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FREDHOLM ALTERNATIVE FOR THE SECOND ORDER DIFFERENTIAL OPERATOR ASSOCIATED TO A CLASS OF BOUNDARY CONDITIONS

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Abstract. This work is concerned with the Fredholm property of the second order differential operator associated to a class of boundary conditions. Several sufficient conditions will be proved along with constructing the generalized inverse for such operator. The result is a basic tool to analysis the boundary value problems at resonance for nonlinear perturbation of such operators.

1. Introduction

The question of the solvability for nonlinear pertubation of differential operators have been extensively studied. To indentify a few, we refer the reader to [1, 2, 3, 4, 7, 8, 10, 12, 13, 16, 17, 18] and references therein. Almost such problems, written in operator form, is of the type

$$\mathscr{L}x = Nx, \tag{1.1}$$

where $\mathscr{L}$ is a linear mapping between two Banach spaces X and Z, while $N : X \to Z$ is a nonlinear mapping. When studying problem (1.1), one is often confronted with the difficulty that the relevant linearized operator is not invertible in suitable function spaces. There have been some methods to overcome this obstacle as the alternative method [1, 17], the pertubation method (the name was proposed by Kannan [9]) or continuation method of Mawhin [4]. One important ingredient to be able to apply these abstract results is proving the Fredholm property of the operator $\mathscr{L}$. Throught this paper we use the terms "Fredholm property" of an operator acting from a Banach space to another Banach space if that operator is a Fredholm operator or a Fredholm operator of index zero or a Fredholm operator of positive index. The definitions of such operators are given as follows.

DEFINITION 1. Let X, Z be two Banach spaces. A linear operator $\mathscr{L} : \operatorname{dom}\mathscr{L} \subset X \to Z$ is called to be a Fredholm operator if the following conditions hold

(a) $\ker\mathscr{L}$ has finite dimension;

(b) $\operatorname{Im}\mathscr{L}$ is a closed subset of Z and has finite codimension.

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If $\mathscr{L}$ is a Fredholm operator, the index of $\mathscr{L}$ is the interger

$$i(\mathscr{L}) := \dim\ker\mathscr{L} - \operatorname{codim}\mathscr{L}.$$

DEFINITION 2. The operator $\mathscr{L}$ is called a Fredholm operator of index zero (resp. a Fredholm operator of positive index) if it is a Fredholm operator and $i(\mathscr{L}) = 0$ (resp. $i(\mathscr{L}) > 0$).

In recent years, the Fredholm properties of differential operators in connection with various problems have been discussed in many great papers, see [5, 6, 11, 14, 15] and references therein. For the differential operators associated to multi-point boundary conditions, this property is often proved by constructing first a continuous projector $Q : Z \to Z$ which satisfies $\ker Q = \operatorname{Im}\mathscr{L}$. And then it follows that $\operatorname{Im}\mathscr{L}$ has finite codimension as well as index of $\mathscr{L}$ is equal to zero (see [2, 3, 7, 10, 12, 13, 18]). However, it seems that the construction a such projector is difficult when $\dim\ker\mathscr{L}$ is large ([10]-[13]). So looking for the sufficient conditions to ensure Fredholm property of $\mathscr{L}$ is quite limited.

The goal of current paper is to study the differential operator $\mathscr{L} : \operatorname{dom}\mathscr{L} \subset X \to Z$ defined by

$$\mathscr{L}x(t) = \frac{d^2x}{dt^2}(t) = x''(t),$$

where $X := C^1\left([0,1];\mathbb{R}^d\right)$, $Z := L^1\left((0,1);\mathbb{R}^d\right)$ endowed with their usual norms and

$$\operatorname{dom}\mathscr{L} = \left\{ x \in X, x'' \in Z \text{ and } \left\{ \begin{array}{l} Ax(0) + Bx'(0) = D\int_0^1 x(s)ds \\ \\ Ex(1) + Fx'(1) = G\int_0^1 x(s)ds \end{array} \right. \right\},$$

with A, B, D and E, F, G are square matrices of order d. Our results mentioned two issues. The first is looking for the conditions of coefficient matrices for which the operator $\mathscr{L}$ is the Fredholm operator of index zero (section 2) and the second is characterizing the set of all right-hand side functions $y \in Z$ for which the equation $\mathscr{L}x = y$ has at least one solution $x \in \operatorname{dom}\mathscr{L}$ (section 3). To the best of our knowledge, the that issues have not been developed in general cases of dimension of the kernel. Furthermore our method involves several new ideas and gives a unified method of attack for many boundary value problems at resonance. Previous paper dealt with one problem at a time whereas our method allows us to solve many problem at once.

We end this section by noting that our results can be used to discuss the solvability of equation

$$\mathscr{L}x(t) = f\left(t, x(t), x'(t)\right), t \in (0,1),$$

on $\operatorname{dom}\mathscr{L}$ by using the Mawhin's continuation theorem. This can be done by standard arguments (see [4, 10, 13]). However we will not state here.

2. Fredholm property of the operator $\mathscr{L}$

In the rest of paper we use the following notations

- ◇ $\mathbb{O}$ is zero matrix of order $d \times d$ and θ is zero element of $\mathbb{R}^d$.
- ◇ I_d : the identity matrix of order d.
- ◇ $\mathbb{I}^\nu z(t) := \int_0^t (t-s)^{\nu-1} z(s)ds$, for all $z \in Z$ and $\nu \in \{1,2\}$.

