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GLOBAL WELLPOSEDNESS TO THE INCOMPRESSIBLE MHD EQUATIONS WITH SOME LARGE INITIAL DATA

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Abstract. In this paper, we mainly study the global wellposedness for the n-dimensional homogeneous and nonhomogeneous incompressible magnetohydrodynamic equations in the critical Besov spaces. By fully using the advantage of weighted function generated by heat kernel and Fourier localization technique, we first get the global wellposedness for the homogeneous incompressible MHD equations with initial data under a nonlinear smallness hypothesis. It is amazing that we can exhibit an initial data satisfying that nonlinear smallness assumption, despite each component of the initial data could be arbitrarily large. Then, as an application of our global well-posedness, we also extend our result to the inhomogeneous incompressible MHD equations.

1. Introduction

In this paper, we consider the following incompressible MHD equations:

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{1}{Re}\Delta u+u\cdot\nabla u+\nabla P-b\cdot\nabla b=0, & \\ \dfrac{\partial b}{\partial t}-\dfrac{1}{Rm}\Delta b+u\cdot\nabla b-b\cdot\nabla u=0, & (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \\ \nabla\cdot u=\nabla\cdot b=0, & \\ u(x,0)=u_0,\ b(x,0)=b_0, & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where u is the velocity field, b is the magnetic field, $P(x,t)$ is the scalar pressure, u_0 and b_0 are the initial velocity field and the initial magnetic field respectively. $Re>0$ is the viscosity coefficient and $Rm>0$ is the magnetic diffusive coefficient which we will assume $Re=Rm=1$ for convenience. The MHD equations are a well-known model which governs the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas, liquid metals, and salt water, etc.

This model has been studied by many mathematicians and made more progress in the past years due to its importance, see [5, 8, 10, 14, 15, 21, 22], [29]-[32], [35]-[39], [42]–[44]. Briefly, Duvaut and Lions [14] established the local existence and uniqueness of solution in the classical Sobolev space $H^s(\mathbb{R}^n), s\geqslant n,$ they also proved

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the global existence of solutions to this system with small initial data. Sermange and Temam [37] proved the global unique solution in $\mathbb{R}^2$. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu [5] proved that such a system is globally well-posed for any data in $H^2(\mathbb{R}^2)$. In a recent remarkable paper Lin, Xu and Zhang [30] or Zhang [43] proved the global existence of smooth solution of the 2-D MHD system around the trivial solution $(x_2,0)$ (see [29] for 3-D case). In [35], Ren, Wu, Xiang and Zhang got the global existence and the decay estimates of small smooth solution for the 2-D MHD equations without magnetic diffusion. Fefferman, McCormick, Robinsonb and Rodrigo [15] established the local-in-time existence and uniqueness of strong solutions in H^s for $s > \frac{n}{2}$ to the viscous, non-resistive MHD equations in $\mathbb{R}^2$ or $\mathbb{R}^3$ by using a new commutator estimate. Chemin, McCormick, Robinsonb and Rodrigo [8] lately extended the result of [15] to some Besov spaces. We also emphasize the partial regularity theory and blowup criteria in [10, 21, 44] (see also the references therein).

By critical, we mean that we want to solve the system (1.1) in functional spaces with invariant norms by the changes of scales which leaves (1.1) invariant. In the case of incompressible MHD fluids, it is easy to see that the transformations: $(u_\lambda, b_\lambda)(t,x) = (\lambda u(\lambda^2\cdot,\lambda\cdot),\lambda b(\lambda^2\cdot,\lambda\cdot))$ have that property, provided that the pressure term has been changed accordingly.

When the magnetic field $b(x,t)$ is identically equal to zero, that is, in the case of the incompressible Navier-Stokes equations, there have been lots of results, see [1, 4, 6, 7, 16, 17, 19, 25, 26, 27, 28, 40]. The well-posedness and the ill-posedness of the initial value problem for (NS) have been considered by many mathematicians in the series of scaling invariant spaces $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$, with $3 < p < \infty$; see for example, Fujita and Kato [16], Kato [26], Kozono and Yamazaki [28], Koch and Tataru [27], Germain [19], Bourgain and Pavlović [4] and Yoneda [40].

Inspired by the work [7], [23], [32], [33], [39], [42], our aim in this paper is to go beyond the smallness condition on the initial data and to exhibit arbitrarily large initial data in critical Besov spaces which generate a unique, global solution. We can construct an example of a family of initial data with very large critical Besov norm which satisfies the nonlinear smallness hypothesis. Before giving our main results, we make some transforms for (1.1).

Let $W^+ = u+b$, $W^- = u-b$, then we change (1.1) into

$$\begin{cases} \partial_t W^+ - \Delta W^+ + W^- \cdot \nabla W^+ + \nabla P = 0, \\ \partial_t W^- - \Delta W^- + W^+ \cdot \nabla W^- + \nabla P = 0, \\ \nabla \cdot W^+ = \nabla \cdot W^- = 0, \\ W^+(x,0) = W_0^+ = u_0 + b_0, \\ W^-(x,0) = W_0^- = u_0 - b_0. \end{cases} \tag{1.2}$$

In the following, we shall spit the solutions (W^+,W^-) to (1.2) as

$$W^+ = W_F^+ + \overline{W}^+, \quad W^- = W_F^- + \overline{W}^- \tag{1.3}$$

with

$$W_F^+ = e^{t\Delta} W_0^+, \quad W_F^- = e^{t\Delta} W_0^-, \tag{1.4}$$

and $(\overline{W}^+, \overline{W}^-)$ satisfying

$$\begin{cases} \partial_t \overline{W}^+ - \Delta \overline{W}^+ + \overline{W}^- \cdot \nabla W_F^+ + \overline{W}^- \cdot \nabla \overline{W}^+ + W_F^- \cdot \nabla \overline{W}^+ + W_F^- \cdot \nabla W_F^+ + \nabla P = 0, \\ \partial_t \overline{W}^- - \Delta \overline{W}^- + \overline{W}^+ \cdot \nabla W_F^- + \overline{W}^+ \cdot \nabla \overline{W}^- + W_F^+ \cdot \nabla \overline{W}^- + W_F^+ \cdot \nabla W_F^- + \nabla P = 0, \\ \nabla \cdot \overline{W}^+ = \nabla \cdot \overline{W}^- = 0, \\ \overline{W}^+(x,0) = 0, \quad \overline{W}^-(x,0) = 0. \end{cases} \tag{1.5}$$

Now we can state our main results in this paper:

THEOREM 1. *Let $n \geqslant 3$, $1 < p < 2n$, for any $(u_0, b_0) \in \dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$, $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, there exist two positive constants c_0 and C_0 such that if:*

$$C_0(\|W_F^- \cdot \nabla W_F^+\|_{L^1(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})} + \|W_F^+ \cdot \nabla W_F^-\|_{L^1(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})}) \\ \times \exp\{C_0(\|W_0^+\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} + \|W_0^-\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}})\} \leqslant c_0, \tag{1.6}$$

then (1.1) *has a global solution*

$$(u,b) \in C([0,+\infty);\dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)) \cap \widetilde{L}^\infty((0,+\infty);\dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)) \cap L^1((0,+\infty);\dot{B}_{p,1}^{1+\frac{n}{p}}(\mathbb{R}^n)), \tag{1.7}$$

and

$$\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} + \|b\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})} + \|b\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})} \\ \lesssim (1 + \|W_F^- \cdot \nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})} + \|W_F^+ \cdot \nabla W_F^-\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}) \\ \times \exp\{C(\|W_0^+\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} + \|W_0^-\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}})\}, \tag{1.8}$$

where $W_0^+ = u_0 + b_0$, $W_0^- = u_0 - b_0$, $W_F^+ = e^{t\Delta} W_0^+$, and $W_F^- = e^{t\Delta} W_0^-$.

