

## ON OSCILLATORY NONLINEAR SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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*Abstract.* In this work, we investigate the oscillation criteria for second order neutral delay differential equations of the form

$$(r(t)[y(t) + p(t)y(\delta(t))]')' + q(t)G(y(\tau(t))) = 0$$

and

$$(r(t)[[y(t) + p(t)y(\delta(t))]']^\alpha)' + q(t)(y^\beta(\tau(t))) = 0,$$

where  $\alpha$  and  $\beta$  are the ratio of odd positive integers.

### 1. Introduction

Consider the nonlinear neutral delay differential equations of the form

$$(r(t)[y(t) + p(t)y(\delta(t))]')' + q(t)G(y(\tau(t))) = 0, \tag{1.1}$$

$$(r(t)[[y(t) + p(t)y(\delta(t))]']^\alpha)' + q(t)(y^\beta(\tau(t))) = 0, \tag{1.2}$$

where  $r, q, \delta, \tau \in C(\mathbb{R}_+, \mathbb{R}_+)$ ,  $p \in C(\mathbb{R}_+, \mathbb{R})$ ,  $G \in C(\mathbb{R}, \mathbb{R})$  such that  $xG(x) > 0$  for  $x \neq 0$ ,  $\alpha, \beta$  are the ratio of odd positive integers and  $\delta(t) \leq t, \tau(t) \leq t$  with  $\delta(t)$  is bijective and  $\lim_{t \rightarrow \infty} \delta(t) = \infty = \lim_{t \rightarrow \infty} \tau(t)$ .

In [5], the authors have considered (1.1) when  $G(x) = x$  and also studied (1.2) in [6], and they have established sufficient conditions for oscillation of all solutions of (1.1) and (1.2) subject to the comparison results. Using double Riccati transformation, Li and Rogovchenko [16] have studied (1.1) and presented new oscillation criteria with restriction on  $p(t) \geq 0$  and also required inter alia  $\tau(t) \leq \delta(t) \leq t$  condition but, the base is the comparison results. A similar observation can be remarked in [7], where Baculikova et al. have studied the neutral equation

$$(r(t)[y(t) + p(t)y(\delta(t))]')' + q(t)y(\tau(t)) + v(t)y(\sigma(t)) = 0. \tag{1.3}$$

In this work, our objective is to establish the sufficient condition results for oscillation of all solutions of (1.1) and (1.2) without the comparison results under the assumptions

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$$\begin{aligned}
 (H_1) \quad & \int_0^\infty \frac{dt}{r(t)} = \infty, \\
 (H_2) \quad & \int_0^\infty \frac{dt}{r(t)} < \infty, \\
 (H'_1) \quad & \int_0^\infty \frac{dt}{r\alpha(t)} = \infty, \\
 \text{and} \\
 (H'_2) \quad & \int_0^\infty \frac{dt}{r\alpha(t)} < \infty
 \end{aligned}$$

for all ranges  $p(t)$  with  $|p(t)| < \infty$ .

Neutral delay differential equations find numerous applications in electric networks. For example, they are frequently used for the study of distributed networks containing lossless transmission lines which arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits (see for e.g. [13]). The problem of obtaining sufficient conditions to ensure the second order differential equations which are special cases of (1.1) and (1.2) are oscillatory has received a great attention. We refer the reader to some of the works [1] - [4],[9] - [12],[15] - [25] and the references cited therein.

By a solution of (1.1) (or (1.2)), we mean a continuously differentiable function  $y(t)$  which is defined for  $t \geq \min\{\delta(t_0), \tau(t_0)\}$  such that  $y(t)$  satisfies (1.1) (or (1.2)) for all  $t \geq t_0$ . In the sequel, it will be always assumed that the solutions of (1.1) (or (1.2)) exist on some half-line  $[t_1, \infty]$  for  $t_1 \geq t_0$ . A solution of (1.1) (or (1.2)) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Eq.(1.1) (or (1.2)) is called oscillatory if all of its solutions are oscillatory.

## 2. Oscillation Criteria for (1.1)

In this section we establish the oscillation criteria for (1.1). We need the following hypotheses for our use in the sequel:

$$\begin{aligned}
 (H_3) \quad & \text{there exists } \lambda > 0 \text{ such that } G(u) + G(v) \geq \lambda G(u+v) \text{ and } G(uv) \leq G(u)G(v) \\
 & \text{for } u, v \in \mathbb{R} \text{ and } u, v > 0 \quad (\text{see for e.g. [14]}); \\
 (H_4) \quad & G(-u) = -G(u), \quad u \in \mathbb{R}; \\
 (H_5) \quad & \int_0^\infty Q(t)dt = \infty, \quad Q(t) = \min\{q(t), q(\delta(t))\}; \\
 (H_6) \quad & \int_0^\infty q(t)dt = \infty; \\
 (H_7) \quad & \delta^n(t) = \delta(\delta^{n-1}(t)), \quad \lim_{n \rightarrow \infty} \delta^n(t) < \infty; \\
 (H_8) \quad & \int_{t_0}^\infty Q(t)G(R(\tau(t)))dt = \infty, \quad R(t) = \int_t^\infty \frac{ds}{r(s)}; \\
 (H_9) \quad & \int_0^\infty \frac{1}{r(t)} \int_T^t Q(s)G(R(\tau(s)))dsdt = \infty \text{ for every } T > 0; \\
 (H_{10}) \quad & \int_0^\infty q(t)G(R(\tau(t)))dt = \infty, \quad R(t) = \int_t^\infty \frac{ds}{r(s)}; \\
 (H_{11}) \quad & \int_0^\infty \frac{1}{r(t)} \int_T^t q(s)G(R(\tau(s)))dsdt = \infty \text{ for every } T > 0.
 \end{aligned}$$

LEMMA 1. [5] Assume that  $(H_1)$  holds. If  $y(t)$  is a positive solution of (1.1) such that the corresponding function  $z(t) = y(t) + p(t)y(\delta(t)) > 0$ , then  $z(t)$  satisfies

$$z(t) > 0, r(t)z'(t) > 0, (r(t)z'(t))' < 0$$

eventually.

LEMMA 2. Assume that  $(H_2)$  holds. Let  $y(t)$  be any continuous function defined on  $[t_0, \infty)$ ,  $t_0 \geq 0$  such that  $(r(t)y'(t))' \leq 0$  on  $[t_0, \infty)$ . If  $y'(t) < 0$  for  $t \geq t_0$ , then  $y(t) \geq -R(t)r(t)y'(t)$ .

*Proof.* For  $s \geq t$ ,  $r(s)y'(s) \leq r(t)y'(t)$  and hence

$$y(s) \leq y(t) + \int_t^s \frac{r(t)y'(t)}{r(\theta)} d\theta = y(t) + r(t)y'(t) \int_t^s \frac{1}{r(\theta)} d\theta.$$

Therefore,

$$0 < y(s) \leq y(t) + r(t)y'(t) \int_t^s \frac{d\theta}{r(\theta)}$$

implies that  $y(t) \geq -R(t)r(t)y'(t)$ .

THEOREM 1. Let  $0 \leq p(t) \leq a < \infty$  and  $\tau(\delta(t)) = \delta(\tau(t))$  for every  $t > 0$ . If  $(H_1)$ ,  $(H_3) - (H_5)$  hold, then (1.1) is oscillatory.

*Proof.* On the contrary, without loss of generality we assume that  $y(t) > 0$  (because of  $(H_4)$ ) is a nonoscillatory solution of (1.1) on  $[t_0, \infty)$ ,  $t_0 > 0$ . Hence, there exists  $t_1 > t_0$  such that  $y(t) > 0$ ,  $y(\delta(t)) > 0$ , and  $y(\tau(t)) > 0$  for  $t \geq t_1$ . Defining  $z(t)$  as in Lemma 2.1 and then taking the lemma into account, it follows that  $z(t) \geq C$  for  $t \geq t_2$ . Using (1.1), it follows that

$$(r(t)z'(t))' + q(t)G(y(\tau(t))) + G(a)(r(\delta(t))z'(\delta(t)))' + G(a)q(\delta(t))G(y(\tau(\delta(t)))) = 0.$$

Using  $(H_3)$  in the above equation, we obtain

$$(r(t)z'(t))' + G(a)(r(\delta(t))z'(\delta(t)))' + \lambda Q(t)G(z(\tau(t))) \leq 0,$$

due to  $\tau(\delta(t)) = \delta(\tau(t))$ . Consequently, there exists  $t_3 > t_2$  such that

$$\lambda G(C)Q(t) \leq -(r(t)z'(t))' - G(a)(r(\delta(t))z'(\delta(t)))' \tag{2.1}$$

for  $t \geq t_3$ . Integrating (2.1) from  $t_3$  to  $+\infty$ , we obtain a contradiction to  $(H_5)$ . This completes the proof of the theorem.

REMARK 1. Equation (1.1) includes a class of nonlinear neutral differential equations when  $p(t) \geq 0$ . It is learnt that  $G$  could be linear, sublinear or superlinear also.

THEOREM 2. Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that  $(H_1)$ ,  $(H_4)$  and  $(H_6)$  hold. Then every unbounded solution of (1.1) is oscillatory.