First it's necessary to note that, for $x \in \mathrm{dom}\mathscr{L}$, we can write

$$x(t) = x(0) + x'(0)t + \mathbb{I}^2 \mathscr{L}x(t). \tag{2.1}$$

So, when $x \in \mathrm{dom}\mathscr{L}$, the boundary conditions $\begin{cases} Ax(0) + Bx'(0) = D\int_0^1 x(s)ds \\ Ex(1) + Fx'(1) = G\int_0^1 x(s)ds \end{cases}$ are equivalent to

$$\mathscr{A}\left(x(0), x'(0)\right)^T = \mathscr{B}\left(\mathscr{L}x\right), \tag{2.2}$$

where

- $\mathscr{A} = \begin{pmatrix} A-D & B-\frac{1}{2}D \\ E-G & E+F-\frac{1}{2}G \end{pmatrix}$
- $\mathscr{B} : Z \to \mathbb{R}^{2d}$ is a continuous linear mapping defined by

$$\mathscr{B}(z) = \left(\underbrace{D\int_0^1 \mathbb{I}^2 z(t)dt}_{\mathscr{B}_1(z)}, \underbrace{G\int_0^1 \mathbb{I}^2 z(t)dt - \left(E\mathbb{I}^2 z(1) + F\mathbb{I}^1 z(1)\right)}_{\mathscr{B}_2(z)} \right), z \in Z. \tag{2.3}$$

Therefore $\mathrm{dom}\mathscr{L}$ can be represented as follows

$$\mathrm{dom}\mathscr{L} = \left\{ x(t) = x(0) + x'(0)t + \mathbb{I}^2\mathscr{L}x(t), t \in [0,1] : \mathscr{A}\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \mathscr{B}\left(\mathscr{L}x\right) \right\}.$$

LEMMA 1. *We have*

$$\ker\mathscr{L} = \{x \in X : x(t) = c_0 + c_1 t, t \in [0,1], (c_0, c_1) \in \ker\mathscr{A}\} \cong Ker\mathscr{A},$$

and $\mathrm{Im}\mathscr{L} = \{z \in Z : \mathscr{B}(z) \in \mathrm{Im}\mathscr{A}\}$.

Proof. The proof of this Lemma is straightforward and we will omit the details.

THEOREM 1. *The operator $\mathscr{L}$ is the Fredholm operator. Moreover the index of $\mathscr{L}$ is*

- *zero if* $\dim(\mathrm{Im}\mathscr{A} + \mathrm{Im}\mathscr{B}) = 2d$.
- *positive if* $\dim(\mathrm{Im}\mathscr{A} + \mathrm{Im}\mathscr{B}) < 2d$.

To prove this Theorem we need the following lemma

LEMMA 2. *Let E,F be two vector spaces on field $\mathscr{K}$, $\mathscr{T}$ be a linear operator from E into F. Assume that $\mathscr{V}$ be a subspaces of F. If $\mathscr{U}$ is an algebraic complement of $\mathscr{T}^{-1}(\mathscr{V})$ in E then $\mathscr{U}$ is isomorphic to any algebraic complement of $\mathscr{V}\cap\mathscr{T}(E)$ in $\mathscr{T}(E)$.*

Proof. Let $\mathscr{T}_{\mathscr{U}}$ be the restriction of $\mathscr{T}$ on $\mathscr{U}$. Since $\mathscr{T}_{\mathscr{U}}^{-1}(0)\subset\mathscr{U}\cap\mathscr{T}^{-1}(\mathscr{V})=\{0\}$ it is easy to see that $\mathscr{T}_{\mathscr{U}}$ is an isomorphism from $\mathscr{U}$ into $\mathscr{T}(\mathscr{U})$ which means that $\mathscr{U}$ is isomorphic to $\mathscr{T}(\mathscr{U})$. So it is sufficient to show that $\mathscr{T}_{\mathscr{U}}(\mathscr{U})\equiv\mathscr{T}(\mathscr{U})$ is an algebraic complement of $\overline{\mathscr{V}}:=\mathscr{V}\cap\mathscr{T}(E)$ in $\mathscr{T}(E)$.

Indeed, for any $y\in\mathscr{T}(E)$, there exist $x_1\in\mathscr{T}^{-1}(\mathscr{V})$ and $x_2\in\mathscr{U}$ such that

$$y=\mathscr{T}x_1+\mathscr{T}x_2.$$

Since $\mathscr{T}x_1\in\overline{\mathscr{V}}$ and $\mathscr{T}x_2\in\mathscr{T}(\mathscr{U})$ the above equality implies that $\mathscr{T}(E)=\overline{\mathscr{V}}+\mathscr{T}(\mathscr{U})$. On the other hand, if $y_0\in\overline{\mathscr{V}}\cap\mathscr{T}(\mathscr{U})$, then $y_0=\mathscr{T}(x_0)$, where

$$x_0\in\mathscr{T}^{-1}(\mathscr{V})\cap\mathscr{U}=\{0\}.$$

Therefore $y_0=\mathscr{T}(0)=0$. This implies $\overline{\mathscr{V}}\cap\mathscr{T}(\mathscr{U})=\{0\}$. So $\mathscr{T}(E)=\overline{\mathscr{V}}\oplus\mathscr{T}(\mathscr{U})$. The proof of Lemma is complete.