REMARK 1. It should be mentioned that, very recently, He, Huang and Wang [22] in $L^3(\mathbb{R}^3)$ (when $Re = Rm$) and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ (when $Re \neq Rm$) respectively proved that if the difference between the magnetic field and the velocity is small initially, there exists a global strong solution without smallness restriction on the size of initial velocity or magnetic field. Noticing that the embedding relation $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ and the method here used which can be easily verified is still valid when $Re \neq Rm$, our global wellposedness thus can be regarded as an extension of [22]. Moreover, our result also implies the positive answer to the open problem given by Remark 2.6 in [22].

REMARK 2. We emphasize that for any $n\geqslant 3$, $p\in(n,2n)$, $\alpha,\varepsilon\in(0,1)$ and ϕ,ψ be in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, our result implies the global wellposedness of (1.1) with initial data of the form

$$W^+_{0,\varepsilon}(x)=W^-_{0,\varepsilon}(x)=\omega_{0,\varepsilon}(x)+v_{0,\varepsilon}(x),$$

which

$$\omega_{0,\varepsilon}(x)\overset{\text{def}}{=}\frac{(-\log\varepsilon)^{\frac14}}{\varepsilon^{1-\frac{n}{p}+\frac{\alpha}{p}-\alpha}}(0,\cdots,0,\partial_2\cdots\partial_{n-2}\partial_n\psi^\varepsilon,-\partial_2\cdots\partial_{n-1}\psi^\varepsilon),$$

$$v_{0,\varepsilon}(x)\overset{\text{def}}{=}\frac{(-\log\varepsilon)^{\frac14}}{\varepsilon^{1-\frac{n}{p}}}(\partial_2\cdots\partial_{n-1}\phi_\varepsilon,\cdots,\partial_1\cdots\partial_{n-3}\partial_{n-1}\phi_\varepsilon,-(n-2)\partial_1\cdots\partial_{n-2}\phi_\varepsilon,0),$$

where

$$\phi_\varepsilon(x)=\cos\left(\frac{x_n}{\varepsilon}\right)\phi(x),\quad \psi^\varepsilon(x)=\cos\left(\frac{x_1}{\varepsilon}\right)\psi(x_1,\cdots,x_{n-2},\frac{x_{n-1}}{\varepsilon^\alpha},x_n). \tag{1.9}$$

Indeed, we can get by a similar computation of [7] that

$$\|e^{t\Delta}W^+_{0,\varepsilon}\cdot\nabla e^{t\Delta}W^-_{0,\varepsilon}\|_{L^1(\mathbb{R}^+;\dot B^{\frac{n}{p}-1}_{p,1})}\leqslant C\varepsilon^\gamma(-\log\varepsilon)^{\frac12}, \tag{1.10}$$

with $\gamma=(2n-\alpha)(\frac1p-\frac1{2n})>0$. Yet

$$C^{-1}(-\log\varepsilon)^{\frac14}\leqslant\|((W^+_{0,\varepsilon})_i,(W^-_{0,\varepsilon})_i)\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\leqslant C(-\log\varepsilon)^{\frac14},\quad i=1,\cdots,n. \tag{1.11}$$

Combining with the above remark, this class of large data allow the initial data u_0,b_0, even W_0^+,W_0^- can be arbitrarily large, which considerably improve the recent results [39] and [42].

As an application of our main Theorem 1.1, we also consider the following non-homogeneous incompressible MHD system [12]:

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\(\rho u)_t+\operatorname{div}(\rho u\otimes u)-\mu\Delta u+\nabla\Pi=b\cdot\nabla b,\\b_t-\nu\Delta b+u\cdot\nabla b-b\cdot\nabla u=0,\\\operatorname{div}u=0,\operatorname{div}b=0,\\(\rho,u,b)|_{t=0}=(\rho_0,u_0,b_0),\end{cases} \tag{1.12}$$

where ρ is the density and u is the velocity field, b is the magnetic field, $\Pi(x,t)$ is the scalar pressure, $\mu>0$ is the viscosity coefficient and $\nu>0$ is the magnetic diffusive coefficient. Moreover, in order to avoid vacuum regions, we will always suppose the initial density to satisfy

$$0<\rho^*\leqslant\rho_0(x),\quad x\in\mathbb{R}^n.$$

By applying maximum principle on the parabolic equation (1.12), one gets a priori that the density ρ (if it exists on the time interval $[0,T]$) keeps the same lower bound as the initial density ρ_0:

$$0<\rho^* \leqslant \rho(t,x), \quad \forall t\in[0,T],\ x\in\mathbb{R}^n.$$

We define $a=1/\rho-1$ and assume $\mu=\nu=1$ for convenience which allows us to work with the following system:

$$\begin{cases} \partial_t a + u\cdot\nabla a = 0, \\ \partial_t u + u\cdot\nabla u + (1+a)(\nabla\Pi - \Delta u) = (1+a)(b\cdot\nabla b), \\ \partial_t b - \Delta b + u\cdot\nabla b - b\cdot\nabla u = 0, \\ \mathrm{div} u = 0, \mathrm{div} b = 0, \\ (a,u,b)|_{t=0} = (a_0,u_0,b_0). \end{cases} \tag{1.13}$$

Compared with the homogeneous MHD equations, the nonhomogeneous incompressible MHD has been also extensively studied (see [2],[9],[13],[18],[20][24]). Gerbeau and Le Bris [18] (see also Desjardins and Le Bris [13]) got the global existence of weak solutions of finite energy in the whole space or in the torus. With the initial data are closed to a constant state, Abidi and Paicu [2] established the global strong solutions in the critical Besov spaces. Moreover, they allowed variable viscosity and conductivity coefficients. Chen, Tan, and Wang [9] extended the local existence in presence of vacuum by using the Galerkin method, energy method and the domain expansion technique. Lately , with initial data satisfies some compatibility conditions, by using a critical Sobolev inequality of logarithmic type, Huang and Wang [24] got the global strong solution to the 2-D nonhomogeneous incompressible MHD system, which improved all the previous results. Recently, Gui [20] studied the Cauchy problem of the 2-D magnetohydrodynamic system with inhomogeneous density and electrical conductivity. He showed that this system with a constant viscosity is globally well-posed for a generic family of the variations of the initial data and an inhomogeneous electrical conductivity. Moreover, he established that the system is globally well-posed in the critical spaces if the electrical conductivity is homogeneous. In this paper, we shall generalize the global results in [2], [41] to a more general case. We get the following theorem:

THEOREM 2. *Let q, p satisfy $1<q\leqslant p<2n$ be such that $\frac{1}{q}-\frac{1}{p}\leqslant\frac{1}{n}$. Suppose that $a_0\in\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)$, $(u_0,b_0)\in\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $\mathrm{div}\, u_0=\mathrm{div}\, b_0=0$ and (u_0,b_0) satisfies the condition (1.6). Moreover, there is a sufficiently small constant $c>0$ such that*

$$\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}} \leqslant c, \tag{1.14}$$

then the system (1.13) has a global solution $(a,u,b,\nabla\Pi)$ with

$$a\in C([0,+\infty);\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n))\cap\widetilde{L}^{\infty}(\mathbb{R}^+;\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n));\ \ \nabla\Pi\in L^1(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n));$$
$$(u,b)\in C([0,+\infty);\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap\widetilde{L}^{\infty}(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1(\mathbb{R}^+;\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)).$$

Before going on, we give an overview of the paper. In the second section, we shall collect some basic facts on Littlewood-Paley analysis and various product laws in Besov spaces; then in Section 3, we prove the global wellposedness of Theorem 1 . Finally in the last section, we present the proof of the global existence part of Theorem 2.

