

OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SECOND ORDER NEUTRAL DYNAMIC EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. In this paper, oscillatory and asymptotic properties of solutions of nonlinear second order neutral dynamic equations of the form

$$\left(r(t) (y(t) + p(t)y(\alpha(t)))^\Delta \right)^\Delta + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = 0$$

and

$$\left(r(t) (y(t) + p(t)y(\alpha(t)))^\Delta \right)^\Delta + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = f(t)$$

are studied under assumptions

$$\int_0^\infty \frac{1}{r(t)} \Delta t < \infty \quad \text{and} \quad \int_0^\infty \frac{1}{r(t)} \Delta t = \infty$$

for various ranges of $p(t)$, where \mathbb{T} is a time scale with $\sup \mathbb{T} = \infty$, $t \in [t_0, \infty)_{\mathbb{T}}$, and $t_0 \geq 0$. Examples illustrating the results are included.

1. Introduction

The study of dynamic equations on time scales goes to seminal work of Stefan Hilger [8] and has received a lot of attention in recent years. Time scales were created to unify the study of continuous and discrete mathematics and is particularly use in differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and allow us to avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale \mathbb{T} , which is a non-empty closed subset of the real numbers \mathbb{R} . In this way the results of this paper not only apply to the set of

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real numbers or set of integers, but also to more general time scales such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{N_0} = \{t : t = q^k, k \in N_0\}$ with $q > 1$, $\mathbb{T} = N_0^2 = \{t^2 : t \in N_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in N_0\}$ e.t.c.. For basic notations on the time scale calculus, we refer the reader to monographs [4, 5] and the references cited therein.

In recent years, there has been increasing interest of obtaining sufficient conditions for the oscillation and nonoscillation of solution of second order neutral dynamic equation on time scales (see [1, 2, 9, 11, 13]) and references cited therein.

Q. Yang et al. [19] studied the oscillation of second-order quasi linear neutral dynamic equation

$$(r(t)|z^\Delta(t)|^{\alpha-1}z^\Delta(t))^\Delta + q(t)|x(\delta(t))|^{\beta-1}x(\delta(t)) = 0, \quad (1.1)$$

on an arbitrary time scale \mathbb{T} , where $z(t) = x(t) + p(t)x(\tau(t))$, $\alpha, \beta > 0$ are constants, and obtained oscillation criteria for the equation when $\beta > \alpha$, $\beta = \alpha$ and $\beta < \alpha$, respectively with the followig assumptions:

- (A₁) $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ with $\int_{t_0}^{\infty} r^{\frac{-1}{\alpha}}(t)\Delta t = \infty$;
 (A₂) $p, q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ with $0 \leq p(t) < 1, q(t) \geq 0$;
 (A₃) $\tau, \delta \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t) \leq t$, and $\delta(t) \leq t$, and $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \delta(t) = \infty$.

Kubiacyk et al. [10] established some sufficient conditions for oscillation of the second-order neutral functional dynamic equation

$$(r(t)[m(t)y(t) + p(t)y(\tau(t))]^\Delta)^\Delta + q(t)f(y(\delta(t))) = 0 \quad (1.2)$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, on a time scale \mathbb{T} which is unbounded above, where p, q, r, τ and δ are real valued rd-continuous positive functions defined on \mathbb{T} . They obtained results by using Riccati substitution and analysis of Riccati dynamic inequality.

Saker and O' Regan [14] considered the second-order nonlinear neutral functional dynamic equation

$$(p(t)([y(t) + r(t)y(\tau(t))]^\Delta)^\gamma)^\Delta + f(t, y(\delta(t))) = 0, \quad (1.3)$$

on a time scale \mathbb{T} and established some new sufficient conditions for oscillation. The results improve oscillation results for neutral dynamic equation on time scales and are new when $\delta(t) > t$ and/or $0 < \gamma < 1$.

In E. Thandapani et al. [16] obtained the oscillation criteria for the second-order nonlinear neutral delay dynamic equation on time scales

$$\left((r(t)(y(t) + p(t)y(t - \tau))^\Delta)^\gamma \right)^\Delta + q(t)y^\beta(t - \delta) = 0, \quad t \in \mathbb{T},$$

where \mathbb{T} is a time scale and

- (H₁) $\gamma \geq 1$, and $\beta > 0$ are quotients of odd positive integers;
 (H₂) τ, δ are fixed nonnegative constants such that the delay functions $\tau(t) = t - \tau < t$ and $\delta(t) = t - \delta < t$ satisfy $\tau(t) : \mathbb{T} \rightarrow \mathbb{T}$ and $\delta(t) : \mathbb{T} \rightarrow \mathbb{T}$ for all $t \in \mathbb{T}$;

(H₃) $p(t)$ is a positive and rd-continuous function on \mathbb{T} such that $0 \leq p(t) < 1$.

Zhang and Wang [20] studied the oscillation criteria for second-order nonlinear dynamic equation

$$\left(r(t) \left(y(t) + p(t)y(\tau(t)^\Delta)^\gamma \right)^\Delta \right)^\Delta + f_1(t, y(\delta_1(t))) + f_2(t, y(\delta_2(t))) = 0,$$

on time scale \mathbb{T} , where $p \in C_{rd}(\mathbb{T}, [0, 1])$, $f_i \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$, $\gamma > 0$ is a quotient of odd positive integers by using the Ricati transformation technique.

In [17], author studied the oscillatory and asymptotic behaviour of solutions of the second order nonlinear delay differential equations of the form

$$(r(t)(y(t) + p(t)y(t - \tau)))' + q(t)G(y(t - \sigma)) - h(t)H(y(t - \delta)) = f(t)$$

and

$$(r(t)(y(t) + p(t)y(t - \tau)))' + q(t)G(y(t - \sigma)) - h(t)H(y(t - \delta)) = 0$$

for various range of $p(t)$ under the assumptions

$$\int_0^\infty \frac{dt}{r(t)} < \infty \quad \text{and} \quad \int_0^\infty \frac{dt}{r(t)} = \infty.$$

In [18], author studied the oscillatory and asymptotic behaviour of solutions of a class of nonlinear second-order neutral difference equations with positive and negative coefficients of the form

$$\Delta(r(n)\Delta(y(n) + p(n)y(n - m))) + q(n)G(y(n - k_1)) - h(n)H(y(n - k_2)) = f(n)$$

and

$$\Delta(r(n)\Delta(y(n) + p(n)y(n - m))) + q(n)G(y(n - k_1)) - h(n)H(y(n - k_2)) = 0$$

under the assumptions

$$\sum_{n=0}^\infty \frac{1}{r(n)} < \infty \quad \text{and} \quad \sum_{n=0}^\infty \frac{1}{r(n)} = \infty$$

for various ranges of $p(n)$.

