the maximum principle and following theorem, they get the existence of viscosity solution to the original problem. Once again for the complete proof, see Theorem 4.1 [49].

THEOREM 12. Let Ω satisfy a uniform exterior cone condition and $\mathcal{C} \subset C(\partial\Omega)$ be compact, $R > 0$ and $B_R = \{f \in L^n(\Omega) \mid \|f\|_{L^n(\Omega)} \leq R\}$. Then the set of all $u \in C(\bar{\Omega})$ such that there exist $f \in L^n(\Omega)$ and $\psi \in \mathcal{C}$ for which u is an L^n -viscosity solution of both

$$P_{\lambda, \Lambda}^-(D^2u) - \gamma|Du| \leq f \text{ and } -f \leq P_{\lambda, \Lambda}^+(D^2u) + \gamma|Du|$$

in Ω and $u = \psi$ on $\partial\Omega$ is precompact in $C(\bar{\Omega})$.

REMARK 11. In Section 3 [49], the authors also constructed the global barriers for the inequation

$$P_{\lambda, \Lambda}^-(D^2u) - \gamma|Du| \geq M$$

by using the results of [134, 135] which further extends the results of [133].

Theorem 11 does not tell us the uniqueness of L^n -viscosity solution but the following theorem tells us about uniqueness of the viscosity solution. Furthermore, it will also give interior $W^{2,p}$ -estimate for the viscosity solution under some more conditions on F . In order to state the next theorem, we need following notation

$$\beta(x, y) = \sup_{X \in S(n)} \frac{|F(x, 0, 0, X) - F(y, 0, 0, X)|}{1 + \|X\|}. \tag{3.19}$$

The following is the existence and uniqueness theorem for the solution of (3.16).

THEOREM 13. *Let Ω satisfy the uniform exterior cone condition. Let F satisfies (SCI) with $\delta|r - s| = \omega(|r - s|)$ and convex (or concave) in D^2u . Let $f \in L^p(\Omega)$ and let $p > p_0(n, \lambda, \Lambda, \gamma)$ and $\text{diam}(\Omega)$ and $\psi \in C(\partial\Omega)$. Then there exists $\theta = \theta(n, p, \lambda, \Lambda)$ such that if for any $y \in \Omega$*

$$\left(\frac{1}{r^n} \int_{B_r(y)} \beta(y, x)^p dx \right)^{\frac{1}{p}} \leq \theta \text{ for } r \leq r_0,$$

for some $r_0 > 0$, then there exists a unique L^p -viscosity solution $u \in W_{loc}^{2,p}(\Omega) \cap C(\bar{\Omega})$ of (3.16) such that for every $\Omega' \subset\subset \Omega$

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \omega(\|u\|_{L^\infty(\Omega)}) + \|f\|_{L^p(\Omega)}),$$

where $C = C(r_0, n, p, \lambda, \Lambda, \gamma, \omega(1), \text{diam}(\Omega), \text{dist}(\Omega', \partial\Omega))$.

Convexity of F in D^2u can be replaced by weaker condition for the details, see Remark 3.3 [156]. The proof of Theorem 13 once again is similar to the proof of Theorem 11. For the details, we refer to Theorem 3.1 [156]. One can obtain a better estimate and existence result for $F(x, u, Du, D^2u) = f$. In [165], author has proved the following theorem.

THEOREM 14. *Assume that F satisfies (SCI) for a.e. x , $F(\cdot, 0, 0, 0) \equiv 0$ in Ω and F is concave (convex) in D^2u . Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \in C^{1,1}$ and consider the (3.16) with $f \in L^p$ and $n - \varepsilon_0 < p < \infty$ with $\phi \in W^{2,p}(\Omega)$. Then there exists a unique L^p -viscosity solution u of (3.16). Moreover, $u \in W^{2,p}(\Omega)$ and*

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left(\|u\|_{L^\infty(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)} + \|f\|_{L^p(\Omega)} \right). \tag{3.20}$$

For the proof of Theorem 14, we refer to Theorem 4.6 [165].

4. Liouville type theorem and some existence results

In the previous section, we presented the existence of viscosity solutions for those proper operators which satisfy the comparison principle and also have a viscosity sub and a viscosity supersolution. Although it covers certain class of the operators but still there is a large class of operators which do not fall in the category mentioned above. We will look for another methods for investigating the existence of solutions. There is a method consisting of converting the differential operator into compact perturbation of the identity operator defined on some Banach space and applying degree theory. For the definition and properties of degree of compact perturbation of identity operator, we refer to [61],[41]. One such method is available in a sequence of articles [35, 36, 86, 156, 73, 159, 165]. In order to apply degree theory, we need the regularity of the viscosity solution, and for that we need an apriori bound, for example, see (3.20) in which we need a bound for $\|u\|_{L^\infty(\Omega)}$ for getting the bound for $\|u\|_{W^{2,p}(\Omega)}$. The most famous classical technique for obtaining an apriori bound is the blow-up method available in [83, 84]. In this method an equation in a bounded domain blows up into another equation in the whole Euclidean space or a half space, see for example Lemma 3.1 [1], and with the help of corresponding Liouville-type theorem in the Euclidean space \mathbb{R}^n or half space \mathbb{R}^n_+ and a contradiction argument, an apriori bounds could be obtained. Thus whole process reduces to find the Liouville-type theorem in the context of the viscosity solutions provided all the mentioned information about the solution are available. Liouville type theorem in the context of the viscosity solution were obtained by A. Cutri, F. Leoni in [55]. They concluded the Liouville type theorem to certain fully nonlinear elliptic PDEs by generalising the classical Hadamard three circle (three sphere) theorem for the Laplace operator given in Chapter 2, Section 12 [145].

THEOREM 15. *Let $u \in C(\mathbb{R}^n)$ be a viscosity solution either of*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) \leq 0 \text{ in } \mathbb{R}^n, \tag{4.1}$$

or of

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2u) \geq 0 \text{ in } \mathbb{R}^n, \tag{4.2}$$

if u is respectively, bounded either from above or from below, and if the parameter α defined by $\alpha = \frac{\Lambda}{\lambda}(n - 1) + 1$ satisfies $\alpha \leq 2$, then u is constant.

For the proof, we refer to Theorem 3.2 [55]. Liouville type theorem for fully nonlinear uniformly elliptic equations having zeroth order term is given below.

THEOREM 16. *Assume that $\beta = \frac{\Lambda}{\lambda}(n - 1) + 1 > 2$ and let $u \in C(\mathbb{R}^n)$, be a viscosity solution of*

$$\begin{cases} F(x, D^2u) - g(x)u^\beta \geq 0 \text{ in } \mathbb{R}^n, \\ u \geq 0, \end{cases} \tag{4.3}$$

where F is uniformly elliptic and $g \in C(\mathbb{R}^n)$, is a nonnegative function such that there exist $G, r_0 > 0$ and $\gamma \geq -2$ satisfying

$$g(x) \geq G|x|^\gamma \text{ for } |x| \geq r_0. \tag{4.4}$$

If $\gamma = -2$ and $0 < p < 1$ or if $\gamma > -2$ and $0 < p \leq \frac{(\beta+\gamma)}{(\beta-\gamma)}$, then $u \equiv 0$.

For the proof, we refer to Theorem 4.1 [55]. Theorem 16 provides optimal result in the cases if $\gamma = -2$ and $p \geq 1$, $\gamma < -2, p > 0$ and $p > \frac{\beta+\gamma}{\beta-2}$ if $\gamma > -2$ for the counter example see remark 7 [55]. In the above theorem, we can replace F by $\mathcal{P}_{\lambda,\Lambda}^+$, to get the following theorem.

THEOREM 17.

$$\begin{cases} \mathcal{P}_{\lambda,\Lambda}^+(D^2u) - u^p \geq 0 \text{ in } \mathbb{R}^n, \\ u \geq 0 \text{ in } \mathbb{R}^n, \end{cases} \tag{4.5}$$

where $p > 1$. Suppose $N \geq 3$ and set $p^- = \frac{\beta}{\beta-2}$, where β is defined in Theorem 15. If $1 < p \leq p^-$ (or $1 < p < \infty$ if $\beta \leq 2$), then the only viscosity supersolution of (4.5) is $u \equiv 0$.

Theorem 17 also holds for $\mathcal{P}_{\lambda,\Lambda}^-$ if we replace β by α and p^- by $p^+ = \frac{\alpha}{\alpha-2}$. In [71], authors considered more general operator(rotationally invariant) and define the critical exponent for that operator. These operators are not, in general, convex or concave but they do not include the gradient term. Next, we would like to present Liouville type theorem for fully nonlinear elliptic operator which depends upon the gradient as well as zeroth order terms. In [39], I. Capuzzo Dolcetta and A. Cutri generalize the Hadamard three circle Lemma to more general fully nonlinear operator and as a consequence got a Liouville type theorem valid for the more general operator. In order to state the precise statement, we need some more hypotheses which are given below.

Suppose that F given by Equation () is uniformly elliptic, satisfies $F(x, 0, 0, 0) = 0$ and

$$F(x, s, p, 0) \leq \rho(|x|)|p| + r(x)s^\alpha \quad \forall (x, s, p) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n,$$

where $\alpha \geq 1$ and r, ρ are continuous functions real valued functions such that

$$|x|\rho(|x|) \geq -\Lambda(n - 1) \text{ and } r(x) \leq 0 \quad \forall x \in \Omega.$$

Under the above assumptions authors proved the following theorems

THEOREM 18. Let w be the viscosity solution of the

$$\begin{cases} F(x, w, Dw, D^2w) \geq 0 \text{ in } \mathbb{R}^n \\ w \geq 0 \text{ in } \mathbb{R}^n, \end{cases} \tag{4.6}$$

where F satisfies the above mentioned conditions. Define

$$\phi(r) = \int_{r_1}^r t^{-\frac{\Lambda}{\lambda}} \exp\left(-\frac{1}{\lambda} \int_{r_1}^t \rho(\tau) d\tau\right) dt.$$

If

$$\lim_{r \rightarrow +\infty} \phi(r) = +\infty,$$

Then u is a constant. Moreover, if $r(x_0) < 0$ for some x_0 then $u \equiv 0$.

THEOREM 19. *Let w be the viscosity supersolution of 4.6 with ρ and $r(x)$ satisfying the following conditions*

$$\sup_{\mathbb{R}^n} |x| \rho(|x|) = K < \infty \text{ and } r(x) \leq -g(|x|) \text{ for } r \text{ large, and}$$

$$\lim_{r \rightarrow +\infty} r^2 g(r) (L - \phi(r))^{\alpha-1} = +\infty.$$

If $\lim_{r \rightarrow +\infty} \phi(r) = L < +\infty$ then $u \equiv 0$.

For the proof of the Theorems 18 and (19), we refer to Theorem 4.1 and Theorem 4.2 [39], respectively.

REMARK 12. For further extension of this idea, see [70].

In [74], P. Felmer, A. Quaas proved the Liouville type theorem for radial solution to (4.5) valid for larger range of p than in (17). In order to state the results of [74], we need to define some terminology.

DEFINITION 10.

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^{\pm}(D^2 u) - u^p = 0 \text{ in } \mathbb{R}^n, \\ u \geq 0 \text{ in } \mathbb{R}^n. \end{cases} \tag{4.7}$$

Assume that u is a radial solution of (4.7) then we say that;

(i) u is a pseudo-slow decaying solution if there exist constants $0 < C_1 < C_2$, such that

$$C_1 = \liminf_{r \rightarrow \infty} r^\alpha u(r) < \limsup_{r \rightarrow \infty} r^\alpha u(r) = C_2.$$

(ii) u is a slow decaying solution if there exists $0 < c^*$, such that

$$\lim_{r \rightarrow \infty} r^\alpha u(r) = c^*.$$

(iii) u is a fast decaying solution if there exists $0 < C$, such that

$$\lim_{r \rightarrow \infty} r^\nu u(r) = C,$$

where $\nu = \alpha$ or $\nu = \beta$ depending on $\mathcal{P}_{\lambda, \Lambda}^+$ or $\mathcal{P}_{\lambda, \Lambda}^-$ appears in (4.7).

Let us consider

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) = u^p \text{ in } \mathbb{R}^n, \\ u \geq 0 \text{ in } \mathbb{R}^n. \end{cases} \tag{4.8}$$

The next theorem deals with the existence and nonexistence of the radial solution to (4.8).

THEOREM 20. *There are critical exponents $1 < p_-^s < p_-^* < p_-^p$, with*

$$p_-^s = \frac{\beta}{\beta - 2},$$

and

$$\frac{\beta + 2}{\beta - 2} < p_-^* < p_-^p = p_n^* = \frac{n + 2}{n - 2},$$

that satisfy:

- (i) *If $1 < p < p_-^*$, then there is no non-trivial radial solution to (4.8).*
- (ii) *If $p = p_-^*$, then there is a unique fast decaying radial solution to (4.8).*
- (iii) *If $p_-^* < p \leq p_-^p$, then there is a unique radial solution to (4.8) which is a slow decaying or a pseudo-slow decaying solution.*
- (iv) *If $p > p_-^p$, then there is a unique slow decaying radial solution to (4.8), where for uniqueness in (ii), (iii) and (iv), we mean uniqueness up to scaling.*

For the proof, we refer to Theorem 1.2 [74]. There are also similar results for $\mathcal{P}_{\lambda, \Lambda}^-$.

THEOREM 21.

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^-(D^2u) = u^p \text{ in } \mathbb{R}^n, \\ u \geq 0 \text{ in } \mathbb{R}^n. \end{cases} \tag{4.9}$$

Suppose that $\alpha > 2$, then there are critical exponents $1 < p_+^s < p_+^* < p_+^p$, with

$$p_+^s = \frac{\alpha}{\alpha - 2}, \quad p_+^* = \frac{\alpha + 2}{\alpha - 2} \text{ and } p_n^* = \frac{n + 2}{n - 2},$$

and

$$\max\{p_+^s, p_n^*\} < p_+^* < p_+^p.$$

- (i) *If $1 < p < p_+^*$, then there is no non-trivial radial solution to (4.9).*
 - (ii) *If $p = p_+^*$, then there is a unique fast decaying radial solution to (4.9).*
 - (iii) *If $p_+^* < p \leq p_+^p$, then there is a unique pseudo-slow decaying radial solution to (4.9).*
 - (iv) *If $p_+^p < p$, then there is a unique slow decaying radial solution to (4.9).*
- Once again for uniqueness in (ii), (iii) and (iv), we mean uniqueness up to scaling.

For the proof, we refer to Theorem 1.1 [74]. In [74], authors also proved the following existence theorem on the ball.

THEOREM 22. *Let $R > 0$ and $B(0, R)$, be a ball of radius R centered at the origin in \mathbb{R}^n . Then the problem*

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^\pm(D^2u) - u^p = 0 \text{ in } B(0, R), \\ u > 0 \text{ in } B(0, R), \\ u = 0 \text{ on } \partial B(0, R), \end{cases} \tag{4.10}$$

has unique solution provided $1 < p < p_\mp^*$.

For the proof of Theorem 22, we refer to Theorem 5.1 [74]. In [75], the same authors define the critical exponent for general uniformly elliptic operators and considered the same problem for that operator. We have taken the next theorem from [1], which is given below.

THEOREM 23. *Let $f : [0, \infty) \rightarrow [0, \infty)$, be a continuous function satisfying the following three assumptions:*

- (i) $f(t) = 0$, if $t = 0, t = 1$, and $f(t) > 0$, if $t \neq 1, t > 0$.
- (ii) There exist constants $\gamma > 0$, and $\sigma \in (1, \frac{\alpha}{\alpha-2})$, such that $f(t) \geq \gamma(t-1)^\sigma$, for $t > 0$.
- (iii) There exists a constant $\bar{b} > 0$, such that $\liminf_{t \rightarrow 0^+} \frac{f(t)}{t} \geq \bar{b}$.

Then any bounded solution of the problem

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^-(D^2w) \leq f(w) \text{ in } \mathbb{R}^n, \\ w \geq 0, \end{cases} \tag{4.11}$$

is either the constant function $w \equiv 0$, or else $w \equiv 1$.

For the proof of above theorem, we refer to Theorem 1.2 [1].

THEOREM 24. *Let w be a viscosity solution of the inequation*

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2w) + f(w) \leq 0 \text{ in } \mathbb{R}^n, \tag{4.12}$$

where f is a continuous nonnegative function. Then either $\inf_{\mathbb{R}^n} w = -\infty$, or $\inf_{\mathbb{R}^n} w$, is a zero of f .

For the proof of above theorem, we refer to Proposition 2.1 [1].