Notations : Let A, B be two operators, we denote $[A,B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leqslant Cb$. For X a Banach space and I an interval of $\mathbb{R}$, we denote by $C(I;X)$ the set of continuous functions on I with values in X. For $q\in[1,+\infty]$, the notation $L^q(I;X)$ stands for the set of measurable functions on I with values in X, such that $t\to\|f(t)\|_X$ belongs to $L^q(I)$. We always denote $(d_j)_{j\in\mathbb{Z}}$ is a generic element of $l^1(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}}d_j=1$.

2. Preliminaries

Let (χ,ϕ) be two smooth radial functions, $0\leqslant(\chi,\phi)\leqslant 1$, such that χ is supported in the ball $\mathscr{B}=\{\xi\in\mathbb{R}^n,|\xi|\leqslant\frac{4}{3}\}$ and φ is supported in the ring $\mathscr{C}=\{\xi\in\mathbb{R}^n,\frac{3}{4}\leqslant|\xi|\leqslant\frac{8}{3}\}$. Moreover, there holds

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\qquad\forall\xi\neq 0.$$

Let $h=\mathscr{F}^{-1}\varphi$ and $\widetilde{h}=\mathscr{F}^{-1}\chi$, then we define the dyadic blocks as follows:

$$\dot{\Delta}_jf=\varphi(2^{-j}D)f=2^{jn}\int_{\mathbb{R}^n}h(2^jy)f(x-y)dy,$$
$$\dot{S}_jf=\chi(2^{-j}D)f=2^{jn}\int_{\mathbb{R}^n}\widetilde{h}(2^jy)f(x-y)dy.$$

By telescoping the series, we thus have the following Littlewood-Paley decomposition

$$u=\sum_{j\in\mathbb{Z}}\dot{\Delta}_ju,\qquad\forall u\in\mathscr{S}'(\mathbb{R}^n)/\mathscr{P}[\mathbb{R}^n],$$

where $\mathscr{P}[\mathbb{R}^n]$ is the set of polynomials (see [34]). Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_k\dot{\Delta}_ju\equiv 0\quad\text{if } |k-j|\geqslant 2\quad\text{and}\quad\dot{\Delta}_k(\dot{S}_{j-1}u\dot{\Delta}_ju)\equiv 0\quad\text{if } |k-j|\geqslant 5.$$

Now we recall the definition of homogeneous Besov spaces.

DEFINITION 1. Let $(p,r)\in[1,+\infty]^2, s\in\mathbb{R}$ and $u\in\mathscr{S}_h'(\mathbb{R}^n)$, which means that $u\in\mathscr{S}'(\mathbb{R}^3)$ and $\lim_{j\to-\infty}\|\dot{S}_ju\|_{L^\infty}=0$ (see Definition 1.26 of [3]), we set

$$\|u\|_{\dot{B}^s_{p,r}}\triangleq(2^{js}\|\dot{\Delta}_ju\|_{L^p})_{l^r},$$

then we define $\dot{B}^s_{p,r}(\mathbb{R}^n)\triangleq\{u\in\mathscr{S}_h'(\mathbb{R}^n)\big|\|u\|_{\dot{B}^s_{p,r}}<\infty\}$.

REMARK 3. Let $1 \leqslant p,r \leqslant \infty, s \in \mathbb{R}$, and $u \in \mathscr{S}_h'(\mathbb{R}^n)$. Then u belongs to $\dot{B}^s_{p,r}$ if and only if there exists $\{d_{j,r}\}_{j\in Z}$ such that $\|d_{j,r}\|_{l^r} = 1$ and

$$\|\dot{\Delta}_j u\|_{L^p} \leqslant C d_{j,r} 2^{-js} \|u\|_{\dot{B}^s_{p,r}} \quad \text{for all } j \in \mathbb{Z}.$$

We are going to define the space of Chemin-Lerner (see[23]) in which we will work, which is a refinement of the space $L^\lambda_T(\dot{B}^s_{p,r}(\mathbb{R}^n))$.

DEFINITION 2. Let $s \leqslant \frac{n}{p}$, $(r,\lambda,p) \in [1,+\infty]^3$ and $T \in (0,+\infty]$. We define $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r}(\mathbb{R}^n))$ as the completion of $C([0,T];\mathscr{S}(\mathbb{R}^n))$ by the norm

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} = \left\{ \sum_{q\in\mathbb{Z}} 2^{rqs} \left(\int_0^T \|\dot{\Delta}_q f(t)\|^\lambda_{L^p} dt \right)^{\frac{r}{\lambda}} \right\}^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$.

REMARK 4. It is easy to observe that for $0 < s_1 < s_2$, $\theta \in [0,1]$, $p,r,\lambda,\lambda_1,\lambda_2 \in [1,+\infty]$, we have the following interpolation inequality in the Chemin-Lerner space (see[3]):

$$\|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \leqslant \|u\|^\theta_{\widetilde{L}^{\lambda_1}_T(\dot{B}^{s_1}_{p,r})} \|u\|^{(1-\theta)}_{\widetilde{L}^{\lambda_2}_T(\dot{B}^{s_2}_{p,r})}$$

with $\frac{1}{\lambda} = \frac{\theta}{\lambda_1} + \frac{1-\theta}{\lambda_2}$ and $s = \theta s_1 + (1-\theta)s_2$.

Let us emphasize that, according to the Minkowski inequality, we have

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \leqslant \|f\|_{L^\lambda_T(\dot{B}^s_{p,r})} \quad \text{if } \lambda \leqslant r, \qquad \|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \geqslant \|f\|_{L^\lambda_T(\dot{B}^s_{p,r})}, \quad \text{if } \lambda \geqslant r.$$

In order to prove main Theorem 1.1, we need to introduce the following weighted Chemin-Lerner type norm from [23, 33]:

DEFINITION 3. Let $f(t) \in L^1_{loc}(\mathbb{R}^+), f(t) \geqslant 0$. We define

$$\|u\|_{\widetilde{L}^q_{T,f}(\dot{B}^s_{p,r})} = \left\{ \sum_{j\in\mathbb{Z}} 2^{rjs} \left(\int_0^T f(t) \|\dot{\Delta}_j u(t)\|^q_{L^p} dt \right)^{\frac{r}{q}} \right\}^{\frac{1}{r}}$$

for $s \in \mathbb{R}, p \in [1,\infty], q,r \in [1,\infty)$, and with the standard modification for $q = \infty$ or $r = \infty$.

The following Bernstein's lemma will be repeatedly used throughout this paper.

LEMMA 1. *Let $\mathscr{B}$ be a ball and $\mathscr{C}$ a ring of $\mathbb{R}^n$. A constant C exists so that for any positive real number λ, any non-negative integer k, any smooth homogeneous* *function σ of degree m, and any couple of real numbers (a,b) with $1 \leqslant a \leqslant b$, there hold*

$$\begin{aligned} \operatorname{Supp}\hat{u} \subset \lambda \mathscr{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leqslant C^{k+1}\lambda^{k+n(1/a-1/b)}\|u\|_{L^a}, \\ \operatorname{Supp}\hat{u} \subset \lambda \mathscr{C} &\Rightarrow C^{-k-1}\lambda^k\|u\|_{L^a} \leqslant \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leqslant C^{k+1}\lambda^k\|u\|_{L^a}, \\ \operatorname{Supp}\hat{u} \subset \lambda \mathscr{C} &\Rightarrow \|\sigma(D)u\|_{L^b} \leqslant C_{\sigma,m}\lambda^{m+n(1/a-1/b)}\|u\|_{L^a}. \end{aligned}$$

On the other hand, it has been demonstrated that the Bony's decomposition [3] is very effective to deal with nonlinear problems. Here, we recall the Bony's decomposition in the homogeneous context:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),$$

where

$$\dot{T}_u v \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v, \;\; \dot{R}(u,v) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v, \;\; \textit{and} \;\; \widetilde{\dot{\Delta}}_j v \stackrel{\text{def}}{=} \sum_{|j'-j|\leqslant 1} \dot{\Delta}_{j'} v.$$

As an application of the above basic facts on Littlewood-Paley theory, we present the following product laws in Besov spaces.