The objective of this paper is to study the oscillatory and asymptotic properties of solutions of the nonlinear second-order neutral delay dynamic equations with positive and negative coefficients of the form

$$(r(t)(y(t) + p(t)y(\alpha(t)))^\Delta)^\Delta + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = 0 \quad (\text{H})$$

and

$$(r(t)(y(t) + p(t)y(\alpha(t)))^\Delta)^\Delta + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = f(t) \quad (\text{NH})$$

on a time scale \mathbb{T} such that $\sup \mathbb{T} = \infty$ and $t_0 \in \mathbb{T}$.

We consider these equations under the assumptions that

$$(H_0) \int_{t_0}^{\infty} \frac{1}{r(s)} \left(\int_s^{\infty} h(t) \Delta t \right) \Delta s < \infty;$$

$$(H_1) \int_0^{\infty} \frac{1}{r(t)} \Delta t < \infty;$$

$$(H_2) \int_0^{\infty} \frac{1}{r(t)} \Delta t = \infty.$$

Here we extended the result of [17, 18] to second order dynamic equations with positive and negative coefficients and the results obtained are new and generalize the earlier work in [17, 18].

For equations (H) and (NH) we will use the notation $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$ and assume that $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $p, f \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $q, h \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $G, H \in (\mathbb{R}, \mathbb{R})$ satisfying $uG(u) > 0$ and $uH(u) > 0$ for $u \neq 0$, G is nondecreasing, H is bounded, and $\alpha, \beta, \gamma \in C_{rd}(\mathbb{T}, \mathbb{T})$ are strictly increasing functions such that

$$\lim_{t \rightarrow \infty} \alpha(t) = \lim_{t \rightarrow \infty} \beta(t) = \lim_{t \rightarrow \infty} \gamma(t) = \infty, \alpha(t), \beta(t), \gamma(t) \leq t$$

and

$$(\alpha \circ \beta)(t) = (\beta \circ \alpha)(t) \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}.$$

The inverse of $\alpha(t)$ will be denoted by $\alpha^{-1}(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$. Whenever we write $t \geq t_1$, we mean $t \in [t_1, \infty) \cap \mathbb{T}$.

Let $t_{-1} = \inf_{t \in [t_0, \infty)_{\mathbb{T}}} \{\alpha(t), \beta(t), \gamma(t)\}$. By a solution of (H) and (NH), we mean a function $y \in C_{rd}([t_{-1}, \infty)_{\mathbb{T}}, \mathbb{R})$, and such that $y(t) + p(t)y(\alpha(t)) \in C_{rd}^1([t_0, \infty), \mathbb{R})$, $r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta} \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and such that (H) ((NH)) is satisfied on $[t_0, \infty)_{\mathbb{T}}$. A solution of (H) or (NH) is called oscillatory if it is neither eventually positive nor eventually negative, and it is nonoscillatory otherwise. In this paper, we do not consider solutions that eventually vanish identically. An equation will be called oscillatory if all its solutions are oscillatory.

2. Preliminary Lemmas

We will need the following lemmas in the sequel.

LEMMA 1. *Let (H_1) hold. Let $u(t)$ be an eventually positive rd-continuously differentiable function such that $r(t)u^{\Delta}(t)$ is rd-continuously differentiable function on $[t_0, \infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta}(t))^{\Delta} \leq 0$ for large t , where $r \in C([0, \infty)_{\mathbb{T}}, (0, \infty))$.*

(i) *If $u^{\Delta}(t) > 0$, then there exist a constant $K > 0$ such that $u(t) \geq KR(t)$, for large t .*

(ii) *If $u^{\Delta}(t) < 0$, then $u(t) > -r(t)u^{\Delta}(t)R(t)$, where $R(t) = \int_t^{\infty} \frac{\Delta s}{r(s)}$.*

Proof. (i) Since $R(t) < \infty, R(t) \rightarrow 0$ as $t \rightarrow \infty$ and $u(t)$ is nondecreasing, we can find a constant $K > 0$ such that $u(t) \geq KR(t)$ for all large t .

(ii) For $s \geq t$, we have $r(s)u^\Delta(s) \leq r(t)u^\Delta(t)$, and hence

$$u(s) \leq u(t) + \int_t^s \frac{r(t)u^\Delta(t)}{r(\theta)} \Delta\theta = u(t) + r(t)u^\Delta(t) \int_t^s \frac{\Delta\theta}{r(\theta)}.$$

Thus,

$$0 < u(s) \leq u(t) + r(t)u^\Delta(t) \int_t^s \frac{\Delta\theta}{r(\theta)}$$

implies that $u(t) \geq -r(t)u^\Delta(t)R(t)$.

LEMMA 2. Assume that (H_2) hold. Let $u(t)$ and $u^\Delta(t)$ be positive rd-continuously differentiable functions with $u^{\Delta^2}(t) \leq 0$ for $t \geq T \geq 0$. Then

$$u(t) \geq (t - T)u^\Delta(t) = \eta(t)r(t)u^\Delta(t)$$

for $t \geq T \geq 0$, where $\eta(t) = \frac{t-T}{r(t)}$.

Proof. The proof is simple and hence the details are omitted.

LEMMA 3. ([12], Lemma 3.5) Let $F, H, P : [t_0, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ satisfy

$$F(t) = H(t) + P(t)H(\alpha(t)) \quad \text{for } t \in [\hat{t}, \infty)_{\mathbb{T}},$$

where $\hat{t} \in [t_0, \infty)_{\mathbb{T}}$ is such that $\alpha(t) \geq t_0$ for all $t \in [\hat{t}, \infty)_{\mathbb{T}}$. Assume that there exist constants $P_1, P_2 \in \mathbb{R}$ such that $P(t)$ is one of the following ranges:

- (1) $-\infty \leq P(t) \leq 0$,
- (2) $0 \leq P(t) \leq P_1 < 1$,
- (3) $1 < P_2 \leq P(t) < \infty$.

If $H(t) > 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$, $\liminf_{t \rightarrow \infty} H(t) = 0$, and $\lim_{t \rightarrow \infty} F(t) = L \in \mathbb{R}$ exists, then $L = 0$.

3. Oscillation properties for (H)

In this section, we study the asymptotic behaviour of solutions of equation (H) under assumptions (H_1) and (H_2) . We will make use of the following conditions on the functions in the equations (H) and (NH):

(H_3) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u + v)$ for $u, v \in \mathbb{R}$ with $u, v > 0$;

(H_4) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$;

(H_5) $G(-u) = -G(u)$; $u \in \mathbb{R}$;

(H_6) $\int_0^{\pm c} \frac{\Delta u}{G(u)} < \infty$.