4.1. Liouville type theorem in exterior domain

In the best of our knowledge, the first result on the nonexistence in the exterior domain in the context of the fully nonlinear elliptic equation appeared in [9]. The proof of results in [55] fully depends upon the classical three circle lemma and radial solution. A subset $\Omega \subset \mathbb{R}^n$ is called exterior domain if $\mathbb{R}^n \setminus B_R(0) \subset \Omega \subset \mathbb{R}^n \setminus \{0\}$. In [9], the authors considered the following problem

$$F(D^2u) = u^p, \tag{4.13}$$

where $p \geq 1$ and F is a positively homogeneous uniformly elliptic operator. In order to present the result, let us introduce some terminology used in the statement and are available in [10]. Theorem 3 of [10] states that there exists a nonconstant solution of

$$F(D^2u) = 0 \text{ in } \mathbb{R}^n \setminus \{0\}, \tag{4.14}$$

which is bounded from below in B_1 and bounded from above in $\mathbb{R}^n \setminus B_1$ see also, [11]. In fact theorem states more that the set of all such solutions is of the form $\{a\Psi + b :$

$a > 0, b \in \mathbb{R}$, where $\Psi \in C_{loc}^{1,\delta}(\mathbb{R}^n \setminus \{0\})$ can be chosen to satisfy one of the following homogeneity relations, for all $s > 0$

$$\Psi(sx) = \Psi(x) - \log s \tag{4.15}$$

or

$$\Psi(sx) = s^{-\alpha^*} \Psi(x), \alpha^* \Psi(x) > 0 \text{ in } \mathbb{R}^n \setminus \{0\}, \tag{4.16}$$

for some $\alpha^* \in (-1, \infty) \setminus \{0\}$. The number α^* is called scaling exponent and is crucial for the following theorem. It is easy to observe that if u is a solution of (4.13) in $\mathbb{R}^n \setminus B_r$, then for each $\tau > 0$ the function defined by

$$u_\tau(x) = \tau^{\beta^*} u(\tau x), \text{ where } \beta^* = \beta^*(p) = \frac{2}{p-1},$$

is also a solution of (4.13) in $\mathbb{R}^n \setminus B_{\frac{r}{\tau}}$.

THEOREM 25. *Assume that F is a positively homogeneous uniformly elliptic operator and $p > 1$. Then the Equation 4.13,*

- (1) *has no nontrivial nonnegative supersolution in any exterior domain if $\alpha^* \leq \beta^*$,*
- (2) *has a positive supersolution in \mathbb{R}^n , if $\alpha^* > \beta^*$,*
- (3) *has a positive solution in $\mathbb{R}^n \setminus \{0\}$ if $\alpha^* > \beta^*$.*

For the proof of Theorem 25, we refer to Theorem 1.4[9]. In [8], S. N. Armstrong and B. Sirakov studied such type of problem almost completely for those operators which share maximum principle. It covers a many operators with divergent as well as nondivergent structure including Issac operator. In this paper authors optimize many previous results by providing the minimal conditions on the nonlinearity of the function as well as many new results in the exterior domain and in the half space. At the same time they also consider the system of elliptic equations. As this paper contains many results so instead of presenting all the results here we will refer to whole paper. Below we are presenting Liouville type theorem in the half space. Here we will not give it to separate section. In [149](see also [117]), A. Quaas and B. Sirakov proved the following Liouville type theorem in the halfspace by using the moving plane method.

THEOREM 26. *Suppose we have a nontrivial classical bounded solution of*

$$\begin{cases} \mathcal{P}_{\lambda,\Lambda}^-(D^2u) = f(u) \text{ in } \mathbb{R}_+^n, \\ u \geq 0 \text{ in } \mathbb{R}_+^n, \\ u = 0 \text{ on } \partial\mathbb{R}_+^n, \end{cases} \tag{4.17}$$

where f is locally Lipschitz continuous function with $f(0) \geq 0$. Then if the problem

$$\mathcal{P}_{\lambda,\Lambda}^-(D^2u) = f(u),$$

has a nontrivial nonnegative bounded solution in \mathbb{R}_+^n such that $u = 0$ on $\partial\mathbb{R}_+^n$, then the same problem has a positive solution in \mathbb{R}^{n-1} .

For Liouville type theorem to the system of fully nonlinear elliptic equation in the half space, see [131] and references therein. See [118, 81, 132], where authors dealt with geometric properties of the solution of fully nonlinear equations by using the moving plane method. In [72], P. Felmer and A. Quaas, have shown the existence of eigenvalue for $\mathcal{P}_{\lambda, \Lambda}^{\pm}$ in the $B(0, R)$ by using the Krein-Rutman theorem that was proved by Rabinowitz [151]. In fact, they characterized all the positive solutions of

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^{\pm}(D^2u) = \mu u \text{ in } B(0, R), \\ u > 0 \text{ in } B(0, R), \\ u = 0 \text{ on } \partial B(0, R). \end{cases} \tag{4.18}$$

The precise statement of the theorem is as follows:

THEOREM 27. *The eigenvalue problem (4.18) has a solution (μ_1^{\pm}, u_1^{\pm}) , with μ_1^{\pm} and u_1^{\pm} are positive and u_1^- negative. Moreover, all positive solutions to (4.18) are of the form $(\alpha\mu_1^+, \alpha u_1^+)$ and all negative solutions are of the form $(\alpha\mu_1^-, \alpha u_1^-)$ with $\alpha > 0$.*

For the proof of Theorem 27, we refer to Section 3 [72].

REMARK 13. For the Monge-Ampère equations, see [127]. Below, we will present some results concerning the eigenvalue with some more information, see Theorem 5.

Theorem 22 was extended by P. Felmer and A. Quaas in [72], allowing more general nonlinearity. In order to state precise statement of the theorem, we need some assumptions, which are following.

(f1) $f \in C([0, +\infty))$ and is locally Lipschitz.

(f2) $f(s) \geq 0$ and there is $1 < p < p_{\pm}^*$ and a constant $C > 0$, such that

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{s^p} = C.$$

(f3) There is a constant $c \geq 0$, such that $c - \gamma < \mu_1^{\pm}$ and

$$\lim_{s \rightarrow 0} \frac{f(s)}{s} = c,$$

where $\mu_1^+(\mu_1^-)$, is the first eigenvalue for $\mathcal{P}_{\lambda, \Lambda}^+(\mathcal{P}_{\lambda, \Lambda}^-)$ in B_R with the Dirichlet boundary condition.

THEOREM 28. *Assume $n \geq 3$ and f satisfies the hypotheses (f1), (f2) and (f3) stated above. Then there exist a positive radially symmetric C^2 solution of*

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^{\pm}(D^2u) + \gamma u = f(u) \text{ in } B(0, R), \\ u = 0 \text{ on } \partial B(0, R). \end{cases} \tag{4.19}$$

For the proof, we refer to Theorem 1.2 [72]. The authors have also extended the Theorem 28 in \mathbb{R}^n , by choosing $\gamma = 1$ and $f(u) = u^p$, more precisely, they proved the following theorem.

THEOREM 29. *Assume $n \geq 3$ and $1 < p < p_{\pm}^*$. There exist a positive radially symmetric C^2 solution of*

$$\mathcal{P}_{\lambda, \Lambda}^{\pm}(D^2u) + u = u^p \text{ in } \mathbb{R}^n. \tag{4.20}$$

5. Eigenvalue problem and some existence results

At the end of the previous section we have presented some results concerning the eigenvalue for the Pucci’s extremal operators in the ball. Here we will present the theorem concerning the existence of eigenvalue in the general domain as well as more general operator. In [148], A. Quaas used the Liouville Theorem 17 to obtain an apriori bound. Having obtained an apriori bound, he used the degree theory in the positive cone as presented in [59] and moving plane method to generalise Theorem 28 for the convex and $C^{2,\alpha}$ domain Ω under the conditions (f1), (f2) and (f3) with p_{\pm} instead of p_{\pm}^* . It is clear that, in general, $p_{\pm} < p_{\pm}^*$, so it is valid for small range of p than in the case of the ball. This is due to lack of the Liouville type theorem for that range. For more on the moving plane method, see [19]. He also proved the existence of the first eigenfunction for $\mathcal{P}_{\lambda, \Lambda}^{\pm}$ in a convex and $C^{2,\alpha}$ domain Ω , see Theorem 3.1 [148]. In [28], J. Busca, M. J. Esteban and A. Quaas considered the following eigenvalue problem

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^{\pm}(D^2u) = \mu u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is any smooth bounded domain. They proved the existence of the eigenvalues, eigenfunction and some more informations concerning it. They considered nonlinear bifurcation problem associated with the extremal Pucci’s operator to show the existence of the solution of

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^{\pm}(D^2u) = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.2}$$

under certain conditions on f . The following proposition gives the characterization of the eigenvalues of $\mathcal{P}_{\lambda, \Lambda}^{\pm}$.

PROPOSITION 5. Let Ω be a regular domain. There exist two positive constants μ_1^+, μ_1^- that we call first half-eigenvalues such that:

(i) There exist two functions $\phi_1^+, \phi_1^- \in C^2(\Omega) \cap C(\bar{\Omega})$, such that (μ_1^+, ϕ_1^+) , (μ_1^-, ϕ_1^-) are solutions to

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^+(D^2u) = \mu u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.3}$$

and $\phi_1^+ > 0$, $\phi_1^- < 0$ in Ω . Moreover, these two half eigenvalues are simple, that is, all positive solutions to (5.3) are of the form $(\mu_1^+, \alpha\phi_1^+)$, with $\alpha > 0$. The same holds for

negative solution.

(ii) The first half eigenvalues also satisfy

$$\mu_1^+ = \inf_{A \in \mathcal{A}} \mu_1(A), \quad \mu_1^- = \sup_{A \in \mathcal{A}} \mu_1(A),$$

where \mathcal{A} is the set of all symmetric measurable matrices such that $0 < \lambda I \leq A(x) \leq \Lambda I$, and μ_1 is the principle eigenvalue of associated nondivergent second order linear elliptic operator associated to A .

(iii) The two first half eigenvalues can also be characterized as

$$\mu_1^+ = \sup_{u > 0} \operatorname{ess\,inf}_{\Omega} \left(\frac{\mathcal{P}_{\lambda, \Lambda}^+(D^2 u)}{u} \right), \quad \mu_1^- = \sup_{u < 0} \operatorname{ess\,inf}_{\Omega} \left(\frac{\mathcal{P}_{\lambda, \Lambda}^+(D^2 u)}{u} \right).$$

The supremum is taken over all functions $u \in W_{loc}^{2,N}(\Omega) \cap C(\bar{\Omega})$.

(iv) The two first half eigenvalues also satisfy following characterization

$$\begin{aligned} \mu_1^+ &= \sup \{ \mu \mid \text{there exists } \phi > 0 \text{ in } \Omega \text{ satisfying } \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi) \geq \mu \phi \}, \\ \mu_1^- &= \sup \{ \mu \mid \text{there exists } \phi < 0 \text{ in } \Omega \text{ satisfying } \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi) \leq \mu \phi \}. \end{aligned}$$

Eigenvalues of the operator $\mathcal{P}_{\lambda, \Lambda}^{\pm}$ form an increasing sequence which can be seen in Theorem 1.2 [28]. In [28], the authors considered the following bifurcation problem:

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) = \mu u + f(u, \mu) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.4}$$

where f is continuous

$$f(s, \mu) = o(|s|) \text{ near } s = 0, \text{ uniformly for } \mu \in \mathbb{R}$$

and Ω is a general bounded domain. It can be observed that if $(\bar{\mu}, 0)$ is a bifurcation point for (5.4), then $\bar{\mu}$ is an eigenvalue of $\mathcal{P}_{\lambda, \Lambda}^+$.

THEOREM 30. *The pair $(\mu_1^+, 0)$ (resp., $(\mu_1^-, 0)$), is a bifurcation point of positive (resp., negative) solutions to (5.4). Moreover, the set of nontrivial solutions of (5.4) whose closure contains $(\mu_1^+, 0)$ (resp., $(\mu_1^-, 0)$), is either unbounded or contains a pair $(\bar{\mu}, 0)$ for some $\bar{\mu}$ eigenvalue of (5.3) with $\bar{\mu} \neq \mu_1^+$ (resp., $\bar{\mu} \neq \mu_1^-$).*

REMARK 14. Bifurcation results to the following problem

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) = \mu g(x, u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.5}$$

with the following assumptions on g :

1. $u \rightarrow g(x, u)$ is nondecreasing and $g(x, 0) = 0$,

2. $u \rightarrow \frac{g(x,u)}{u}$ decreasing, and
3. $\lim_{u \rightarrow 0} \frac{g(x,u)}{u} = 1, \lim_{u \rightarrow \infty} \frac{g(x,u)}{u} = 0$

can be found in [120] for the Bellman equation, see also [125]. In [120], author used the method of sub and supersolutions so the conditions imposed there were very crucial, but in [28] results were proved by using degree theory.

There are various results concerning eigenvalues and eigenfunctions for fully non-linear elliptic equations, for details see [28]. More generally in [150], they considered the more general fully nonlinear elliptic operator $F(x, u, Du, D^2u)$ and found out its eigenvalues under certain conditions on F . In fact, they developed a theory of eigenvalues and eigenfunctions as H. Berestycki, L. Nirenberg, S.R.S. Varadhan, did in the case of linear operators in [20]. Let us suppose that $\gamma, \delta > 0$ and all $M, N \in S(n), p, q \in \mathbb{R}^n, u, v \in \mathbb{R}, x \in \Omega, F$ satisfies (SC1)

$$F(x, tu, tp, tM) = tF(x, u, p, M) \text{ for all } t \geq 0. \tag{5.6}$$

Suppose also that

$$F(x, 0, 0, M) \text{ is continuous in } S(n) \times \bar{\Omega}. \tag{5.7}$$

One more important condition that will play an important role is given by

$$\begin{aligned} F(x, u - v, p - q, M - N) &\leq H(x, u, p, M) - H(x, v, q, N) \\ &\leq -F(x, -(u - v), -(p - q), -(M - N)). \end{aligned} \tag{5.8}$$

The above condition (5.8) tells us that how far an operator $H(x, u, p, M)$ is from linear. If F satisfies (5.6), then (5.8) is equivalent to concavity of F in (u, p, M) . The existence of eigenvalues and eigenfunctions are given in following theorem.

THEOREM 31. *Suppose that F satisfies (5.6)-(5.8) and (SC1). Then there exist functions $\phi_1^+, \phi_1^- \in W_{loc}^{2,p}(\Omega) \cap C(\bar{\Omega})$, for all $p < \infty$ and $\phi_1^+ > 0, \phi_1^- < 0$ in Ω , such that*

$$\begin{cases} F(x, \phi_1^+, D\phi_1^+, D^2\phi_1^+) = \mu_1^+ \phi_1^+ \text{ in } \Omega, \\ \phi_1^+ = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.9}$$

$$\begin{cases} F(x, \phi_1^-, D\phi_1^-, D^2\phi_1^-) = \mu_1^- \phi_1^- \text{ in } \Omega, \\ \phi_1^- = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.10}$$

For the proof of Theorem 31, see Theorem 1.1 [150].

REMARK 15. One can also obtain minimax formula for μ_1^+ . For this, we refer to Theorem 1.1 of [4].

The principle eigenfunctions are simple, in fact, A. Quaas and B. Sirakov proved the following theorem.

THEOREM 32. *Suppose that F satisfies hypothesis of Theorem 31. Assume that there exists an L^n -viscosity solution $u \in C(\bar{\Omega})$ of*

$$\begin{cases} F(x, u, Du, D^2u) = \mu_1^+ \phi_1^+ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.11}$$

or of

$$\begin{cases} F(x, u, Du, D^2u) \leq \mu_1^+ \phi_1^+ \text{ in } \Omega, \\ u(x_0) > 0, u \leq 0 \text{ on } \partial\Omega, \end{cases} \tag{5.12}$$

for some $x_0 \in \Omega$. Then $u \equiv t\phi_1^+$, for some $t \in \mathbb{R}$. Similarly, if $v \in C(\Omega)$, satisfies either (5.11) or the reverse inequalities in (5.12) with μ_1^+ replaced by μ_1^- , then $v \equiv t\phi_1^-$, for some $t \in \mathbb{R}$.

For the proof of Theorem 32, we refer to Theorem 1.2 [150]. These eigenvalues of F are isolated, that is, under a similar conditions as in Theorem 32, there exists $\varepsilon_0 > 0$, depending on $n, \Omega, \lambda, \Lambda, \gamma$ and δ , such that the problem

$$\begin{cases} F(x, u, Du, D^2u) = \mu u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.13}$$

has no solution $u \neq 0$ for $\mu \in (-\infty, \mu_1^- + \varepsilon_0) \setminus \{\mu_1^+, \mu_1^-\}$. The A. Quaas and B. Sirakov also give the necessary and sufficient conditions for the positivity of the principle eigenvalues. The following theorem also shows that the existence of a positive L^n -viscosity supersolution implies the existence of a positive uniformly bounded strong supersolution. More precisely, we have following theorem.