LEMMA 2. *Let $1 \leqslant p,r \leqslant \infty$, $s_1 < \frac{n}{p}, s_2 < \frac{n}{p}$ with $s_1+s_2 > n\max(0,\frac{2}{p}-1)$ ($s_1 \leqslant \frac{n}{p}, s_2 \leqslant \frac{n}{p}$ if $r=1$). Assume that $a \in \dot{B}^{s_1}_{p,r}(\mathbb{R}^n), b \in \dot{B}^{s_2}_{p,r}(\mathbb{R}^n)$. Then $ab \in \dot{B}^{s_1+s_2-\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ and*

$$\|ab\|_{\dot{B}^{s_1+s_2-\frac{n}{p}}_{p,r}} \leqslant C\|a\|_{\dot{B}^{s_1}_{p,r}}\|b\|_{\dot{B}^{s_2}_{p,r}}.$$

The proof of this lemma is standard which can be found in [1], [33], we omit its proof here.

LEMMA 3. (Lemma 2.100 from Bahouri et al. (2011)). *Let $\sigma \in \mathbb{R}, 1 \leqslant r \leqslant \infty$, and $1 \leqslant p \leqslant p_1 \leqslant \infty$. Let v be a vector field over $\mathbb{R}^n$. Assume that*

$$\sigma > -n\min\{\frac{1}{p_1},\frac{1}{p'}\} \quad \text{or} \quad \sigma > -1-n\min\{\frac{1}{p_1},\frac{1}{p'}\} \quad \text{if} \quad \operatorname{div} v = 0. \tag{2.1}$$

Define $R_j \stackrel{\text{def}}{=} [v\cdot\nabla,\Delta_j]f$ (or $R_j \stackrel{\text{def}}{=} \operatorname{div}[v,\Delta_j]f$, if $\operatorname{div} v = 0$). There exists a constant C depending continuously on p, p_1, σ, and n, such that

$$\|(2^{j\sigma}\|R_j\|_{L^p})_j\|_{l^r} \leqslant C\|\nabla v\|_{B^{\frac{n}{p}}_{p_1,\infty}\cap L^\infty}\|f\|_{B^\sigma_{p,r}}, \quad \text{if} \quad \sigma < 1+\frac{n}{p_1}. \tag{2.2}$$

Further, if $\sigma > 0$ (or $\sigma > -1$, if $\operatorname{div} v = 0$) and $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$, then

$$\|(2^{j\sigma}\|R_j\|_{L^p})_j\|_{l^r} \leqslant C(\|\nabla v\|_{L^\infty}\|f\|_{B^\sigma_{p,r}} + \|\nabla v\|_{B^{\sigma-1}_{p_1,r}}\|\nabla f\|_{L^{p_2}}). \tag{2.3}$$

In the limit case $\sigma = -\min(\frac{n}{p_1}, \frac{n}{p'})$ *[or* $\sigma = -1-\min(\frac{n}{p_1}, \frac{n}{p'})$ *if* $\mathrm{div} v = 0$ *], we have*

$$\sup_{j\geqslant -1} 2^{j\sigma}\|R_j\|_{L^p} \leqslant C\|\nabla v\|_{B^{\frac{n}{p_1}}_{p_1,1}} \|f\|_{B^{\sigma}_{p,\infty}}. \tag{2.4}$$

REMARK 5. The estimates (2.2) – (2.4) are also valid in the homogeneous framework (i.e., with $\dot{\Delta}_j$ instead of Δ_j and with homogenous Besov norms instead of nonhomogeneous ones), provided

$$\sigma < \frac{n}{p}, \quad \text{or} \quad \sigma = \frac{n}{p} \quad \text{and} \quad r = 1.$$

Finally, we recall the solvability of the following Cauchy problem of the heat equation in the Chemin-Lerner type space:

$$\partial_t u - \mu\Delta u = f(x,t), \qquad u(x,0) = u_0(x). \tag{2.5}$$

LEMMA 4. *(see [3]) Let* $s \in \mathbb{R}, 1 \leqslant p, r, \rho \leqslant \infty$ *and* $0 < T \leqslant \infty$*. Assume that* $u_0 \in \dot{B}^s_{p,r}(\mathbb{R}^n)$ *and* $f \in \widetilde{L}^{\rho}_T(\dot{B}^{s-2+\frac{2}{\rho}}_{p,r}(\mathbb{R}^n))$*. Then the heat equation (2.5) has a unique solution* $u \in \widetilde{L}^{\infty}_T(\dot{B}^s_{p,r}(\mathbb{R}^n)) \cap \widetilde{L}^{\rho}_T(\dot{B}^{s+\frac{2}{\rho}}_{p,r}(\mathbb{R}^n))$*. In addition, there exists a constant* $C > 0$ *such that for all* $\rho_1 \in [\rho, \infty]$*, we have*

$$\mu^{\frac{1}{\rho_1}}\|u\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{s+\frac{2}{\rho_1}}_{p,r})} \leqslant C(\|u_0\|_{\dot{B}^s_{p,r}} + \mu^{\frac{1}{\rho}-1}\|f\|_{\widetilde{L}^{\rho}_T(\dot{B}^{s-2+\frac{2}{\rho}}_{p,r})}).$$

If furthermore r *is finite then* $u \in C([0,T], \dot{B}^s_{p,r}(\mathbb{R}^n))$*.*

3. Global wellposedness of Theorem 1

3.1. The estimate of the pressure

In this subsection, we will give the estimate of the pressure function in the framework of weighted Chemin-Lerner type space. Taking divergence to the first equation of (1.5) yields that

$$-\Delta P = \mathrm{div}(\overline{W}^{-}\cdot\nabla W_F^{+} + \overline{W}^{-}\cdot\nabla\overline{W}^{+} + W_F^{-}\cdot\nabla\overline{W}^{+} + W_F^{-}\cdot\nabla W_F^{+}). \tag{3.1}$$

The following proposition concerning the estimate of the pressure .

PROPOSITION 3. Let $1 < p < 2n$ with $(W^+, W^-) \in \widetilde{L}^{\infty}_t(\dot{B}^{-1+\frac{n}{p}}_{p,1}) \cap L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})$. Denote

$$f(t) = \|W_F^{+}(t)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}} + \|W_F^{-}(t)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}},$$

and

$$P_\lambda = P\exp\{-\lambda\int_0^t f(\tau)d\tau\}, \quad \text{for} \quad \lambda>0,$$

and similar notations for $\overline{W}_\lambda^+$, $\overline{W}_\lambda^-$. Then (3.1) has a unique solution $\nabla P\in L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})$ and there holds

$$\begin{aligned}
&\|\nabla P_\lambda\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\\
&\quad\lesssim \varepsilon\|\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\overline{W}_\lambda^-\|_{L_{t,f}^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\overline{W}_\lambda^+\|_{L_{t,f}^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\\
&\qquad+\|\overline{W}^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|W_F^-\cdot\nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}.
\end{aligned}\tag{3.2}$$