*Proof.* Let  $y(t)$  be an unbounded nonoscillatory solution of (1.1). Proceeding as in the proof of Theorem 1, we can find a  $t_1 > t_0$  such that  $y(t) > 0$ ,  $y(\delta(t)) > 0$  and  $y(\tau(t)) > 0$  for  $t \geq t_1$ . Since  $z(t)$  is monotonic, then either  $z(t) > 0$  or  $< 0$  for  $t \geq t_1$ .

Clearly, Lemma 1 holds when  $z(t) > 0$  for  $t \geq t_1$ . Using the fact that  $z(t) \leq y(t)$  for  $t \geq t_1$ , it follows from (1.1) that

$$(r(t)z'(t))' + q(t)G(z(\tau(t))) \leq 0.$$

The rest of this case follows from Theorem 1.

Suppose that  $z(t) < 0$  for  $t \geq t_1$ . Since  $y(t)$  is unbounded, then there exists  $\{v_n\}_{n=1}^\infty \subset [t_3, \infty)$  such that  $v_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $y(v_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Indeed,

$$z(v_n) \geq y(v_n) - by(\tau(v_n)) \geq (1 - b)y(v_n)$$

implies that  $z(t) > 0$ , which is absurd. This completes the proof of the theorem.

REMARK 2. We may note that  $G$  could be linear, superlinear or sublinear in Theorem 2. If  $z(t) < 0$  for  $t \geq t_2$ , then  $y(t) < y(\delta(t))$  implies that

$$y(t) < y(\delta(t)) < y(\delta^2(t)) < \dots < y(\delta^n(t)) < \dots$$

holds. Consequently,  $y(t)$  is bounded due to  $(H_7)$  and hence  $z(t)$  is bounded. When  $z'(t) < 0$  for  $t \geq t_2$ , it happens that  $r(t)z'(t) \leq r(t_2)z'(t_2)$  which then implies that  $\lim_{t \rightarrow \infty} z(t) = -\infty$ . Therefore, this case doesn't arise in Theorem 2.

THEOREM 3. Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that  $(H_1), (H_4), (H_6)$  and  $(H_7)$  hold. Then every solution of (1.1) either oscillates or converges to zero.

*Proof.* The proof of the theorem follows from Theorem 2 and Remark 2.6. In case  $z(t) < 0$ ,

$$\begin{aligned} 0 &\geq \lim_{t \rightarrow \infty} z(t) = \limsup_{t \rightarrow \infty} (y(t) + p(t)y(\delta(t))) \\ &\geq \limsup_{t \rightarrow \infty} (y(t) - by(\delta(t))) \\ &\geq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (-by(\delta(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) - b \limsup_{t \rightarrow \infty} y(\delta(t)) \\ &= (1 - b) \limsup_{t \rightarrow \infty} y(t) \end{aligned}$$

implies that  $\limsup_{t \rightarrow \infty} y(t) = 0$ . Hence,  $\lim_{t \rightarrow \infty} y(t) = 0$ . This completes the proof of the theorem.

THEOREM 4. Let  $-\infty < -b \leq p(t) \leq -1$ ,  $b > 1$ . Assume that  $(H_1), (H_4)$  and  $(H_6)$  hold. Then every bounded solution of (1.1) either oscillates or converges to zero.

*Proof.* The proof of the theorem follows from the proof of Theorem 3. In case

$$z(t) < 0, \quad r(t)z'(t) > 0, \quad (r(t)z'(t))' < 0,$$

we assert that  $\liminf_{t \rightarrow \infty} y(t) = 0$ . Otherwise, let there exist  $a > 0$  and  $t_4 > t_3$  such that  $y(\delta(t)) \geq a$  for  $t \geq t_4$ . Integrating (1.1) from  $t_4$  to  $\infty$  we obtain a contradiction to  $(H_6)$ . Therefore, our assertion is true. Hence, there exists  $\{u_n\}_{n=1}^\infty \subset [t_4, \infty)$  such that  $u_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} y(u_n) = 0$ . Let  $\lim_{t \rightarrow \infty} z(t) = L, L \in (-\infty, 0]$ . For  $t \geq t_4$ , we have

$$z(\delta^{-1}(t)) - z(t) = y(\delta^{-1}(t)) + [p(\delta^{-1}(t)) - 1]y(t) - p(t)y(\delta(t))$$

which then implies that

$$\lim_{t \rightarrow \infty} [y(\delta^{-1}(t)) + \{p(\delta^{-1}(t)) - 1\}y(t) - p(t)y(\delta(t))] = 0.$$

Also, it is true that

$$\lim_{n \rightarrow \infty} [y(\delta^{-1}(u_n)) + \{p(\delta^{-1}(u_n)) - 1\}y(u_n) - p(u_n)y(\delta(u_n))] = 0,$$

that is,

$$\lim_{n \rightarrow \infty} [y(\delta^{-1}(u_n)) - p(u_n)y(\delta(u_n))] = 0.$$

Since

$$y(\delta^{-1}(u_n)) - p(u_n)y(\delta(u_n)) \geq -p(u_n)y(\delta(u_n)),$$

then it follows that  $\limsup_{n \rightarrow \infty} [-p(u_n)y(\delta(u_n))] = 0$ , that is,  $\lim_{n \rightarrow \infty} [-p(u_n)y(\delta(u_n))] = 0$ .

