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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SECOND ORDER NEUTRAL DYNAMIC EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. In this paper, oscillatory and asymptotic properties of solutions of nonlinear second order neutral dynamic equations of the form

$$\left(r(t)\left(y(t)+p(t)y(\alpha(t))\right)^{\Delta}\right)^{\Delta}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0$$

and

$$\left(r(t)\left(y(t)+p(t)y(\alpha(t))\right)^{\Delta}\right)^{\Delta}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t)$$

are studied under assumptions

$$\int_{0}^{\infty}\frac{1}{r(t)}\Delta t<\infty \quad \text{and} \quad \int_{0}^{\infty}\frac{1}{r(t)}\Delta t=\infty$$

for various ranges of $p(t)$, where $\mathbb{T}$ is a time scale with $\sup\mathbb{T}=\infty$, $t\in[t_0,\infty)_{\mathbb{T}}$, and $t_0\geqslant 0$. Examples illustrating the results are included.

1. Introduction

The study of dynamic equations on time scales goes to seminal work of Stefan Hilger [8] and has received a lot of attention in recent years. Time scales were created to unify the study of continuous and discrete mathematics and is particularly use in differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and allow us to avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale $\mathbb{T}$, which is a non-empty closed subset of the real numbers $\mathbb{R}$. In this way the results of this paper not only apply to the set of

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real numbers or set of integers, but also to more general time scales such as $\mathbb{T} = hN$, $\mathbb{T} = q^{N_0} = \{t : t = q^k, k \in N_0\}$ with $q > 1$, $\mathbb{T} = N_0^2 = \{t^2 : t \in N_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in N_0\}$ e.t.c,. For basic notations on the time scale calculus, we refer the reader to monographs [4, 5] and the references cited therein.

In recent years, there has been increasing interest of obtaining sufficient conditions for the oscillation and nonoscillation of solution of second order neutral dynamic equation on time scales (see [1, 2, 9, 11, 13]) and references cited therein.

Q. Yang et al. [19] studied the oscillation of second-order quasi linear neutral dynamic equation

$$(r(t)|z^{\Delta}(t)|^{\alpha-1}z^{\Delta}(t))^{\Delta} + q(t)|x(\delta(t))|^{\beta-1}x(\delta(t)) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $z(t) = x(t) + p(t)x(\tau(t)), \alpha, \beta > 0$ are constants, and obtained oscillation criteria for the equation when $\beta > \alpha, \beta = \alpha$ and $\beta < \alpha$, respectively with the followig assumptions:

(A_1) $r \in C_{rd}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ with $\int_{t_0}^{\infty} r^{\frac{-1}{\alpha}}(t)\Delta t = \infty$;
(A_2) $p, q \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ with $0 \leqslant p(t) < 1, q(t) \geqslant 0$;
(A_3) $\tau, \delta \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{T}), \tau(t) \leqslant t$, and $\delta(t) \leqslant t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$.

Kubiaczyk et al. [10] established some sufficient conditions for oscillation of the second-order neutral functional dynamic equation

$$(r(t)[m(t)y(t) + p(t)y(\tau(t))]^{\Delta}]^{\Delta} + q(t)f(y(\delta(t))) = 0 \tag{1.2}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, on a time scale $\mathbb{T}$ which is unbounded above, where p, q, r, τ and δ are real valued rd-continuous positive functions defined on $\mathbb{T}$. They obtained results by using Riccati substitution and analysis of Riccati dynamic inequality.

Saker and O' Regan [14] considerd the second-order nonlinear neutral functional dynamic equation

$$(p(t)([y(t) + r(t)y(\tau(t))]^{\Delta})^{\gamma})^{\Delta} + f(t, y(\delta(t))) = 0, \tag{1.3}$$

on a time scale $\mathbb{T}$ and established some new sufficient conditions for oscillation. The results improve oscillation results for neutral dynamic equation on time scales and are new when $\delta(t) > t$ and/or $0 < \gamma < 1$.

In E. Thandapani et al. [16] obtained the oscillation criteria for the second-order nonlinear neutral delay dynamic equation on time scales

$$\left((r(t)\,(y(t) + p(t)y(t-\tau))^{\Delta})^{\gamma}\right)^{\Delta} + q(t)y^{\beta}(t-\delta) = 0, \qquad t \in \mathbb{T},$$

where $\mathbb{T}$ is a time scale and

(H_1) $\gamma \geqslant 1$, and $\beta > 0$ are quotients of odd positive integers;

(H_2) τ, δ are fixed nonnegative constants such that the delay functions $\tau(t) = t - \tau < t$ and $\delta(t) = t - \delta < t$ satisfy $\tau(t) : \mathbb{T} \to \mathbb{T}$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$;

(H_3) $p(t)$ is a positive and rd-continuous function on $\mathbb{T}$ such that $0 \leqslant p(t) < 1$.

Zhang and Wang [20] studied the oscillation criteria for second-order nonlinear dynamic equation

$$\left(r(t)\left(y(t)+p(t)y(\tau(t)^{\Delta})^{\gamma}\right)^{\Delta}\right)^{\Delta}+f_1(t,y(\delta_1(t)))+f_2(t,y(\delta_2(t)))=0,$$

on time scale $\mathbb{T}$, where $p \in C_{rd}(\mathbb{T},[0,1])$, $f_i \in C(\mathbb{T}\times\mathbb{R},\mathbb{R}), i=1,2, \gamma>0$ is a quotient of odd positive integers by using the Ricati transformation technique.

In [17], author studied the oscillatory and asymptotic behaviour of solutions of the second order nonlinear delay differential equations of the form

$$\left(r(t)\left(y(t)+p(t)y(t-\tau)\right)'\right)'+q(t)G(y(t-\sigma))-h(t)H(y(t-\delta))=f(t)$$

and

$$\left(r(t)\left(y(t)+p(t)y(t-\tau)\right)'\right)'+q(t)G(y(t-\sigma))-h(t)H(y(t-\delta))=0$$

for various range of $p(t)$ under the assumptions

$$\int_0^{\infty}\frac{dt}{r(t)}<\infty \quad \text{and} \quad \int_0^{\infty}\frac{dt}{r(t)}=\infty.$$

In [18], author studied the oscillatory and asymptotic behaviour of solutions of a class of nonlinear second-order neutral difference equations with positive and negative coefficients of the form

$$\Delta\left(r(n)\Delta(y(n)+p(n)y(n-m))\right)+q(t)G(y(n-k_1))-h(t)H(y(n-k_2))=f(t)$$

and

$$\Delta\left(r(n)\Delta(y(n)+p(n)y(n-m))\right)+q(t)G(y(n-k_1))-h(t)H(y(n-k_2))=0$$

under the assumptions

$$\sum_{n=0}^{\infty}\frac{1}{r(n)}<\infty \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{1}{r(n)}=\infty$$

for various ranges of $p(n)$.