Proof of Theorem 1 Since $\mathscr{B}$ is continuous from Z into $\mathbb{R}^{2d}$ and $\operatorname{Im}\mathscr{A}$ is closed in $\mathbb{R}^{2d}$ it is clear that $\operatorname{Im}\mathscr{L}$ is a closed subspace of Z. Further we have $\dim\ker\mathscr{L}=\dim\ker\mathscr{A}<\infty$. So it remains to show that $\operatorname{codim}\operatorname{Im}\mathscr{L}=\dim\ker\mathscr{L}$. Indeed, by using Lemma 2, if Z_0 is an algebraic complement of $\operatorname{Im}\mathscr{L}$ in Z then Z_0 isomorphic to any algebraic complement of $\operatorname{Im}\mathscr{A}\cap\operatorname{Im}\mathscr{B}$ in $\operatorname{Im}\mathscr{B}$. So

$$\begin{aligned}\operatorname{codim}\operatorname{Im}\mathscr{L}&=\dim(Z/\operatorname{Im}\mathscr{L})=\dim Z_0\\&=\dim(\operatorname{Im}\mathscr{B}/(\operatorname{Im}\mathscr{A}\cap\operatorname{Im}\mathscr{B}))\\&=\dim\operatorname{Im}\mathscr{B}-\dim(\operatorname{Im}\mathscr{A}\cap\operatorname{Im}\mathscr{B})\\&=\dim(\operatorname{Im}\mathscr{A}+\operatorname{Im}\mathscr{B})-\dim\operatorname{Im}\mathscr{A}\end{aligned}$$

This implies that $\operatorname{codim}\operatorname{Im}\mathscr{L}<\infty$ and so $\mathscr{L}$ is the Fredholm operator. The remains is evident. The proof of Lemma is complete. □

Next we will provide some sufficient conditions for $\mathscr{L}$ to be a Fredholm operator of index zero. First we need the following lemma:

LEMMA 3. *The image of $\mathscr{B}$ can be defined by*

$$\operatorname{Im}\mathscr{B}=\left\{(D\alpha,\,E\beta+F\gamma+G\alpha)\in\mathbb{R}^d\times\mathbb{R}^d:\alpha,\beta,\gamma\in\mathbb{R}^d\right\}.$$

Consequently, if $\operatorname{Im}G\subset\operatorname{Im}E+\operatorname{Im}F$ *then* $\operatorname{Im}\mathscr{B}=\operatorname{Im}D\times(\operatorname{Im}E+\operatorname{Im}F)$.

Proof. Recall that $\mathscr{B}(z) = (\mathscr{B}_1(z), \mathscr{B}_2(z))$, where

$$\mathscr{B}_1(z) = D\int_0^1 dt \int_0^t (t-s)z(s)ds,$$
$$\mathscr{B}_2(z) = G\int_0^1 dt \int_0^t (t-s)z(s)ds - E\int_0^1 (1-s)z(s)ds - F\int_0^1 z(s)ds.$$

Clearly it's suffit to prove the inclusion supset $(\supset)$. Let (ξ, ζ) be an element of the set

$$\left\{(D\alpha, E\beta + F\gamma + G\alpha) \in \mathbb{R}^d \times \mathbb{R}^d : \alpha, \beta, \gamma \in \mathbb{R}^d\right\}.$$

Then we can write $\xi = D\alpha_1$, and $\zeta = E\alpha_2 + F\alpha_3 + G\alpha_1$, where $\alpha_i = (\alpha_{i1}, \alpha_{i2}, ..., \alpha_{id}) \in \mathbb{R}^d$. We consider the function

$$z(t) = (z_1(t), z_2(t), ..., z_d(t)),\ t \in [0,1],$$

where $z_j(t) = a_j + b_j t + c_j t^2$ with the coefficients a_j, b_j, c_j be choosen such that

$$\begin{cases} \int_0^1 dt \int_0^t (t-s)z(s)ds = \alpha_{1j}, \\ \int_0^1 (1-s)z_j(s)ds = -\alpha_{2j}, \\ \int_0^1 z_j(s)ds = -\alpha_{3j}. \end{cases}$$

It is note that by some simple calculations we can see that the above system of linear equation has a unique solution. Hence we can deduce that $\xi = D\int_0^1 dt \int_0^t (t-s)z(s)ds = \mathscr{B}_1(z)$ and

$$\zeta = G\int_0^1 dt \int_0^t (t-s)z(s)ds - E\int_0^1 (1-s)z(s)ds - F\int_0^1 z(s)ds = \mathscr{B}_2(z),$$

which implies $(\xi, \zeta) \in \operatorname{Im}\mathscr{B}$. So

$$\operatorname{Im}\mathscr{B} = \left\{(D\alpha, E\beta + F\gamma + G\alpha) \in \mathbb{R}^d \times \mathbb{R}^d : \alpha, \beta, \gamma \in \mathbb{R}^d\right\}.$$

Now assume that $\operatorname{Im}G \subset \operatorname{Im}E + \operatorname{Im}F$. If $(\xi, \zeta) \in \operatorname{Im}D \times (\operatorname{Im}E + \operatorname{Im}F)$ then there exist $\alpha, \beta, \gamma \in \mathbb{R}^d$ such that

$$\xi = D\alpha \quad \text{and} \quad \zeta = E\beta + F\gamma.$$

On the other hand, it's clear that there are $\alpha_1, \alpha_2 \in \mathbb{R}^d$ such that $E\alpha_1 + F\alpha_2 = G\alpha$. So we can write

$$\xi = D\alpha \quad \text{and} \quad \zeta = E(\beta - \alpha_1) + F(\gamma - \gamma_2) + G\alpha,$$

which implies $(\xi, \zeta) \in \operatorname{Im}\mathscr{B}$. Hence it is easy to see that $\operatorname{Im}\mathscr{B} = \operatorname{Im}D \times (\operatorname{Im}E + \operatorname{Im}F)$. The Lemma has been proved.

COROLLARY 1. *The operator $\mathscr{L}$ is the Fredholm operator of index zero provided that one of following conditions holds*

(a) $\det(D) \neq 0$ *and* $\det(E) \neq 0$,

(b) $\det(D) \neq 0$ *and* $\det(F) \neq 0$,

(c) $\det(D) \neq 0$ *and* $\det(E+F) \neq 0$,

Proof. Using lemma 3 it's easy to prove that if one of above conditions holds then $\operatorname{Im}\mathscr{B} = \mathbb{R}^{2d}$. So by theorem 1 $\mathscr{L}$ is a Fredholm operator of index zero.