Ultimately,

$$L = \lim_{n \rightarrow \infty} z(u_n) = \lim_{n \rightarrow \infty} [y(u_n) + p(u_n)y(\delta(u_n))] = 0.$$

Therefore,

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} z(t) = \liminf_{t \rightarrow \infty} (y(t) + p(t)y(\delta(t))) \\ &\leq \liminf_{t \rightarrow \infty} (y(t) - b y(\delta(t))) \\ &\leq \limsup_{t \rightarrow \infty} y(t) + \liminf_{t \rightarrow \infty} (-b y(\delta(t))) \\ &= \limsup_{t \rightarrow \infty} y(t) - b \limsup_{t \rightarrow \infty} y(\delta(t)) \\ &= (1 - b) \limsup_{t \rightarrow \infty} y(t), \end{aligned}$$

implies that  $\limsup_{t \rightarrow \infty} y(t) = 0$ . Hence the proof of the theorem is complete.

**THEOREM 5.** *Let  $0 \leq p(t) \leq a < \infty$ . Assume that  $\tau(\delta(t)) = \delta(\tau(t))$  holds for every  $t > 0$ . If  $(H_2) - (H_4), (H_8)$  and  $(H_9)$  hold, then (1.1) is oscillatory.*

*Proof.* Proceeding as in the proof of Theorem 1 we can find  $t_1 > t_0$  such that  $y(t) > 0$ ,  $y(\delta(t)) > 0$  and  $y(\tau(t)) > 0$  for  $t \geq t_1$ . It follows that  $z(t)$  and  $r(t)z'(t)$  are of one sign on  $[t_1, \infty)$ . We consider two cases upon the sign of  $r(t)z'(t)$ , that is,  $z'(t) > 0$  and  $z'(t) < 0$  for  $t \geq t_1$ . Suppose the former holds. Then there exist  $t_2 > t_1$  and a constant  $C_1 > 0$  such that  $z(t) \geq C_1$  for  $t > t_2$ . It is easy to verify that there exist,  $C > 0$  and  $t_3 > t_2$  such that  $z(t) \geq CR(t)$  for  $t \geq t_3$  ( $\because \lim_{t \rightarrow \infty} R(t) = 0$  and  $R(t)$  is bounded). Consequently, we can write the inequality similar to (1.3) as

$$\lambda G(CR(\tau(t)))Q(t) \leq -(r(t)z'(t))' - G(a)(r(\delta(t))z'(\delta(t)))' \tag{2.2}$$

for  $t \geq t_3$ . Integrating (2.2) from  $t_3$  to  $\infty$ , we obtain a contradiction. Ultimately, the latter holds. From Lemma 2, we have that  $z(t) \geq -R(t)r(t)z'(t)$  for  $t \geq t_2 > t_1$ . Since  $r(t)z'(t)$  is nonincreasing, then we can find a constant  $C > 0$  and  $t_3 > t_2$  such that  $r(t)z'(t) \leq -C$  and  $z(t) \geq CR(t)$  for  $t \geq t_3$ . Consequently, (2.2) holds for  $t \geq t_3$ . Integrating (2.2) from  $t_3$  to  $t$ , we obtain that

$$\lambda G(C) \int_{t_3}^t Q(s)G(R(\tau(s)))ds \leq -(1 + G(a))r(t)z'(t)$$

due to nonincreasing  $r(t)z'(t)$ . Hence

$$\frac{\lambda G(C)}{r(t)} \int_{t_3}^t Q(s)G(R(\tau(s)))ds \leq -(1 + G(a))z'(t).$$

Since  $\lim_{t \rightarrow \infty} z(t)$  exists, then it follows from the above inequality that

$$\lambda G(C) \int_{t_3}^{\infty} \frac{1}{r(t)} \int_{t_3}^t Q(s)G(R(\tau(s)))dsdt < \infty,$$

a contradiction to  $(H_9)$ . Thus the proof of the theorem is complete.