THEOREM 33. *Suppose F satisfies the assumptions of Theorem 31.*

(i) *Assume there is a function $u \in C(\bar{\Omega})$, such that*

$$\begin{cases} F(x, u, Du, D^2u) \geq 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{5.14}$$

respectively

$$\begin{cases} F(x, u, Du, D^2u) \leq 0 \text{ in } \Omega, \\ u < 0 \text{ in } \Omega, \end{cases} \tag{5.15}$$

in the L^n -viscosity sense. Then either $\mu_1^+ > 0$ or $\mu_1^+ = 0$ with $u \equiv t\phi_1^+$, for some $t > 0$ (resp., $\mu_1^- > 0$ or $\mu_1^- = 0$ with $u \equiv t\phi_1^-$, for some $t > 0$).

(ii) *Conversely, if $\mu_1^+ > 0$, then there exists a function $u \in W^{2,p}(\Omega)$, $p < \infty$, such that $F(x, u, Du, D^2u) \leq 0$, $u \geq 1$ in Ω and $\|u\|_{W^{2,p}(\Omega)} \leq C$, where C depends on $p, n, \Omega, \lambda, \Lambda, \gamma, \delta$, and μ_1^+ .*

REMARK 16. When F is proper, $u \equiv 1$, satisfies the conditions of Theorem 33 and hence proper operators have positive eigenvalues.

It follows from the Theorem 33 that the eigenvalues are strictly decreasing with respect to domain, that is, if $\Omega' \subset \Omega$, then $\mu_1^+(\Omega) < \mu_1^+(\Omega')$ and $\mu_1^-(\Omega) < \mu_1^-(\Omega')$, in fact, μ_1^+, μ_1^- are continuous with respect to domain, for the details, see Proposition 4.10 [150].

REMARK 17. If the operator F satisfies the assumptions of Theorem 31, then the necessary and sufficient condition for F to satisfy comparison principle is that $\mu_1^+ > 0$. However $\mu_1^- > 0$, does not imply the comparison principle to hold.

REMARK 18. A. Quaas and B. Sirakov also obtained that $\mu_1^+ > 0$, is a sufficient condition for the existence of unique solution $u \in W^{2,p}(\Omega) \cap C(\bar{\Omega})$, of the following Dirichlet problem

$$\begin{cases} F(x, u, Du, D^2u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.16}$$

where $f \in L^p$ for $p \geq n$ and F satisfies the same conditions as in the Theorem 33. In addition to this, authors also obtained that for any $\Omega' \subset \subset \Omega$ following estimate holds

$$\|u\|_{W^{2,p}(\Omega')} \leq C\|f\|_{L^p(\Omega)},$$

where C depends on $p, \Omega', \Omega, \lambda, \Lambda, \gamma, \delta$, and μ_1^+ . On the other hand, if $\mu_1^+ = 0$ or $\mu_1^- > 0 \geq \mu_1^+$, then the problem (5.16) does not possess a solution in $C(\bar{\Omega})$ provided $f \leq 0, f \not\equiv 0$. For the details, see Theorem 1.8 [150].

The above remark states that if only one of the two eigenvalues are positive, the Dirichlet problem (5.16) may not have a solution. The precise statement is given in the following theorem.

THEOREM 34. Suppose that F satisfies (5.6)-(5.8) and (SCI). If $\mu_1^-(F) > 0$, then for any $f \in L^p(\Omega)$ $p \geq n$, such that $f \geq 0$ in Ω , there exists a nonpositive solution $u \in W^{2,p}(\Omega) \cap C(\bar{\Omega})$, of (5.16).

REMARK 19. The uniqueness of the solution obtained in Theorem 34 is not guaranteed. For the proof, we refer to [150].

The following existence result is applicable to Isaac’s operators. It says that an Isaac’s equation is solvable provided the operator is controlled, in the sense of (5.8), by an operator with positive eigenvalues.

THEOREM 35. Assume F satisfies (5.6)-(5.8) and (SCI), and H satisfies (5.8) and (5.6). If $\mu_1^+(F) > 0$, then the problem

$$\begin{cases} H(x, u, Du, D^2u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.17}$$

is solvable in the L^p -viscosity sense for any $f \in L^p(\Omega)$, $p \geq n$. Further, if H is convex in M then $u \in W^{2,p}(\Omega) \cap C(\bar{\Omega})$, and u is unique.

For the proof, see Theorem 1.10 [150]. In [149], A. Quaas and B. Sirakov proved a Liouville type theorem in half space that is valid for larger range of p than Theorem 17. In order to state the theorem, let us introduce a notation $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$.

THEOREM 36. *Suppose $N \geq 3$ and set*

$$\tilde{p}^+ = \frac{\lambda(n-2) + \Lambda}{\lambda(n-2) - \Lambda}.$$

Then the problem

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^+(D^2u) - u^p = 0 \text{ in } \mathbb{R}_+^n, \\ u = 0 \text{ in } \partial\mathbb{R}_+^n, \end{cases} \tag{5.18}$$

does not have a nontrivial nonnegative bounded solution, provided $1 < p \leq \tilde{p}^+$ (or $1 < p < \infty$ if $\lambda(n-2) \leq \Lambda$).

In [149], A. Quaas and B. Sirakov studied the problem

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^+(D^2u) = f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.19}$$

by using the Liouville type Theorem 26, degree theory for a compact operator in closed cone with non-empty interior in a Banach space. In order to state the precise statement of the existence results we need the following assumptions on f ;

1. f is a Hölder continuous function on $\bar{\Omega} \times [0, \infty)$, such that $f(x, 0) = 0$ and $f(x, s) \geq -\gamma s$ for some $\gamma \geq 0$ and all $s \geq 0, x \in \bar{\Omega}$.
2. $\limsup_{u \rightarrow \infty} \frac{f(x, u)}{u} < \mu_1^+ < \liminf_{u \rightarrow 0} \frac{f(x, u)}{u} \leq \infty$, uniformly in $x \in \bar{\Omega}$.

THEOREM 37. *Suppose (1) and (2) hold. Then problem (5.19) has a positive classical solution.*

For the proof, see Theorem 1.1 [149]. Again, let us define another condition which turns to the problem (5.19) into a superlinear equations, that is, $f(x, u)$ satisfies the following condition

$$(3) \limsup_{u \rightarrow 0} \frac{f(x, u)}{u} < \mu_1^+ < \liminf_{u \rightarrow \infty} \frac{f(x, u)}{u} \leq \infty, \text{ uniformly in } x \in \bar{\Omega}.$$

In order to state existence theorem for superlinear equations, they considered the family of problems obtained from (5.19) by replacing $f(x, u)$ by $f(x, u + t)$, for $t \geq 0$. Set U_t denotes the set of nonnegative classical solutions for any such problem and let $\mathcal{B}_t = \bigcup_{0 \leq s \leq t} U_s$.

THEOREM 38. *Let us assume that conditions (i) and (iii) hold and also that for each $t \geq 0$, there is a constant C depending only on t, Ω and f such that*

$$\|u\|_{L^\infty(\Omega)} \leq C, \text{ for all } u \in \mathcal{B}_t. \tag{5.20}$$

Then the problem (5.19) has a classical positive solution.

For the proof, see Theorem 1.2 [149]. As we mentioned in the Remark 18, it was shown by A. Quaas and B. Sirakov that the Dirichlet problem

$$\begin{cases} F(x, u, Du, D^2u) = f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.21}$$

has a unique solution if $\mu_1^+ > 0$, while if $\mu_1^- > 0 \geq \mu_1^+$, then (5.21) has a solution for $f \geq 0$ but (5.21) does not have solution for $f \leq 0, f \not\equiv 0$. The question of uniqueness in the last case was left open, since in Remark 17 we have seen that $\mu_1^- > 0$ alone does not imply the comparison principle. In [154], B. Sirakov showed that uniqueness fails only when one of the two eigenvalues is positive. In order to state the precise statement, we need some more structure conditions that are given below. Let us consider

$$\begin{cases} H(x, u, Du, D^2u) = f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.22}$$

where again Ω is a smooth bounded domain in \mathbb{R}^n . We suppose that H in (5.22) satisfies the following hypothesis;

for all $M \in S(n), p \in \mathbb{R}^n, u \in \mathbb{R}, x \in \Omega$, and for some constants A_0, γ, δ

$$F(x, u, p, M) - A_0 \leq H(x, u, p, M) \leq \mathcal{P}_{\lambda, \Lambda}^+(M) + \gamma|p| + \delta|u| + A_0, \tag{5.23}$$

where F is some (u, p, M) -convex nonlinear operator satisfying (5.6), (5.7) and

$$\mathcal{P}_{\lambda, \Lambda}^-(M) - \gamma|p| - \delta|r| \leq F(x, r, p, M) \leq \mathcal{P}_{\lambda, \Lambda}^+(M) + \gamma|p| + \delta|r|. \tag{5.24}$$

Suppose also that for each $R \in \mathbb{R}$, there exists c_R such that (4.3) is satisfied with $\delta = c_R$ and for all $M, N \in S(n), p, q \in \mathbb{R}^n, x \in \Omega, u, v \in [-R, R]$. B. Sirakov also assumed that

$$\begin{cases} H(x, v, Du, D^2u) - u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.25}$$

has at most one solution u , for each $v \in C(\bar{\Omega})$. Further, he has also used the following decomposition of

$$f(x) = -t\phi(x) + g(x), \tag{5.26}$$

where $t \in \mathbb{R}, \phi_1^+(F_0, \Omega)$ is the first positive normalized eigenfunction of the operator

$$F_0(x, p, M) = F(x, 0, p, M)$$

so that $\max_{\Omega} \phi = 1$. The existence of $\phi \in W^{2,p}(\Omega) \cap C(\bar{\Omega}), p < \infty, \phi > 0$ in Ω , satisfying

$$F_0(x, D\phi, D^2\phi) = \mu_0^+ \phi \text{ in } \Omega,$$

has been proved in [150]. Furthermore F_0 is proper so $\mu_0^+ = \mu_1^+ > 0$, see (16).

THEOREM 39. Suppose F and H satisfy (5.23), (5.24), (5.7) and (5.8). Further, F is also convex in (u, p, M) and

$$\mu_1^+(F, \Omega) < 0 < \mu_1^-(F, \Omega). \tag{5.27}$$

Then for each $g \in L^\infty(\Omega)$ in (5.26), there exists a number $t^*(g) \in \mathbb{R}$, such that:

- (i) if $t < t^*(g)$ then (5.22) has at least two solutions,
- (ii) if $t = t^*(g)$ then (5.22) has at least one solution,
- (iii) if $t > t^*(g)$ then (5.22) has no solution.

Further, the map $g \rightarrow t^*(g)$ is continuous from $L^\infty(\Omega)$ to \mathbb{R} .

For the proof of Theorem 39, we refer to Theorem 1 [154]. The value of $t^*(g)$ is computed by S. N. Armstrong in [4] in terms of H and g . Now we will present the existence results of [5] which are also in the sequel of results presented above. In this paper S. N. Armstrong considered the C -viscosity solution. In order to state the result precisely, we need the following hypothesis given below.

(A) For each $K > 0$, there exist an increasing continuous function $\omega_K : [0, \infty) \rightarrow [0, \infty)$ for which $\omega_K(0) = 0$, and a positive constant $\frac{1}{2} < \alpha \leq 1$, depending on K , such that

$$|F(x, r, p, M) - F(y, r, p, M)| \leq \omega_K(|x - y|^\alpha(|M| + 1)), \tag{5.28}$$

for all $M \in S(n)$, $p \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $x, y \in \Omega$ satisfying $|p|, |r| \leq K$.

Armstrong considered the following problem

$$\begin{cases} H(x, v, Du, D^2u) = \mu u + f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.29}$$

and proved the following existence results.

THEOREM 40. Suppose that F satisfies (SC1), (5.6) and (A) stated above, also assumed that $f \in C(\Omega) \cap L^p(\Omega)$, for some $p > n$.

1. If $f \geq 0$ and $\mu < \mu_1^+(F, \Omega)$, then the Dirichlet problem (5.29) has a unique nonnegative solution $u \in C(\bar{\Omega})$. Moreover, $u \in C^{1,\nu}(\Omega)$.
2. If $f \leq 0$ and $\mu < \mu_1^-(F, \Omega)$, then the Dirichlet problem (5.29) has a nonpositive solution $u \in C^{1,\nu}(\Omega)$.
3. If $\mu < \min\{\mu_1^+(F, \Omega), \mu_1^-(F, \Omega)\}$, then the Dirichlet problem (5.29) has a solution $u \in C^{1,\nu}(\Omega)$.

For the proof, see Theorem 2.3 [5]. It can be seen that no μ satisfying

$$\min\{\mu_1^+(F, \Omega), \mu_1^-(F, \Omega)\} < \mu < \max\{\mu_1^+(F, \Omega), \mu_1^-(F, \Omega)\}$$

is an eigenvalue for F in Ω , neither for such μ do we have general existence or uniqueness of solutions of (5.29). See Section 6 [5] for some nonexistence results, and [154]

for the nonexistence and failure of the uniqueness of the solutions of (5.29), for μ between two half eigenvalues. Next, we will present some existence results for the values of μ satisfying

$$\mu > \max\{\mu_1^+(F, \Omega), \mu_1^-(F, \Omega)\}.$$

Let us define

$$\mu_2(F, \Omega) = \inf\{\rho > \max\{\mu_1^+(F, \Omega), \mu_1^-(F, \Omega)\} : \rho \text{ is an eigenvalue of } F \text{ in } \Omega\}.$$

The $\mu_2(F, \Omega)$ is finite, see Lemma 3.3 [5].

THEOREM 41. *Suppose that F satisfies (SC1), (5.6) and (A) stated above, also assume that $f \in C(\Omega) \cap L^p(\Omega)$ for some $p > n$, and*

$$\max\{\mu_1^+(F, \Omega), \mu_1^-(F, \Omega)\} < \mu < \mu_2(F, \Omega).$$

Then there exists a solution $u \in C^{1,\nu}(\Omega)$ of the Dirichlet problem (5.29).

For the proof, see Theorem 2.4 [5]. In [76], authors studied the Dirichlet problem

$$\begin{cases} F(x, v, Du, D^2u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.30}$$

at the resonance case, that is, one of the semi-eigenvalue (i.e either $\mu_1^+ = 0$ or $\mu_1^- = 0$). Once again, we suppose that F satisfies the conditions (SC1), (5.6), (5.7) and (5.8), and f is of the form

$$f = t\phi_1^+ + g, \quad t \in \mathbb{R}, \tag{5.31}$$

where ϕ_1^+ , is the first half eigenfunction and $g \in L^p(\Omega)$. In order to state the precise statement, we introduce some notations. Let \mathcal{S} be the set of solutions of (5.30) in the space $C(\bar{\Omega}) \times \mathbb{R}$ as follows;

$$(u, t) \in \mathcal{S} \text{ iff } u \text{ is a solution of (5.30), with } f = t\phi_1^+ + g.$$

Further, given a subset $\mathcal{B} \subset C(\bar{\Omega}) \times \mathbb{R}$ and $t \in \mathbb{R}$, define

$$\mathcal{B}_t = \{u \in C(\bar{\Omega}) : (u, t) \in \mathcal{B}\}$$

and

$$\mathcal{B}_I = \cup_{t \in I} \mathcal{B}_t$$

if I is an interval. Below, we will present the results in the three cases

- (i) $\mu_1^+(F) = 0$
- (ii) $\mu_1^-(F) = 0$
- (iii) $\mu_1^-(F) < 0$, separately.

THEOREM 42. Assume $\mu_1^+(F) = 0$, then the following hold.