REMARK 1. (H_4) implies (H_5) . Indeed $G(1)G(1) = G(1)$, so that $G(1) = 1$. Further, $G(-1)G(-1) = G(1) = 1$ gives $(G(-1))^2 = 1$. Since $G(-1) < 0$, then $G(-1) = -1$. Consequently, $G(-u) = G(-1)G(u) = -G(u)$. On the other hand, $G(uv) = G(u)G(v)$ for $u > 0, v > 0$ and $G(-u) = -G(u)$ implies that

$$G(uv) = G(u)G(v) \text{ for every } u, v \in \mathbb{R}.$$

REMARK 2. The prototype of G satisfying (H_3) and (H_5) is

$$G(u) = (a + b|u|^\mu) |u|^\nu \operatorname{sgnu},$$

where $a \geq 1, b \geq 1, \mu \geq 0$, and $\nu \geq 0$. However, the prototype of G satisfying (H_3) and (H_4) is $G(u) = |u|^\gamma \operatorname{sgnu}$, where $\gamma > 0$. This G also satisfies H_3 and (H_5) .

REMARK 3. Notice that if $y(t)$ is a solution of (H) , then $x(t) = -y(t)$ is also a solution of (H) provided that G and H satisfies (H_4) or (H_5) .

THEOREM 1. Let $0 \leq p(t) \leq p_1 < \infty$, and assume that conditions $(H_0), (H_1), (H_3)$ - (H_5) hold. If

(H_7)

$$\int_0^\infty Q(t)G(R(\beta(t)))\Delta t = \infty,$$

where $Q(t) = \min\{q(t), q(\alpha(t))\}$ and $R(t) = \int_t^\infty \frac{\Delta s}{r(s)}$, then any solution of (H) is either oscillatory or converges to zero as $t \rightarrow \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (H) on $[t_0, \infty)_{\mathbb{T}}$, say $y(t)$ is eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t), y(\alpha(t)), y(\beta(t)), y(\gamma(t))$ and $y(\alpha(\beta(t)))$ are all positive for $t \geq t_1$. Define

$$z(t) = y(t) + p(t)y(\alpha(t)), \tag{3.1}$$

and

$$k(t) = \int_t^\infty \frac{1}{r(s)} \int_s^\infty h(\theta)H(y(\gamma(\theta)))\Delta\theta\Delta s. \tag{3.2}$$

Notice that condition (H_0) and the fact that H is a bounded function imply that $k(t)$ exists for all t . Now if we let

$$w(t) = z(t) - k(t) = y(t) + p(t)y(\alpha(t)) - k(t), \tag{3.3}$$

then

$$(r(t)w^\Delta(t))^\Delta = -q(t)G(y(\beta(t))) \leq 0, \tag{3.4}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Clearly, $r(t)w^\Delta(t)$ is a monotonic function on $[t_1, \infty)_{\mathbb{T}}$. Let $w^\Delta(t) < 0$ for $t \geq t_1$. Suppose that $w(t) < 0$, then $y(t) \leq z(t) \leq k(t)$, $t \geq t_1$. We note that $k(t)$ is bounded with $\lim_{t \rightarrow \infty} k(t) = 0$ and hence there exists a constant $M > 0$ such that $y(t) \leq M$ for $t \geq t_2$ for some $t_2 > t_1$. So, $w(t)$ is bounded and $\lim_{t \rightarrow \infty} w(t)$ exists. This is contradicts to the fact that $\lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} z(t) \neq 0$ implies that $z(t) < 0$ for $t \geq t_3$ for some $t_3 > t_2$.

Assume that $w(t) > 0$ for $t \geq t_1$. Successive integrating inequality (3.4) from t_1 to t , we can find a constant $\eta > 0$ such that $w(t) \leq \eta$ for $t \geq t_2 > t_1$. By Lemma 1 (ii), we get $w(t) \geq -r(t)w^\Delta(t)R(t)$ and hence $z(t) \geq -r(t)w^\Delta(t)R(t)$ for $t \geq t_2$. Since $w(t)$ is bounded, $R(t)$ is bounded and $r(t)w^\Delta(t)$ is monotonic implies that $\lim_{t \rightarrow \infty} (r(t)w^\Delta(t))$ exist. By using (H_3) , and (H_4) in equation (H) gives

$$\begin{aligned}
 0 &= (r(t)w^\Delta(t))^\Delta + q(t)G(y(\beta(t))) + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta \\
 &\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
 &= (r(t)w^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta + q(t)G(y(\beta(t))) \\
 &\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
 &\geq (r(t)w^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta + \lambda Q(t)G(y(\beta(\alpha(t))) \\
 &\quad + py(\alpha(\beta(t)))) \tag{3.5} \\
 &\geq (r(t)w^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta + \lambda Q(t)G(z(\beta(t))) \\
 &\geq (r(t)w^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta \\
 &\quad + \lambda Q(t)G(-r(\beta(t))w^\Delta(\beta(t))R(\beta(t))) \\
 &= (r(t)w^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta \\
 &\quad + \lambda Q(t)G(R(\beta(t)))G(-r(\beta(t))w^\Delta(\alpha(t)))
 \end{aligned}$$

for $t \geq t_3 > t_2$. Since $-r(t)w^\Delta(t)$ is nondecreasing, we can find a constant $c > 0$, and $t_4 > t_3$ such that $-r(t)w^\Delta(t) \geq c$, for $t \geq t_4$. From (3.5), we have

$$\lambda Q(t)G(c)G(R(\beta(t))) \leq - (r(t)w^\Delta(t))^\Delta - G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta \tag{3.6}$$

for $t \geq t_5 > t_4$. Integrating (3.6) from t_5 to ∞ , we get

$$\int_{t_5}^{\infty} Q(t)G(R(\beta(t)))\Delta t < \infty,$$

which contradicts (H_7) .

Next, we suppose that $w^\Delta(t) > 0$ for $t \geq t_1$. If $w(t) < 0$, then $w(t)$ exists and $0 \neq \lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} z(t)$ which implies $z(t) < 0$, for large t , which is a contradiction to the fact that $z(t) > 0$. Hence $\lim_{t \rightarrow \infty} w(t) = 0$. So also, $\lim_{t \rightarrow \infty} z(t) = 0$ implies that $\lim_{t \rightarrow \infty} y(t) = 0$ since $y(t) \leq z(t)$ for $t \geq t_2 > t_1$.