The objective of this paper is to study the oscillatory and asymptotic properties of solutions of the nonlinear second-order neutral delay dynamic equations with positive and negative coefficients of the form

$$\left(r(t)\left(y(t)+p(t)y(\alpha(t))\right)^{\Delta}\right)^{\Delta}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0 \qquad \text{(H)}$$

and

$$\left(r(t)\left(y(t)+p(t)y(\alpha(t))\right)^{\Delta}\right)^{\Delta}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t) \qquad \text{(NH)}$$

on a time scale $\mathbb{T}$ such that $\sup\mathbb{T} = \infty$ and $t_0 \in \mathbb{T}$.

We consider these equations under the assumptions that

(H$_0$) $\int\limits_{t_0}^{\infty} \frac{1}{r(s)} \left(\int\limits_{s}^{\infty} h(t)\Delta t \right) \Delta s < \infty$;

(H$_1$) $\int\limits_{0}^{\infty} \frac{1}{r(t)} \Delta t < \infty$;

(H$_2$) $\int\limits_{0}^{\infty} \frac{1}{r(t)} \Delta t = \infty$.

Here we extended the result of [17, 18] to second order dynamic equations with positive and negative coefficients and the results obtained are new and generalize the earlier work in [17, 18].

For equations (H) and (NH) we will use the notation $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$ and assume that $r \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $p, f \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $q,h \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $G,H \in (\mathbb{R},\mathbb{R})$ satisfying $uG(u) > 0$ and $uH(u) > 0$ for $u \neq 0$, G is nondecreasing, H is bounded, and $\alpha,\beta,\gamma \in C_{rd}(\mathbb{T},\mathbb{T})$ are strictly increasing functions such that

$$\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = \infty, \alpha(t), \beta(t), \gamma(t) \leqslant t$$

and

$$(\alpha \circ \beta)(t) = (\beta \circ \alpha)(t) \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}.$$

The inverse of $\alpha(t)$ will be denoted by $\alpha^{-1}(t) \in C_{rd}(\mathbb{T},\mathbb{T})$. Whenever we write $t \geqslant t_1$, we mean $t \in [t_1,\infty) \cap \mathbb{T}$.

Let $t_{-1} = \inf_{t\in[t_0,\infty)_{\mathbb{T}}} \{\alpha(t),\beta(t),\gamma(t)\}$. By a solution of (H) and (NH), we mean a function $y \in C_{rd}([t_{-1},\infty)_{\mathbb{T}},\mathbb{R})$, and such that $y(t) + p(t)y(\alpha(t)) \in C^1_{rd}([t_0,\infty),\mathbb{R})$, $r(t)\left(y(t)+p(t)y(\alpha(t))\right)^{\Delta} \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and such that (H) ((NH)) is satisfied on $[t_0,\infty)_{\mathbb{T}}$. A solution of (H) or (NH) is called oscillatory if it is neither eventually positive nor eventually negative, and it is nonoscillatory otherwise. In this paper, we do not consider solutions that eventually vanish identically. An equation will be called oscillatory if all its solutions are oscillatory.

2. Preliminary Lemmas

We will need the following lemmas in the sequel.

LEMMA 1. *Let (H$_1$) hold. let $u(t)$ be an eventually positive rd-continuously differentiable function such that $r(t)u^{\Delta}(t)$ is rd-continuously differentiable function on $[t_0,\infty)_{\mathbb{T}}$ such that $\left(r(t)u^{\Delta}(t)\right)^{\Delta} \leqslant 0$ for large t, where $r \in C([0,\infty)_{\mathbb{T}},(0,\infty))$.*

(i) If $u^{\Delta}(t) > 0$, then there exist a constant $K > 0$ such that $u(t) \geqslant KR(t)$, for large t.

(ii) If $u^{\Delta}(t) < 0$, then $u(t) > -r(t)u^{\Delta}(t)R(t)$, where $R(t) = \int\limits_{t}^{\infty} \frac{\Delta s}{r(s)}$.

Proof. (i) Since $R(t) < \infty, R(t) \to 0$ as $t \to \infty$ and $u(t)$ is nondecreasing, we can find a constant $K > 0$ such that $u(t) \geqslant KR(t)$ for all large t.

(ii) For $s \geqslant t$, we have $r(s)u^{\Delta}(s) \leqslant r(t)u^{\Delta}(t)$, and hence

$$u(s) \leqslant u(t) + \int_t^s \frac{r(t)u^{\Delta}(t)}{r(\theta)} \Delta\theta = u(t) + r(t)u^{\Delta}(t) \int_t^s \frac{\Delta\theta}{r(\theta)}.$$

Thus,

$$0 < u(s) \leqslant u(t) + r(t)u^{\Delta}(t) \int_t^s \frac{\Delta\theta}{r(\theta)}$$

implies that $u(t) \geqslant -r(t)u^{\Delta}(t)R(t)$.

LEMMA 2. *Assume that (H_2) hold. Let $u(t)$ and $u^{\Delta}(t)$ be positive rd-continuously differentiable functions with $u^{\Delta^2}(t) \leqslant 0$ for $t \geqslant T \geqslant 0$. Then*

$$u(t) \geqslant (t-T)u^{\Delta}(t) = \eta(t)r(t)u^{\Delta}(t)$$

for $t \geqslant T \geqslant 0$, where $\eta(t) = \frac{t-T}{r(t)}$.

Proof. The proof is simple and hence the details are omitted.