Proof of Proposition 3. As both the existence and uniqueness parts of Proposition 3 basically follow from the uniform estimate (3.2) for approximate solutions of (3.1). For simplicity, we just prove (3.2) for smooth enough solutions of (3.1). Multiplying (3.1) by $\exp\{-\lambda\int_0^t f(\tau)d\tau\}$, we arrive at

$$\nabla P_\lambda=\nabla(-\Delta)^{-1}\mathrm{div}(\overline{W}_\lambda^-\cdot\nabla W_F^+ +\overline{W}^-\cdot\nabla\overline{W}_\lambda^+ + W_F^-\cdot\nabla\overline{W}_\lambda^+ +(W_F^-\cdot\nabla W_F^+)_\lambda).$$

Applying the operator $\dot{\Delta}_j$ to the above equation and taking $L_t^1(L^p)$-norm yield that

$$\begin{aligned}
\|\nabla P_\lambda\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\lesssim&\|\overline{W}_\lambda^-\cdot\nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}^-\cdot\nabla\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\\
&+\|W_F^-\cdot\nabla\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|(W_F^-\cdot\nabla W_F^+)_\lambda\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}.
\end{aligned}\tag{3.3}$$

By Lemma 1, 2 and Young's inequality, we have

$$\|(W_F^-\cdot\nabla W_F^+)_\lambda\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\leqslant C\|W_F^-\cdot\nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})},\tag{3.4}$$

$$\|\overline{W}^-\cdot\nabla\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\lesssim\|\overline{W}^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})},\tag{3.5}$$

$$\|\overline{W}_\lambda^-\cdot\nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\lesssim\int_0^t\|\overline{W}_\lambda^-\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}\|W_F^+\|_{\dot{B}_{p,1}^{1+\frac{n}{p}}}d\tau\lesssim\|\overline{W}_\lambda^-\|_{L_{t,f}^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})},\tag{3.6}$$

$$\begin{aligned}
\|W_F^-\cdot\nabla\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\lesssim&\int_0^t\|W_F^-\|_{\dot{B}_{p,1}^{\frac{n}{p}}}\|\overline{W}_\lambda^+\|_{\dot{B}_{p,1}^{\frac{n}{p}}}d\tau\\
\lesssim&\int_0^t\|W_F^-\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}^{1/2}\|W_F^-\|_{\dot{B}_{p,1}^{1+\frac{n}{p}}}^{1/2}\|\overline{W}_\lambda^+\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}^{1/2}\|\overline{W}_\lambda^+\|_{\dot{B}_{p,1}^{1+\frac{n}{p}}}^{1/2}d\tau\\
\lesssim&\,\varepsilon\|\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|W_F^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\overline{W}_\lambda^+\|_{L_{t,f}^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}.
\end{aligned}\tag{3.7}$$

Inserting the estimates (3.4)-(3.7) into (3.3), we can finally get (3.2) which implies the result of our proposition.

3.2. Complete the proof of Theorem 1.1

In fact, when $(u_0,b_0)\in \dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$, by the proof of local existence of Theorem 1.1 , there exists a positive time T so that the MHD equations have a unique solution (u,b) with

$$(u,b)\in C([0,T];\dot{B}_{p,1}^{-1+\frac{n}{p}})\cap\widetilde{L}^{\infty}([0,T];\dot{B}_{p,1}^{-1+\frac{n}{p}})\cap L^1([0,T];\dot{B}_{p,1}^{1+\frac{n}{p}}). \tag{3.8}$$

Denote T^* to be the largest time so that there holds (3.8), hence to prove Theorem 1.1, we only need to prove that $T^*=\infty$ and (1.7) holds .

Let $f(t)$, P_λ, $\overline{W}_\lambda^+$, $\overline{W}_\lambda^-$ be given by Proposition 3, it follows from the first equation of (1.5) that

$$\partial_t\overline{W}_\lambda^+ +\lambda f\overline{W}_\lambda^+ -\Delta\overline{W}_\lambda^+ +\overline{W}_\lambda^-\cdot\nabla W_F^+ +\overline{W}^-\cdot\nabla\overline{W}_\lambda^+ + W_F^-\cdot\nabla\overline{W}_\lambda^+ \\ +(W_F^-\cdot\nabla W_F^+)_\lambda+\nabla P_\lambda=0.$$

Applying the operator $\dot{\Delta}_j$ to the above equation and taking L^2 inner product with $|\dot{\Delta}_j\overline{W}_\lambda^+|^{p-2}\dot{\Delta}_j\overline{W}_\lambda^+$ (when $1<p<2$, we need to make some modification see [11]), we obtain

$$\frac{d}{dt}\|\dot{\Delta}_j\overline{W}_\lambda^+\|_{L^p}+\lambda f(t)\|\dot{\Delta}_j\overline{W}_\lambda^+\|_{L^p}+c_1 2^{2j}\|\dot{\Delta}_j\overline{W}_\lambda^+\|_{L^p} \\ \lesssim\|\dot{\Delta}_j(\overline{W}_\lambda^-\cdot\nabla W_F^+)\|_{L^p}+\|\dot{\Delta}_j(\overline{W}^-\cdot\nabla\overline{W}_\lambda^+)\|_{L^p}+\|\dot{\Delta}_j(W_F^-\cdot\nabla\overline{W}_\lambda^+)\|_{L^p} \\ +\|\dot{\Delta}_j((W_F^-\cdot\nabla W_F^+)_\lambda)\|_{L^p}+\|\dot{\Delta}_j(\nabla P_\lambda)\|_{L^p}.$$

By Proposition 3, we can get

$$\|\overline{W}_\lambda^+\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\lambda\|\overline{W}_\lambda^+\|_{L^1_{t,f}(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+c_1\|\overline{W}_\lambda^+\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})} \\ \lesssim\varepsilon\|\overline{W}_\lambda^+\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\overline{W}_\lambda^-\|_{L^1_{t,f}(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\overline{W}_\lambda^+\|_{L^1_{t,f}(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \\ +\|\overline{W}^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\overline{W}_\lambda^+\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{n}{p}})}. \tag{3.9}$$

Similar arguments as in deriving (3.9) can be used to conclude from the second equation of (1.5) that

$$\|\overline{W}_\lambda^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\lambda\|\overline{W}_\lambda^-\|_{L^1_{t,f}(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+c_2\|\overline{W}_\lambda^-\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})} \\ \lesssim\varepsilon(\|\overline{W}_\lambda^+\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\overline{W}_\lambda^-\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})})+\|\overline{W}^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\overline{W}_\lambda^+\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})} \\ +(1+\|W_F^+\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})})(\|\overline{W}_\lambda^-\|_{L^1_{t,f}(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}_\lambda^+\|_{L^1_{t,f}(\dot{B}_{p,1}^{-1+\frac{n}{p}})}) \\ +(\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^+\cdot\nabla W_F^-\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{n}{p}})}). \tag{3.10}$$

Summing up the estimates (3.9) and (3.10), choosing

$$\varepsilon \ \textit{small enough}, \quad \overline{C}=\min\{c_1,c_2\}$$

and

$$\lambda \geqslant C(1+\|W_F^+\|_{L^\infty(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^-\|_{L^\infty(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})}),$$

we have that

$$\begin{aligned}
&\|\overline{W}_\lambda^+\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}_\lambda^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\overline{C}(\|\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\overline{W}_\lambda^-\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})})\\
&\leqslant C(\|W_F^-\cdot\nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^+\cdot\nabla W_F^-\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})})\\
&+C_1(\|\overline{W}^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})})(\|\overline{W}_\lambda^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}). \quad (3.11)
\end{aligned}$$

Now let C_2 be a small enough positive constant, which will be determined later on, we define T^{**} by

$$\begin{aligned}
T^{**}=\sup\Big\{t\in[0,T^*):\|\overline{W}^+\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\\
+\|\overline{W}^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\overline{W}^-\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}\leqslant C_2\Big\}. \quad (3.12)
\end{aligned}$$