Now we suppose that $w(t) > 0$ for $t \geq t_2 > t_1$. By Lemma 1 (i), it follows that $w(t) \geq KR(t)$ and $z(t) \geq w(t) \geq KR(t)$ for $t \geq t_2$. From (3.5), we get

$$\lambda Q(t)G(K)G(R(\beta(t))) \leq - (r(t)w^\Delta(t))^\Delta - G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta$$

for $t \geq t_3 > t_2$. Integrating the above inequality implies

$$\int_{t_5}^{\infty} Q(t)G(R(\beta(t)))\Delta t < \infty,$$

is a contradiction. This completes the proof of the theorem.

THEOREM 2. *Let $-1 < p_2 \leq p(t) \leq 0$. If (H_0) , (H_1) , (H_4) and (H_8)*

$$\int_0^{\infty} q(t)G(R(\beta(t)))\Delta t = \infty,$$

where $R(t) = \int_t^{\infty} \frac{\Delta s}{r(s)}$ hold, then any solution of **(H)** is either oscillatory or tends to zero as $t \rightarrow \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of **(H)**, say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ are positive for all $t_1 \in [t_0, \infty)_{\mathbb{T}}$, $t_1 \geq t_0$. Setting $z(t)$, $k(t)$ and $w(t)$ as in (3.1), (3.2) and (3.3), we obtain (3.4) for $t \geq t_1$. Hence, $w^\Delta(t)$ is monotonic for large $t \in [t_1, \infty)_{\mathbb{T}}$ which implies that either $w(t) > 0$ or $w(t) < 0$ for $t \geq t_2 > t_1$.

Suppose that $w^\Delta(t) < 0$ and $w(t) < 0$ for $t \geq t_2$. Then $0 \neq \lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} z(t)$ implies that $z(t) < 0$ for $t \geq t_2$. Hence, $y(t) < y(\alpha(t))$ for $t \geq t_3$ for some $t_3 > t_2$, that is, $y(t)$ is bounded on $[t_3, \infty)_{\mathbb{T}}$. Consequently, $w(t)$ is bounded and $\lim_{t \rightarrow \infty} (r(t)w^\Delta(t))$ exists. Since, $w(t)$ is monotonic, then $\lim_{t \rightarrow \infty} w(t) = L, L \in (-\infty, 0)$ gives $\lim_{t \rightarrow \infty} z(t) = L$. We claim that $\liminf_{t \rightarrow \infty} y(t) = 0$. If not, there exists a constant $M > 0$ and $t_4 > t_3$ such that $y(t) \geq M$ for $t > t_4$. Integrating (3.4), we get

$$\int_{t_4}^{\infty} q(t)\Delta t < \infty,$$

a contradiction to the fact that $R(t) \rightarrow 0$ as $t \rightarrow \infty$ and (H_8) implies that

$$\int_0^{\infty} q(t)\Delta t = \infty. \tag{3.7}$$

So, our claim holds. By Lemma 3, $L = 0$. Hence,

$$\begin{aligned} 0 = \lim_{t \rightarrow \infty} z(t) &= \limsup_{t \rightarrow \infty} [y(t) + p(t)y(\alpha(t))] \\ &\geq \limsup_{t \rightarrow \infty} [y(t) + p_2 y(\alpha(t))] \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2 y(\alpha(t))) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t), \end{aligned}$$

which implies that $\limsup_{t \rightarrow \infty} y(t) = 0$, that is, $\lim_{t \rightarrow \infty} y(t) = 0$.

Next, we consider the case $w(t) > 0$ for $t \geq t_2$. Let $\lim_{t \rightarrow \infty} w(t) = l, l \in [t_0, \infty)$. We claim that $y(t)$ is bounded. If not, there exists an increasing sequence $\{\tau_n\}_{n=1}^\infty \subset [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \rightarrow \infty, y(\tau_n) \rightarrow \infty$ as $n \rightarrow \infty$, and $y(\tau_n) = \max\{y(t) : t_2 \leq t \leq \tau_n\}$. we choose τ_1 large enough so that $\alpha(\tau_1) \geq t_2$. Hence,

$$0 \geq w(\tau_n) \geq y(\tau_n) + p(\tau_n)y(\alpha(\tau_n)) - k(\tau_n) \geq (1 + p_2)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1 + p_2 > 0$, we have $w(\tau_n) > 0$ for large n , which is a contradiction to the fact that $\lim_{t \rightarrow \infty} w(t)$ exists, so our claim is true. Hence, $\lim_{t \rightarrow \infty} (r(t)w^\Delta(t))$ exists. Using Lemma 1 (ii), we get $w(t) \geq -r(t)w^\Delta(t)R(t)$ and hence $y(t) \geq w(t) \geq -r(t)w^\Delta(t)R(t), t \geq t_3 > t_2$. Hence, (3.4) becomes

$$q(t))G(R(\beta(t)))G\left(-r(\beta(t))w^\Delta(\beta(t))\right) \leq -\left(r(t)w^\Delta(t)\right)^\Delta,$$

for $t \geq t_4 > t_3$. Since $r(t)w^\Delta(t)$ is nonincreasing, we can find a constant $b > 0$ such that $r(\beta(t))w^\Delta(\beta(t)) \leq -b$ for $t \geq t_5$ for some $t_5 > t_4$. Integrating the last inequality, we get

$$\int_{t_5}^\infty q(t)G(R(\beta(t)))\Delta t < \infty,$$

is a contradiction to (H_8) .

Suppose that $w^\Delta(t) > 0$ for $t \geq t_1$. So, we have two cases, $w(t) > 0$ or $w(t) < 0$ for $t \geq t_1$. Let $w(t) > 0$ for $t \geq t_1$, then by Lemma 1 (i), $y(t) \geq w(t) \geq KR(t)$ for $t \geq t_3 > t_2$ and hence equation (3.4) becomes

$$q(t))G(KR(\beta(t))) \leq -\left(r(t)w^\Delta(t)\right)^\Delta$$

for $t \geq t_3 > t_2$. Integrating the above inequality from t_3 to ∞ , we get

$$\int_{t_3}^\infty q(t)G(R(\beta(t)))\Delta t < \infty,$$

is a contradiction to (H_8) . Hence, $w(t) < 0$ for $t \geq t_1$. Therefore, $\lim_{t \rightarrow \infty} w(t)$ exists and $0 \neq \lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} z(t)$ implies that $z(t) < 0$ for $t \geq t_2 > t_1$. So, $y(t)$ is bounded on $[t_3, \infty)_{\mathbb{T}}$ for some $t_3 > t_2$. Using the same type of reasoning mentioned above, we obtain $\lim_{t \rightarrow \infty} y(t) = 0$. If $0 = \lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} z(t)$, then $y(t)$ is bounded. Otherwise, this is a contradiction that $w(t) > 0$ for large t . Proceeding as above again we obtain $\lim_{t \rightarrow \infty} y(t) = 0$.