LEMMA 3. *([12], Lemma 3.5]) Let $F, H, P : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ satisfy*

$$F(t) = H(t) + P(t)H(\alpha(t)) \quad \text{for} \quad t \in [\hat{t}, \infty)_{\mathbb{T}},$$

where $\hat{t} \in [t_0, \infty)_{\mathbb{T}}$ is such that $\alpha(t) \geqslant t_0$ for all $t \in [\hat{t}, \infty)_{\mathbb{T}}$. Assume that there exist constants $P_1, P_2 \in \mathbb{R}$ such that $P(t)$ is one of the following ranges:

(1) $-\infty \leqslant P(t) \leqslant 0$,

(2) $0 \leqslant P(t) \leqslant P_1 < 1$,

(3) $1 < P_2 \leqslant P(t) < \infty$.

If $H(t) > 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$, $\liminf_{t\to\infty} H(t) = 0$, and $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$ exists, then $L = 0$.

3. Oscillation properties for (H)

In this section, we study the asymptotic behaviour of solutions of equation (H) under assumptions (H_1) and (H_2). We will make use of the following conditions on the functions in the equations (H) and (NH):

(H_3) there exists $\lambda > 0$ such that $G(u) + G(v) \geqslant \lambda G(u+v)$ for $u, v \in \mathbb{R}$ with $u, v > 0$;

(H_4) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$;

(H_5) $G(-u) = -G(u)$; $u \in \mathbb{R}$;

(H_6) $\int_0^{\pm c} \frac{\Delta u}{G(u)} < \infty$.

REMARK 1. (H_4) implies (H_5). Indeed $G(1)G(1) = G(1)$, so that $G(1) = 1$. Further, $G(-1)G(-1) = G(1) = 1$ gives $(G(-1))^2 = 1$. Since $G(-1) < 0$, then $G(-1) = -1$. Consequently, $G(-u) = G(-1)G(u) = -G(u)$. On the other hand, $G(uv) = G(u)G(v)$ for $u > 0, v > 0$ and $G(-u) = -G(u)$ implies that

$$G(uv) = G(u)G(v) \text{ for every } u, v \in \mathbb{R}.$$

REMARK 2. The prototype of G satisfying (H_3) and (H_5) is

$$G(u) = \left(a + b\,|u|^{\mu}\right)|u|^{\upsilon}\, sgnu,$$

where $a \geqslant 1, b \geqslant 1, \mu \geqslant 0$, and $\upsilon \geqslant 0$. However, the prototype of G satisfying (H_3) and (H_4) is $G(u) = |u|^{\gamma} sgnu$, where $\gamma > 0$. This G also satisfies H_3 and (H_5).

REMARK 3. Notice that if $y(t)$ is a solution of (H), then $x(t) = -y(t)$ is also a solution of (H) provided that G and H satisfies (H_4) or (H_5).

THEOREM 1. *Let $0 \leqslant p(t) \leqslant p_1 < \infty$, and assume that conditions (H_0), (H_1), (H_3)-(H_5) hold. If*

(H_7)

$$\int_0^{\infty} Q(t)G(R(\beta(t)))\Delta t = \infty,$$

where $Q(t) = \min\{q(t), q(\alpha(t))\}$ and $R(t) = \int_t^{\infty} \frac{\Delta s}{r(s)}$, then any solution of (H) is either oscillatory or converges to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (H) on $[t_0, \infty)_{\mathbb{T}}$, say $y(t)$ is eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\alpha(\beta(t)))$ are all positive for $t \geqslant t_1$. Define

$$z(t) = y(t) + p(t)y(\alpha(t)), \tag{3.1}$$

and

$$k(t) = \int_t^{\infty} \frac{1}{r(s)} \int_s^{\infty} h(\theta)H(y(\gamma(\theta)))\Delta\theta\Delta s. \tag{3.2}$$

Notice that condition (H_0) and the fact that H is a bounded function imply that $k(t)$ exists for all t. Now if we let

$$w(t) = z(t) - k(t) = y(t) + p(t)y(\alpha(t)) - k(t), \tag{3.3}$$

then

$$\left(r(t)w^{\Delta}(t)\right)^{\Delta} = -q(t)G(y(\beta(t))) \leqslant 0, \tag{3.4}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Clearly, $r(t)w^{\Delta}(t)$ is a monotonic function on $[t_1,\infty)_{\mathbb{T}}$. Let $w^{\Delta}(t) < 0$ for $t \geqslant t_1$. Suppose that $w(t) < 0$, then $y(t) \leqslant z(t) \leqslant k(t)$, $t \geqslant t_1$. We note that $k(t)$ is bounded with $\lim_{t\to\infty} k(t) = 0$ and hence there exists a constant $M > 0$ such that $y(t) \leqslant M$ for $t \geqslant t_2$ for some $t_2 > t_1$. So, $w(t)$ is bounded and $\lim_{t\to\infty} w(t)$ exists. This is contradicts to the fact that $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) \neq 0$ implies that $z(t) < 0$ for $t \geqslant t_3$ for some $t_3 > t_2$.

Assume that $w(t) > 0$ for $t \geqslant t_1$. Successive integrating inequality (3.4) from t_1 to t, we can find a constant $\eta > 0$ such that $w(t) \leqslant \eta$ for $t \geqslant t_2 > t_1$. By Lemma 1 (ii), we get $w(t) \geqslant -\ r(t)w^{\Delta}(t)R(t)$ and hence $z(t) \geqslant -\ r(t)w^{\Delta}(t)R(t)$ for $t \geqslant t_2$. Since $w(t)$ is bounded, $R(t)$ is bounded and $r(t)w^{\Delta}(t)$ is monotonic implies that $\lim_{t\to\infty}\left(r(t)w^{\Delta}(t)\right)$ exist. By using (H_3), and (H_4) in equation (H) gives

$$\begin{aligned}
0 &= \left(r(t)w^{\Delta}(t)\right)^{\Delta} + q(t)G(y(\beta(t))) + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} \\
&\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
&= \left(r(t)w^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} + q(t)G(y(\beta(t))) \\
&\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
&\geqslant \left(r(t)w^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} + \lambda Q(t)G(y(\beta(\alpha(t))) \\
&\quad + py(\alpha(\beta(t))))) \\
&\geqslant \left(r(t)w^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} + \lambda Q(t)G(z(\beta(t)) \\
&\geqslant \left(r(t)w^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} \\
&\quad + \lambda Q(t)G(-r(\beta(t))w^{\Delta}(\beta(t))R(\beta(t))) \\
&= \left(r(t)w^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} \\
&\quad + \lambda Q(t)G(R(\beta(t)))G(-r(\beta(t))w^{\Delta}(\alpha(t)))
\end{aligned} \tag{3.5}$$