In what follows, we shall prove that $T^{**}=\infty$ under the assumption of (1.6). Otherwise, taking $C_1C_2\leqslant\frac{1}{2}\overline{C}$, we deduce from (3.11) that

$$\begin{aligned}
\|\overline{W}_\lambda^+\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}_\lambda^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\overline{C}(\|\overline{W}_\lambda^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\overline{W}_\lambda^-\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})})\\
\leqslant C(\|W_F^-\cdot\nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^+\cdot\nabla W_F^-\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}),
\end{aligned}$$

which gives rise to

$$\begin{aligned}
&\|\overline{W}^+\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\overline{W}^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\overline{C}(\|\overline{W}^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\overline{W}^-\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})})\\
&\quad\leqslant C(\|W_F^-\cdot\nabla W_F^+\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^+\cdot\nabla W_F^-\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})})\\
&\qquad\times\exp\Big\{C\int_0^t(\|W_F^+\|_{\dot{B}_{p,1}^{1+\frac{n}{p}}}+\|W_F^-\|_{\dot{B}_{p,1}^{1+\frac{n}{p}}})d\tau\Big\}. \quad (3.13)
\end{aligned}$$

Noting (1.4) and Lemma 4, we have

$$\begin{aligned}
\|W_F^+\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^-\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|W_F^+\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|W_F^-\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}\\
\lesssim\|W_0^+\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}+\|W_0^-\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}. \quad (3.14)
\end{aligned}$$

Therefore, we can deduce from (3.13) that

$$\begin{aligned}
&\|\overline{W}^+\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\overline{W}^-\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\overline{C}(\|\overline{W}^+\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\overline{W}^-\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})})\\
&\quad\leqslant C(\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|W_F^+\cdot\nabla W_F^-\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})})\\
&\qquad\times\exp\Big\{C(\|W_0^+\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}+\|W_0^-\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}})\Big\}.
\end{aligned}\tag{3.15}$$

In particular, (3.15) implies that if we take C_0 large enough and c_0 sufficiently small in (1.6), there holds

$$\|\overline{W}^+\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\overline{W}^-\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\overline{W}^+\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\overline{W}^-\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\leqslant\frac{C_2}{2}$$

for $t\leqslant T^{**}$, which contradicts with (3.12). Whence we conclude that $T^{**}=\infty$, and there holds

$$\begin{aligned}
&\|W^+\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|W^-\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\overline{C}(\|W^+\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|W^-\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})})\\
&\quad\lesssim(1+\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|W_F^+\cdot\nabla W_F^-\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})})\\
&\qquad\times\exp\Big\{C(\|W_0^+\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}+\|W_0^-\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}})\Big\},
\end{aligned}\tag{3.16}$$

which and (3.14) also imply that

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|b\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\overline{C}(\|u\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|b\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})})\\
&\quad\lesssim(1+\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|W_F^+\cdot\nabla W_F^-\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})})\\
&\qquad\times\exp\Big\{C(\|W_0^+\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}+\|W_0^-\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}})\Big\}.
\end{aligned}\tag{3.17}$$

Consequently, we complete the proof our main Theorem 1.

4. Proof of the Theorem 2

The strategy to the proof of Theorem 2 is to seek a solution of (1.13) with the form $u=u_R+\bar{u}$, $b=b_R+\bar{b}$, $\nabla\Pi=\nabla\Pi_R+\nabla\overline{\Pi}$ with $(u_R,b_R,\nabla\Pi_R)$ solving the classical MHD system:

$$\begin{cases}
\partial_t u_R-\Delta u_R+u_R\cdot\nabla u_R+\nabla\Pi_R-b_R\cdot\nabla b_R=0,\\
\partial_t b_R-\Delta b_R+u_R\cdot\nabla b_R-b_R\cdot\nabla u_R=0,\\
\mathrm{div}u_R=0,\quad \mathrm{div}b_R=0,\\
u_R|_{t=0}=u_0,\quad b_R|_{t=0}=b_0,
\end{cases}\tag{4.1}$$

and $(a,\bar{u},\bar{b},\nabla\overline{\Pi})$ solving

$$\begin{cases}\partial_t a+(\bar{u}+u_R)\cdot\nabla a=0,\\ \partial_t\bar{u}+\bar{u}\cdot\nabla\bar{u}+u_R\cdot\nabla\bar{u}+\bar{u}\cdot\nabla u_R-(1+a)\Delta\bar{u}-a\Delta u_R+(1+a)\nabla\overline{\Pi}+a\nabla\Pi_R\\ \qquad\qquad -(1+a)(\bar{b}\cdot\nabla\bar{b}+\bar{b}\cdot\nabla b_R+b_R\cdot\nabla\bar{b})-a(b_R\cdot\nabla b_R)=0,\\ \partial_t\bar{b}-\Delta\bar{b}+\bar{u}\cdot\nabla\bar{b}+\bar{u}\cdot\nabla b_R+u_R\cdot\nabla\bar{b}-\bar{b}\cdot\nabla\bar{u}-\bar{b}\cdot\nabla u_R-b_R\cdot\nabla\bar{u}=0,\\ \mathrm{div}\bar{u}=0,\quad \mathrm{div}\bar{b}=0,\\ (a,\bar{u},\bar{b})|_{t=0}=(a_0,0,0).\end{cases}\tag{4.2}$$

From the Theorem 1 we have just proved, we can get the following estimate for (u_R,b_R) under the assumption (1.6):

$$\begin{aligned}&\|u_R\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|b_R\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|u_R\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|b_R\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}++\|\nabla\Pi_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\ &\quad\lesssim(1+\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|W_F^+\cdot\nabla W_F^-\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})})\\ &\qquad\times\exp\Big\{C(\|W_0^+\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}+\|W_0^-\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}})\Big\}.\end{aligned}\tag{4.3}$$

In what following, we mainly discuss the equation (4.2). In order to close the energy estimates, we will begin from the estimates of transport equation and pressure function.

4.1. The estimate of the transport equation

The goal of this section is to investigate the following free transport equation in the framework of weighted Chemin-Lerner type norm:

$$\partial_t a+u\cdot\nabla a=0,\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^n,\qquad a|_{t=0}=a_0,\quad x\in\mathbb{R}^n.\tag{4.4}$$

PROPOSITION 4. Let $p\geqslant q>1$ with $\frac{1}{q}-\frac{1}{p}\leqslant\frac{1}{n}$, and λ be a positive number. Let $u\in\widetilde{L}^\infty_T(\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))$ and $a_0\in\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)$. Denote

$$g(t)\triangleq\|u_R(t)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\quad\text{and}\quad a_\lambda\triangleq a\exp\Big\{-\lambda\int_0^t g(t')dt'\Big\}.$$

Then (4.4) has a unique solution $a\in C([0,T];\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n))$ satisfying

$$\begin{aligned}&\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}+(\lambda-C)\|a_\lambda\|_{\widetilde{L}^1_{t,g}(\dot{B}^{\frac{n}{q}}_{q,1})}\\ &\qquad\leqslant C(\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}}+\|\bar{u}\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\end{aligned}\tag{4.5}$$

for any $t\in(0,T]$ and λ large enough.

The proof of this proposition is similar to [33], here, we omit the details.