- (i) There exists a number $t_+^* = t_+^*(g)$ such that if $t < t_+^*$, then there is no solution of (5.30), while for $t > t_+^*$, (5.30) has a solution.
- (ii) The set \mathcal{S} is a continuous curve such that \mathcal{S}_t has a singleton for all $t > t_+^*$, that is, solution is unique for $t > t_+^*$. If $t_+^* \leq t < s$ and $(u_t, t), (u_s, s) \in \mathcal{S}$, then $u_t > u_s$ in Ω . The map $t \rightarrow u_t(x)$ is convex for each $x \in \Omega$.
- (iii) There exists $t^+ = t^+(g) > t_+^*$ such that if $t \geq t^+$, then $u_t < 0$ in Ω , and for every compact subset $K \subset \subset \Omega$, we have $\lim_{t \rightarrow +\infty} \max_{x \in K} u_t(x) = -\infty$.
- (iv) If $t = t_+^*$, then either
 - (a) (5.30) does not have solution (that is, $\mathcal{S}_{t_+^*}$ is empty), $\lim_{t \searrow t_+^*} \min_{x \in K} u_t = +\infty$ for every $K \subset \subset \Omega$, and there exists $\varepsilon = \varepsilon(g) > 0$ such that if $t \in (t_+^*, t_+^* + \varepsilon)$, then $u_t > 0$. or
 - (b) there exists a function u^* such that $\mathcal{S}_{t_+^*} = \{u^* + s\phi_1^+ | s \geq 0\}$.

Next, P. Felmer, A. Quaas, and B. Sirakov considered the second case of the resonance, that is, $\mu_1^-(F) = 0$.

THEOREM 43. Assume $\mu_1^-(F) = 0$, then there exists a number $t_-^* = t_-^*(g)$ such that the following hold.

- (i) If $t < t_-^*$, then there is no solution of (5.30).
- (ii) There is a closed connected set $\mathcal{C} \subset \mathcal{S}$ such that $\mathcal{C}_t \neq \emptyset$ for all $t > t_-^*$.
- (iii) The set S_I is bounded in $W^{2,p}(\Omega)$ for each compact subset $I \subset (t_-^*, \infty)$.
- (iv) If we denote $\alpha_t = \inf\{\sup_{\Omega} u \mid u \in \mathcal{S}_t\}$, we have $\lim_{t \rightarrow +\infty} \alpha_t = +\infty$.
- (v) The set $\mathcal{C}_{[t_-^*, t_-^* + \varepsilon)}$ is unbounded in $L^\infty(\Omega)$ for all $\varepsilon > 0$; there exists $C = C(g) > 0$, such that if $u \in \mathcal{S}_{[t_-^*, t_-^* + \varepsilon)}$ and $\|u\|_{L^\infty(\Omega)} \geq C$, then $u < 0$ in Ω ; if $u_n \in \mathcal{S}_{[t_-^*, t_-^* + \varepsilon)}$ and $\|u_n\|_{L^\infty(\Omega)} \rightarrow \infty$, then $\max_K u_n \rightarrow -\infty$ for each compact subset $K \subset \subset \Omega$.
- (vi) If $\mathcal{S}_{t_-^*}$ is unbounded in $L^\infty(\Omega)$, then there exists a function u_* such that $\mathcal{S}_{t_-^*} = \{u_* + s\phi_1^- \mid s \geq 0\}$.

Finally, they also considered the case $\mu_1^-(F) < 0$ but very little is known in this case. If $\mu_1^-(F)$ is small and negative, then the following theorem holds.

THEOREM 44. There exists $0 < L \leq \infty$, such that if $\mu_1^-(F) \in (-L, 0)$, then

- (i) there exists a closed connected set $\mathcal{C} \subset \mathcal{S}$ such that $\mathcal{C}_t \neq \emptyset$ for each $t \in \mathbb{R}$ (S_I is bounded in $W^{2,p}(\Omega)$ for each bounded $I \subset \mathbb{R}$);
- (ii) setting $\alpha_t = \inf\{\sup_{\Omega} u \mid u \in \mathcal{S}_t\}$, and $\bar{u}_t(x) = \sup\{u(x) \mid u \in \mathcal{S}_t\}$, we have

$$\lim_{t \rightarrow +\infty} \alpha_t = +\infty \text{ and } \lim_{t \rightarrow -\infty} \sup_K \bar{u}_t(x) = -\infty,$$

for each $K \subset \subset \Omega$ and $\bar{u}_t < 0$ in Ω , for all t below some number $t^-(g)$.

In [78], P. Felmer, A. Quaas and B. Sirakov obtained some existence results for following equation

$$\begin{cases} H(x, v, Du, D^2u) = \mu u + f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.32}$$

where Ω is a bounded and smooth domain in \mathbb{R}^n , under Landesman-Lazer type assumptions on f . For the Landesman-Lazer type results in the linear case, see [58] and [89]. Note that in (5.29), f depends only on x .

Suppose that $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$, is continuous and sublinear in u at infinity, that is,

$$(A0) \quad \lim_{|s| \rightarrow \infty} \frac{f(x, s)}{s} = 0 \text{ uniformly in } x \in \bar{\Omega}.$$

$$(A1) \quad f(x, 0) \geq 0 \text{ and } f(x, 0) \not\equiv 0 \text{ in } \Omega.$$

$$(A2) \quad f(x, \cdot) \text{ is locally Lipschitz, that is, for each } R \in \mathbb{R} \text{ there is } L_R \text{ such that } |f(x, s_1) - f(x, s_2)| \leq L_R |s_1 - s_2| \text{ for all } s_1, s_2 \in (-R, R) \text{ and } x \in \bar{\Omega}.$$

Let us define functions which depend only on x

$$f^s(x) := \limsup_{s \rightarrow +\infty} f(x, s), \quad f^i(x) := \liminf_{s \rightarrow +\infty} f(x, s)$$

and

$$f_s(x) := \limsup_{s \rightarrow -\infty} f(x, s), \quad f_i(x) := \liminf_{s \rightarrow -\infty} f(x, s).$$

Below, we write the critical t -values at the resonance as follows:

$$t_+^* = t_+^*(g) = t_{\mu_1^+, F}^*(g) \text{ and } t_-^* = t_-^*(g) = t_{\mu_1^-, F}^*(g),$$

and $p > n$ is a fixed number. We assume that there are

$$(A^i) \quad \text{a function } a^i \in L^p(\Omega), \text{ such that } a^i(x) \leq f^i(x) \text{ in } \Omega \text{ and } t_+^*(a^i) < 0,$$

$$(A_s) \quad \text{a function } a_s \in L^p(\Omega), \text{ such that } a_s(x) \geq f_s(x) \text{ in } \Omega \text{ and } t_-^*(a_s) > 0,$$

$$(A^s) \quad \text{a function } a^s \in L^p(\Omega), \text{ such that } a^s(x) \leq f^s(x) \text{ in } \Omega \text{ and } t_+^*(a^s) > 0,$$

$$(A_i) \quad \text{a function } a_i \in L^p(\Omega), \text{ such that } a_i(x) \leq f_i(x) \text{ in } \Omega \text{ and } t_-^*(a_i) < 0.$$

THEOREM 45. *Suppose that F satisfies (SCI), (5.6), (5.7) and (5.8), and f satisfies (A0) and (A^i) . Then there exist $\delta > 0$ and two disjoint closed connected sets of solutions of (5.32), $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S}$ such that*

1. $\mathcal{C}_1(\mu) \neq \emptyset$ for all $\mu \in (-\infty, \mu_1^+]$,
2. $\mathcal{C}_1(\mu) \neq \emptyset$ and $\mathcal{C}_2(\mu) \neq \emptyset$ for all $\lambda \in (\mu_1^+, \mu_1^+ + \delta)$.

The set \mathcal{C}_2 is a branch of solutions “bifurcating from plus infinity to the right of μ_1^+ ”, that is, $\mathcal{C}_2 \subset C(\bar{\Omega}) \times (\mu_1^+, \infty)$ there is a sequence $\{(u_n, \mu_n)\} \in \mathcal{C}$, such that $\mu_n \rightarrow \mu_1^+$ and $\|u_n\|_{L^\infty(\Omega)} \rightarrow \infty$. Moreover, for every sequence $\{(u_n, \lambda_n)\} \in \mathcal{C}_2$, such that $\mu_n \rightarrow \mu_1^+$ and $\|u_n\|_{L^\infty(\Omega)} \rightarrow \infty$, u_n is a positive in Ω , for n large enough.

If we assume (A_s) holds, then there is a branch of solutions of (5.32) “bifurcating from minus infinity to the right of μ_1^- ”, that is, a connected set $\mathcal{C}_3 \subset \mathcal{S}$ such that $\mathcal{C}_3 \subset C(\bar{\Omega}) \times (\mu_1^-, \infty)$, for which there is a sequence $\{(u_n, \lambda_n)\} \in \mathcal{C}_3$ such that $\mu_n \rightarrow \mu_1^-$ and $\|u_n\|_{L^\infty(\Omega)} \rightarrow \infty$. Moreover, for every sequence $\{(u_n, \mu_n)\} \in \mathcal{C}_n$ such that $\mu_n \rightarrow \mu_1^-$ and $\|u_n\|_{L^\infty(\Omega)} \rightarrow \infty$, u_n is negative in Ω for n large enough.

For the proof of the Theorem 45, see Theorem 1.1 [78].

THEOREM 46. *Suppose that F satisfies (SCI), (5.6), (5.7) and (5.8), and f satisfies (A0), (A1), (A2), (Aⁱ) and (A_s). Then there exist a constant $\delta > 0$ and three disjoint closed connected sets of solutions $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset \mathcal{S}$, such that (5.32)*

1. $\mathcal{C}_1(\mu) \neq \emptyset$ for all $\mu \in (-\infty, \mu_1^+]$,
2. $\mathcal{C}_i(\mu) \neq \emptyset, i = 1, 2$, for all $\mu \in (\mu_1^+, \mu_1^-]$,
3. $\mathcal{C}_i(\mu) \neq \emptyset, i = 1, 2, 3$ for all $\mu \in (\mu_1^-, \mu_1^- + \delta)$.

The sets \mathcal{C}_2 and \mathcal{C}_3 have the same bifurcation from infinity properties as in the previous theorem.

For the proof of the Theorem 46, we refer to Theorem 1.2 [78].

THEOREM 47. *Assume that F satisfies the same hypotheses as in Theorem 46, and f satisfies (A0), (A1), (A2), (A^s) and (A_i). Then there exist $\delta > 0$ and disjoint closed connected sets of solutions $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S}$, such that*

1. $\mathcal{C}_1(\mu) \neq \emptyset$, for all $\mu \in (-\infty, \mu_1^+ - \delta]$,
2. $\mathcal{C}_1(\mu) \neq \emptyset, \mathcal{C}_2(\mu)$ contains at least one elements for all $\mu \in (\mu_1^+ - \delta, \mu_1^+)$ and \mathcal{C}_2 is a branch bifurcating from plus infinity to the left of μ_1^+ ,
3. $\mathcal{C}_1(\mu) \neq \emptyset$ and $\mathcal{C}_2 \neq \emptyset$. for all $\mu \in [\mu_1^+, \mu_1^-)$, and either:
 - (a) \mathcal{C}_1 is a branch bifurcating from minus infinity to the left of μ_1^- ,
 - (b) there is a closed connected set of solutions $\mathcal{C}_3 \subset \mathcal{S}$, disjoint to \mathcal{C}_1 and \mathcal{C}_2 bifurcating from minus infinity to the left of μ_1^- , such that \mathcal{C}_3 has at least two elements for all $\mu \in (\mu_1^- - \delta, \mu_1^-)$,
4. $\mathcal{C}_2(\mu) \neq \emptyset$, for all $\mu \in [\mu_1^-, \mu_1^- + \delta]$. In case (b) in (3), $\mathcal{C}_2(\mu) \neq \emptyset$ and $\mathcal{C}_3(\mu) \neq \emptyset$, for all $\mu \in [\mu_1^-, \mu_1^- + \delta]$.

For the proof of the Theorem 47, we refer to Theorem 1.3 [78]. A slightly different type of nonlinearity in the gradient terms has been appeared in the literature. The first result in this direction was appeared in [80]. Further, using the result in [80], S. Koike, A. Świech, allowed the quadratic nonlinearity in the gradient term. There are various difficulties arise by considering this nonlinearity due to lack of the maximum principle. They considered the following equation

$$\begin{cases} F(x, Du(x), D^2u(x)) = f(x) \text{ in } \Omega, \\ u = \psi \text{ on } \partial\Omega, \end{cases} \tag{5.33}$$

with the assumptions that F is measurable in x . This equation is uniformly elliptic, that is,

$$\mathcal{P}_{\lambda, \Lambda}^-(M - N) \leq F(x, p, M) - F(x, q, N) \leq \mathcal{P}_{\lambda, \Lambda}^+(M - N).$$

In order to deal with a wide class of PDEs with quadratic nonlinearities in the gradient of u , they assumed that there exist $\gamma_1, \gamma_2 > 0$, such that

$$\begin{cases} |F(x, u, p, M)| \leq \gamma_1 |p| + \gamma_2 |q|^2, \\ |F(x, p, M) - F(x, q, M)| \leq \gamma_1 |p - q| + \gamma_2 (|p| + |q|) |p - q|, \end{cases} \quad (5.34)$$

for $x \in \Omega$, $p, q \in \mathbb{R}^n$ and $M \in S(n)$. Under this structural assumptions on F , S. Koike, A. Świech proved the following theorem.

THEOREM 48. *Assume that F is elliptic and measurable in x and also satisfies (5.34), and Ω satisfy a uniform exterior cone condition. Assume also that $f \in L^p(\Omega)$ for $p > p_0$. Then there exists $\delta = \delta(n, \lambda, \Lambda, p, \gamma_1, \gamma_2, \text{diam}(\Omega)) > 0$, such that if*

$$\gamma_2 \|f\|_{L^p(\Omega)} \text{diam}(\Omega)^{2-\frac{n}{p}} < \delta,$$

for any $\psi \in C(\partial\Omega)$, there is an L^p -viscosity solution of (5.33).

For the proof, see Theorem 5.1 [104]. A slightly general result in this direction was proved by B. Sirakov in [155], which extends the existence results in [47]. He considered the following equation

$$\begin{cases} F(x, u, Du, D^2u) + c(x)u = f(x) \text{ in } \Omega, \\ u = \psi \text{ on } \partial\Omega, \end{cases} \quad (5.35)$$

and proved the existence and uniqueness results for (5.35). Let us first define the structure conditions

$$\begin{cases} F(x, u, p, M) - F(x, v, q, N) \leq \mathcal{P}_{\lambda, \Lambda}^+(M - N) + \mu(|p| + |q|)|p - q| + b(x)|p - q| \\ \hspace{15em} + d(x)\bar{h}(u, v), \\ F(x, u, p, M) - F(x, v, q, N) \geq \mathcal{P}_{\lambda, \Lambda}^-(M - N) - \mu(|p| + |q|)|p - q| - b(x)|p - q| \\ \hspace{15em} - d(x)\underline{h}(u, v), \end{cases} \quad (5.36)$$

where $0 < \lambda \leq \Lambda$, $\mu \in \mathbb{R}^+$, $b \in L^p(\Omega)$, for some $p > n$, $d \in L^n(\Omega)$, $b, d \geq 0$ and $\bar{h}, \underline{h} \in C(\mathbb{R}^2)$. One example that satisfies the assumptions given by (5.36) is

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2u) + \mu(x)|Du|^2 + b(x)|Du| + c(x)u = f(x),$$

with $\mu \in L^\infty(\Omega)$, $b \in L^p(\Omega)$, $p > n$, $c, f \in L^n(\Omega)$. From the above example, it is clear that the results of B. Sirakov are also true for the equations in divergence form.

THEOREM 49. *Suppose F satisfies (5.36) with $\underline{h} = h((u - v)^+)$, $\bar{h} = ((v - u)^+)$, for some continuous function h , such that $h(0) = 0$. Let $c \in L^n(\Omega)$, then*
 (i) *if $c(x) \leq -\bar{c}$ almost everywhere in Ω , for some constant $\bar{c} > 0$, then for any data $f \in L^n$ and $\psi \in C(\partial\Omega)$ there exists a solution $u \in C(\bar{\Omega})$ of*

$$\begin{cases} F(x, u, Du, D^2u) + c(x)u = f(x) \text{ in } \Omega, \\ u = \psi \text{ on } \partial\Omega. \end{cases} \quad (5.37)$$

(ii) There exists a positive constant δ_0 , depending on $\lambda, \Lambda, n, \|b\|_{L^p(\Omega)}$, and $\text{diam}(\Omega)$ such that if $f \in L^n(\Omega)$ and $\psi \in C(\partial\Omega)$ satisfy

$$\|\mu|f| + \mu M c^+ + c^+\|_{L^n(\Omega)} < \delta_0,$$

then there exists a solution $u \in C(\bar{\Omega})$ of (5.37).

(iii) If problem (5.37) with $c^+ \equiv 0$ or with $\mu = 0$ and $\|c^+\|_{L^p(\Omega)} < \delta_0$ has a strong solution then this solution is unique viscosity solution of (5.37).