for $t \geqslant t_3 > t_2$. Since $-r(t)w^{\Delta}(t)$ is nondecreasing, we can find a constant $c > 0$, and $t_4 > t_3$ such that $-r(t)w^{\Delta}(t) \geqslant c$, for $t \geqslant t_4$. From (3.5), we have

$$\lambda Q(t)G(c)G(R(\beta(t))) \leqslant -\left(r(t)w^{\Delta}(t)\right)^{\Delta} - G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} \tag{3.6}$$

for $t \geqslant t_5 > t_4$. Integrating (3.6) from t_5 to ∞, we get

$$\int_{t_5}^{\infty} Q(t)G(R(\beta(t)))\Delta t < \infty,$$

which contradicts (H_7).

Next, we suppose that $w^{\Delta}(t) > 0$ for $t \geqslant t_1$. If $w(t) < 0$, then $w(t)$ exists and $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ which implies $z(t) < 0$, for large t, which is a contradiction to the fact that $z(t) > 0$. Hence $\lim_{t\to\infty} w(t) = 0$. So also, $\lim_{t\to\infty} z(t) = 0$ implies that $\lim_{t\to\infty} y(t) = 0$ since $y(t) \leqslant z(t)$ for $t \geqslant t_2 > t_1$.

Now we suppose that $w(t) > 0$ for $t \geqslant t_2 > t_1$. By Lemma 1 (i), it follows that $w(t) \geqslant KR(t)$ and $z(t) \geqslant w(t) \geqslant KR(t)$ for $t \geqslant t_2$. From (3.5), we get

$$\lambda Q(t)G(K)G(R(\beta(t))) \leqslant -\left(r(t)w^{\Delta}(t)\right)^{\Delta} - G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta}$$

for $t \geqslant t_3 > t_2$. Integrating the above inequality implies

$$\int_{t_5}^{\infty} Q(t)G(R(\beta(t)))\Delta t < \infty,$$

is a contradiction. This completes the proof of the theorem.

THEOREM 2. *Let* $-1 < p_2 \leqslant p(t) \leqslant 0$. *If* ($H_0$), ($H_1$), ($H_4$) *and* ($H_8$)

$$\int_{0}^{\infty} q(t)G(R(\beta(t)))\Delta t = \infty,$$

where $R(t) = \int_t^{\infty} \frac{\Delta s}{r(s)}$ *hold, then any solution of* (H) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (H), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ are positive for all $t_1 \in [t_0,\infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Setting $z(t), k(t)$ and $w(t)$ as in (3.1), (3.2) and (3.3), we obtain (3.4) for $t \geqslant t_1$. Hence, $w^{\Delta}(t)$ is monotonic for large $t \in [t_1,\infty)_{\mathbb{T}}$ which implies that either $w(t) > 0$ or $w(t) < 0$ for $t \geqslant t_2 > t_1$.

Suppose that $w^{\Delta}(t) < 0$ and $w(t) < 0$ for $t \geqslant t_2$. Then $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ implies that $z(t) < 0$ for $t \geqslant t_2$. Hence, $y(t) < y(\alpha(t))$ for $t \geqslant t_3$ for some $t_3 > t_2$, that is, $y(t)$ is bounded on $[t_3,\infty)_{\mathbb{T}}$. Consequently, $w(t)$ is bounded and $\lim_{t\to\infty} \left(r(t)w^{\Delta}(t)\right)$ exists. Since, $w(t)$ is monotonic, then $\lim_{t\to\infty} w(t) = L, L \in (-\infty,0)$ gives $\lim_{t\to\infty} z(t) = L$. We claim that $\liminf_{t\to\infty} y(t) = 0$. If not, there exists a constant $M > 0$ and $t_4 > t_3$ such that $y(t) \geqslant M$ for $t > t_4$. Integrating (3.4), we get

$$\int_{t_4}^{\infty} q(t)\Delta t < \infty,$$

a contradiction to the fact that $R(t) \to 0$ as $t \to \infty$ and (H_8) implies that

$$\int_{0}^{\infty} q(t)\Delta t = \infty. \tag{3.7}$$

So, our claim holds. By Lemma 3, $L = 0$. Hence,

$$\begin{aligned}
0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} \left[y(t) + p(t)y(\alpha(t))\right] \\
&\geqslant \limsup_{t\to\infty} \left[y(t) + p_2 y(\alpha(t))\right] \\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} \left(p_2 y(\alpha(t))\right) \\
&= (1+p_2)\limsup_{t\to\infty} y(t),
\end{aligned}$$

which implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$.

Next, we consider the case $w(t) > 0$ for $t \geqslant t_2$. Let $\lim_{t\to\infty} w(t) = l, l \in [t_0, \infty)$. We claim that $y(t)$ is bounded. If not, there exists an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2,\infty)_{\mathbb{T}}$ such that $\tau_n \to \infty, y(\tau_n) \to \infty$ as $n \to \infty$, and $y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}$. we choose τ_1 large enough so that $\alpha(\tau_1) \geqslant t_2$. Hence,

$$0 \geqslant w(\tau_n) \geqslant y(\tau_n) + p(\tau_n)y(\alpha(\tau_n)) - k(\tau_n) \geqslant (1+p_2)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1+p_2 > 0$, we have $w(\tau_n) > 0$ for large n, which is a contradiction to the fact that $\lim_{t\to\infty} w(t)$ exists, so our claim is true. Hence, $\lim_{t\to\infty} \left(r(t)w^{\Delta}(t)\right)$ exists. Using Lemma 1 (ii), we get $w(t) \geqslant -\ r(t)w^{\Delta}(t)R(t)$ and hence $y(t) \geqslant w(t) \geqslant -r(t)w^{\Delta}(t)R(t)$, $t \geqslant t_3 > t_2$. Hence, (3.4) becomes

$$q(t))G(R(\beta(t)))G\left(-r(\beta(t))w^{\Delta}(\beta(t))\right) \leqslant -\left(r(t)w^{\Delta}(t)\right)^{\Delta},$$

for $t \geqslant t_4 > t_3$. Since $r(t)w^{\Delta}(t)$ is nonincreasing, we can find a constant $b > 0$ such that $r(\beta(t))w^{\Delta}(\beta(t)) \leqslant -b$ for $t \geqslant t_5$ for some $t_5 > t_4$. Integrating the last inequality, we get

$$\int_{t_5}^{\infty} q(t)G(R(\beta(t)))\Delta t < \infty,$$

is a contradiction to (H_8).