COROLLARY 2. *The operator $\mathscr{L}$ is the Fredholm operator of index zero provided that one of following conditions hold*

(a) $2B = D$, $\det\left(E+F-\dfrac{G}{2}\right) \neq 0$, *and* $\operatorname{Im}(A-D) + \operatorname{Im}D = \mathbb{R}^d$,

(b) $\det(A-D) \neq 0$ *and one of determinants* $\det(E)$, $\det(F)$, $\det(E+F)$ *is not equal to zero,*

(c) $\det(2B-D) \neq 0$ *and one of determinants* $\det(E)$, $\det(F)$, $\det(E+F)$ *is not equal to zero.*

Proof. (a) In this case it's clear that $\operatorname{Im}\mathscr{A} = \operatorname{Im}(A-D) \times \mathbb{R}^d$. By combining this equality and lemma 3 we get

$$\operatorname{Im}\mathscr{A} + \operatorname{Im}\mathscr{B} = (\operatorname{Im}(A-D) + \operatorname{Im}D) \times \mathbb{R}^d$$

and so $\dim(\operatorname{Im}\mathscr{A} + \operatorname{Im}\mathscr{B}) = 2d$. From theorem 1 we deduce that $\mathscr{L}$ is the Fredholm operator of index zero.

(b) We will check that $\operatorname{Im}\mathscr{A} + \operatorname{Im}\mathscr{B} = \mathbb{R}^d \times \mathbb{R}^d$. Indeed, for $u, v \in \mathbb{R}^d$, the system

$$\begin{cases} (A-D)x + \left(B - \frac{1}{2}D\right)y + D\alpha = u \\ (E-G)x + \left(E+F-\frac{1}{2}G\right)y + G\alpha + E\beta + F\gamma = v \end{cases}$$

has at least one solution defined by $x = (A-D)^{-1}u$, $y = \alpha = 0$ and

$$\begin{cases} \beta = E^{-1}(v-(E-G)x),\ \gamma = 0, & \text{if } \det(E) \neq 0, \\ \beta = 0,\ \gamma = F^{-1}(v-(E-G)x), & \text{if } \det(F) \neq 0, \\ \beta = \gamma = (E+F)^{-1}(v-(E-G)x), & \text{if } \det(E+F) \neq 0. \end{cases}$$

This show that $(u,v) \in \operatorname{Im}\mathscr{A} + \operatorname{Im}\mathscr{B}$ which implies $\operatorname{Im}\mathscr{A} + \operatorname{Im}\mathscr{B} = \mathbb{R}^d \times \mathbb{R}^d$. So $\mathscr{L}$ is the Fredholm operator of index zero by using theorem 1.

(c) This case can be proved similarly.

REMARK 1. By using lemma 3 and the analysis of matrix $\mathscr{A}$ we can obtain some various sufficient conditions which are similar the corollaries 1 and 2.

3. The generalized inverse of operator $\mathscr{L}$

Assume that $\dim(\operatorname{Im}\mathscr{A}+\operatorname{Im}\mathscr{B})=2d$. It follows from Theorem 1 that there exist continuous projectors $P:X\to X$ and $Q:Z\to Z$ such that

$$\operatorname{Im}P=\operatorname{Ker}\mathscr{L},\quad \operatorname{Ker}Q=\operatorname{Im}\mathscr{L},\quad X=\operatorname{Ker}\mathscr{L}\oplus\operatorname{Ker}P,\quad Z=\operatorname{Im}\mathscr{L}\oplus\operatorname{Im}Q.$$

Moreover the restriction of $\mathscr{L}$ on $\operatorname{dom}\mathscr{L}\cap\ker P$, $\mathscr{L}_P$, is an isomorphism from $\operatorname{dom}\mathscr{L}\cap\ker P$ onto $\operatorname{Im}\mathscr{L}$. Then the generalized inverse of $\mathscr{L}$ is defined by $\mathscr{K}_{P,Q}=\mathscr{L}_P^{-1}(I-Q)$.

The construction such projectors as well as the generalized inverse of $\mathscr{L}$ is very important to study the linear equation

$$\mathscr{L}x=y,$$

on $\operatorname{dom}\mathscr{L}$ as well as the correspondent nonlinear boundary value problems at resonance by using the Mawhin's continuation theorem (see [4, 10, 18, 13] and references therein). In the following we will present a general way to construct the projectors Q, P and pseudo inverse $\mathscr{K}_{P,Q}$. For this aim we denote the orthogonal complement of $\operatorname{Im}\mathscr{A}\cap\operatorname{Im}\mathscr{B}$ in $\operatorname{Im}\mathscr{B}$ by Σ and assume that $\{e_i : i=1,2,\dots,m\}$ is an its basis. Denoted by $\mathscr{M}$ the matrix whose j^{th} column is e_j. It is well known that

$$P_{\mathscr{M}}=\mathscr{M}\left(\mathscr{M}^T\mathscr{M}\right)^{-1}\mathscr{M}^T$$

is the orthogonal projection matrix with $\operatorname{Im}P_{\mathscr{M}}=\Sigma$.