(iv) The strong solutions of (5.37) are not unique if $\mu > 0$ and $c(x) \equiv c > 0$ (arbitrarily small), even for $\lambda = \Lambda, b = d = f = \psi \equiv 0$.

For the proof of Theorem 49, see Theorem 1 [155]. Finally, in this direction S. Koike, A. Świech, proved the existence of L^p -strong solutions concerning similar equations. In [105], the authors considered the equation allowing superlinear growth in Du of m -th order. More precisely, they considered the following equation

$$\begin{cases} \mathcal{P}^\pm(D^2u) \mp \gamma(x)|Du| \mp \mu(x)|Du|^m = f(x) \text{ in } \Omega, \\ u = \psi \text{ on } \partial\Omega, \end{cases} \tag{5.38}$$

and proved the following theorem which generalises the above results in some sense.

THEOREM 50. Let $p_0 < p \leq q_1, q_1 > n, f \in L^p(\Omega), \gamma \in L^{q_1}$ and $\psi \in W^{2,p}(\Omega)$. Assume that one of the following conditions holds.

$$\begin{cases} (i) \ q = \infty, p_0 < p, n > m(n-p), \\ (ii) \ n < p \leq n < q < \infty, \\ (iii) \ p_0 < p \leq n < q < \infty, mq(n-p) < n(q-p). \end{cases} \tag{5.39}$$

Let

$$\begin{cases} r = mp \text{ for (i)} \\ r = \infty \text{ for (ii) with } p = q, \\ r = \frac{mpq}{q-p} \text{ for (ii) with } p < q \text{ or (iii)}. \end{cases} \tag{5.40}$$

Set $\varepsilon_1 = (2\bar{C}D)^{-m} > 0$, where \bar{C} is some constant. If

$$\|\mu\|_{L^q} (\|f\|_{L^p(\Omega)} + \|\psi\|_{2,p})^{m-1} < \varepsilon_1,$$

then there exist L^p -strong solutions $u \in W^{2,p}(\Omega)$, of (5.38). Moreover

$$\|u\|_{W^{2,p}(\Omega)} \leq \hat{C} (\|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}),$$

for some $\hat{C} = \hat{C}(n, \lambda, \Lambda, p, q_1, q, m, \|\gamma\|_{q_1}, \Omega) > 0$.

Next, we want to present existence results concerning the singular perturbed equations and some asymptotic behaviour of the solutions. In the fully nonlinear setting, singular

perturbed problems are appeared in [1]. S. Alarcón, L. Iturriaga and A. Quaas studied the problem

$$\begin{cases} \varepsilon^2 \mathcal{P}_{\lambda, \Lambda}^+(D^2u) = f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.41}$$

where Ω is a smooth bounded domain in \mathbb{R}^n , $n > 2$. They proved that it possesses non-trivial solutions for small values of ε provided f is nonnegative continuous function which has a positive zero. In order to state the precise statement, we need to put some conditions on f :

1. $f : \bar{\Omega} \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous function and $f(x, \cdot)$ is locally Lipschitz in $(0, \infty)$ for all $x \in \bar{\Omega}$, $f(x, 0) = f(x, 1) = 0$ and $f(x, t) > 0$ for $t \in \{0; 1\}$.
2. $\liminf_{t \rightarrow +0} \frac{f(x, t)}{t} = 1$ uniformly for $x \in \bar{\Omega}$
3. There exist a continuous function $a : \bar{\Omega} \rightarrow (0, \infty)$ and $\sigma \in (1, \frac{\alpha}{\alpha-2})$ such that

$$\lim_{t \rightarrow 1} \frac{f(x, t)}{|t-1|^\sigma} = a(x).$$

4. There exist $k > 0$, and $T > 1$, such that the map $t \rightarrow f(x, t) + kt$, is increasing for $t \in [0, T]$ and $x \in \Omega$.

THEOREM 51. *Assume that Ω is a bounded smooth domain. Then, under the hypotheses (1)-(4), there exists $\varepsilon^* > 0$, such that the problem (5.41) has at least two positive viscosity solutions $u_{1, \varepsilon}$, $u_{2, \varepsilon}$ for $0 < \varepsilon < \varepsilon^*$.*

Moreover, these solutions satisfy $\|u_{1, \infty}\|_{L^\infty} \rightarrow 1^-$ and $\|u_{2, \infty}\|_{L^\infty} \rightarrow 1^+$ as $\varepsilon \rightarrow 0$.

6. Singular elliptic equations

In this section, we continue our earlier discussion for Singular fully nonlinear elliptic equations. There are mainly two type of singularities appeared in literature in the context of viscosity solutions of the fully nonlinear elliptic equations. First type contains the singular nonlinearity and second type contains the singularities in the operator. The first article, in the context of the viscosity solutions for first type singular fully nonlinear elliptic PDEs is [78]. The study of second type of singular fully nonlinear elliptic PDEs was started by I. Birindelli and F. Demengel in a sequence of papers [21, 22, 23, 24, 25]. By taking into account the work of Evans and Spruck [67] and Juutinen, Lindquist and Manfredi [101], I. Birindelli and F. Demengel noticed that one can not take test function whose gradient is zero at the test point since the operator may not be defined at that point. So in this case, there will be a small change in the definition of the viscosity solutions.

First Type of Singularity:

As we have mentioned that the first article in this direction is [78], where authors considered the following problem

$$\begin{cases} F(x, u, Du, D^2u) = f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.1}$$

where F is a positively homogeneous fully nonlinear elliptic operator and f has singularities at $u = 0$. In this paper, authors proved the existence, uniqueness and regularities of viscosity solutions to (6.1). The basic classical work in this direction for semilinear equations can be found in [54]. The precise statement, in the fully nonlinear case is the following.

THEOREM 52. *Suppose F is of the form (2.18) and satisfies (SCI), (5.6), (5.7) and (5.8). Assume $\rho > 0$ and $p \in L^n(\Omega)$, further, $p \geq 0$ in Ω and $p > 0$ on a subset of Ω with positive measure. Then the problem*

$$\begin{cases} F(x, u, Du, D^2u) = p(x)u^{-\rho} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.2}$$

has unique L^n -viscosity solution u in $W_{loc}^{2,n} \cap C(\bar{\Omega})$.

The authors proved Theorem 52 by the method of sub and supersolution, with the combinations of the results on the existence of eigenvalues and eigenfunctions of fully nonlinear operators in [150]. Further, they also proved that if p in the above theorem behaves like a power of the distance function as in [86], [62], then the following theorem also holds.

THEOREM 53. *Suppose that F satisfies hypotheses of Theorem 52, assume there are constants $c_1, c_2 > 0$ such that*

$$c_1d(x)^\eta \leq p(x) \leq c_2d(x)^\eta, \text{ for some } \eta \geq 0, \tag{6.3}$$

where $d(x) = d(x, \partial\Omega)$. Then for a solution u of Equation(6.2), we have:

(i) If $\rho < 1 + \eta$ then $u \in C^{1,\beta}(\Omega)$, for some β which depends only on $\eta, \rho, \lambda, \Lambda, \gamma, \delta, n$, and Ω . (ii) If $\rho = 1 + \eta$ then $u \in C^\beta(\Omega)$ for all $\beta < 1$, and there exist constants $a_1, a_2, D > 0$ such that

$$a_1d(x)(D - \log d(x))^{\frac{1}{1+\rho}} \leq u(x) \leq a_2d(x)(D - \log d(x))^{\frac{1}{1+\rho}}.$$

(iii) If $\rho > 1 + \eta$ then $u \in C^{\frac{\eta+2}{1+\rho}}(\bar{\Omega})$ and for some constants $a_1, a_2 > 0$,

$$a_1d(x)^{\frac{\eta+2}{1+\rho}} \leq u(x) \leq a_2d(x)^{\frac{\eta+2}{1+\rho}}, x \in \Omega.$$

In course of proving Theorem 52, P. Felmer, A. Quaas and B. Sirakov also proved the existence of first eigenvalue for fully nonlinear elliptic operator with a non-negative weight. More precisely, they considered the following eigenvalue problem

$$\begin{cases} F(x, u, Du, D^2u) = \mu p(x)u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.4}$$

and proved the existence of $(\mu_1^+, \phi^+) \in \mathbb{R} \times W^{2,n}(\Omega)$ satisfying (6.4) with $\mu = \mu_1^+$ such that $\phi^+ > 0$ in Ω . Here μ_1^+ also depends upon the weight function $p(x)$. They also proved the comparison principle for

$$H(w) := F(x, w, Dw, D^2w) + p(x)(w + \delta)^{-p} = 0 \text{ in } \Omega, \tag{6.5}$$

for details, we refer to Theorem 7 [78]. In this direction, [3] is also recently appeared, where the authors use some geometric approach to find the viscosity solution.

Second Type of Singularity:

Before proceeding further, we would like to comment that in this subsection we are following the definition of the viscosity solution, ellipticity, etc., from [21] for making my presentation easier. Let us consider an operator

$$F : \mathbb{R}^n \times (\mathbb{R}^n)^* \times S(n) \longrightarrow \mathbb{R}, \tag{6.6}$$

where $(\mathbb{R}^n)^* = \mathbb{R}^n \setminus \{0\}$, which satisfies the following conditions

(SF1) $F(x, p, 0) = 0, \forall (x, p) \in \mathbb{R}^n \times (\mathbb{R}^n)^*$.

(SF2) There exists a continuous function $\omega, \omega(0) = 0$, such that if $X, Y \in S(n)$ and ξ_j satisfy

$$-\xi_j \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq 4\xi_j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{6.7}$$

where I is the identity matrix in $S(n)$, then for all $x, y \in \mathbb{R}^n$, we have

$$F(x, \xi_j(x - y), X) - F(y, \xi_j(x - y), -Y) \leq \omega\left(\xi_j|x - y|^2 + \frac{1}{j}\right).$$

(SF3) There exist $\sigma, \theta \in \mathbb{R}$ satisfying $\sigma \geq \theta > -1$, and $\lambda, \Lambda \in \mathbb{R}^+$, such that for all $x \in \mathbb{R}^n, p \in (\mathbb{R}^n)^*, M, N \in S(n), N \geq 0$, we have

$$|p|^\theta \lambda \text{trace}(N) \leq F(x, p, M + N) - F(x, p, M) \leq \left(\frac{|p|^\sigma + |p|^\theta}{2}\right) \Lambda \text{trace}(N).$$

In [21], authors proved the comparison principle for the singular fully nonlinear equations satisfying certain conditions. The comparison principle reads as follows.

THEOREM 54. *Let Ω be a bounded open set in \mathbb{R}^n , whose boundary is piecewise C^1 . Suppose that F satisfies conditions (SF1), (SF2) and (SF3), and b is some continuous and increasing function on \mathbb{R} such that $b(0) = 0$. Suppose that $w \in C(\bar{\Omega})$ is a*

viscosity sub-solution of $F = b$ and $v \in C(\bar{\Omega})$ is a viscosity supersolution of $F = b$. If $w \leq v$ on $\partial\Omega$, then $w \leq v$ in Ω .

If b is nondecreasing, the same conclusion holds when v is a strict supersolution or vice versa when w is a strict subsolution.

For the proof, we refer to Theorem 1.1 [21]. Finally, the condition $\sigma \geq \theta > -1$, is optimal since it is possible to construct a counter example for $\sigma = -1$.

REMARK 20. The case where b is nonzero but satisfies some increasing behaviour at infinity can be found in [164].

There is a strong maximum principle in the case where $b = 0$, that is, which is given below.

PROPOSITION 6. Suppose that Ω be a bounded open set in \mathbb{R}^n , which is piecewise C^1 . Suppose that F satisfies (SF1) and (SF3) with $\sigma = \theta$. Let $v \in C(\bar{\Omega})$, $v \geq 0$ on $\partial\Omega$, be a supersolution of $F(x, Dv, D^2v) = 0$. Then, either $v > 0$ in Ω , or $v \equiv 0$.

For the proof of Proposition 6, we refer to Proposition 2.2 [21]. Using the Theorem 54 and Proposition 6, I. Birindelli, F. Demengel [21], have also proved the following Liouville’s type theorem.

THEOREM 55. Suppose that $u \in C(\mathbb{R}^n)$, is a nonnegative viscosity solution of

$$-F(x, Du, D^2u) \geq h(x)u^{\bar{q}} \text{ in } \mathbb{R}^n, \tag{6.8}$$

with h satisfying $h(x) = a|x|^\gamma$, for large $|x|$, $a > 0$ and $\gamma > -(\sigma + 2)$. Let $\beta = \frac{\Lambda}{\lambda}(n - 1) - 1$ and suppose that

$$0 < \bar{q} \leq \frac{1 + \gamma + (\sigma + 1)(\beta + 1)}{\beta},$$

then $u \equiv 0$.

For the proof, we refer to Theorem 3.1 [21].

6.1. Eigenvalues and Eigenfunctions

We have remarked that the definition of the viscosity solution is different for singular operators. So we consider the eigenvalue problem for the operators which are singular separately. In [22], the authors considered the operator which does not explicitly depend on x and satisfies the following homogeneous type condition.

(SF4) $F(tp, \tau X) = |t|^\sigma \tau F(p, X)$ for all $t \in \mathbb{R}$, $\tau \geq 0$, $\sigma > -1$ and $F(p, X) \leq F(p, Y)$ for any $p \neq 0$, and $X \leq Y$.

Note that when F satisfies (SF3), then it satisfies monotonicity condition defined in (SF4).

Let us consider the following set

$$E = \{ \mu \in \mathbb{R} : \exists \phi, \phi_* > 0 \text{ in } \bar{\Omega}, F(D\phi, D^2\phi) + \mu \phi^{\sigma+1} \leq 0 \text{ in the viscosity sense } \}, \tag{6.9}$$

where ϕ_* is lower semicontinuous envelope of ϕ and

$$\bar{\mu} = \sup E. \tag{6.10}$$

THEOREM 56. *Suppose that Ω is a bounded open subset of \mathbb{R}^n . Suppose that $\kappa < \bar{\mu}$ and F satisfies (SF4), then every viscosity solutions of*

$$\begin{cases} F(Du, D^2u) + \kappa|u|^\sigma u \geq 0 \text{ in } \Omega, \\ u \leq 0 \text{ on } \partial\Omega, \end{cases} \tag{6.11}$$

satisfies $u \leq 0$ in Ω .

LEMMA 2. *Suppose that $\Omega = B(0, R)$, $l = \frac{\sigma+2}{\sigma+1}$ and*

$$u(x) = \frac{1}{2l} \left(|x|^l - R^l \right)^2.$$

Let F satisfies (SF3), then there exists some constant C depending on n, σ, λ and Λ such that

$$\sup_{x \in B(0, R)} \left\{ \frac{-F(Du, D^2u)}{u^{\sigma+1}} \right\} \leq \frac{C}{R^{\sigma+2}}. \tag{6.12}$$

For the proofs of Proposition 56 and Theorem 2, we refer to Proposition 3.2 and Lemma 3.5 [22], respectively. Using (56), (2), we obtain the following proposition.

PROPOSITION 7. *Suppose that R is the radius of the largest ball contained in Ω and that F satisfies (SF3) with $\sigma = \theta$. Then, there exists some constant C depending on n, σ, λ and Λ such that $\bar{\mu} \leq \frac{C}{R^{\sigma+2}}$.*

Next, we will present the extension of comparison Theorem 54 in the context of F satisfying (SF4) and is independent of x .

THEOREM 57. *Suppose that $\kappa < \bar{\mu}$, f is a nonpositive upper semi continuous and g is lower semi continuous with $f \leq g$ and either $f < 0$ in Ω or $g(\bar{x}) > 0$ on every points \bar{x} satisfying $f(\bar{x}) = 0$. Suppose that there exist v bounded and nonnegative and w bounded, respectively satisfying*

$$F(Dv, D^2v) + \kappa v^{1+\sigma} \leq f, F(Dw, D^2w) + \kappa|w|^\sigma w \geq g,$$

in the viscosity sense, with $w \leq v$, on $\partial\Omega$. Then $w \leq v$ in Ω .

For the proof of Theorem 57, we refer to Theorem 3.6 [22]. The following corollary is a consequence of Theorem 57.

COROLLARY 1. *Suppose that $\kappa < \bar{\mu}$, there exists at most one nonnegative viscosity solution of*

$$\begin{cases} F(Du, D^2u) + \kappa u^{\sigma+1} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.13}$$

for $f < 0$ and continuous.

Once again for the proofs of Proposition 7, Theorem 57 and Corollary 1, we refer to [22]. There are also some Hölder and Lipschitz regularity results for the viscosity solution of

$$\begin{cases} F(Du, D^2u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.14}$$

in Section 4 [22]. Next, we will present some existence results from Section 5 [22] under certain conditions on F and f . We will consider two separate cases : $\kappa < \bar{\mu}$; $\kappa = \bar{\mu}$.