Suppose that $w^{\Delta}(t) > 0$ for $t \geqslant t_1$. So, we have two cases, $w(t) > 0$ or $w(t) < 0$ for $t \geqslant t_1$. Let $w(t) > 0$ for $t \geqslant t_1$, then by Lemma 1 (i), $y(t) \geqslant w(t) \geqslant KR(t)$ for $t \geqslant t_3 > t_2$ and hence equation (3.4) becomes

$$q(t))G(KR(\beta(t))) \leqslant -\left(r(t)w^{\Delta}(t)\right)^{\Delta}$$

for $t \geqslant t_3 > t_2$. Integrating the above inequality from t_3 to ∞, we get

$$\int_{t_3}^{\infty} q(t)G(R(\beta(t)))\Delta t < \infty,$$

is a contradiction to (H_8). Hence, $w(t) < 0$ for $t \geqslant t_1$. Therefore, $\lim_{t\to\infty} w(t)$ exists and $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ implies that $z(t) < 0$ for $t \geqslant t_2 > t_1$. So, $y(t)$ is bounded on $[t_3,\infty)_{\mathbb{T}}$ for some $t_3 > t_2$. Using the same type of reasoning mentioned above, we obtain $\lim_{t\to\infty} y(t) = 0$. If $0 = \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$, then $y(t)$ is bounded. Otherwise, this is a contradiction that $w(t) > 0$ for large t. Proceeding as above again we obtain $\lim_{t\to\infty} y(t) = 0$.

THEOREM 3. *Let* $-\infty < p_3 \leqslant p(t) \leqslant p_2 < -1$. *If* ($H_0$), ($H_1$), ($H_4$) *and* ($H_8$) *hold, then every bounded solution of (H) is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (H) on $[t_0,\infty)_{\mathbb{T}}$, say $y(t)$ is an eventually positive solution. There exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\alpha(\beta(t)))$ all are positive for $t \geqslant t_1$. Setting $z(t), k(t)$ and $w(t)$ as in (3.1), (3.2) and (3.3), we obtain (3.4) for $t \geqslant t_1$. From (3.4), it follows that $w^{\Delta}(t) > 0$ or $w^{\Delta}(t) < 0$ for $t \geqslant t_1$. Proceeding as in the proof of Theorem 2, we obtain $L = 0$. Hence,

$$\begin{aligned}
0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} \left[y(t) + p(t)y(\alpha(t))\right] \\
&\leqslant \liminf_{t\to\infty} \left[y(t) + p_2 y(\alpha(t))\right] \\
&\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} \left(p_2 y(\alpha(t))\right) \\
&\leqslant \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} \left(y(\alpha(t))\right) \\
&= (1+p_2)\limsup_{t\to\infty} y(t),
\end{aligned}$$

which implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$ since $(1+p_2) < 0$. The remaining part of the proof can be followed from the proof of the Theorem 2. This completes the proof of the theorem.

THEOREM 4. *Let* $0 \leqslant p(t) \leqslant p < \infty$. *If* (H_0), (H_2)–(H_4) *and*

$$(H_9) \qquad \int_0^{\infty} Q(t)\Delta t = \infty,$$

hold, then any solution of (H) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (H), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ are positive for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Setting $z(t), k(t)$ and $w(t)$ as in (3.1), (3.2) and (3.3), we obtain (3.4) for $t \geqslant t_1$. Hence, $(r(t)w^{\Delta}(t))$ is a monotonic function on $[t_1,\infty)_{\mathbb{T}}$. Let $w^{\Delta}(t) < 0$ for $t \geqslant t_1$. Integrating (3.4) from T to t, we obtain

$$w(t) \leqslant w(T) + r(T)w^{\Delta}(T)\int_T^t \frac{\Delta s}{r(s)}.$$

Hence, $w(t) < 0$ due to (H_2). Proceeding as in the proof of the Theorem 1, we obtain a contradiction if $w(t) < 0$ for $t \geqslant t_2 > t_1$. Hence, $w^{\Delta}(t) > 0$ for $t \geqslant t_1$. First assume that $w(t) < 0$ for $t \geqslant t_2$, then $\lim_{t\to\infty} w(t)$ exists, that is, either $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ or $\lim_{t\to\infty} w(t) = 0$. In both these cases using the same type of argument as in the proof of Theorem 1, we obtain $\lim_{t\to\infty} y(t) = 0$.

Suppose that $w(t) > 0$ for $t \geqslant t_1$. Consequently, there exists a constant $\alpha > 0$ such that $w(t) \geqslant \alpha$ for $t \geqslant t_2 > t_1$, that is, $z(t) \geqslant w(t) \geqslant \alpha$, with which (3.5) yields

$$\int_{t_3}^{\infty} Q(t)\Delta t < \infty,$$

for $t_3 > t_2$, contradicting (H_9). This completes the proof of the theorem.