**THEOREM 6.** *Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that  $(H_2), (H_4), (H_{10})$  and  $(H_{11})$  hold. Then every unbounded solution of (1.1) oscillates.*

*Proof.* Proceeding as in the proof of Theorem 2, we conclude that  $z(t)$  is monotonic on  $[t_1, \infty)$ . Hence there exists  $t_2 > t_1$  such that  $z(t) > 0$  or  $z(t) < 0$  for  $t \geq t_2$ . Consider that  $z(t) > 0$  on  $[t_2, \infty)$ . Using the same type of reasoning as in the proof of Theorem 5, we can find  $C > 0$  and  $t_3 > t_2$  such that  $z(t) \geq CR(t)$  for  $t \geq t_3$ . Since  $z(t) \leq y(t)$ , then it follows that  $y(t) \geq CR(t)$  on  $[t_3, \infty)$ , if we assume that  $z'(t) > 0$ . Therefore (1.1) becomes

$$(r(t)z'(t))' + G(C)G(R(\tau(t)))q(t) \leq 0 \tag{2.3}$$

for  $t \geq t_3$ . Integrating (2.3) from  $t_3$  to  $\infty$ , we obtain a contradiction to  $(H_{10})$ . Hence  $z'(t) < 0$  for  $t \geq t_2$ . Rest of this case follows from the proof of Theorem 5.

Next, we suppose that  $z(t) < 0$  on  $[t_2, \infty)$ . Let there exist  $t_3 > t_2$  such that  $z'(t) > 0$  or  $z'(t) < 0$  for  $t \geq t_3$ . Rest of this case follows from the proof of Theorem 2. Thus the proof of the theorem is complete.

**THEOREM 7.** *Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that  $(H_2)$ ,  $(H_4)$ ,  $(H_7)$ ,  $(H_{10})$  and  $(H_{11})$  hold. Then every solution of (1.1) either oscillates or converges to zero.*

*Proof.* The proof of the theorem follows from the proof of Theorems 3 and 6. Due to  $(H_7)$ ,  $y(t)$  is bounded and hence  $z(t)$  is bounded. Therefore,  $\lim_{t \rightarrow \infty} z(t)$  exists when  $z(t) < 0$ . Hence, the theorem is proved.

**THEOREM 8.** *Let  $-\infty < -b \leq p(t) \leq -1$ ,  $b > 1$ . Assume that  $(H_2)$ ,  $(H_4)$ ,  $(H_{10})$  and  $(H_{11})$  hold. Then every bounded solution of (1.1) either oscillates or converges to zero.*

*Proof.* The proof of the theorem can be followed from the proof of Theorems 7 and 4. Hence, the details are omitted.

**EXAMPLE 1.** Consider (1.1) on  $[\pi, \infty)$ , where  $p(t) = (1 + 1/t)$ ,  $\delta(t) = t - \pi$ ,  $\tau(t) = t - 3\pi$ ,  $q(t) = t$ ,  $r(t) = t^2$  and  $G(u) = u$ . Clearly, all conditions of Theorem 5 are satisfied for (1.1) when  $G(u) = u$ . Hence every solution of (1.1) oscillates. In particular,  $y(t) = \sin t$  is an oscillatory solution of (1.1).

**EXAMPLE 2.** Consider (1.1) on  $[\pi, \infty)$ , where  $p(t) = -e^{-\pi}$ ,  $\delta(t) = t - \pi$ ,  $\tau(t) = t - \frac{3\pi}{2}$ ,

$$q(t) = \frac{4}{(e^{-\frac{3\pi}{2}} + |e^{-t} \sin t|)} \geq \frac{4}{(1 + e^{-\frac{3\pi}{2}})},$$

$r(t) = 1$  and  $G(u) = u(1 + |u|)$ . Clearly, all conditions of Theorem 3 are satisfied for (1.1). Hence every solution of (1.1) either oscillates or converges to zero. In particular,  $y(t) = e^{-t} \sin t$  is such a solution of (1.1).

### 3. Oscillation Criteria for (1.2)

In this section, we establish sufficient condition for oscillation of all solutions of (1.2), where  $\alpha$  and  $\beta$  are the quotient of odd positive integers. We need the following lemmas for our use in the sequel:

**LEMMA 3.** [6] *Assume that  $A \geq 0$ ,  $B \geq 0$  and  $\lambda \geq 0$ . Then*

$$(A + B)^\lambda \leq 2^{\lambda-1}(A^\lambda + B^\lambda).$$

*If  $0 \leq \lambda \leq 1$ , then*

$$(A + B)^\lambda \leq (A^\lambda + B^\lambda).$$

**LEMMA 4.** [6] *Let  $p(t) \geq 0$ . Assume that  $\int_0^\infty r^{-1/\alpha}(t)dt = \infty$ . If  $y(t)$  is a positive solution of (1.2), then the corresponding function  $z(t) = y(t) + p(t)y(\delta(t))$  satisfies*

$$z(t) > 0, \quad z'(t) > 0, \quad (r(t)(z'(t))^\alpha)' < 0$$

*for any large value of  $t$ .*

LEMMA 5. Assume that  $\int_0^\infty r^{-1/\alpha}(t)dt < \infty$ . Let  $R_\alpha(t) = \int_t^\infty r^{-1/\alpha}(t)dt$ . Let  $y(t)$  be an eventually positive solution of (1.2) such that

$$z(t) = y(t) + p(t)y(\delta(t)) \quad (3.1)$$

is of one sign, for large value of  $t$ . Then the following are true:

1. If  $z(t) > 0$  and  $z'(t) > 0$  for large  $t$ , then  $z(t) \geq CR_\alpha(t)$ .
2. If  $z(t) > 0$  and  $z'(t) < 0$  for large  $t$ , then

$$z(t) \geq -R_\alpha(t)r^{1/\alpha}(t)z'(t) \text{ and } \left(\frac{z(t)}{R_\alpha(t)}\right)' \geq 0.$$

*Proof.* The proof of 1 is immediate. Proof of 2 follows from the proof of Lemma 2. Hence, the details are omitted.

THEOREM 9. Let  $0 \leq p(t) \leq a < \infty$ . Assume that  $\tau(\delta(t)) = \delta(\tau(t))$  for every  $t > 0$  If

$$(H_{12}) \quad \int_0^\infty r^{-1/\alpha}(t)dt = \infty = \int_0^\infty Q(t)dt, \quad Q(t) = \min\{q(t), q(\delta(t))\}$$

holds, then every solution of (1.2) oscillates.

*Proof.* Proceeding as in Theorem 1 and taking Lemma 4 into account, it follows that  $z(t)$  is nondecreasing on  $[t_1, \infty)$  and hence  $z(t) \geq C$  for  $t \geq t_2 > t_1$ . Using (1.2), it follows that

$$(r(t)(z'(t))^\alpha)' + q(t)y^\beta(\tau(t)) + a^\beta(r(\delta(t))(z'(\delta(t))))^\alpha' + a^\beta q(\delta(t))y^\beta(\tau(\delta(t))) = 0. \quad (3.2)$$

Due to Lemma 3, (3.2) becomes

$$(r(t)(z'(t))^\alpha)' + a^\beta(r(\delta(t))(z'(\delta(t))))^\alpha' + 2^{1-\lambda}Q(t)z^\beta(\tau(t)) \leq 0$$

( $\because \tau(\delta(t)) = \delta(\tau(t))$ ) for  $t \geq t_2$ . Hence, there exists  $t_3 > t_2$  such that

$$2^{1-\lambda}Q(t)C^\beta \leq -(r(t)(z'(t))^\alpha)' - a^\beta(r(\delta(t))(z'(\delta(t))))^\alpha' \quad (3.3)$$

for  $t \geq t_3$ . Integrating (3.3) from  $t_3$  to  $+\infty$ , we obtain a contradiction to  $(H_{12})$ . This completes the proof of the theorem.

THEOREM 10. Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that

$$(H_{13}) \quad \int_0^\infty r^{-1/\alpha}(t)dt = \infty = \int_0^\infty q(t)dt$$

hold. Then every unbounded solution of (1.2) is oscillatory.

*Proof.* On the contrary, we proceed as in Theorem 2 and obtain

$$(r(t)(z'(t))^\alpha)' + q(t)z^\beta(\tau(t)) \leq 0 \quad (3.4)$$

$t \geq t_1$  due to (3.1). In this case, Lemma 4 is applicable for  $t \geq t_1$ . Proceeding as in the proof of Theorem 9, we obtain a contradiction to  $(H_{13})$ . The proof for the case  $z(t) < 0$  for  $t \geq t_1$  follows from Theorem 2. Hence the theorem is proved.



**THEOREM 11.** *Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that  $(H_7)$  and  $(H_{13})$  hold. Then every solution of (1.2) either oscillates or converges to zero.*

*Proof.* The proof of the theorem follows from the proof of Theorem 3. Hence the details are omitted.

**THEOREM 12.** *Let  $-\infty < -b \leq p(t) \leq -1$ ,  $b > 1$ . If  $(H_{13})$  holds, then every bounded solution of (1.2) either oscillates or converges to zero.*

**THEOREM 13.** *Let  $0 \leq p(t) \leq a < \infty$ . Assume that  $\tau(\delta(t)) = \delta(\tau(t))$  hold for every  $t > 0$ . If*

$$(H_{14}) \quad \int_0^\infty r^{-\frac{1}{\alpha}}(t)dt < \infty, \quad \int_{t_0}^\infty Q(t)R_\alpha^\beta(\tau(t))dt = \infty$$

and

$$(H_{15}) \quad \int_0^\infty \left[ \frac{1}{r(t)} \int_T^t Q(s)R_\alpha^\beta(\tau(s))ds \right]^{\frac{1}{\alpha}} dt = \infty \text{ for every } T > 0$$

hold, then (1.2) is oscillatory.