In order to construct the projector Q we first note that if $z=c_0+c_1t+c_2t^2$ then $\mathscr{B}(z)=\mathscr{D}\begin{pmatrix}c_0\\c_1\\c_2\end{pmatrix}$, where

$$\mathscr{D}=\begin{pmatrix} D/6 & D/24 & D/60\\ G/6-E/2-F & G/24-E/6-F/2 & G/60-E/12-F/3\end{pmatrix}$$

By some lengthy calculations we can prove the equality $\operatorname{Im}\mathscr{B}=\operatorname{Im}\mathscr{D}$. On the other hand it's clear that the restriction $\mathscr{D}_*$ of $\mathscr{D}$ on the orthogonal complement $\ker\mathscr{D}^\perp$ of $\ker\mathscr{D}$ is an isomorphism from $\ker\mathscr{D}^\perp$ onto $\operatorname{Im}\mathscr{D}$. For $i=1,2,\dots,m$, we put

$$\omega_i(t)=\omega_i^0+\omega_i^1t+\omega_i^2t^2,\ \text{where}\ \left(\omega_i^0,\omega_i^1,\omega_i^2\right)=\mathscr{D}_*^{-1}e_i,$$

and consider the subspace Z_0 of Z which is spanned by the vectors $\{\omega_i\}_{i=1}^m$. Then it's evident that

$$\dim Z_0=\dim(\operatorname{Im}\mathscr{A}\cap\operatorname{Im}\mathscr{B})^\perp=\dim\operatorname{Im}\mathscr{B}-\dim(\operatorname{Im}\mathscr{A}\cap\operatorname{Im}\mathscr{B})=\dim\ker\mathscr{A}.$$

We can also show that Z_0 is an complement of $\operatorname{Im}\mathscr{L}$ in Z. Moreover,

$$z\in\operatorname{Im}\mathscr{L}\Leftrightarrow\langle\mathscr{B}z,e_i\rangle=0,\ \forall i\in\{1,2,\dots,m\}.$$

Now we define the linear map $Q: Z \to Z$ by

$$Q(z) = \sum_{i=1}^{m} \lambda_i \omega_i(t), \ \forall z \in Z, \tag{3.1}$$

where $(\lambda_1, \lambda_2, ..., \lambda_m)$ is the unique solution of the system linear equation $\sum_{i=1}^{m} \lambda_i e_i = P_{\mathscr{M}} \mathscr{B}(z)$. It's not difficult to prove that Q is a continuous projector on Z and

$$\operatorname{Im} Q = Z_0, \quad \ker Q = \operatorname{Im} \mathscr{L} \quad \text{and} \quad Z = \operatorname{Im} \mathscr{L} \oplus \operatorname{Im} Q.$$

Now we shall construct the projector P and the map $\mathscr{L}_P^{-1}$. Let $\mathscr{A}^+$ be the Moore-Penrose pseudoinverse of $\mathscr{A}$. For $x \in X$ we put

$$Px = (1,t)\left(\mathrm{I}_{2d} - \mathscr{A}^+ \mathscr{A}\right)\left(x(0), x'(0)\right)^T, \tag{3.2}$$

Here if $(\alpha, \beta) \in \mathbb{R}^d \times \mathbb{R}^d$ then the notation $(1,t)(\alpha,\beta)^T$ stand for $\alpha + \beta t$. Since $\mathrm{I}_{2d} - \mathscr{A}^+ \mathscr{A}$ is an orthogonal projector on $\ker \mathscr{A}$ it's not difficult to check that P is a continuous projector on X. Furthermore we have

$$\operatorname{Im} P = \ker \mathscr{L}, \quad X = \ker \mathscr{L} \oplus \ker P.$$

LEMMA 4. *The operator $\mathscr{L}$ on $\operatorname{dom} \mathscr{L} \cap \ker P$ is invertible and its inverse $\mathscr{L}_P^{-1}$ is defined by*

$$\mathscr{L}_P^{-1} z(t) = (1,t)\mathscr{A}^+ \left(\mathscr{B}(z)\right)^T + \mathbb{I}^2 z(t), \ t \in [0,1], \quad z \in \operatorname{Im} \mathscr{L}. \tag{3.3}$$

Moreover, there exists positive constant C such that $\mathscr{L}_P^{-1}$ satisfies the following estimate

$$\left\| \mathscr{L}_P^{-1} z \right\| \leqslant C \|z\|, \forall z \in \operatorname{Im} \mathscr{L}. \tag{3.4}$$

Proof. Let $z \in \operatorname{Im} \mathscr{L}$ then it follows from (3.3) that

$$\left(\mathscr{L}_P^{-1} z(0), (\mathscr{L}_P^{-1} z)'(0)\right) = \left(\mathscr{A}^+ \mathscr{B}(z)\right)^T. \tag{3.5}$$

Hence it follows that $P \mathscr{L}_P^{-1} z = \theta$, that is, $\mathscr{L}_P^{-1} z \in \ker P$. Moreover by combining (3.3) and (3.5) we obtain

$$\mathscr{A}\left(\mathscr{L}_P^{-1} z(0), \left(\mathscr{L}_P^{-1} z\right)'(0)\right)^T = \mathscr{A}\mathscr{A}^+ \mathscr{B}(z) = \mathscr{B}(z).$$

This show that $\mathscr{L}_P^{-1} z \in \operatorname{dom} \mathscr{L}$ and so $\mathscr{L}_P^{-1}$ is well-defined.

On the other hand it's clear that $\mathscr{L}\mathscr{L}_P^{-1} z(t) = z(t)$, for all $t \in [0,1]$ and $z \in \operatorname{Im} \mathscr{L}$. Moreover, if $x \in \operatorname{dom} \mathscr{L} \cap \ker P$ then we have

$$\mathscr{L}_P^{-1} \mathscr{L} x(t) = (1,t)\mathscr{A}^+ \mathscr{B}(\mathscr{L} x) + \mathbb{I}^2 \mathscr{L} x(t) \tag{3.6}$$

for all $t \in [0,1]$. Since $x \in \operatorname{dom} \mathscr{L} \cap \ker P$ it follows that

$$\mathscr{A}\left(x(0), x'(0)\right)^T = \mathscr{B}(\mathscr{L} x) \quad \text{and} \quad \mathscr{A}^+ \mathscr{A}\left(x(0), x'(0)\right)^T = \left(x(0), x'(0)\right)^T. \tag{3.7}$$

By combining (3.6), (3.7) we get $\mathscr{L}_P^{-1}\mathscr{L}x(t) = x(t), \forall t \in [0,1]$.