Let us consider the case $\kappa < \bar{\mu}$.

THEOREM 58. *Assume that F satisfies (SF4), (SF3). Suppose that f is bounded and $f \leq 0$ on $\bar{\Omega}$. Then, for $\kappa < \bar{\mu}$ there exists a nonnegative viscosity solution u of*

$$\begin{cases} F(Du, D^2u) + \kappa u^{\sigma+1} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.15}$$

Furthermore, the solution is unique.

For the proof of Theorem 58, we refer to Theorem 5.1 [22]

PROPOSITION 8. *Suppose that f is bounded, continuous and nonpositive, and $\kappa \in \mathbb{R}$. Suppose that there exist w and $v \geq 0$, respectively a subsolution and supersolution of*

$$\begin{cases} F(Du, D^2u) + \kappa u^{\sigma+1} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.16}$$

with $w \leq v$. Then there exists a viscosity solution u of (6.16), such that $w \leq u \leq v$. Moreover, if $f < 0$ in Ω the solution is unique.

For the proof of Proposition 8, we refer to Proposition 5.2 [22].

PROPOSITION 9. *Suppose that F satisfies (SF4), (SF3) with $\sigma = \theta$. For any bounded and nonpositive f in $\bar{\Omega}$, there exists a viscosity solution u of*

$$\begin{cases} F(Du, D^2u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.17}$$

Of course, u is nonnegative by the maximum principle and Hölder continuous.

For the proof of Proposition 9, we refer to Proposition 5.3 [22].
 Let us consider the case $\kappa = \bar{\mu}$.

THEOREM 59. *Let F satisfy (SF3) and (SF4) with $\sigma = \theta$. Then, there exists a viscosity solution $u > 0$, in Ω of*

$$\begin{cases} F(Du, D^2u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.18}$$

Moreover, u is γ -Hölder continuous for all $\gamma \in (0, 1)$ and locally Lipschitz if F satisfies one additional condition given below.

There exists $\eta \in (\frac{1}{2}, 1]$ such that for all $|p| = 1$, q , $|q| < \frac{1}{2}$, and $B \in S(n)$

$$|F(q + p, B) - F(p, B)| \leq \nu |q|^\eta |B|.$$

For the proof, we refer to Theorem 5.5 [22]. Eigenvalues and eigenfunctions for general singular fully nonlinear second order elliptic operator were considered in [23]. I. Birindelli, F. Demengel considered the operators of the form

$$G(x, u, Du, D^2u) = F(x, Du, D^2u) + b(x) \cdot Du |Du|^\sigma + c(x) |u|^\sigma u, \tag{6.19}$$

where F is continuous on $\Omega \times (\mathbb{R}^n)^* \times S(n)$. Let us also define some conditions that will be needed in the statement of theorems, below.

(SF6) There exists a continuous function $\tilde{\omega}$, $\tilde{\omega}(0) = 0$ such that for all $x, y \in \Omega$, $p \neq 0 \in \mathbb{R}^n$ and $X, Y \in S(n)$ satisfy

$$|F(x, p, X) - F(y, p, X)| \leq \tilde{\omega}(\xi |x - y|) |p|^\sigma |X|, \text{ for } \xi \in \mathbb{R}.$$

(SF7) There exists a continuous function ϑ with $\vartheta(0) = 0$ such that

$$F(x, \xi(x - y), X) - F(y, \xi(x - y), -Y) \leq \vartheta(\xi |x - y|^2),$$

holds for all $x, y \in \mathbb{R}^n$, $x \neq y$, whenever $X, Y \in S(n)$ and $\xi \in \mathbb{R}$ satisfy (6.7) with $\xi_j = \xi$. Further, assume that $b : \Omega \rightarrow \mathbb{R}^n$ is a continuous, bounded function satisfying:

(SF8) Either $\sigma < 0$ and b is Hölder continuous with exponent $1 + \sigma$, or $\sigma \geq 0$ and for all x and y , $\langle b(x) - b(y), x - y \rangle \leq 0$. As it is clear from the definition of the operator that the authors are considering the explicit dependence of F on x as well as the lower order terms. The results of [23] are also general than the results in [22] in the sense of the definition of $\bar{\mu}$. In order to avoid the confusion with the notation we will use $\tilde{\mu}$ instead of $\bar{\mu}$ for the operator G .

$$\tilde{\mu} = \sup\{\mu : \exists \phi < 0 \text{ in } \Omega, F(x, D\phi, D^2\phi) + b(x) \cdot D\phi |D\phi|^\sigma + c(x) |\phi|^\sigma \phi \geq 0 \text{ in } \Omega\}. \tag{6.20}$$

In the similar way, one can also define

$$\tilde{\tilde{\mu}} = \sup\{\mu : \exists \phi > 0 \text{ in } \Omega, F(x, D\phi, D^2\phi) + b(x) \cdot D\phi |D\phi|^\sigma + c(x) |\phi|^\sigma \phi \leq 0 \text{ in } \Omega\}. \tag{6.21}$$

In [23], the authors also generalised the comparison principle 54.

THEOREM 60. *Suppose that F satisfies (SF3), (SF7) and for all $x \in \Omega$, $p \in \mathbb{R}^n \setminus \{0\}$, $M, N \in S(n)$ with $N \geq 0$,*

$$F(x, p, N + M) \geq F(x, p, M),$$

and suppose that b is a bounded continuous function and satisfies (SF8). Let f and g be respectively, upper and lower semicontinuous functions. Suppose that β is some continuous function on \mathbb{R}^+ such that $\beta(0) = 0$. Suppose that $w > 0$ in Ω lower semicontinuous and v upper semicontinuous functions satisfying

$$F(x, Dw, D^2w) + b(x).Dw(x)|Dw|^\alpha - \beta(w) \leq f,$$

and

$$F(x, Dv, D^2v) + b(x).Dv(x)|Dv|^\alpha - \beta(v) \geq g,$$

respectively in the viscosity sense. Suppose that β is increasing on \mathbb{R}^+ and $f \leq g$ or β is nondecreasing and $f < g$. If $w \leq v$ on $\partial\Omega$, then $w \leq v$ in Ω .

The proof of above theorem follows in two steps. For the proof, we refer to Theorem 1 [23]. Next, we state the strong maximum principle.

THEOREM 61. *Suppose that F satisfies (SF3) with $\sigma = \theta$, b and c are continuous, bounded and b satisfies (SF8). Let v be a non-negative lower semicontinuous viscosity supersolution of*

$$F(x, Dv, D^2v) + b(x).Dv|Dv|^\sigma + c(x)v^{\sigma+1} \leq 0. \tag{6.22}$$

Then either $v \equiv 0$ or $v > 0$ in Ω .

For the proof, we refer to Theorem 2 [23]. There is also a Höpf type result, for the statement and proof, we refer to Corollary 1 [23].

THEOREM 62. *Let Ω be a bounded domain in \mathbb{R}^n . Suppose that F satisfies (SF4), (SF7) and for $x \in \Omega$, $p \in \mathbb{R}^n \setminus \{0\}$, $M, N \in S(n)$ with $N \geq 0$*

$$F(x, p, N + M) \geq F(x, p, M),$$

and that b and c are continuous and b satisfies (SF8). Suppose that $\kappa < \tilde{\mu}$ and that w is a viscosity sub solution of

$$F(x, Dw, D^2w) + b(x).Dw|Dw|^\sigma + c(x)|w|^\sigma w + \kappa|w|^\sigma w \geq 0 \text{ in } \Omega,$$

with $w \leq 0$ on $\partial\Omega$, then $w \leq 0$ in Ω .

PROPOSITION 10. Suppose that F satisfies (SF4) and (SF3) with $\sigma = \theta$, b and c are bounded; furthermore let c be nonpositive in $\bar{\Omega}$. Then there exists a function v which is a nonnegative viscosity super solution of

$$\begin{cases} F(x, Dv, D^2v) + b(x).Dv|Dv|^\sigma + c(x)|v|^\sigma v \leq -1 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.23}$$

For the proof, we refer to Proposition 7 [23].

THEOREM 63. Suppose that F satisfies (SF4) and (SF3) with $\sigma = \theta$ and that b and c are continuous with $c \leq 0$.

(i) If f is a bounded continuous, $f \leq 0$ on $\bar{\Omega}$, then there exists a nonnegative viscosity solution u of

$$\begin{cases} F(x, Du, D^2u) + b(x).Du|Du|^{\sigma+1} + c(x)u^{\sigma+1} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.24}$$

(ii) For any bounded continuous function with $f < -M < 0$ for some positive constant M and any $0 \leq c_0 \leq \left(\frac{M}{\|c\|_{L^\infty(\Omega)}}\right)^{\frac{1}{1+\sigma}}$, there exists u a non negative solution of

$$\begin{cases} F(x, Du, D^2u) + b(x).Du|Du|^{\sigma+1} + c(x)u^{\sigma+1} = f \text{ in } \Omega, \\ u = c_0 \text{ on } \partial\Omega. \end{cases} \tag{6.25}$$

For the proof, we refer to Theorem 6 [23].

THEOREM 64. Suppose that F satisfies (SF3), (SF7) and for all $x \in \Omega$, $p \in \mathbb{R}^n \setminus \{0\}$, $M, N \in S(n)$ with $N \geq 0$,

$$F(x, p, N + M) \geq F(x, p, M).$$

Let b be a bounded, continuous and satisfies (SF8), and $\mu < \tilde{\mu}$.

(i) If f is bounded, continuous, and $f \leq 0$ on $\bar{\Omega}$, then there exists a nonnegative viscosity solution u of

$$\begin{cases} F(x, Du, D^2u) + b(x).Du|Du|^\sigma + (c(x) + \mu)u^{\sigma+1} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.26}$$

(ii) For any bounded, continuous function f with $f < -M < 0$, for some positive constant M and any $0 \leq c_0 \leq \left(\frac{M}{\|c\|_{L^\infty(\Omega)}}\right)^{\frac{1}{1+\sigma}}$, there exists a nonnegative viscosity solution of

$$\begin{cases} F(x, Du, D^2u) + b(x).Du|Du|^{\sigma+1} + c(x)u^{\sigma+1} = f \text{ in } \Omega, \\ u = c_0 \text{ on } \partial\Omega. \end{cases} \tag{6.27}$$

For the proof, we refer to Theorem 7 [23]. The next theorem gives the existence of eigenfunction and justifies that $\tilde{\mu}$ is an eigenvalue of the operator defined by (6.19).

THEOREM 65. *Suppose that F satisfies (SF4), (SF3) and (SF7) and that b, c are bounded, continuous and b also satisfies (SF8). Then there exists $0 < \phi$ in Ω such that ϕ is a viscosity solution of*

$$\begin{cases} F(x, D\phi, D^2\phi) + b(x) \cdot D\phi |D\phi|^\sigma + (c(x) + \tilde{\mu}) |\phi|^{\sigma+1} \phi = 0 \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.28}$$

For the proof of Theorem 65, we refer to Theorem 8 [23]. Note that in the proof of Theorem 65, I. Birindelli, F. Demengel, have used the fact that Ω is C^2 . Later on, the same authors proved the existence of the eigenvalue and eigenfunction for same operator in more general bounded domain which satisfies the uniform exterior cone condition, which we will present below. In order to state the precise results of [24], we need to define two functions which satisfy certain conditions.

$$(SF5) \begin{cases} \text{For } i = 1, 2, \text{ let } h_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R} \text{ such that } h_i(\cdot, t) \in L^\infty(\Omega) \text{ for all } t, \\ h_i(x, \cdot) \text{ is non-increasing and continuous for all } x \in \Omega, h_i(x, 0) = 0 \\ \text{and } \lim_{t \rightarrow \infty} \frac{h_2(x, t)}{t^{\sigma+1}} = 0. \end{cases} \tag{6.29}$$

Whenever h_i satisfy (6.29), we say that it satisfy (SF5). In [24], I. Birindelli and F. Demengel proved the following existence theorem.

THEOREM 66. *Suppose that $\mu < \inf\{\tilde{\mu}, \tilde{\mu}\}$ where $\tilde{\mu}$ and $\tilde{\mu}$ are given by (6.21) and (6.20), respectively. Suppose further that $h_1(x, t)$ and $h_2(x, t)$ satisfy (SF5), F and b satisfy (SF4), (SF3), (SF6), (SF7) and (SF8), respectively. Then for $g \in W^{2,\infty}(\partial\Omega)$ and $f \in C(\bar{\Omega})$, there exists a solution u of*

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du |Du|^\sigma + (c(x) + \mu) |u|^\sigma u + h_1(x, u) = f(x) + h_2(x, u) \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{6.30}$$

The proof of Theorem 66 follows by the several steps consisting of theorems and propositions. For details, we refer to Theorem 1.1 [24]. We would like to state the theorems and propositions in the steps of proof of the Theorem 66. In the next, we will assume that F, h, b, c satisfy the same conditions as in the above theorem, unless otherwise stated.

THEOREM 67. *Let Ω be a bounded domain of \mathbb{R}^n . Suppose that F satisfies (SF4), (SF3) with $\sigma = \theta$ and (SF7) and that b and c are continuous and b satisfies (SF8). Suppose also that h is a continuous function such that $h(x, \cdot)$ is nonincreasing, $h(x, 0) = 0$. Suppose that $\kappa < \tilde{\mu}$ and that w is a viscosity subsolution of*

$$F(x, Dw, D^2w) + b(x) \cdot Dw |Dw|^\sigma + (c(x) + \kappa) |w|^\sigma w + h(x, w) \geq 0 \text{ in } \Omega,$$

and $w \leq 0$ on $\partial\Omega$, then $w \leq 0$ in Ω . If $\kappa < \bar{\mu}$ and v is a supersolution of

$$F(x, Dv, D^2v) + b(x) \cdot Dv |Dv|^\sigma + (c(x) + \kappa) |v|^\sigma v + h(x, v) \leq 0 \text{ in } \Omega$$

and $v \geq 0$ on $\partial\Omega$ then, $v \geq 0$ in Ω .

THEOREM 68. Suppose that F satisfies (SF4), (SF3) with $\sigma = \theta$, and (SF7) and that b and c are bounded, continuous and b satisfies (SF8). Suppose that h is such that, for all $x \in \Omega$, $t \rightarrow \frac{-h(x,t)}{t^{\sigma+1}}$ is nondecreasing on \mathbb{R}^+ . Suppose that $\kappa < \bar{\mu}$, $f \leq 0$, f is upper semicontinuous and g is lower semicontinuous with $f \leq g$. If w is an upper semicontinuous subsolution of

$$F(x, Dw, D^2w) + b(x) \cdot Dw |Dw|^\sigma + (c(x) + \kappa) |w|^\alpha w + h(x, w) \geq g \text{ in } \Omega,$$

and v is a non-negative lower semicontinuous supersolution of

$$F(x, Dv, D^2v) + b(x) \cdot Dv |Dv|^\sigma + (c(x) + \kappa) |v|^\sigma v + h(x, v) \leq f \text{ in } \Omega,$$

such that $w \leq v$ on $\partial\Omega$, then $w \leq v$ in Ω in each of the following two cases:

- (1) If $v > 0$ on $\bar{\Omega}$ and either $f < 0$ in Ω , or if $f(\bar{x}) = 0$ then $g(\bar{x}) > 0$.
- (2) if $v > 0$ in Ω , $f < 0$ and $f < g$ on $\bar{\Omega}$.

For the proof of Theorem 67 and Theorem 11, we refer to Theorem 2 and Theorem 3 [24], respectively.

PROPOSITION 11. Let $q < \sigma$, $\eta > 0$ and $\kappa > 0$. Then there exists $\varepsilon > 0$ and $M > 0$, such that for any continuous function f satisfying $0 > f(x) > -\varepsilon$, there exist two solutions u and v of

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du |Du|^\sigma - \beta u^{1+q} + \mu u^{1+\sigma} = f(x) \text{ in } \Omega, \\ u = M \text{ on } \partial\Omega. \end{cases} \tag{6.31}$$

with $u \leq M \leq v$, $u \not\equiv M$ and $v \not\equiv M$.

For the proof, we refer to Proposition 1 [24].

PROPOSITION 12. Suppose that $g \in W^{2,\infty}(\partial\Omega)$, $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $h(x, \cdot)$ is non increasing, continuous and $f \in L^\infty$. Then there exists a viscosity solution u of

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du |Du|^\sigma + h(x, u) = f(x) \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{6.32}$$

For the proof, we refer to Proposition 2 [24].