THEOREM 5. *Let* $0 \leqslant p(t) \leqslant p_1 < \infty$, $r^{\Delta}(t) \geqslant 0$ *and* $\beta(t) \leqslant \alpha(t)$. *If* ($H_0$), ($H_2$)-($H_4$), ($H_6$) *and*

$$(H_{10}) \qquad \int^{\infty} Q(t)G(\eta(\beta(t)))\Delta t = \infty,$$

hold, then any solution of (H) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 4, we consider the case $w^{\Delta}(t) > 0$ and $w(t) > 0$ for $t \geqslant t_1$. Since $r^{\Delta}(t) \geqslant 0$ implies $w^{\Delta^2}(t) \leqslant 0$ for $t \geqslant t_1$. From (3.5), and Lemma 2 it follows that

$$\begin{aligned} 0 &\geqslant \left(r(t)w^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} + \lambda Q(t)G(z(\beta(t))) \\ &\geqslant \left(r(t)w^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta} \\ &\quad + \lambda Q(t)G(\eta(\beta(t)))G(r(\beta(t))w^{\Delta}(\beta(t))), \end{aligned}$$

for $t \geqslant t_2 > t_1$. Hence

$$\begin{aligned} \lambda Q(t)G(\eta(\beta(t))) \leqslant &- \left[G(r(\beta(t))w^{\Delta}(\beta(t)))\right]^{-1}\left(r(t)w^{\Delta}(t)\right)^{\Delta} \\ &- G(p)\left[G(r(\beta(t))w^{\Delta}(\beta(t)))\right]^{-1}\left(r(\alpha(t))w^{\Delta}(\alpha(t))\right)^{\Delta}. \end{aligned} \tag{3.8}$$

Since $\lim_{t\to\infty}\left(r(t)w^{\Delta}(t)\right)$ exists, then by using (H_6) in (3.6), we get

$$\int_{t_2}^{\infty} Q(t)G(\eta(\beta(t)))\Delta t < \infty,$$

a contradiction to (H_{10}). Hence the proof of theorem is complete.

THEOREM 6. *Let* $-1 \leqslant p_2 \leqslant p(t) \leqslant 0$. *If* ($H_0$), ($H_2$), ($H_4$) *and* (3.7) *hold, then any solution of* (H) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 4, we obtain $w(t) < 0$ for $t \geqslant t_2 > t_1$ when $w^{\Delta}(t) < 0$. Hence, $w(t)$ is monotonic function on $[t_2,\infty)_{\mathbb{T}}$ and $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ exists. Following the argument in Theorem 2, we obtain $\lim_{t\to\infty} y(t) = 0$.

Assume that $w^{\Delta}(t) > 0$ for $t \geqslant t_1$. If $w(t) < 0$ for $t \geqslant t_2$ for some $t_2 > t_1$, then by using same arguments as in Theorem 2, we obtain $\lim_{t\to\infty} y(t) = 0$. Suppose that $w(t) > 0$ for $t \geqslant t_2 > t_1$. Then there exists a constant $\gamma > 0$ and $t_3 > t_2$ such that $w(t) \geqslant \gamma$ for $t \geqslant t_3$. Consequently, $y(t) \geqslant w(t) \geqslant \gamma$ for $t \geqslant t_3$. Integrating (3.4) from for $t_4 \geqslant t_3$ to ∞, yields

$$\int_{t_4}^{\infty} q(s)\Delta s < \infty,$$

which is a contradiction (3.7). Hence the theorem is proved.

THEOREM 7. *Let* $-\infty \leqslant p_3 \leqslant p(t) \leqslant p_4 < -1$. *If* ($H_0$), ($H_2$), ($H_4$) *and* (3.7) *hold, then every bounded solution of* (H) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. The proof of the theorem is follows from Theorem 3 and Theorem 6.

4. Oscillation properties of (NH)

This section is concerned with the oscillatory and asymptotic behaviour of solutions of equation (NH) for suitable forcing functions $f(t)$. We restrict our forcing functions to those that change signs. We will use the following condition:

(H_{11}) There exists $F \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $rF^{\Delta} \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $(rF^{\Delta})^{\Delta} = f$ and

$$-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty.$$

THEOREM 8. *Let* $0 \leqslant p(t) < p < \infty$. *Assume that If* ($H_0$), ($H_3$)-($H_5$) *and* ($H_{11}$) *hold. If*

(H_{12})

$$\limsup_{t\to\infty}\int_{t_0}^{t} Q(s)G(F(\beta(s)))\Delta s = +\infty \ \textit{and} \ \liminf_{t\to\infty}\int_{t_0}^{t} Q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then every bounded solution of (NH) *is oscillates.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (NH) on $[t_0,\infty)_{\mathbb{T}}$ so that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ and $y(\alpha(\beta(t)))$ are all positive on $[t_1,\infty)_{\mathbb{T}}$, for some $t_1 \geqslant t_0$. With $z(t), k(t)$ and $w(t)$ as in (3.1)-(3.3), let

$$v(t) = w(t) - F(t) = z(t) - k(t) - F(t) \tag{4.1}$$

for $t \geqslant t_1$. Then (NH) becomes

$$\left(r(t)v^{\Delta}(t)\right)^{\Delta} = -q(t)G(y(\beta(t))) \leqslant 0. \tag{4.2}$$

Thus, $v(t)$ and $v^{\Delta}(t)$ are monotonic on $[t_2,\infty)_{\mathbb{T}}$, for some $t_2 > t_1$. Suppose that $v^{\Delta}(t) < 0$ for $t \geqslant t_1$. If $v(t) < 0$ for $t \geqslant t_2 > t_1$, then $z(t) < k(t) + F(t)$. Hence,

$$\begin{aligned} 0 = \liminf_{t\to\infty} z(t) &\leqslant \liminf_{t\to\infty} \left(k(t) + F(t)\right) \\ &\leqslant \limsup_{t\to\infty} k(t) + \liminf_{t\to\infty} F(t) \\ &= \lim_{t\to\infty} k(t) + \liminf_{t\to\infty} F(t) < 0, \end{aligned}$$

is a contradiction to the fact that $z(t) > 0$. Thus, $v(t) > 0$ for $t \geqslant t_2$, that is, $z(t) > k(t) + F(t) > F(t)$ for $t \geqslant t_2$. In view of (NH), (H_3), and (H_4) it is easy see that

$$\begin{aligned} 0 &= \left(r(t)v^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))v^{\Delta}(\alpha(t))\right)^{\Delta} + q(t)G(\beta(t)) \\ &\quad + G(p_1)q(\alpha(t))G(\beta(\alpha(t))) \\ &\geqslant \left(r(t)v^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))v^{\Delta}(\alpha(t))\right)^{\Delta} + \lambda Q(t)G(z(\beta(t))) \\ &\geqslant \left(r(t)v^{\Delta}(t)\right)^{\Delta} + G(p_1)\left(r(\alpha(t))v^{\Delta}(\alpha(t))\right)^{\Delta} + \lambda Q(t)G(F(\beta(t))), \end{aligned} \tag{4.3}$$