THEOREM 3. *Let $-\infty < p_3 \leq p(t) \leq p_2 < -1$. If $(H_0), (H_1), (H_4)$ and (H_8) hold, then every bounded solution of (H) is either oscillatory or tends to zero as $t \rightarrow \infty$.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (H) on $[t_0, \infty)_{\mathbb{T}}$, say $y(t)$ is an eventually positive solution. There exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t), y(\alpha(t)), y(\beta(t)), y(\gamma(t))$ and $y(\alpha(\beta(t)))$ all are positive for $t \geq t_1$. Setting $z(t), k(t)$ and $w(t)$ as in (3.1), (3.2) and (3.3), we obtain (3.4) for $t \geq t_1$. From (3.4), it follows that $w^\Delta(t) > 0$ or $w^\Delta(t) < 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 2, we obtain $L = 0$. Hence,

$$\begin{aligned} 0 = \lim_{t \rightarrow \infty} z(t) &= \liminf_{t \rightarrow \infty} [y(t) + p(t)y(\alpha(t))] \\ &\leq \liminf_{t \rightarrow \infty} [y(t) + p_2y(\alpha(t))] \\ &\leq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (p_2y(\alpha(t))) \\ &\leq \limsup_{t \rightarrow \infty} y(t) + p_2 \limsup_{t \rightarrow \infty} (y(\alpha(t))) \\ &= (1 + p_2) \limsup_{t \rightarrow \infty} y(t), \end{aligned}$$

which implies that $\limsup_{t \rightarrow \infty} y(t) = 0$, that is, $\lim_{t \rightarrow \infty} y(t) = 0$ since $(1 + p_2) < 0$. The remaining part of the proof can be followed from the proof of the Theorem 2. This completes the proof of the theorem.

THEOREM 4. *Let $0 \leq p(t) \leq p < \infty$. If $(H_0), (H_2)–(H_4)$ and*

$$(H_0) \quad \int_0^\infty Q(t)\Delta t = \infty,$$

hold, then any solution of (H) is either oscillatory or converges to zero as $t \rightarrow \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (H), say $y(t), y(\alpha(t)), y(\beta(t)), y(\gamma(t))$ are positive for all $t \in [t_1, \infty)_{\mathbb{T}}, t_1 \geq t_0$. Setting $z(t), k(t)$ and $w(t)$ as in (3.1), (3.2) and (3.3), we obtain (3.4) for $t \geq t_1$. Hence, $(r(t)w^\Delta(t))$ is a monotonic function on $[t_1, \infty)_{\mathbb{T}}$. Let $w^\Delta(t) < 0$ for $t \geq t_1$. Integrating (3.4) from T to t , we obtain

$$w(t) \leq w(T) + r(T)w^\Delta(T) \int_T^t \frac{\Delta s}{r(s)}.$$

Hence, $w(t) < 0$ due to (H_2) . Proceeding as in the proof of the Theorem 1, we obtain a contradiction if $w(t) < 0$ for $t \geq t_2 > t_1$. Hence, $w^\Delta(t) > 0$ for $t \geq t_1$. First assume that $w(t) < 0$ for $t \geq t_2$, then $\lim_{t \rightarrow \infty} w(t)$ exists, that is, either $0 \neq \lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} z(t)$ or $\lim_{t \rightarrow \infty} w(t) = 0$. In both these cases using the same type of argument as in the proof of Theorem 1, we obtain $\lim_{t \rightarrow \infty} y(t) = 0$.

Suppose that $w(t) > 0$ for $t \geq t_1$. Consequently, there exists a constant $\alpha > 0$ such that $w(t) \geq \alpha$ for $t \geq t_2 > t_1$, that is, $z(t) \geq w(t) \geq \alpha$, with which (3.5) yields

$$\int_{t_3}^\infty Q(t)\Delta t < \infty,$$

for $t_3 > t_2$, contradicting (H_9) . This completes the proof of the theorem.

THEOREM 5. *Let $0 \leq p(t) \leq p_1 < \infty$, $r^\Delta(t) \geq 0$ and $\beta(t) \leq \alpha(t)$. If (H_0) , (H_2) - (H_4) , (H_6) and*

$$(H_{10}) \quad \int_{t_2}^{\infty} Q(t)G(\eta(\beta(t)))\Delta t = \infty,$$

hold, then any solution of (H) is either oscillatory or tends to zero as $t \rightarrow \infty$.

Proof. Proceeding as in the proof of Theorem 4, we consider the case $w^\Delta(t) > 0$ and $w(t) > 0$ for $t \geq t_1$. Since $r^\Delta(t) \geq 0$ implies $w^{\Delta^2}(t) \leq 0$ for $t \geq t_1$. From (3.5), and Lemma 2 it follows that

$$\begin{aligned} 0 &\geq (r(t)w^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta + \lambda Q(t)G(z(\beta(t))) \\ &\geq (r(t)w^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta \\ &\quad + \lambda Q(t)G(\eta(\beta(t)))G(r(\beta(t))w^\Delta(\beta(t))), \end{aligned}$$

for $t \geq t_2 > t_1$. Hence

$$\begin{aligned} \lambda Q(t)G(\eta(\beta(t))) &\leq - [G(r(\beta(t))w^\Delta(\beta(t)))]^{-1} (r(t)w^\Delta(t))^\Delta \\ &\quad - G(p) [G(r(\beta(t))w^\Delta(\beta(t)))]^{-1} (r(\alpha(t))w^\Delta(\alpha(t)))^\Delta. \end{aligned} \tag{3.8}$$

Since $\lim_{t \rightarrow \infty} (r(t)w^\Delta(t))$ exists, then by using (H_6) in (3.6), we get

$$\int_{t_2}^{\infty} Q(t)G(\eta(\beta(t)))\Delta t < \infty,$$

a contradiction to (H_{10}) . Hence the proof of theorem is complete.

THEOREM 6. *Let $-1 \leq p_2 \leq p(t) \leq 0$. If (H_0) , (H_2) , (H_4) and (3.7) hold, then any solution of (H) is either oscillatory or tends to zero as $t \rightarrow \infty$.*

Proof. Proceeding as in the proof of Theorem 4, we obtain $w(t) < 0$ for $t \geq t_2 > t_1$ when $w^\Delta(t) < 0$. Hence, $w(t)$ is monotonic function on $[t_2, \infty)_{\mathbb{T}}$ and $0 \neq \lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} z(t)$ exists. Following the argument in Theorem 2, we obtain $\lim_{t \rightarrow \infty} y(t) = 0$.