4.2. The estimate of the pressure

For $\lambda > 0$, we denote

$$f(t) = \|u_R(t)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}} + \|u_R(t)\|^2_{\dot{B}^{\frac{n}{p}}_{p,1}} + \|b_R(t)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}} + \|b_R(t)\|^2_{\dot{B}^{\frac{n}{p}}_{p,1}} + \|\nabla\Pi_R(t)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}},$$

and

$$a_\lambda = a\exp\left\{-\int_0^t \lambda f(t'))dt'\right\}, \quad \bar{u}_\lambda = \bar{u}\exp\left\{-\int_0^t \lambda f(t'))dt'\right\},$$

$$\bar{b}_\lambda = \bar{b}\exp\left\{-\int_0^t \lambda f(t'))dt'\right\}, \quad \overline{\Pi}_\lambda = \overline{\Pi}\exp\left\{-\int_0^t \lambda f(t'))dt'\right\}.$$

We now begin to estimate the pressure. Taking the divergence to the second equation of (1.13) and using $\mathrm{div} u = \mathrm{div} b = 0$, we have

$$\begin{aligned}\nabla\overline{\Pi} =&\nabla(-\Delta)^{-1}\mathrm{div}\Big[a\nabla\overline{\Pi} + \bar{u}\cdot\nabla\bar{u} + u_R\cdot\nabla\bar{u} + \bar{u}\cdot\nabla u_R - a\Delta\bar{u} - a\Delta u_R \\ &+ a\nabla\Pi_R - (1+a)(\bar{b}\cdot\nabla\bar{b} + \bar{b}\cdot\nabla b_R + b_R\cdot\nabla\bar{b}) - a(b_R\cdot\nabla b_R)\Big] \\ =&\nabla(-\Delta)^{-1}\mathrm{div}\Big[a\nabla\overline{\Pi} + \bar{u}\cdot\nabla\bar{u} + \bar{u}\cdot\nabla u_R - a\Delta\bar{u} - a\Delta u_R \\ &+ a\nabla\Pi_R - (1+a)(\bar{b}\cdot\nabla\bar{b} + \bar{b}\cdot\nabla b_R) - a(b_R\cdot\nabla b_R)\Big]. \end{aligned} \tag{4.6}$$

Multiplying by $\exp\left\{-\int_0^t \lambda f(t'))dt'\right\}$ on both side of (4.6), it follows that

$$\begin{aligned}\nabla\overline{\Pi}_\lambda =&\nabla(-\Delta)^{-1}\mathrm{div}\Big[a\nabla\overline{\Pi}_\lambda + \bar{u}\cdot\nabla\bar{u}_\lambda + \bar{u}_\lambda\cdot\nabla u_R - a\Delta\bar{u}_\lambda - a_\lambda\Delta u_R \\ &+ a_\lambda\nabla\Pi_R - (1+a)(\bar{b}\cdot\nabla\bar{b}_\lambda + \bar{b}_\lambda\cdot\nabla b_R) - a_\lambda(b_R\cdot\nabla b_R)\Big]. \end{aligned} \tag{4.7}$$

By the fact that the Riesz transform $\mathscr{R}$ maps continuously from homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^n)$ to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ with the uniform operator norm, we yields that

$$\begin{aligned}\left\|\nabla\overline{\Pi}_\lambda\right\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \lesssim&\|\bar{u}\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|\bar{u}_\lambda\cdot\nabla u_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ &+ \|a\Delta\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|a_\lambda\Delta u_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \left\|a_\lambda\nabla\Pi_R\right\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ &+ \|(1+a)(\bar{b}_\lambda\cdot\nabla b_R)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|(1+a)(\bar{b}\cdot\nabla\bar{b}_\lambda)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ &+ \|a(b_R\cdot\nabla b_R)_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|a\nabla\overline{\Pi}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}. \end{aligned} \tag{4.8}$$

By the Besov product laws Lemma 2 and Definition 3, we get

$$\begin{aligned}\|\bar{u}\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} &\lesssim \|\bar{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})}\\ &\lesssim \|\bar{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\|\bar{u}_\lambda\cdot\nabla u_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} &\lesssim \int_0^t \|u_R(t')\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda(t')\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\,dt'\\ &\lesssim \|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\|a\Delta\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} &\lesssim \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\|\Delta\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\ &\lesssim \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\|a_\lambda\Delta u_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} &\lesssim \int_0^t \|a_\lambda(t')\|_{\dot{B}^{\frac{n}{q}}_{q,1}}\|u_R(t')\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\,dt'\\ &\lesssim \|a_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{\frac{n}{q}}_{q,1})},\end{aligned} \tag{4.12}$$

$$\left\|a\nabla\overline{\Pi}_\lambda\right\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \lesssim \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\left\|\nabla\overline{\Pi}_\lambda\right\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}, \tag{4.13}$$

$$\begin{aligned}\|a(b_R\cdot\nabla b_R)_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} &\lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{q}}_{q,1}}\|b_R\|^2_{\dot{B}^{\frac{n}{p}}_{p,1}}\,dt'\\ &\lesssim \|a_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{\frac{n}{q}}_{q,1})},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\|(1+a)(\bar{b}_\lambda\cdot\nabla b_R)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} &\lesssim (1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}_\lambda\cdot\nabla b_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\ &\lesssim (1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})},\end{aligned} \tag{4.15}$$

$$\begin{aligned}\|(1+a)(\bar{b}\cdot\nabla\bar{b}_\lambda)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} &\lesssim (1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}\cdot\nabla\bar{b}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\ &\lesssim (1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.\end{aligned} \tag{4.16}$$

Taking the above estimates (4.9)- (4.16)into (4.8), we can get the following inequality under the assumption that $C\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}$ is small enough:

$$\begin{aligned}\big\|\nabla\overline{\Pi}_\lambda\big\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \lesssim & \|\bar{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})} + \|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ & + \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})} + (1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ & + (1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})} + \|a_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{\frac{n}{q}}_{q,1})}.\end{aligned} \tag{4.17}$$

4.3. Complete the proof of Theorem 2

Indeed for given $a_0 \in \dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)$, $u_0 \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $b_0 \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ with $\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}}$ sufficiently small and p, q satisfying the conditions in Theorem 2, it follows by a similar argument in [2] that there exists a positive time T so that (1.13) has a local solution (a,u,b) with

$$\begin{aligned}& a \in C([0,T];\dot{B}^{\frac{n}{q}}_{q,1}) \cap \widetilde{L}^\infty((0,T);\dot{B}^{\frac{n}{q}}_{q,1});\ \nabla\Pi \in L^1((0,T);\dot{B}^{-1+\frac{n}{p}}_{p,1}); \\ & (u,b) \in C([0,T];\dot{B}^{-1+\frac{n}{p}}_{p,1}) \cap \widetilde{L}^\infty((0,T);\dot{B}^{-1+\frac{n}{p}}_{p,1}) \cap L^1((0,T);\dot{B}^{1+\frac{n}{p}}_{p,1}),\end{aligned} \tag{4.18}$$

in addition, if $\frac{1}{p}+\frac{1}{q} \geqslant \frac{2}{n}$, then the solution is unique. We denote T^* to be the supremum of T so that (4.18) holds. Hence, to prove Theorem 2, we only need to prove that $T^* = \infty$.