PROPOSITION 13. Suppose that $g \in W^{2,\infty}(\partial\Omega)$, and that $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is such that, $h(x, \cdot)$ is non increasing continuous and $m \in \mathbb{R}^+$. Then there exists a viscosity subsolution w of

$$\begin{cases} F(x, Dw, D^2w) + b(x) \cdot Dw |Dw|^\sigma + h(x, w) \geq m \text{ in } \Omega, \\ w = g \text{ on } \partial\Omega. \end{cases} \tag{6.33}$$

PROPOSITION 14. Suppose that $m > 0$, $g \in W^{2,\infty}(\partial\Omega)$ and $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $h(x, \cdot)$ is non increasing and continuous. Then there exists a viscosity supersolution v of (6.32) with $f \equiv -m$.

For the proof of (13) and (14), we refer to Proposition 3 and Proposition 4 [24], respectively.

Next, there is regularity result for the solution to (6.32) with $h = 0$.

THEOREM 69. Suppose that $g \in W^{2,\infty}(\partial\Omega)$, then every viscosity solutions to (6.32) with $h \equiv 0$ satisfies:

$$|u(x) - u(y)| \leq C|x - y|^\gamma \forall x, y \in \Omega,$$

for any $\gamma \in (0, 1)$, where C is a constant depending on γ , $\|g\|_{W^{2,\infty}(\partial\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$.

THEOREM 70. Suppose $\kappa < \mu_1 = \min\{\tilde{\mu}, \tilde{\mu}\}$. Then for all $g, f \in \mathbb{R}^+$ there exist viscosity solutions $\bar{u} \geq 0$ $\underline{u} \leq 0$ of

$$\begin{cases} F(x, D\bar{u}, D^2\bar{u}) + b(x).D\bar{u}|D\bar{u}|^\sigma + (c(x) + \kappa)|\bar{u}|^\sigma \bar{u} + h_1(x, \bar{u}) = -f(x) \\ \hspace{15em} + h_2(x, \bar{u}) \text{ in } \Omega, \\ \bar{u} = g \text{ on } \partial\Omega. \end{cases} \tag{6.34}$$

and

$$\begin{cases} F(x, D\underline{u}, D^2\underline{u}) + b(x).D\underline{u}|D\underline{u}|^\sigma + (c(x) + \kappa)|\underline{u}|^\sigma \underline{u} + h_1(x, \underline{u}) = f(x) \\ \hspace{15em} + h_2(x, \underline{u}) \text{ in } \Omega, \\ \underline{u} = -g \text{ on } \partial\Omega, \end{cases} \tag{6.35}$$

respectively.

For the proof of Theorems (69) and (70), we refer to Theorem 4 and Theorem 5 [24], respectively. As after Theorem 65, we had mentioned that the authors have also proved the existence of eigenvalue and eigenfunction for the singular fully nonlinear operator in case of non-smooth domains which satisfy the uniform exterior cone condition. Here, we will consider that problem. Note that if a domain is C^2 , then it satisfies the uniform exterior cone condition. Let us consider the operator

$$F(x, Du, D^2u) + b(x).Du|Du|^\sigma + (c(x) + \mu)|u|^\sigma u,$$

with F satisfying (SF4), (SF3) with $\sigma = \theta$, (SF6) and (SF7) and b, c are bounded and continuous functions on $\bar{\Omega}$ and in addition to this b also satisfies (SF8).

THEOREM 71. Suppose that Ω satisfies the exterior cone condition and F satisfies (SF3) with $\sigma = \theta$, (SF4), (SF6) and (SF7) and b is continuous and bounded

functions on $\bar{\Omega}$ and in addition to this b also satisfies (SF8). There exists u_0 a nonnegative viscosity solution of

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du |Du|^\sigma = -1 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.36}$$

which is Hölder continuous.

Further, the solution of above theorem also satisfies:

- (i) $\forall \delta$, there exists K , a compact set in Ω such that $\sup_{\Omega \setminus K} |u_0| \leq \delta$,
- (ii) $\forall M > 0$, there exists K some compact subset of Ω , large enough, such that

$$\tilde{\mu}(\Omega - K) > M.$$

For the proof of Theorem 71, we refer to Proposition 3.2 [25]. Theorem 71 will be the first step in the proof of the maximum principle and the construction of the principal eigenfunction in non-smooth bounded domains. The global barrier approach is given in following proposition.

PROPOSITION 15. For all $z \in \partial\Omega$, there exists a continuous function W_z on $\bar{\Omega}$, such that $W_z(z) = 0$, $W_z > 0$ in $\Omega \setminus \{z\}$ which is supersolution of (6.36).

For the proof of Proposition 15, we refer to Proposition 3.3 [25]. The next proposition deals with the Hölder regularity of a sequence of bounded solutions of equation similar to (6.36).

PROPOSITION 16. Let H_j be a sequence of bounded open regular sets such that $H_j \subset \bar{H}_j \subset H_{j+1}$, $j \geq 1$ whose union equals Ω . Let u_j be a sequence of bounded solutions of

$$\begin{cases} F(x, Du_j, D^2u_j) + b(x) \cdot Du_j |Du_j|^\sigma = f_j \text{ in } H_j, \\ u_j = 0 \text{ on } \partial H_j, \end{cases} \tag{6.37}$$

with f_j uniformly bounded in H_j . Then, for $\gamma \in (0, 1)$, there exists C independent of j such that

$$|u_j(x) - u_j(y)| \leq C|x - y|^\gamma,$$

for all $x, y \in \Omega$.

For the details about γ and the proof of Proposition 16, we refer to Proposition 3.6 [25].

COROLLARY 2. Given $f \in C(\bar{\Omega})$, there exists a γ -Hölder continuous viscosity solution of

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du |Du|^\sigma = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.38}$$

with

$$|u(x)| \leq \|f\|_{L^\infty}^{\frac{1}{1+\sigma}} \sup\{u_0(x), -u'_0(x)\}.$$

Further, if $f \leq 0$, $u \geq 0$, and if $f \geq 0$, $u \leq 0$.

For more information about u'_0 and proof, we refer to Remark 3.7 and Corollary 3.8 [25], respectively. The authors in [25], proved the following maximum principle.

PROPOSITION 17. Let $\beta(x, \cdot)$ be a nondecreasing continuous function such that $\beta(x, 0) = 0$. Suppose also that w is upper semicontinuous, bounded above and satisfies

$$F(x, Dw, D^2w) + b(x).Dw|Dw|^\sigma - \beta(x, w) \geq 0, \tag{6.39}$$

with $\limsup w(x_j) \leq 0$ for all $x_j \rightarrow \partial\Omega$. Then $w \leq 0$ in Ω .

For the proof, we refer to Proposition 3.14 [25]. Now we take the main result which is generalisation of (65) in the domain which satisfy the exterior cone condition.

THEOREM 72. Let Ω be a bounded domain which satisfies the uniform exterior cone condition, F and b satisfy conditions mentioned above. There exists a positive function ϕ satisfying

$$\begin{cases} F(x, D\phi, D^2\phi) + b(x).D\phi|D\phi|^\sigma + (c(x) + \tilde{\mu})\phi^{\sigma+1} = 0 \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.40}$$

which is Hölder continuous.

In the last section of [25], the authors also gave some other maximum principle under certain conditions on F , b and Ω , which we are going to state below. Let us first define

$$\mu_e = \sup\{\tilde{\mu}(\Omega') \mid \Omega \subset\subset \Omega', \Omega' \text{ is } C^2 \text{ and bounded}\}$$

and

$$\bar{\mu} = \sup\{\mu, \exists \phi > 0 \text{ in } \bar{\Omega}, F(x, D\phi, D^2\phi) + b(x).D\phi|D\phi|^\sigma + (c(x) + \mu)\phi^{\sigma+1} \leq 0\}.$$

They proved that $\mu_e = \bar{\mu}$ and it is an eigenvalue in the sense that there exists some $\phi_e > 0$, which satisfies

$$\begin{cases} F(x, D\phi_e, D^2\phi_e) + b(x).D\phi_e|D\phi_e|^\sigma + (c(x) + \mu_e)\phi_e^{\sigma+1} = 0 \text{ in } \Omega, \\ \phi_e = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.41}$$

Let us state the maximum principle.

PROPOSITION 18. For $\mu < \bar{\mu}$, if w is a viscosity subsolution of

$$F(x, Dw, D^2w) + b(x).Dw|Dw|^\sigma + (c(x) + \mu)w^{\sigma+1} \geq 0, \tag{6.42}$$

satisfying $w(x) \leq 0$ on $\partial\Omega$, then $w \leq 0$ in Ω .

In the next proposition, the authors stated that μ_e is an eigenvalue of (6.41) and also proved that $\tilde{\mu} = \mu_e$. Further, they also proved the following Proposition.

PROPOSITION 19. Let $\mu < \mu_e$, then for any continuous nonpositive function f there exists a viscosity solution u of

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du |Du|^\sigma + (c(x) + \mu)u^{\sigma+1} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.43}$$

Further, $u \geq 0$ and is Hölder continuous.

7. Regularity of the Viscosity solution

There are various ways to demonstrate the existence of solutions to PDEs. Out of these methods, one is the method of continuity which is applied once some estimates for the solutions up to the boundary are available (in the classical sense). For the method of continuity in the context of the existence of the classical solutions to fully nonlinear elliptic equations, we refer to Section 17.2 [85] and Section 1.2.3 [41]. Below, we will present some regularity and estimates for the viscosity solutions to fully nonlinear elliptic equations. One of the major contributions in the direction of obtaining the regularity of the classical solutions to fully nonlinear elliptic equations was made by the Krylov and Safonov in [113, 114], by obtaining the estimate for the solutions to the non-divergence form elliptic operators with measurable coefficients and of course, Aleksandrov-Bakelman-Pucci maximum principle was applied for the same. These results collectively presented in Chapter 9 [85], see also [142, 143, 152, 44]. With the help of estimate obtained by Krylov and Safonov, regularity of the classical solutions to fully nonlinear elliptic equations was obtained by L. C Evans and N. V. Krylov in two independent works [65] and [111, 112], respectively. The authors proved that the Hessian of the solutions of

$$F(D^2u) = 0, \tag{7.1}$$

are Hölder continuous under the assumption of convexity of F in D^2u . The main idea of the proof was to differentiate (7.1) twice in the direction $|e| = 1$ and to show that $\frac{\partial^2 u}{\partial e^2}$ is a subsolution of

$$-a^{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{\partial^2 u}{\partial e^2} \right) \leq 0, \tag{7.2}$$

where

$$a^{i,j} = \frac{\partial F(D^2u(x))}{\partial \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)},$$

by using the fact that F is a concave. Further, since (7.2) is a linear equation in the non-divergence form with measurable coefficients so the C^α -estimates can be obtained as a consequence of Krylov-Safonov Harnack inequality, see [113, 114].

THEOREM 73. When the operator F is concave or convex, then classical solutions of $F(D^2u) = 0$ satisfy the following $C^{2,\alpha}$ estimate

$$\|u\|_{C^{2,\alpha}(\bar{B}_{\frac{1}{2}})} \leq C(\|u\|_{L^\infty(\bar{B}_1)} + |F(0)|). \tag{7.3}$$

The above theorem also remains true if we replace classical solution by the “viscosity solution”. For the proof, we refer to Theorem 6.6 [36]. Next, we would like to present the generalised maximum principle for viscosity solutions for fully nonlinear elliptic equations, also known as Alexandrov-Bakelman-Pucci(ABP inequality). This inequality in the context of the viscosity solutions first of all appeared in [35] for Pucci maximal operator, see Lemma 1 [35]. We are taking slightly general form than [45]. In order to present the statement of *ABP*-inequality, we need some notations. For $u \in C(\bar{\Omega})$, we define the upper contact set of u as follows:

$$\Gamma^+(u) = \{x \in \Omega : \exists p \in R^n \text{ such that } u(y) \leq u(x) + \langle p, y - x \rangle \text{ for } y \in \Omega\}$$

and

$$\{u > 0\} = \{x \in \Omega \mid u(x) > 0\}.$$

There exists a constant $C = C(\lambda, \gamma, n)$ depending only on the quantities indicated such that if $f \in L^n(\Omega) \cap C(\Omega)$, $0 \leq \gamma$, and $u \in C(\bar{\Omega})$ is a C -viscosity solution of

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2u) - \gamma|Du| \leq f(x) \quad \text{on } \{0 < u\}, \tag{7.4}$$

then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \text{diam}(\Omega)C(\lambda, \gamma, n)\|f^+\|_{L^n(\Gamma^+(u^+))}.$$

Similarly, if u is a C -viscosity solution of

$$f(x) \leq \mathcal{P}_{\lambda, \Lambda}^+(D^2u) + \gamma|Du| \quad \text{on } \{u < 0\} \tag{7.5}$$

then

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + \text{diam}(\Omega)C(\lambda, \gamma, n)\|f^-\|_{L^n(\Gamma^+(u^-))}.$$

For the proof, we refer to Appendix A [45]. There are many consequences of the *ABP*-estimates, for example, the following maximum principle in small domains. Suppose that $u \in C(\bar{\Omega})$ satisfies

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^-(D^2u) - \gamma|Du| - \delta|u| \leq 0 \text{ in } \{u > 0\} \\ u \geq 0 \text{ on } \partial\Omega \end{cases} \tag{7.6}$$

Then there exists a constant $\varepsilon > 0$, depending only on $n, \gamma, \delta, \lambda$ and $\text{diam}(\Omega)$, such that $|\Omega| < \varepsilon$ implies that $u \geq 0$ in Ω . For the details about the *ABP* maximum principle and related results for linear and fully nonlinear elliptic equations, see [88, 20, 31, 40, 136]. Now we would like to present the generalisation of the Krylov-Safonov Harnack inequality to the fully nonlinear elliptic operators in the viscosity sense. It was proved by L. Wang in [162] and further extended by J. Busca and B. Sirakov in [29]. Here we have taken the result from [150].

THEOREM 74. *Suppose that $u \in C(\Omega)$, $f \in C(\Omega) \cap L^n(\Omega)$ and $u \geq 0$ in Ω and*

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^+(D^2u) + \delta|Du| + \delta_0|u| \geq -|f| \text{ in } \Omega \\ \mathcal{P}_{\lambda, \Lambda}^+(D^2u) + \delta|Du| + \delta_0|u| \leq |f| \text{ in } \Omega. \end{cases} \tag{7.7}$$

Then for any compact subset K of Ω ,

$$\sup_K u \leq C(\inf_K u + \|u\|_{L^n(\Omega)}),$$

where the constant C depends only on $\Omega, K, n, \Lambda, \lambda, \delta$ and δ_0 .

For the proof, we refer to Theorem 3.6 [150]. The next result concerns with the interior Hölder regularity of the viscosity solutions of fully nonlinear elliptic PDEs. Such type of the result in the context of the viscosity solution first of all appeared in Section VII.1 [98]. These results are also appeared in [80, 162]. We have taken the following interior Hölder estimate from [80] (Theorem 5.21).

THEOREM 75. *Assume that $f \in C(\Omega) \cap L^n(\Omega)$ and $u \in C(\bar{\Omega})$ satisfies Equation 7.7. Then there is a constant $\alpha > 0$ depending only on $n, \delta, \delta_0, \Lambda$ such that for any $\Omega' \subset\subset \Omega$,*

$$\|u\|_{C^\alpha(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \|u\|_{L^n(\Omega)}),$$

where C depends only on $n, \delta, \delta_0, \Lambda, \text{diam}(\Omega)$ and $\text{dist}(\Omega', \partial\Omega)$.

N. Winter proved the boundary Hölder regularity for the viscosity solutions of fully nonlinear elliptic PDEs. The author proved the weak Harnack Inequality at the boundary, see Theorem 1.9 [165] and combined the above interior Hölder regularity for the viscosity solutions to obtain the following boundary C^α estimate.

THEOREM 76. *Assume that Ω satisfies a uniform exterior cone condition, $f \in C(\Omega) \cap L^n(\Omega)$, and $\phi \in C^\gamma(\partial\Omega)$. Suppose further that $u \in C(\bar{\Omega})$ satisfies Equation (7.7) with $u = \phi$ on $\partial\Omega$. Then there exists a constant $\alpha > 0$ depending only on $n, \gamma, \Lambda, \delta$ and exterior cone condition such that*

$$\|u\|_{C^\alpha(\Omega)} \leq C(\|u\|_{L^\infty(\Omega)} + \|\phi\|_{C^\gamma(\partial\Omega)} + \|u\|_{L^n(\Omega)}), \tag{7.8}$$

where C depends only on $n, \gamma, \Lambda, \delta, \delta_0$ and $\text{diam}(\Omega)$.