Finally, by using (3.3) and the continuity of $\mathscr{B}$ we can obtain the estimate (3.4) and the proof of Lemma is complete.

The main result is presented as below. Its proof is directly corollary of above analysis.

THEOREM 2. *Let $\mathscr{H} = \{z \in Z : \langle \mathscr{B}(z), e_i \rangle = 0, \forall i \in \{1,2,...,m\}\}$. The the equation $\mathscr{L}x = y$ has*

(i) no solution if $y \notin \mathscr{H}$,

(ii) at least one solution defined by $x = \mathscr{L}_P^{-1}(y)$ if $h \in \mathscr{H}$.

4. Examples

This section presents some examples which is to illustrate our results.

EXAMPLE 1. (Second order differential operators with Sturm-Liouville type conditions) Let $d = 1$ and

$$A = \alpha,\ B = \beta,\ E = \gamma,\ F = \delta \quad \text{and} \quad D = G = 0,$$

where $\alpha, \beta, \gamma, \delta$ are real numbers which satisfy the conditions $\gamma^2 + \delta^2 \neq 0$ and $\alpha(\gamma + \delta) = \beta\gamma$. We have

$$\mathscr{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \gamma+\delta \end{bmatrix} \quad \text{and} \quad \mathscr{B}z = \left(0, -\gamma\mathbb{I}^2 z(1) - \delta\mathbb{I}^1 z(1)\right).$$

Without loss of generality we can assume $\gamma \neq 0$. Then it's not difficult to prove that $\operatorname{Im}\mathscr{B} = \{0\} \times \mathbb{R}$,

$$\operatorname{Ker}\mathscr{A} = \left\{ \left(-\frac{\gamma+\delta}{\gamma}\xi, \xi \right) : \xi \in \mathbb{R} \right\} \quad \text{and} \quad \operatorname{Im}\mathscr{A} = \left\{ \left(\zeta, \frac{\alpha}{\gamma}\zeta \right) : \zeta \in \mathbb{R} \right\}.$$

This implies $\dim(\operatorname{Im}\mathscr{B} + \operatorname{Im}\mathscr{A}) = 2$ and so $\mathscr{L}$ is the Fredholm operator of index zero by theorem (1).

Next we note that $\Sigma = \operatorname{Im}\mathscr{B} = \{0\} \times \mathbb{R}$ has a basis $\{e_1 = (0,1)\}$ and $P_{\mathscr{M}}$ is the identity mapping on $\operatorname{Im}\mathscr{B}$. On the other hand, since

$$\mathscr{D} = \begin{pmatrix} 0 & 0 & 0 \\ -\rho_1 & -\rho_2 & -\rho_3 \end{pmatrix},$$

with $\rho_1 = \frac{\gamma}{2} + \delta, \rho_2 = \frac{\gamma}{6} + \frac{\delta}{2}, \rho_3 = \frac{\gamma}{6} + \frac{\delta}{2}$, we can show easily that

$$\ker \mathscr{D}^\perp = \{\lambda\,(\rho_1, \rho_2, \rho_3) : \lambda \in \mathbb{R}\},$$

and $\mathscr{D}_*^{-1} : \{0\} \times \mathbb{R} \to \ker \mathscr{D}^{\perp}$ defined by

$$\mathscr{D}_*^{-1}(0,\zeta) = \frac{-\zeta}{\rho_1^2+\rho_2^2+\rho_3^2}(\rho_1,\rho_2,\rho_3), \forall \zeta \in \mathbb{R}.$$

Hence $\omega_1(t) = -\dfrac{\rho_1+\rho_2 t+\rho_3 t^2}{\rho_1^2+\rho_2^2+\rho_3^2}$, for all $t \in [0,1]$. So the projector Q be defined by following formula

$$Qz(t) = \frac{\gamma \mathbb{I}^2 z(1) + \delta \mathbb{I}^1 z(1)}{\rho_1^2+\rho_2^2+\rho_3^2}\left(\rho_1+\rho_2 t+\rho_3 t^2\right).$$

Now it's not difficult to show that $\mathscr{A}^+ = \begin{pmatrix} 0 & \gamma^{-1} \\ 0 & 0 \end{pmatrix}$ and $I_2 - \mathscr{A}^+\mathscr{A} = \begin{pmatrix} 0 & \gamma^{-1}(\gamma+\delta) \\ 0 & 1 \end{pmatrix}$. By using (3.2) we can define the projector P by

$$Px(t) = x'(0)\left(-\frac{\gamma+\delta}{\gamma}+t\right), \forall t \in [0,1].$$

Moreover, since $\mathscr{B}(z) = (0,0)$ for all $z \in \operatorname{Im}\mathscr{L}$ it follows from lemma 4 that

$$\mathscr{L}_P^{-1} z(t) = \int_0^t (t-s)z(s)ds,\ t \in [0,1],\ z \in \operatorname{Im}\mathscr{L}.$$

From above analyses we get

Claim 1. The equation $\mathscr{L}x = y$ has at least one solution defined by

$$x(t) = \mathscr{L}_P^{-1} y(t) = \int_0^t (t-s)y(s)ds,\ t \in [0,1]$$

if and only if

$$\gamma \int_0^1 (1-s)y(s)ds + \delta \int_0^1 y(s)ds = 0.$$

EXAMPLE 2. (Second order differential operators with nonlocal boundary conditions) Let

$$A = D = F = \mathbb{O} \quad \text{and} \quad B = E = G = \mathrm{I}_d.$$

In this case we have $\mathscr{A} = \begin{bmatrix} \mathbb{O} & \mathrm{I}_d \\ \mathbb{O} & \frac{1}{2}\mathrm{I}_d \end{bmatrix}$ and $\mathscr{B}(z) = \left(0, \frac{1}{2}\int_0^1 s^2 z(s)ds - \frac{1}{2}\int_0^1 z(s)ds\right)$. It is easy to prove that

$$\operatorname{Im}\mathscr{B} = \{\theta\} \times \mathbb{R}^d, \quad \operatorname{Ker}\mathscr{A} = \mathbb{R}^d \times \{\theta\} \quad \text{and} \quad \operatorname{Im}\mathscr{A} = \{(2\xi,\xi) : \xi \in \mathbb{R}^d\}.$$

Since $\dim(\operatorname{Im}\mathscr{A} + \operatorname{Im}\mathscr{B}) = 2d$ it follows from Theorem 1 that the operator $\mathscr{L}$ is the Fredholm operator of index zero.