Assume that $w^\Delta(t) > 0$ for $t \geq t_1$. If $w(t) < 0$ for $t \geq t_2$ for some $t_2 > t_1$, then by using same arguments as in Theorem 2, we obtain $\lim_{t \rightarrow \infty} y(t) = 0$. Suppose that $w(t) > 0$ for $t \geq t_2 > t_1$. Then there exists a constant $\gamma > 0$ and $t_3 > t_2$ such that $w(t) \geq \gamma$ for $t \geq t_3$. Consequently, $y(t) \geq w(t) \geq \gamma$ for $t \geq t_3$. Integrating (3.4) from for $t_4 \geq t_3$ to ∞ , yields

$$\int_{t_4}^{\infty} q(s)\Delta s < \infty,$$

which is a contradiction (3.7). Hence the theorem is proved.

THEOREM 7. *Let $-\infty \leq p_3 \leq p(t) \leq p_4 < -1$. If (H_0) , (H_2) , (H_4) and (3.7) hold, then every bounded solution of (H) is either oscillatory or tends to zero as $t \rightarrow \infty$.*

Proof. The proof of the theorem is follows from Theorem 3 and Theorem 6.

4. Oscillation properties of (NH)

This section is concerned with the oscillatory and asymptotic behaviour of solutions of equation (NH) for suitable forcing functions $f(t)$. We restrict our forcing functions to those that change signs. We will use the following condition:

(H₁₁) There exists $F \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $rF^\Delta \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $(rF^\Delta)^\Delta = f$ and

$$-\infty < \liminf_{t \rightarrow \infty} F(t) < 0 < \limsup_{t \rightarrow \infty} F(t) < \infty.$$

THEOREM 8. *Let $0 \leq p(t) < p < \infty$. Assume that If (H_0) , (H_3) - (H_5) and (H_{11}) hold. If*

(H₁₂)

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t Q(s)G(F(\beta(s)))\Delta s = +\infty \text{ and } \liminf_{t \rightarrow \infty} \int_{t_0}^t Q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then every bounded solution of (NH) is oscillates.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (NH) on $[t_0, \infty)_{\mathbb{T}}$ so that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ and $y(\alpha(\beta(t)))$ are all positive on $[t_1, \infty)_{\mathbb{T}}$, for some $t_1 \geq t_0$. With $z(t)$, $k(t)$ and $w(t)$ as in (3.1)-(3.3), let

$$v(t) = w(t) - F(t) = z(t) - k(t) - F(t) \tag{4.1}$$

for $t \geq t_1$. Then (NH) becomes

$$\left(r(t)v^\Delta(t) \right)^\Delta = -q(t)G(y(\beta(t))) \leq 0. \tag{4.2}$$

Thus, $v(t)$ and $v^\Delta(t)$ are monotonic on $[t_2, \infty)_{\mathbb{T}}$, for some $t_2 > t_1$. Suppose that $v^\Delta(t) < 0$ for $t \geq t_1$. If $v(t) < 0$ for $t \geq t_2 > t_1$, then $z(t) < k(t) + F(t)$. Hence,

$$\begin{aligned} 0 &= \liminf_{t \rightarrow \infty} z(t) \leq \liminf_{t \rightarrow \infty} (k(t) + F(t)) \\ &\leq \limsup_{t \rightarrow \infty} k(t) + \liminf_{t \rightarrow \infty} F(t) \\ &= \liminf_{t \rightarrow \infty} k(t) + \liminf_{t \rightarrow \infty} F(t) < 0, \end{aligned}$$

is a contradiction to the fact that $z(t) > 0$. Thus, $v(t) > 0$ for $t \geq t_2$, that is, $z(t) > k(t) + F(t) > F(t)$ for $t \geq t_2$. In view of (NH), (H₃), and (H₄) it is easy see that

$$\begin{aligned} 0 &= (r(t)v^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))v^\Delta(\alpha(t)))^\Delta + q(t)G(\beta(t)) \\ &\quad + G(p_1)q(\alpha(t))G(\beta(\alpha(t))) \\ &\geq (r(t)v^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))v^\Delta(\alpha(t)))^\Delta + \lambda Q(t)G(z(\beta(t))) \\ &\geq (r(t)v^\Delta(t))^\Delta + G(p_1) (r(\alpha(t))v^\Delta(\alpha(t)))^\Delta + \lambda Q(t)G(F(\beta(t))), \end{aligned} \tag{4.3}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. We note that $\lim_{t \rightarrow \infty} v(t)$ exists. If $y(t)$ is unbounded, then

$$v(t) = z(t) - k(t) - F(t) > y(t) - F(t) - k(t)$$

implies that $v(t)$ is unbounded. Thus, $y(t)$ is bounded on $[t_4, \infty)_{\mathbb{T}}$, $t_4 > t_3$, that is, $\lim_{t \rightarrow \infty} (r(t)v^\Delta(t))^\Delta$ exists. Integrating the inequality (4.3), we obtain

$$\limsup_{t \rightarrow \infty} \int_{t_4}^t Q(s)G(F(\beta(s)))\Delta s < \infty$$

contradicting (H₁₂).

Next, we suppose that $v^\Delta(t) > 0$ for $t \geq t_1$. Then $\lim_{t \rightarrow \infty} (r(t)v^\Delta(t))$ exists. Similar contradictions hold for the case $v(t) > 0$ and $v(t) < 0$ for $t \geq t_2 > t_1$.

THEOREM 9. *Let $0 \leq p(t) \leq p_1 < \infty$. If (H₀), (H₁), (H₃)-(H₅), (H₇) and (H₁₁) hold, then (NH) is oscillatory.*