for $t \geqslant t_3$ for some $t_3 \geqslant t_2$. We note that $\lim_{t\to\infty} v(t)$ exists. If $y(t)$ is unbounded, then

$$v(t) = z(t) - k(t) - F(t) > y(t) - F(t) - k(t)$$

implies that $v(t)$ is unbounded. Thus, $y(t)$ is bounded on $[t_4, \infty)_{\mathbb{T}}$, $t_4 > t_3$, that is, $\lim_{t\to\infty} \left(r(t)v^{\Delta}(t)\right)^{\Delta}$ exists. Integrating the inequality (4.3), we obtain

$$\limsup_{t\to\infty} \int_{t_4}^{t} Q(s)G(F(\beta(s)))\Delta s < \infty$$

contradicting (H_{12}).

Next, we suppose that $v^{\Delta}(t) > 0$ for $t \geqslant t_1$. Then $\lim_{t\to\infty} \left(r(t)v^{\Delta}(t)\right)$ exists. Similar contradictions hold for the case $v(t) > 0$ and $v(t) < 0$ for $t \geqslant t_2 > t_1$.

THEOREM 9. *Let* $0 \leqslant p(t) \leqslant p_1 < \infty$. *If* ($H_0$), ($H_1$), ($H_3$)-($H_5$), ($H_7$) *and* ($H_{11}$) *hold, then (NH) is oscillatory.*

Proof. Proceeding as in the proof of Theorem 8, $v(t) < 0$ is not possible when $v^{\Delta}(t) < 0$ for $t \geqslant t_1$. Hence $v(t) > 0$, for some $t \geqslant t_2$ for some $t_2 > t_1$. By using Lemma 1 (ii) with $u(t)$ is replaced by $v(t)$, we get $v(t) \geqslant -r(t)v^{\Delta}(t)R(t)$ for $t \geqslant t_2$ and hence

$$\begin{aligned} z(t) &\geqslant -r(t)v^{\Delta}(t)R(t) + k(t) + F(t) \\ &\geqslant -r(t)v^{\Delta}(t)R(t) + k(t) + F^{+}(t) \\ &> -r(t)v^{\Delta}(t)R(t), \end{aligned}$$

for $t \geqslant t_2$, where $F^{+}(t) = \max\{F(t), 0\}$. Further, $r(t)v^{\Delta}(t)$ is nondecreasing, so we can find a constant $c > 0$ and $t_3 > t_2$ such that $-\left(r(t)v^{\Delta}(t)\right)^{\Delta} \geqslant -c$ for $t \geqslant t_3$. Hence, inequality (3.5) becomes

$$-\lambda Q(t)G(-c)G(R(\beta(t))) \leqslant -\left(r(t)v^{\Delta}(t)\right)^{\Delta} - G(p_1)\left(r(t)v^{\Delta}(t)\right)^{\Delta}, \tag{4.4}$$

where $w(t)$ is replaced by $v(t)$ for $t \geqslant t_4 > t_3$. Since $\lim_{t\to\infty} v(t)$ exists, we claim that $y(t)$ is bounded. Otherwise, following the same argument as in Theorem 8, $v(t)$ is unbounded. Hence, $\lim_{t\to\infty}\left(r(t)v^{\Delta}(t)\right)^{\Delta}$ exists. Integrating the inequality (4.4) from t_4 to ∞, we obtain

$$\limsup_{t\to\infty}\int_{t_4}^{t} Q(s)G(\beta(s))\Delta s<\infty,$$

a contradiction to (H_7).

Let $v^{\Delta}(t)>0$ for $t\geqslant t_1$. The argument for the case $v(t)<0$ for $t\geqslant t_2>t_1$ is same as mentioned in Theorem 8. Hence, $v(t)<0$ for $t\geqslant t_2>t_1$. By Lemma 1 (i), it follows that $v(t)\geqslant KR(t)$, that is, $z(t)>KR(t)+k(t)+F^{+}(t)>KR(t)$ for $t\geqslant t_2$. Using same type of reasoning as in the proof of Theorem 1, we obtain a contradiction to (H_7). This completes the proof of theorem.

THEOREM 10. *Let* $0\leqslant p(t)\leqslant p_1<\infty$. *If* ($H_0$), ($H_2$)-($H_4$), ($H_9$) *and* ($H_{11}$) *hold, then every solution of* (NH) *is oscillates.*

Proof. Proceeding as in the proof of Theorem 8, we assume that $v^{\Delta}(t)<0$ for $t\geqslant t_1$. So, $v(t)<0$ for $t\geqslant t_2>t_1$ due to (H_2). Using the same type of argument as in the proof of Theorem 8, $v(t)<0$ is a contradiction. Thus, $v^{\Delta}(t)>0$ for $t\geqslant t_1$. Hence, $v(t)>0$ for $t\geqslant t_2>t_1$. Since, $v(t)$ is nondecreasing, there exists a constant $\alpha>0$ and $t_3>t_2$ such that $v(t)\geqslant\alpha$, for $t\geqslant t_3$. Thus,

$$z(t)>\alpha+k(t)+F(t)\geqslant\alpha+k(t)+F^{+}(t)>\alpha,$$

for $t\geqslant t_3$, where $F^{+}(x)=\max\{F(t),0\}$. Using the last inequality and then integrating (4.3) from t_4 to ∞, we get

$$\int_{t_4}^{t} Q(t)\Delta t<\infty,$$

for $t_4\geqslant t_3$, a contradiction to (H_9). This completes the proof of the theorem.