Next it's easy to see that $\Sigma = \operatorname{Im}\mathscr{B}$ and $P_{\mathscr{M}}$ is the identity mapping on $\operatorname{Im}\mathscr{B}$. Further by some simple calculations we obtain

$$\mathscr{D} = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \mathbb{O} \\ -\frac{\mathrm{I}_d}{3} & -\frac{\mathrm{I}_d}{8} & -\frac{\mathrm{I}_d}{15} \end{pmatrix},$$

$$\ker \mathscr{D}^{\perp} = \left\{ \left(\alpha, \frac{3\alpha}{8}, \frac{\alpha}{5} \right) \in \mathbb{R}^{3d} : \alpha \in \mathbb{R}^d \right\},$$

and $\mathscr{D}_*^{-1} : \{\theta\} \times \mathbb{R}^d \to \ker \mathscr{D}^{\perp}$ defined by $\mathscr{D}_*^{-1}(\theta,\alpha) = -\frac{4800}{1889}\left(\alpha, \frac{3}{8}\alpha, \frac{1}{5}\alpha\right)$. So by using (3.1) we get the formula of projector Q as follow

$$Qz(t) = -\frac{4800}{1889}\mathscr{B}_2(z)\left(1 + \frac{3}{8}t + \frac{1}{5}t^2\right), t \in [0,1].$$

On the other hand since $\mathscr{A}^+ = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathrm{I}_d & \mathbb{O} \end{pmatrix}$ we can deduce that $Px(t) = x(0)$ for all $t \in [0,1]$. Further it follows easily from Lemma 4 that

$$\mathscr{L}_P^{-1} z(t) = \int_0^t (t-s)z(s)ds, \, t \in [0,1], \, z \in \operatorname{Im}\mathscr{L}.$$

Finally we obtain the following result

Claim 2. The equation $\mathscr{L}x = y$ has at least one solution given by $x(t) = \int_0^t (t-s)y(s)ds$ if and only if

$$\int_0^1 s^2 y(s)ds - \int_0^1 y(s)ds = 0.$$

EXAMPLE 3. Let following matrices

$$\Lambda_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{and} \quad \Lambda_3 = \begin{bmatrix} 1/2 & 2 \\ 2 & 3 \end{bmatrix}.$$

Consider the operator $\mathscr{L}$ on $\operatorname{dom}\mathscr{L}$ with

$$A = E = G = \Lambda_1, \, B = \Lambda_2, \, F = \Lambda_3, \, D = \mathbb{O}.$$

In this case we have

$$\mathscr{A} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix} \text{ and } \mathscr{B}(z) = \begin{pmatrix} 0 \\ 0 \\ \phi(z) \\ \psi(z) \end{pmatrix},$$

where $z(t) = (z_1(t), z_2(t))$ and

$$\phi(z) = \frac{1}{2}\int_0^1 s^2 z_1(s)ds - \int_0^1 z_1(s)ds - 2\int_0^1 z_2(s)ds,$$

$$\psi(z) = \int_0^1 s^2 z_2(s)ds - 2\int_0^1 z_1(s)ds - 4\int_0^1 z_2(s)ds.$$

It's not difficult to show that

- $\ker \mathscr{A} = \{\lambda(0,-1,2,-1) : \lambda \in \mathbb{R}\}$,
- $\operatorname{Im} \mathscr{A} = \{(\alpha,\beta,\gamma,2\gamma) : \alpha,\beta,\gamma \in \mathbb{R}\}$,
- $\operatorname{Im} \mathscr{B} = \{(0,0)\} \times \mathbb{R}^2$.

Since $\operatorname{Im}\mathscr{B} \cap \operatorname{Im}\mathscr{A} = \{(0,0,\gamma,2\gamma) : \gamma \in \mathbb{R}\}$ it follows that $\dim(\operatorname{Im}\mathscr{B} + \operatorname{Im}\mathscr{A}) = 4$ and $\mathscr{L}$ so is the Fredholm operator of index zero.

Construction of the projector Q:

- The orthogonal complement of $\operatorname{Im}\mathscr{A} \cap \operatorname{Im}\mathscr{B}$ in $\operatorname{Im}\mathscr{B}$ is $\Sigma = \{(0,0,-2\gamma,\gamma) : \gamma \in \mathbb{R}\}$. A basis of this subspace is $\{e_1 = (0,0,-2,1)\}$, and so we get

$$P_{\mathscr{M}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 0 & -\frac{2}{5} & \frac{1}{5} \end{pmatrix}.$$

- $$\mathscr{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{6} & -2 & -\frac{3}{8} & -1 & -\frac{7}{30} & -\frac{2}{3} \\ -2 & -\frac{11}{3} & -1 & -\frac{7}{4} & -\frac{2}{3} & -\frac{17}{15} \end{pmatrix}$$