*Proof.* On the contrary, we proceed as in Theorem 9 and Theorem 5. It follows from (3.2) that

$$-(r(t)(z'(t)^\alpha))' - a^\beta (r(\delta(t))(z'(\delta(t)))^\alpha)' \geq 2^{1-\lambda} C^\beta R_\alpha^\beta(\tau(t))Q(t) \tag{3.5}$$

for  $t \geq t_3$  due to Lemma 3 and Lemma 5(1). Integrating (3.5) from  $t_3$  to  $\infty$ , we obtain a contradiction to  $(H_{14})$ . Hence  $z'(t) < 0$  for  $t \geq t_1$ . Applying Lemma 5(2), it happens that  $z(t) \geq -R_\alpha(t)r^{\frac{1}{\alpha}}(t)z'(t)$  for  $t \geq t_2 \geq t_1$ . Since  $r(t)(z'(t)^\alpha)$  is nonincreasing, then we can find a constant  $C^\alpha > 0$  and  $t_3 > t_2$  such that  $r(t)(z'(t)^\alpha) \leq -C^\alpha$  and  $z(t) \geq CR_\alpha(t)$  for  $t \geq t_3$ . Consequently, (3.5) holds for  $t \geq t_3$ . Integrating (3.5) from  $t_3$  to  $t$ , we obtain that

$$2^{1-\lambda} C^\beta \int_{t_3}^t Q(s)R_\alpha^\beta(\tau(s))ds \leq -(1+a^\beta)r(t)(z'(t)^\alpha)$$

due to nondecreasing  $(r(t)(z'(t)^\alpha))^\alpha$ . Hence

$$\frac{2^{1-\lambda} C^\beta}{r(t)} \int_{t_3}^t Q(s)R_\alpha^\beta(\tau(s))ds \leq -(1+a^\beta)(z'(t)^\alpha).$$

Since  $\lim_{t \rightarrow \infty} z(t)$  exists, then it follows from the above inequality that

$$\left[ \frac{2^{1-\lambda} C^\beta}{(1+a^\beta)} \right]^{\frac{1}{\alpha}} \int_{t_3}^\infty \left[ \frac{1}{r(t)} \int_{t_3}^t Q(s)R_\alpha^\beta(\tau(s))ds \right]^{\frac{1}{\alpha}} dt < \infty,$$

a contradiction to  $(H_{15})$ . This completes the proof of the theorem.

**THEOREM 14.** *Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that*

$$(H_{16}) \quad \int_0^\infty r^{-\frac{1}{\alpha}}(t)dt < \infty, \quad \int_0^\infty q(t)R_\alpha^\beta(\tau(t))dt = \infty$$

and

$$(H_{17}) \quad \int_0^\infty \left[ \frac{1}{r(t)} \int_T^t q(s) R_\alpha^\beta(\tau(s)) ds \right]^{\frac{1}{\alpha}} dt = \infty \text{ for every } T > 0$$

hold. Then every unbounded solution of (1.2) oscillates.

*Proof.* Let  $y(t)$  be an unbounded nonoscillatory solution of (1.2). Without loss of generality, we may assume that  $y(t) > 0$  for  $t \geq t_0$ . Hence there exists  $t_1 > t_0$  such that  $y(t) > 0$ ,  $y(\delta(t)) > 0$  and  $y(\tau(t)) > 0$  for  $t \geq t_1$ . Using (3.1) in (1.2), it follows that  $z(t)$ ,  $r(t)(z'(t))^\alpha$  are of one sign on  $[t_2, \infty)$ ,  $t_2 > t_1$ . Suppose that  $z(t) > 0$  for  $t \geq t_2$ . Let  $z'(t) > 0$  for  $t \geq t_2$ . Since  $z(t) \leq y(t)$ , then (1.2) reduces to (3.4). Using Lemma 5(1) in (3.4) and then integrating from  $t_3 (> t_2)$  to  $\infty$ , we get a contradiction to  $(H_{16})$ . Ultimately,  $z'(t) < 0$  for  $t \geq t_2$ . The rest of this case follows from Theorem 13.

The proof for the case  $z(t) < 0$  for  $t \geq t_1$  follows from Theorem 2. Hence the theorem is proved.

**THEOREM 15.** Let  $-1 < -b \leq p(t) \leq 0$ ,  $b > 0$ . Assume that  $(H_7)$ ,  $(H_{16})$  and  $(H_{17})$  hold. Then every solution of (1.2) either oscillates or converges to zero.