See also [15] for the C^α -regularity of the viscosity solutions. Let us state the global $C^{1,\alpha}$ estimate for the viscosity solution which first of all appeared in [157].

THEOREM 77. *Assume F satisfies (SCI) and (A) (given below at Theorem 39), and Ω is smooth. Suppose that u is a viscosity solution of*

$$\begin{cases} F(x, u, Du, D^2u) = 0 \text{ in } \Omega, \\ u = \phi \text{ on } \partial\Omega, \end{cases} \tag{7.9}$$

where $\phi \in C^{1,\tau}(\partial\Omega)$. Then $u \in C^{1,\nu}(\Omega)$ for some $\nu > 0$ depending only on n, Λ, ν, τ , and we have the estimate

$$\|u\|_{C^{1,\nu}(\Omega)} \leq C(\|u\|_{L^\infty(\Omega)} + \|\phi\|_{1,\gamma}), \tag{7.10}$$

where C depends on $n, \Lambda, \delta, \gamma, \omega_K, \alpha$ and Ω .

For the proof, we refer to Lemma 3.1 and Theorem 3.2 [157]. The following theorem is a consequence of Theorem 3 [35].

THEOREM 78. *Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and F satisfies (SCI) and (A), below Theorem 39, and F is also concave or convex in M . Suppose that $f \in C^\tau(\Omega)$ for some $\tau > 0$, and u is a viscosity solution of the equation*

$$F(x, u, Du, D^2u) = f \text{ in } \Omega. \tag{7.11}$$

Then $u \in C_{loc}^{2,\alpha}(\Omega)$ for some $\alpha > 0$.

Interior $W^{2,p}$ estimates in [35] for the viscosity solutions of concave, uniformly elliptic equations were extended to include the gradient term and first order term by A. Świech, in [156]. In the proof, he first obtained the gradient estimate and used this to prove Theorem 13.

THEOREM 79. *Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $p \geq n$, and F satisfies (SCI) and (A) (given below in Theorem 39). Further, F concave or convex in M , and $F(0, 0, 0, x) \equiv 0$. Suppose that $f \in L^p(\Omega)$, $\phi \in W^{2,p}(\Omega)$ and u is an L^p – viscosity solution of the following Dirichlet problem*

$$\begin{cases} F(x, u, Du, D^2u) = f \text{ in } \Omega, \\ u = \phi \text{ on } \partial\Omega. \end{cases} \tag{7.12}$$

Then $u \in W^{2,p}(\Omega)$, and we have the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{L^\infty(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where $C = C(n, \lambda, \Lambda, \delta, \gamma, p, \Omega)$.

For the proof, we refer to Theorem 4.6 [165]. Let us recall the ABP-estimate for singular or degenerate fully nonlinear equations. In [56], authors proved the ABP-estimate for singular or degenerate fully nonlinear elliptic equations of the form

$$F(Du, D^2u) + b(x) \cdot Du |Du|^\sigma + c(x) |u|^\sigma u = f \text{ in } \Omega, \tag{7.13}$$

where $\sigma > -1$. In this case, the proof of ABP-estimate is based on the regularization procedures (sub-convolution and standard mollification).

THEOREM 80. *Let us consider Equation (7.13), where F is continuous and satisfying (SF1) and (SF4) with $\sigma = \theta$, and additionally with $\sigma > -1$, $c \leq 0$, and $|b|, |c| \leq \gamma$, there exists $C = C(n, \alpha, \lambda, \gamma, \text{diam}(\Omega))$ such that for any $u \in C(\bar{\Omega})$ viscosity subsolution (resp., supersolution) of (7.13) in $\{x \in \Omega \mid 0 < u(x)\}$ (resp., $\{x \in \Omega \mid 0 > u(x)\}$), satisfies*

$$\begin{aligned} \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \cdot \text{diam}(\Omega) \|f^-\|_{L^n(\Gamma^+(u^+))}^{\frac{1}{1+\sigma}}, \\ \left(\text{resp., } \sup_{\Omega} u \leq \sup_{\partial\Omega} u^- + C \cdot \text{diam}(\Omega) \|f^+\|_{L^n(\Gamma^+(u^-))}^{\frac{1}{1+\sigma}} \right). \end{aligned}$$

For the proof of (80), we refer to Theorem 1 [56]. One of the most important application of Theorem 80 is the following maximum principle in a small domain without any restriction on the sign of c . Such a type of result has been widely used in [19].

COROLLARY 3. *Under the hypotheses of Theorem 80 except any restriction on the sign of c , there exists $C = C(n, \sigma, \lambda, \gamma, \text{diam}(\Omega))$, $\varepsilon > 0$, such that for any $u \in C(\bar{\Omega})$ viscosity subsolution of (7.13), with $c \leq \varepsilon$, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \cdot \text{diam}(\Omega) \|f^-\|_{L^n(\Gamma^+(u^+))}^{\frac{1}{1+\sigma}}.$$

For the proof of Theorem 80 and its Corollary 3, we refer to Theorem 1 and Corollary 2 [56]. Next, we will present Harnack inequality which is a consequence of the ABP-estimate (39).

THEOREM 81. *Assume $\sigma \in (-1, 0)$ and F satisfies (SF1) and (SF4) with $\sigma = \theta$. If $u \in C(\Omega)$ is a nonnegative viscosity solution of (7.13), with b, c and f continuous functions in Ω , then for every $\Omega' \subset\subset \Omega$ we have*

$$\sup_{\Omega'} u \leq C \left(\inf_{\Omega'} u + \|u\|_{L^n(\Omega')}^{\frac{1}{1+\sigma}} \right),$$

where the constant C depends on $\lambda, \Lambda, \sigma, b, c, n, \Omega'$ and Ω .

THEOREM 82. *Assume the hypotheses of Theorem 81 and additionally that Ω is bounded and satisfies a uniform exterior cone condition, the functions b, c and f are continuous in $\bar{\Omega}$, $c \leq 0$ and $\phi \in C^\sigma(\partial\Omega)$, $\sigma \in (0, 1)$, then equation*

$$\begin{cases} F(Du, D^2u) + b(x) \cdot Du |Du|^\sigma + c(x) |u|^\sigma u = f \text{ in } \Omega, \\ u = \phi \text{ on } \partial\Omega. \end{cases} \tag{7.14}$$

possesses at least one solution. Moreover, there are constants $C > 0$ and $\beta \in (0, 1)$ such that

$$\|u\|_{C^\beta(\bar{\Omega})} \leq (\|\phi\|_{C^\sigma(\partial\Omega)} + \|u\|_{L^n(\bar{\Omega})}^{\frac{1}{1+\sigma}}).$$

For the proof of Theorems 81 and 82, we refer to Theorem 1.1 and 1.2 [57], respectively. See, also Section 4 [90], Section 3 [26]. In [26], authors obtained the Harnack inequality in the two dimensional case and using the other Harnack inequality obtained in [57, 90], they also obtained the positive eigenfunction for singular fully nonlinear elliptic PDEs in the unbounded domain.

7.1. Regularity for nonconvex fully nonlinear equations

Let us consider the following Dirichlet problem

$$\begin{cases} F(D^2u) = f \text{ in } \Omega, \\ u = \phi \text{ on } \partial\Omega, \end{cases} \tag{7.15}$$

where Ω is a unit ball in \mathbb{R}^n . We have seen that when F is convex or concave then in view of Evans-Krylov regularity theorem, every viscosity solutions of (7.15) coincides with classical solutions. However, for the general F , the problem of the coincidence of viscosity solutions of (7.15) with the classical solutions remains open. In [137], N. Nadirashvili and S. Vlăduț proved that there is a nonconvex smooth F for which the Dirichlet problem (7.15) have a viscosity solution that is not a classical solution. The precise result of [137] is the following.

THEOREM 83. *Suppose that $\Omega \subset \mathbb{R}^{12}$ is a unit ball and $\phi = \omega$ on $\partial\Omega$. Then there exists a smooth uniformly elliptic F such that the Dirichlet problem (7.17) has no classical solution.*

For the details about ω and proof of Theorem 83, we refer to Theorem 1 and its Corollary [137], see also [141]. Note that under the uniform ellipticity conditions on F , existence of the viscosity solutions to (7.17) is guaranteed. So in this context, there is one question that comes naturally, under what condition weaker than the convexity on F , solutions of (7.17) are classical?. The first attempt to understand the regularity of solution to nonconvex (nonconcave) fully nonlinear elliptic PDEs was made by X. Cabrè and L. A Caffarelli in [32]. The authors proved interior $C^{2,\alpha}$ regularity results as well as the existence of $C^{2,\alpha}$ solutions for a class of nonconvex fully nonlinear elliptic equations $F(D^2u, x) = f(x)$, for $x \in B_1 \subset \mathbb{R}^n$. They considered the following type of nonconvex fully nonlinear elliptic operator

$$\begin{cases} F(M) = \min\{F^\cap(M), F^\cup(M)\} \text{ for all } M \in S(n), \\ F(0) = 0, F^\cap(M) \text{ and } F^\cup(M) \text{ are uniformly elliptic,} \\ F^\cap(M) \text{ is concave and } F^\cup(M) \text{ is convex.} \end{cases} \tag{7.16}$$

THEOREM 84. *Let $u \in C^2(B_1)$ be a solution of $F(D^2u) = 0$ in $B_1 \subset \mathbb{R}^n$, where F is of the form (7.16). Then $u \in C^{2,\alpha}(\bar{B}_{1/2})$ and*

$$\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \leq C \|u\|_{L^\infty(B_1)},$$

where $0 < \alpha < 1$ and C is constant depending on n, λ, Λ .

Note that the interior $C^{2,\alpha}$ estimate obtained in Theorem 84 is valid for the C^2 solutions. The following theorem tells about the existence of the classical solutions to the Dirichlet problem associated with F defined by (7.16).

THEOREM 85. *Let F be of the form (7.16). Then there exists a constant $\bar{\alpha} \in (0, 1)$ depending on n, λ, Λ such that for every $\alpha \in (0, \bar{\alpha})$, $f \in C^\alpha(\bar{B}_1)$ and $\phi \in C(\partial B_1)$, the following Dirichlet problem*

$$\begin{cases} F(D^2u) = f \text{ in } B_1, \\ u = \phi \text{ on } \partial B_1 \end{cases} \tag{7.17}$$

admits a unique solution $u \in C^{2,\alpha}(B_1) \cap C(\bar{B}_1)$. Moreover, one have

$$\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \leq C \left(\|f\|_{C^\alpha(\bar{B}_1)} + \|\phi\|_{L^\infty(\partial B_1)} \right),$$

for some constant C_α depending on n, λ, Λ and α . For the proofs of Theorem 84 and 85, we refer to Theorem 1.1 and 1.2 [32], respectively.

In [32], L. A. Caffarelli, X. Cabré also obtained $W^{2,p}$ estimate for the solutions of Equation (7.17). In fact, they stated the following result, which is a corollary of Theorems 84 and 85.

THEOREM 86. *Let $u \in C(B_1)$ be a viscosity solution of*

$$F(D^2u) = f \text{ in } B_1, \tag{7.18}$$

where f is a continuous function in B_1 and F is an operator of the form (7.16). Then: (i) If $f \in C^\alpha(B_1)$ for some $0 < \alpha < \bar{\alpha}$, where $\bar{\alpha} \in (0, 1)$ is a constant depending on n, λ, Λ , then $u \in C^{2,\alpha}(B_1)$ and

$$\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\bar{B}_{3/4})} \right),$$

for some constant C depending on n, λ, Λ and α .

(ii) If $f \in L^p(B_1)$ and $n \leq p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C_\alpha \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right),$$

for some constant C_α depending on n, λ, Λ and p .

For the proof of Theorem 86, we refer to Corollary 1.3 [32]. Next, we would like to present the result of O. Savin [153], in which he considered more general operator depending on gradient and zero-th order term, which states that if F is uniformly elliptic in a neighbourhood of the origin and $F(x, 0, 0, 0) \equiv 0$, then there exists a constant C depending on F such that if u is a solution to Equation 7.15 with mentioned F and $f = 0$ and $\|u\|_{L^\infty(B_1)} \leq C$ then $u \in C^{2,\alpha}(B_{\frac{1}{2}})$ and satisfies the following estimate

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq \delta.$$

For the details, we refer to [153] and for proof of Theorem 1.3, see [153]. Further, using this result and estimate obtained in [119], S. N. Armstrong, L. E. Silvestre and C. K. Smart [7] obtained the $C^{2,\alpha}$ -estimate for the solution outside a closed set (say C), and also proved that this set has the Hausdorff dimension less than $n - \gamma$ for some $\gamma > 0$ depending on λ, Λ, n and closed set C . This closed set is the singular set of the solution, that is, at each point $x \in C$ solution fails to be $C^{2,\alpha}$ in any neighbourhood of that point. The precise statement of the theorem is as follows.

THEOREM 87. *Suppose that F in (7.15) is uniformly elliptic and differentiable with uniformly continuous derivative and $f = 0$. Let $u \in C(\Omega)$ be the viscosity solution of (7.15) in Ω then there is constant $\gamma > 0$ depending on λ, Λ, n and closed set C such that $u \in C^{2,\alpha}(\Omega \setminus C)$ for every $0 < \alpha < 1$.*

For the proof, we refer to Theorem 1 [7]. In the same paper, the authors have also made some comments on γ , and $W^{2,\gamma}$ estimates are obtained in [119]. In view of [153], C. Imbert and L. Silvestre [91], obtained the Hölder estimate for the solution of elliptic equation with large gradient. In fact, they defined the following Pucci type operator

$$\mathcal{P}^+(D^2u, Du) = \begin{cases} -\lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i + \Lambda |Du|, & \text{if } |Du| \geq t \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$\mathcal{P}^-(D^2u, Du) = \begin{cases} -\Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i + \lambda |Du| & \text{if } |Du| \geq t \\ -\infty, & \text{otherwise} \end{cases}$$

and proved that any $u : \bar{B}_1 \rightarrow \mathbb{R}$, satisfying $\mathcal{P}^-(D^2u, Du) \leq C$ and $\mathcal{P}^+(D^2u, Du) \geq -C$ in that ball and $\|u\|_{L^\infty(B_{1/2})} \leq C$ then $u \in C^\alpha(B_{1/2})$ and also satisfies the estimate $\|u\|_{C^\alpha(B_{1/2})} \leq CK$, where K is another constant. For the proof, we refer to Theorem 1.1 [91], see also Theorem 1.3 [91] for the Harnack’s inequality. Next, we would like to present the results of [106], in which J. Kovats considered Isaac’s operator, which were not covered in [32]. Let us consider the following Isaac’s operator

$$F(D^2u) = \Delta u + \left(\frac{\partial^2 u}{\partial x_1^2}\right)^+ - \left(\frac{\partial^2 u}{\partial x_2^2}\right)^- = 0. \tag{7.19}$$

Note that, F given by Equation 7.19 can not be written in the form (7.16), for details, see Example 1 [106]. In fact, the author considered the Isaac’s operators which satisfy the minmax principle i.e,

$$F(M) = \max_{\alpha \in \mathcal{A}} \min_{\beta \in \mathcal{B}} [\text{trace}(A^{\alpha\beta} M)] \text{ for all } M \in S(n), \tag{7.20}$$

where \mathcal{A} and \mathcal{B} are compact subsets of \mathbb{R}^n , and prove the following theorem.

THEOREM 88. *Let $u \in C^2(\Omega)$ be a solution of Equation 7.19 in a bounded domain Ω of \mathbb{R}^n . Then $\forall x_0 \in \Omega$ such that $B_r(x_0) \subset \Omega$ and any $0 < p < \infty$,*

$$\|D^2u\|_{L^p(B_{7/8}(x_0))} \leq Cr^{-2+\frac{n}{p}} \|u\|_{L^\infty(B_r(x_0))},$$

where C is a constant depending n, p .

For the proof of Theorem 88, we refer to Theorem 1 [106]. The author in [106] also proved the interior $W^{2,p}$ estimate for the C^2 solutions to the equations $F = 0$, where F is given by (7.20), for the details, see Theorem 2 [106]. In this direction, there are also regularity results for the solutions to Hessian equation, see [138, 139, 140].

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J. Tyagi

*Indian Institute of Technology Gandhinagar
Vishwakarma Government Engineering College Complex
Chandkheda, Visat-Gandhinagar Highway, Ahmedabad
Gujarat, India - 382424
e-mail: jtyagi@iitgn.ac.in*

R.B. Verma

*Indian Institute of Technology Gandhinagar
Vishwakarma Government Engineering College Complex
Chandkheda, Visat-Gandhinagar Highway, Ahmedabad
Gujarat, India - 382424
e-mail: ram.verma@iitgn.ac.in*