Proof. Proceeding as in the proof of Theorem 8, $v(t) < 0$ is not possible when $v^\Delta(t) < 0$ for $t \geq t_1$. Hence $v(t) > 0$, for some $t \geq t_2$ for some $t_2 > t_1$. By using Lemma 1 (ii) with $u(t)$ is replaced by $v(t)$, we get $v(t) \geq -r(t)v^\Delta(t)R(t)$ for $t \geq t_2$ and hence

$$\begin{aligned} z(t) &\geq -r(t)v^\Delta(t)R(t) + k(t) + F(t) \\ &\geq -r(t)v^\Delta(t)R(t) + k(t) + F^+(t) \\ &> -r(t)v^\Delta(t)R(t), \end{aligned}$$

for $t \geq t_2$, where $F^+(t) = \max\{F(t), 0\}$. Further, $r(t)v^\Delta(t)$ is nondecreasing, so we can find a constant $c > 0$ and $t_3 > t_2$ such that $-(r(t)v^\Delta(t))^\Delta \geq -c$ for $t \geq t_3$. Hence, inequality (3.5) becomes

$$-\lambda Q(t)G(-c)G(R(\beta(t))) \leq -\left(r(t)v^\Delta(t)\right)^\Delta - G(p_1)\left(r(t)v^\Delta(t)\right)^\Delta, \tag{4.4}$$

where $w(t)$ is replaced by $v(t)$ for $t \geq t_4 > t_3$. Since $\lim_{t \rightarrow \infty} v(t)$ exists, we claim that $y(t)$ is bounded. Otherwise, following the same argument as in Theorem 8, $v(t)$ is

unbounded. Hence, $\lim_{t \rightarrow \infty} (r(t)v^\Delta(t))^\Delta$ exists. Integrating the inequality (4.4) from t_4 to ∞ , we obtain

$$\limsup_{t \rightarrow \infty} \int_{t_4}^t Q(s)G(\beta(s))\Delta s < \infty,$$

a contradiction to (H_7) .

Let $v^\Delta(t) > 0$ for $t \geq t_1$. The argument for the case $v(t) < 0$ for $t \geq t_2 > t_1$ is same as mentioned in Theorem 8. Hence, $v(t) < 0$ for $t \geq t_2 > t_1$. By Lemma 1 (i), it follows that $v(t) \geq KR(t)$, that is, $z(t) > KR(t) + k(t) + F^+(t) > KR(t)$ for $t \geq t_2$. Using same type of reasoning as in the proof of Theorem 1, we obtain a contradiction to (H_7) . This completes the proof of theorem.

THEOREM 10. *Let $0 \leq p(t) \leq p_1 < \infty$. If (H_0) , (H_2) - (H_4) , (H_9) and (H_{11}) hold, then every solution of (NH) is oscillates.*

Proof. Proceeding as in the proof of Theorem 8, we assume that $v^\Delta(t) < 0$ for $t \geq t_1$. So, $v(t) < 0$ for $t \geq t_2 > t_1$ due to (H_2) . Using the same type of argument as in the proof of Theorem 8, $v(t) < 0$ is a contradiction. Thus, $v^\Delta(t) > 0$ for $t \geq t_1$. Hence, $v(t) > 0$ for $t \geq t_2 > t_1$. Since, $v(t)$ is nondecreasing, there exists a constant $\alpha > 0$ and $t_3 > t_2$ such that $v(t) \geq \alpha$, for $t \geq t_3$. Thus,

$$z(t) > \alpha + k(t) + F(t) \geq \alpha + k(t) + F^+(t) > \alpha,$$

for $t \geq t_3$, where $F^+(x) = \max\{F(t), 0\}$. Using the last inequality and then integrating (4.3) from t_4 to ∞ , we get

$$\int_{t_4}^t Q(t)\Delta t < \infty,$$

for $t_4 \geq t_3$, a contradiction to (H_9) . This completes the proof of the theorem.

THEOREM 11. *Let $-1 < p_2 \leq p(t) \leq 0$. Assume that If (H_0) , (H_4) , (H_{11}) and*

$$(H_{13}) \quad \int_{t_4}^\infty q(t)G(F^+(y(\beta(t))))\Delta t = \infty,$$

hold. Then any solution $y(t)$ of (NH) is either oscillatory or satisfies $\limsup_{t \rightarrow \infty} |y(t)| = 0$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (NH) on $[t_0, \infty)_{\mathbb{T}}$ so that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ and $y(\alpha(\beta(t)))$ are all positive on $[t_1, \infty)_{\mathbb{T}}$ for some $t_1 \geq t_0$. With $z(t), k(t)$ and $w(t)$ as in (3.1)-(3.2) and (4.1), we get (4.2). Thus, $v(t)$ and $v^\Delta(t)$ are monotonic on $[t_2, \infty)_{\mathbb{T}}$, for some $t_2 > t_1$. Let $v^\Delta(t) < 0$ for $t \geq t_1$. Hence, $\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} (z(t) - F(t))$ implies that $z(t) - F(t) < 0$ when $v(t) < 0$, that is, $\liminf_{t \rightarrow \infty} z(t) = -\infty$. So $\limsup_{t \rightarrow \infty} y(t) = +\infty$.

Next, we assume that $v(t) > 0$ for $t \geq t_2 > t_1$. Hence, $\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} (z(t) - k(t) - F(t))$ implies that $z(t) - F(t) > 0$. If $\lim_{t \rightarrow \infty} v(t) \neq 0$, then $z(t) > F(t)$ for $t \geq t_2$. Hence, $y(t) > F^+(t)$ for $t \geq t_3 > t_2$. We claim that $y(t)$ is bounded. If not, there exists an increasing sequence $\{\tau_n\}_{n=1}^\infty \subset [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \rightarrow \infty$, $x(\tau_n) \rightarrow \infty$ as $n \rightarrow \infty$, and

$$x(\tau_n) = \max \{x(t) : t_3 \leq t \leq \tau_n\}.$$

We may choose n large enough so that $\alpha(\tau_n) \geq t_2$. Hence,

$$\begin{aligned} v(\tau_n) &\geq y(\tau_n) + py(\alpha(\tau_n)) - k(\tau_n) - F(\tau_n) \\ &\geq (1 + p_2)y(\tau_n) - k(\tau_n) - F(\tau_n) \end{aligned}$$

implies that $v(\tau_n) \rightarrow \infty$ as $n \rightarrow \infty$, a contradiction to the fact that $\lim_{t \rightarrow \infty} v(t)$ exists. So our claim holds and $\lim_{t \rightarrow \infty} (r(t)v^\Delta(t))$ exists. Integrating (4.2) from t_3 to ∞ , we obtain

$$\int_{t_3}^\infty q(t)G(F^+(y(\beta(t))))\Delta t < \infty,$$

a contradiction to (H_{13}) . If $\lim_{t \rightarrow \infty} v(t) = 0$, then $z(t) - F(t) > 0$ or $z(t) - F(t) < 0$ for all $t \geq t_2$. If $z(t) - F(t) > 0$ for $t \geq t_2$, then we have a contradiction where as if $z(t) - F(t) < 0$ for $t \geq t_2$, then $\limsup_{t \rightarrow \infty} y(t) = +\infty$.

Assume that $v^\Delta(t) > 0$ for $t \geq t_1$. Then $\lim_{t \rightarrow \infty} (r(t)v^\Delta(t))$ exists. Proceeding as the proof above, we obtain a contradiction for case $v(t) > 0$. If $v(t) < 0$, then $\limsup_{t \rightarrow \infty} y(t) = +\infty$. This completes the proof of theorem.