Multiplying $(1.13)_2$ and $(1.13)_3$ by $\exp\{-\int_0^t \lambda f(t'))dt'\}$, applying $\dot{\Delta}_j$ to the equations and taking the L^2 inner product of the resulting equation with $|\dot{\Delta}_j\bar{u}_\lambda|^{p-2}\dot{\Delta}_j\bar{u}_\lambda$, $|\dot{\Delta}_j\bar{b}_\lambda|^{p-2}\dot{\Delta}_j\bar{b}_\lambda$ respectively, we get from the basic energy method that

$$\begin{aligned}& \|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|\bar{b}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \bar{c}(\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})} + \|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}) \\ & \quad + \lambda(\|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}) \\ & \lesssim \big\|(1+a)\nabla\overline{\Pi}_\lambda\big\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|a_\lambda\Delta u_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|\bar{u}\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ & \quad + \|\bar{u}_\lambda\cdot\nabla u_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|u_R\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|a\Delta\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ & \quad + \|(1+a)(b_R\cdot\nabla\bar{b}_\lambda)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|(1+a)(\bar{b}_\lambda\cdot\nabla b_R)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ & \quad + \big\|a_\lambda\nabla\Pi_R\big\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|(1+a)(\bar{b}\cdot\nabla\bar{b}_\lambda)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|\bar{u}\cdot\nabla\bar{b}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \\ & \quad + \|a(b_R\cdot\nabla b_R)_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|\bar{u}_\lambda\cdot\nabla b_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})} + \|u_R\cdot\nabla\bar{b}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\end{aligned}$$

$$+\|\bar{b}\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\cdot\nabla u_R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|b_R\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}. \tag{4.19}$$

By Lemma 2, interpolation inequality and Young's inequality, we obtain

$$\begin{aligned}\|u_R\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}&\lesssim\int_0^t\|u_R(t')\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda(t')\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\,dt'\\&\lesssim\int_0^t\|u_R(t')\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda(t')\|^{\frac12}_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda(t')\|^{\frac12}_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\,dt'\\&\lesssim C(\varepsilon)\|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\varepsilon\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.\end{aligned} \tag{4.20}$$

Similarly,

$$\|u_R\cdot\nabla\bar{b}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim C(\varepsilon)\|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\varepsilon\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}, \tag{4.21}$$

$$\|b_R\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim C(\varepsilon)\|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\varepsilon\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}, \tag{4.22}$$

$$\begin{aligned}&\|(1+a)(b_R\cdot\nabla\bar{b}_\lambda)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\&\qquad\lesssim(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{3}{q}}_{q,1})})(C(\varepsilon)\|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\varepsilon\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}).\end{aligned} \tag{4.23}$$

By the product laws in Besov spaces and (4.17), we have

$$\begin{aligned}&\|(1+a)\nabla\bar{\Pi}\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\&\leqslant C\Big(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\Big)\Big\{\|\bar{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\&\quad+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\&\quad+(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|a_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{\frac{n}{q}}_{q,1})}\Big\}.\end{aligned} \tag{4.24}$$

The other terms in (4.20) can be estimated by a similar way of (4.9)- (4.16) and be controlled by the right hand side of (4.17). Substituting the above estimates (4.20)-(4.24) into (4.19), taking ε small enough and under the assumption (1.14) we yield that

$$\begin{aligned}&\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\bar{c}(\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})})\\&\quad+\lambda(\|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})})\\&\lesssim\|\bar{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\bar{u}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\end{aligned}$$

$$+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}$$
$$+(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|a_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{\frac{n}{q}}_{q,1})}. \tag{4.25}$$

On the other hand, taking $g(t)=f(t)$ in Proposition 4 implies

$$\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}+(\lambda-C)\|a_\lambda\|_{\widetilde{L}^1_{t,f}(\dot{B}^{\frac{n}{q}}_{q,1})}\leqslant C(\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}}+\|\bar{u}\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}). \tag{4.26}$$

Summing up (4.25) and (4.26) and taking λ large enough yields that

$$\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}+\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}$$
$$+\bar{c}(\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})})$$
$$\lesssim\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}}+\|\bar{u}\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}+\|\bar{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}$$
$$+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})})\|\bar{b}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}. \tag{4.27}$$

Now let η be a small enough positive constant to be determined later on, we define T^{**} by

$$T^{**}\triangleq\sup\Big\{t\in[0,T^*):\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}+\|\bar{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{b}\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{u}\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\bar{b}\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\leqslant 2\eta\Big\}. \tag{4.28}$$

(4.28) implies that

$$\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}\leqslant 2\eta,$$

in particular, if we take $\eta\leqslant\frac{1}{2c}$, where c is the constant in (1.14). Then (1.14) automatically holds for $t<T^{**}$. In what follows, we shall prove that $T^{**}=\infty$ under the assumption (1.6) and (1.14). Otherwise, if $T^{**}<\infty$, taking

$$\eta\leqslant\bar{\eta}\triangleq\min(\frac{\bar{c}}{8C},\frac{\bar{c}}{4(1+c)C},\frac{1}{2c})\ \text{ for } t<T^{**},$$

we would deduce from (4.27) that

$$\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{q}}_{q,1})}+\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\bar{b}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}$$
$$\lesssim\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}}. \tag{4.29}$$

On the other hand, it's easy to observe from (4.29) that

$$\begin{aligned}&\|a\|_{\widetilde{L}^\infty_t(\dot B^{\frac{n}{q}}_{q,1})}+\|\bar u\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar u\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}\\&\quad\lesssim\Big(\|a_\lambda\|_{\widetilde{L}^\infty_t(\dot B^{\frac{n}{q}}_{q,1})}+\|\bar u_\lambda\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar b_\lambda\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar u_\lambda\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}\\&\qquad+\|\bar b_\lambda\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}\Big)\exp\Big\{\int_0^t\lambda f(t')dt'\Big\}.\end{aligned}$$

As a consequence,

$$\begin{aligned}&\|a\|_{\widetilde{L}^\infty_t(\dot B^{\frac{n}{q}}_{q,1})}+\|\bar u\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar u\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}\\&\quad\lesssim\|a_0\|_{\dot B^{\frac{n}{q}}_{q,1}}\exp\Big\{\int_0^t\lambda f(t')dt'\Big\}\quad\text{for } t<T^{**}.\end{aligned}\tag{4.30}$$

Thanks to the estimate (1.8) in Theorem 1, there exists a constant C such that

$$\begin{aligned}\int_0^t\lambda f(t')dt'\leqslant C\Big(&(1+\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|W_F^+\cdot\nabla W_F^-\|_{L^1_t(\dot B^{-1+\frac{n}{p}}_{p,1})})^2\\&\times\exp\{C(\|W_0^+\|_{\dot B^{-1+\frac{n}{p}}_{p,1}}+\|W_0^-\|_{\dot B^{-1+\frac{n}{p}}_{p,1}})\}\Big).\end{aligned}\tag{4.31}$$

Combining (4.30) with (4.31), we reach

$$\begin{aligned}&\|a\|_{\widetilde{L}^\infty_t(\dot B^{\frac{n}{q}}_{q,1})}+\|\bar u\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar u\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}\\&\leqslant C\|a_0\|_{\dot B^{\frac{n}{q}}_{q,1}}\exp\Big((1+\|W_F^-\cdot\nabla W_F^+\|_{L^1_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|W_F^+\cdot\nabla W_F^-\|_{L^1_t(\dot B^{-1+\frac{n}{p}}_{p,1})})^2\\&\qquad\times\exp\{C(\|W_0^+\|_{\dot B^{-1+\frac{n}{p}}_{p,1}}+\|W_0^-\|_{\dot B^{-1+\frac{n}{p}}_{p,1}})\}\Big)\end{aligned}$$

which implies that if we take $\|a_0\|_{\dot B^{\frac{n}{q}}_{q,1}}$ sufficiently small, the following estimate would hold

$$\begin{aligned}&\|a\|_{\widetilde{L}^\infty_t(\dot B^{\frac{n}{q}}_{q,1})}+\|\bar u\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{\widetilde{L}^\infty_t(\dot B^{-1+\frac{n}{p}}_{p,1})}+\|\bar u\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}+\|\bar b\|_{L^1_t(\dot B^{1+\frac{n}{p}}_{p,1})}\\&\qquad\leqslant Cc_0\leqslant\eta\quad\text{for } t<T^{**},\end{aligned}$$

and the solution would exist beyond T^{**}. This contradicts (4.28). Whence we conclude that $T^{**}=\infty$. This completes the proof of Theorem 2.

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