- $\ker \mathscr{D}^{\perp}$ has a basis as follow

$$\left\{\varepsilon_1 = \left(-\frac{5}{6},-2,-\frac{3}{8},-1,-\frac{7}{30},-\frac{2}{3}\right), \varepsilon_2 = \left(-2,-\frac{11}{3},-1,-\frac{7}{4},-\frac{2}{3},-\frac{17}{15}\right)\right\}.$$

- $\mathscr{D}_*^{-1} : \{(0,0)\} \times \mathbb{R}^2 \to \ker \mathscr{D}^{\perp}$ defined by

$$\mathscr{D}_*^{-1}(0,0,\alpha,\beta) = \left(\frac{1204545600}{119586041}\alpha - \frac{623952000}{119586041}\beta\right)\varepsilon_1 + \left(-\frac{623952000}{119586041}\alpha + \frac{328352400}{119586041}\beta\right)\varepsilon_2$$

- the formula of projector Q: $\quad Qz(t) = \left(-\frac{2}{5}\phi(z) + \frac{1}{5}\psi(z)\right)\omega_1(t)$, for all $z \in Z$, where

$$\omega_1(t) = \begin{pmatrix} -\frac{624976800}{119586041} \\ \frac{286479600}{119586041} \end{pmatrix} + \begin{pmatrix} -\frac{25815600}{7034473} \\ \frac{274954500}{119586041} \end{pmatrix} t + \begin{pmatrix} -\frac{343127520}{119586041} \\ \frac{235604880}{119586041} \end{pmatrix} t^2.$$

Construction of the projector P*:* It is noted that

$$\mathscr{A}^{+} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1/2 & -3/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $x(t) = (x_1(t), x_2(t)),\, t \in [0,1]$ we have

$$Px(t) = x_2'(0)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} t\right).$$

The generalized inverse: By the same arguments as above examples we have

$$\mathscr{K}_P z(t) = \phi(z)\left(\begin{pmatrix} -1 \\ -3/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\right) + \int_0^t (t-s)z(s)ds,\, t \in [0,1],\, z \in \operatorname{Im}\mathscr{L}.$$

Claim 3. The equation $\mathscr{L}x = y$ with $y(t) = (y_1(t), y_2(t))$, has at least one solution given by

$$x(t) = \begin{pmatrix} \phi(y)(-1+t) + \int_0^t (t-s)y_1(s)ds \\ -\frac{3}{2}\phi(y) + \int_0^t (t-s)y_2(s)ds \end{pmatrix},\, t \in [0,1]$$

if and only if

$$\int_0^1 s^2\left(y_2(s) - y_1(s)\right)ds = 0.$$

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REFERENCES

[1] L. CESARI, *Functional analysis, nonlinear differential equations and the alternative method*, Proc. Conf., Mich. State Univ., East Lansing, Mich., 1975, pp. 1–197.

[2] W. FENG AND J. R. L. WEBB, *Solvability of three-point boundary value problems at resonance*, Nonlinear Analysis TMA, **30** (1997), 3227–3238.

[3] W. FENG, J. R. L. WEBB, *Solvability of m-point boundary value problems with nonlinear growth*, J. Math. Anal. Appl., **212** (1997), 467–480.

[4] R. E. GAINES, J. MAWHIN, *Coincidence Degree and Nonlinear Differential Equations*, **568**, Lecture Notes in Math., Springer-Verlag, Berlin, 1977.

[5] H. G. GEBRAN, C. A. STUART, *Exponential decay and Fredholm properties in second-order quasilinear elliptic systems*, J. Differential Equations, **249** (2010), 94–117.

[6] D. GHEORGHE, F. H. VASILESCU, *Quotient morphisms, compositions, and Fredholm index*, Linear Algebra and its Applications, **431** (2009), 2049–2061.

[7] C. P. GUPTA, *Existence theorems for a second order m-point boundary value problem at resonance*, Int. J. Math. & Math. Sci., **18** (1995), 705–710.

[8] V. A. IL'IN, E. I. MOISEEV, *Nonlocal boundary value problem of the first kind for a Sturm–Liouville operator*, J. Differential Equations, **23** (1987), 803–810.

[9] R. KANNAN, *Perturbation methods for nonlinear problems at resonance*, Proc. Conf., Mich. State Univ., East Lansing, Mich., 1975, 209–225.

[10] N. KOSMATOV, *A singular non-local problem at resonance*, J. Math. Anal. Appl., **394** (2012), 425–431.
[11] Y. LATUSHKIN AND Y. TOMILOV, *Fredholm differential operators with unbounded coefficients*, J. Differential Equations, **208** (2005), 388–429.
[12] R. MA, *Existence results of a m-point boundary value problem at resonance*, J. Math. Anal. Appl., **294** (2004), 147–157.
[13] P. D. PHUNG, L. X. TRUONG, *On the existence of a three point boundary value problem at resonance in* $\mathbb{R}^n$, J. Math. Anal. Appl., **416** (2014), 522–533.
[14] A. POGAN AND A. SHELL, *Fredholm properties of radially symmetric, second order differential operators*, Int. J. Dyn. Sys. Diff. Eqns., **3** (2011), 289–327.
[15] P. J. RABIER, *Fredholm operators, semigroups and the asymptotic and boundary behavior of solutions of PDEs*, J. Differential Equations, **193** (2003), 460–480.
[16] V. VOUGALTER AND V. VOLPERT, *On the existence of stationary solutions for some non-Fredholm integro-differential equations*, Documenta Mathematica, **16** (2011), 561–580.
[17] S. A. WILLIAMS, *A sharp sufficient condition for solution of nonlinear elliptic boundary value problem*, J. Differential Equations., **8** (1970), 580–586.
[18] X. ZHANG, M. FENG, W. GE, *Existence result of second-order differential equations with integral boundary conditions at resonance*, J. Math. Anal. Appl., **353** (2009), 311–319.

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