THEOREM 11. *Let* $-1<p_2\leqslant p(t)\leqslant 0$. *Assume that If* ($H_0$), ($H_4$), ($H_{11}$) *and*

$$(H_{13})\qquad\qquad \int^{\infty} q(t)G(F^{+}(y(\beta(t))))\Delta t=\infty,$$

hold. Then any solution $y(t)$ *of* (NH) *is either oscillatory or satisfies* $\limsup_{t\to\infty}|y(t)|=0$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (NH) on $[t_0,\infty)_{\mathbb{T}}$ so that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ and $y(\alpha(\beta(t)))$ are all positive on $[t_1,\infty)_{\mathbb{T}}$ for some $t_1\geqslant t_0$. With $z(t),k(t)$ and $w(t)$ as in (3.1)-(3.2) and (4.1), we get (4.2). Thus, $v(t)$ and $v^{\Delta}(t)$ are monotonic on $[t_2,\infty)_{\mathbb{T}}$, for some $t_2>t_1$. Let $v^{\Delta}(t)<0$ for $t\geqslant t_1$. Hence, $\lim_{t\to\infty}v(t)=\lim_{t\to\infty}(z(t)-F(t))$ implies that $z(t)-F(t)<0$ when $v(t)<0$, that is, $\liminf_{t\to\infty}\ z(t)=-\infty$. So $\limsup_{t\to\infty}\ y(t)=+\infty$.

Next, we assume that $v(t) > 0$ for $t \geqslant t_2 > t_1$. Hence, $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty}(z(t) - k(t) - F(t))$ implies that $z(t) - F(t) > 0$. If $\lim\limits_{t\to\infty} v(t) \neq 0$, then $z(t) > F(t)$ for $t \geqslant t_2$. Hence, $y(t) > F^+(t)$ for $t \geqslant t_3 > t_2$. We claim that $y(t)$ is bounded. If not, there exists an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2,\infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $x(\tau_n) \to \infty$ as $n \to \infty$, and

$$x(\tau_n) = \max\{x(t) : t_3 \leqslant t \leqslant \tau_n\}.$$

We may choose n large enough so that $\alpha(\tau_n) \geqslant t_2$. Hence,

$$\begin{aligned} v(\tau_n) &\geqslant y(\tau_n) + py(\alpha(\tau_n)) - k(\tau_n) - F(\tau_n) \\ &\geqslant (1+p_2)y(\tau_n) - k(\tau_n) - F(\tau_n) \end{aligned}$$

implies that $v(\tau_n) \to \infty$ as $n \to \infty$, a contradiction to the fact that $\lim\limits_{t\to\infty} v(t)$ exists. So our claim holds and $\lim\limits_{t\to\infty}\big(r(t)v^{\Delta}(t)\big)$ exists. Integrating (4.2) from t_3 to ∞, we obtain

$$\int_{t_3}^{\infty} q(t)G(F^+(y(\beta(t))))\Delta t < \infty,$$

a contradiction to (H_{13}). If $\lim\limits_{t\to\infty} v(t) = 0$, then $z(t) - F(t) > 0$ or $z(t) - F(t) < 0$ for all $t \geqslant t_2$. If $z(t) - F(t) > 0$ for $t \geqslant t_2$, then we have a contradiction where as if $z(t) - F(t) < 0$ for $t \geqslant t_2$, then $\limsup\limits_{t\to\infty} y(t) = +\infty$.

Assume that $v^{\Delta}(t) > 0$ for $t \geqslant t_1$. Then $\lim\limits_{t\to\infty}\big(r(t)v^{\Delta}(t)\big)$ exists. Proceeding as the proof above, we obtain a contradiction for case $v(t) > 0$. If $v(t) < 0$, then $\limsup\limits_{t\to\infty} y(t) = +\infty$. This completes the proof of theorem.

THEOREM 12. *Let $-\infty < p_3 \leqslant p(t) \leqslant -1$. If all conditions of Theorem 11 hold, then every bounded solution of (NH) is either oscillatory or satisfies* $\limsup\limits_{t\to\infty} |y(t)| = 0$.

The proof follows from Theorem 11 and hence details are omitted.

EXAMPLE 1. For $\mathbb{T} = \mathbb{R}$, we consider the equation

$$\begin{aligned} \big(y(t) + e^{-t}y(t-\pi)\big)'' &+ (1+e^{-t})y(t-2\pi) \\ &- e^{-t}(1+\sin^2 t)\frac{y(t-4\pi)}{1+y^2(t-4\pi)} = 2e^{-t}\sin t, \end{aligned} \tag{4.5}$$

where,

$$p(t) = e^{-t},\ q(t) = e^{-t} + 1,\ h(t) = e^{-t}(1+\sin^2 t).$$

If we choose $F(t) = e^{-t}\cos t$, then $(r(t)F'(t))' = 2e^{-t}\sin t = f(t)$. Clearly, ($H_0$), ($H_2$)-($H_4$), ($H_9$), and ($H_{11}$) are satisfied. Hence by Theorem 10, every solution of (4.5) are oscillatory. In particular, $y(t) = \sin t$ is such an oscillatory solution of (4.5).

EXAMPLE 2. For $\mathbb{T} = \mathbb{Z}$, consider the equation

$$\Delta(e^n \Delta(y(n) + p(n)y(n-1))) + q(n)y(n-2) \\ - h(n)\frac{y(n-4)}{1+y^2(n-4)} = (-1)^n e^n, \ n \geqslant 0, \quad (4.6)$$

where

$$p(n) = 2 + (-1)^n, \ \ q(n) = (2e+3)e^n - e^{-n}, \ \ h(n) = 2e^{-n}.$$

Indeed,

$$Q(n) = (2e+3)e^{n-1} - e^{-(n-1)}, \ \ R(n) = \frac{e}{e-1}e^{-n}.$$

If we choose $F(n) = [2(1+e)^{-1}(-1)^n]$, then $f(n) = \Delta(e^n \Delta F(n))$. Clearly, ($H_0$), ($H_1$), ($H_3$)-($H_5$), and ($H_{11}$) are satisfied. Moreover, (H_7) is given by

$$\sum_{n=0}^{\infty} Q(n)G(R(n-2)) = \sum_{n=0}^{\infty}\left[(2e+3)e^{n-1} - e^{-(n-1)}\right]\frac{e}{e-1}e^{-(n-2)} \\ = \frac{e^2(2e+3)}{e-1}\sum_{n=0}^{\infty} 1 - \frac{e^4}{e-1}\sum_{n=0}^{\infty} e^{-2n} = \infty.$$

Hence, by Theorem 9, equation (4.6) oscillates. In particular, $y(n) = (-1)^n$ is one of such solution.

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