*Proof.* The proof of the theorem follows from the proof of Theorems 3 and 14. Due to  $(H_7)$ ,  $y(t)$  is bounded and hence  $z(t)$  is bounded. Therefore,  $\lim_{t \rightarrow \infty} z(t)$  exists when  $z(t) < 0$ . Thus, the proof of the theorem is complete.

**THEOREM 16.** Let  $-\infty < -b \leq p(t) \leq -1$ ,  $b > 1$ . Assume that  $(H_{16})$  and  $(H_{17})$  hold. Then every bounded solution of (1.2) either oscillates or converges to zero.

*Proof.* The proof of the theorem can be followed from the proof of Theorems 15 and 4. Hence, the details are omitted.

**EXAMPLE 3.** Consider  $\alpha = \beta = 1$ . If we choose  $p(t) = e^{-2\pi}$ ,  $\delta(t) = t - 2\pi$ ,  $\tau(t) = t - 3\pi$ ,  $q(t) = 4e^{3\pi}$  and  $r(t) = 1$ , then all conditions of Theorem 9 are satisfied for (1.2) on  $[2\pi, \infty)$ . Hence every solution of (1.2) oscillates. In particular,  $y(t) = e^{-t} \cos t$  is an oscillatory solution of (1.2).

**EXAMPLE 4.** Consider  $\alpha = \beta = 1$ . If we choose  $p(t) = (1 + 1/t)$ ,  $\delta(t) = t - \pi$ ,  $\tau(t) = t - 3\pi$ ,  $q(t) = t$  and  $r(t) = t^2$ , then all conditions of Theorem 13 are satisfied for (1.2) on  $[3\pi, \infty)$ . Hence every solution of (1.2) oscillates. In particular,  $y(t) = \sin t$  is an oscillatory solution of (1.2).

#### 4. Summary

Our method suggests that the comparison results are not necessary to study the oscillatory behaviour of solutions of the equations like (1.1), (1.2) and (1.3). It is interesting to apply our method to the nonlinear form of (1.3) as

$$(r(t)[y(t) + p(t)y(\delta(t))]')' + q(t)G(y(\tau(t))) + v(t)H(y(\sigma(t))) = 0, \quad (4.1)$$

where  $r, q, v, \delta, \tau, \sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ ,  $p \in C(\mathbb{R}_+, \mathbb{R})$ ,  $G, H \in C(\mathbb{R}, \mathbb{R})$  such that

$$xG(x) > 0, xH(x) > 0 \text{ for } x \neq 0, \text{ and } \delta(t) \leq t, \tau(t) \leq t, \sigma(t) \leq t$$

with  $\lim_{t \rightarrow \infty} \delta(t) = \infty = \lim_{t \rightarrow \infty} \tau(t) = \infty = \lim_{t \rightarrow \infty} \sigma(t)$ .

Using our method, we state here two results without proofs:

**THEOREM 17.** *Let  $0 \leq p(t) \leq a < \infty$  for every  $t > 0$ . Assume that  $(H_1)$  holds. If the following assumptions hold:*

$(H_{18}) \tau(\delta(t)) = \delta(\tau(t)), \sigma(\delta(t)) = \delta(\sigma(t))$  for every  $t > 0$ ,

$(H_{19})$  there exist  $\lambda, \mu > 0$  such that

$$G(u) + G(v) \geq \lambda G(u + v), H(u) + H(v) \geq \mu H(u + v)$$

for  $u, v \in \mathbb{R}$  and  $u, v > 0$  (see for e.g. [14]),

$(H_{20}) G(uv) \leq G(u)G(v), H(uv) \leq H(u)H(v), G(-u) = -G(u), H(-u) = -H(u)$  for all  $u, v \in \mathbb{R}$ ,

and

$(H_{21}) \int_{t_0}^{\infty} [Q(t) + kV(t)] dt = \infty, k > 0$ , where  $Q(t) = \min\{q(t), q(\delta(t))\}$  and  $V(t) = \min\{v(t), v(\delta(t))\}$ ,

then every solution of (4.1) oscillates.

**THEOREM 18.** *Let  $0 \leq p(t) \leq a < \infty$  for every  $t > 0$ . Assume that  $(H_2)$  and  $(H_{18}) - (H_{20})$  hold. If:*

$(H_{22}) \int_{t_0}^{\infty} [Q(t)G(R(\tau(t))) + kV(t)H(R(\sigma(t)))] dt = \infty, R(t) = \int_t^{\infty} \frac{ds}{r(s)}, k > 0$ ,

and

$(H_{23}) \int_0^{\infty} \frac{1}{r(t)} \int_T^t [Q(s)G(R(\tau(s))) + kV(s)H(R(\sigma(s)))] ds dt = \infty, \forall T > 0, k > 0$ ,

then every solution of (4.1) oscillates.

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