THEOREM 12. *Let $-\infty < p_3 \leq p(t) \leq -1$. If all conditions of Theorem 11 hold, then every bounded solution of (NH) is either oscillatory or satisfies $\limsup_{t \rightarrow \infty} |y(t)| = 0$.*

The proof follows from Theorem 11 and hence details are omitted.

EXAMPLE 1. For $\mathbb{T} = \mathbb{R}$, we consider the equation

$$\begin{aligned} (y(t) + e^{-t}y(t - \pi))'' + (1 + e^{-t})y(t - 2\pi) \\ - e^{-t}(1 + \sin^2 t) \frac{y(t - 4\pi)}{1 + y^2(t - 4\pi)} = 2e^{-t} \sin t, \end{aligned} \quad (4.5)$$

where,

$$p(t) = e^{-t}, \quad q(t) = e^{-t} + 1, \quad h(t) = e^{-t}(1 + \sin^2 t).$$

If we choose $F(t) = e^{-t} \cos t$, then $(r(t)F'(t))' = 2e^{-t} \sin t = f(t)$. Clearly, (H_0) , (H_2) - (H_4) , (H_9) , and (H_{11}) are satisfied. Hence by Theorem 10, every solution of (4.5) are oscillatory. In particular, $y(t) = \sin t$ is such an oscillatory solution of (4.5).

EXAMPLE 2. For $\mathbb{T} = \mathbb{Z}$, consider the equation

$$\Delta(e^n \Delta(y(n) + p(n)y(n-1))) + q(n)y(n-2) - h(n) \frac{y(n-4)}{1+y^2(n-4)} = (-1)^n e^n, \quad n \geq 0, \quad (4.6)$$

where

$$p(n) = 2 + (-1)^n, \quad q(n) = (2e+3)e^n - e^{-n}, \quad h(n) = 2e^{-n}.$$

Indeed,

$$Q(n) = (2e+3)e^{n-1} - e^{-(n-1)}, \quad R(n) = \frac{e}{e-1} e^{-n}.$$

If we choose $F(n) = [2(1+e)^{-1}(-1)^n]$, then $f(n) = \Delta(e^n \Delta F(n))$. Clearly, (H_0) , (H_1) , (H_3) - (H_5) , and (H_{11}) are satisfied. Moreover, (H_7) is given by

$$\begin{aligned} \sum_{n=0}^{\infty} Q(n)G(R(n-2)) &= \sum_{n=0}^{\infty} \left[(2e+3)e^{n-1} - e^{-(n-1)} \right] \frac{e}{e-1} e^{-(n-2)} \\ &= \frac{e^2(2e+3)}{e-1} \sum_{n=0}^{\infty} 1 - \frac{e^4}{e-1} \sum_{n=0}^{\infty} e^{-2n} = \infty. \end{aligned}$$

Hence, by Theorem 9, equation (4.6) oscillates. In particular, $y(n) = (-1)^n$ is one of such solution.

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REFERENCES

- [1] R. P. Agarwal, M. Bohner, T. Li, C. Zhang, *A new approach in the study of oscillatory behaviour of even-order neutral delay differential equations*, Appl. Math. Comput. **225** (2013), 787–794.
- [2] D. K. Anderson and A. Zafer, *Nonlinear oscillation of second order dynamic equations on time scales*, Appl. Math. Lett. **22** (2009), 1591–1597.
- [3] D. D. Bainov, D.P. Mishev, *Oscillation Theory for Neutral Differential Equation with Delay*, Hilger, Bristol, 1991.
- [4] M. Bohner, *Dynamic Equations on Time Scales*, Birkhäuser, 2001.
- [5] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scale*, Birkhäuser, 2002.
- [6] L.H. Erbe, Q. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equation*, Dekker, NewYork, 1995.
- [7] I. Gyori, G. Ladas, *Oscillation Theory for Delay Differential Equations with Applications*, Oxford Univ. Press, London, 1991.
- [8] S. Hilger, *Analysis on measure chainis: a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.
- [9] B. Karpuz, J. V. Manojlovic, O. Ocalan, Y. Shoukaku, *Oscillation criteria for a class of second-order neutral delay differential equations*, Appl. Math. Comput. **210** (2009), 303–312.
- [10] I. Kubiacyzyk, S. H. Saker, A. Sikorska-Nowak, *Oscillation criteria for nonlinear neutral functional dynamic equations on time scales*, Math. Slovaca. **63** (2013), 263–290.
- [11] T. Li, Y. V. Rogovvchenko, *Oscillation theorems for second-order nonlinear neutral delay differential equations*, Abst. Appl. Anal. 2014 (2014), Article ID 594190, 5pages.

- [12] S. Panigrahi and P.R. Reddy, *On oscillatory fourth order nonlinear neutral delay dynamic equations*, Comput. Math. Appl. **62** (2011), 4258–4271.
- [13] H. Qin, N. Shang, Y. Lu, *A note on oscillation criteria of second order nonlinear neutral delay differential equations*, Comput. Math. Appl. **56** (2008), 2987–2992.
- [14] S. H. Saker and D. O'Regan, *New oscillation criteria for second-order neutral dynamic equations on time scales via Riccati substitution*, Hiroshima Math. J. **42** (1) (2012), 109–122.
- [15] E. Thandapani, V. Piramanantham and S. Pinelas, *Oscillation criteria for second-order neutral delay dynamic equations with mixed nonlinearities*, Advances in Difference Equations, **2011**, Article ID594190, 5pages.
- [16] E. Thandapani and V. Piramanantham, *Oscillation criteria for second order nonlinear neutral dynamic equations on time scales*, Tamkang J. Math. **43** (2012), 109–122.
- [17] A. K. Tripathy, *Oscillation properties of a class of neutral differential equations with positive and negative coefficients*, Fasciculi Mathematici, **45** (2010), 133–155.
- [18] A. K. Tripathy, *Oscillatory behaviour of a class of nonlinear second order mixed difference equations*, Elect. J. Qual. Theory of Diff. Eqs. **48** (2010), 1–19.
- [19] Q. Yang, Z. Xu, *Oscillation criteria for second order quasilinear neutral delay differential equations on time scales*, Comput. Math. Appl. **62** (2011), 3682–3691.
- [20] Shao-Yan Zhang and Qi-Ru Wang, *Oscillation criteria for second-order nonlinear dynamic equations on time scales*, Abstract and Applied Analysis, **2012** (2012), Article ID 743469, 20 pages.

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