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EXISTENCE AND STABILITY RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY AND IMPULSES

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Abstract. In this paper, we establish the existence, uniqueness and Ulam stability of solutions for a class of problem for nonlinear implicit fractional differential equations with impulse and Caputo fractional derivative. The arguments are based upon the Banach contraction principle, and Schaefer's fixed point theorem. We present two examples to show the applicability of our results.

1. Introduction

In this paper, we establish existence, uniqueness and stability results to the following nonlinear implicit fractional differential equation with finite delay and impulses

$$^{c}D^{\alpha}_{t_k}y(t) = f(t, y_t, {}^{c}D^{\alpha}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \dots, m, \ 0 < \alpha \leqslant 1, \quad (1.1)$$

$$\Delta y|_{t_k} = I_k(y_{t_k^-}), \ k = 1, \dots, m, \quad (1.2)$$

$$y(t) = \varphi(t), \ t \in [-r, 0], \ r > 0, \quad (1.3)$$

where $^{c}D^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : PC([-r,0],\mathbb{R}) \to \mathbb{R}$, and $\varphi \in PC([-r,0],\mathbb{R}), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. $PC([-r,0],\mathbb{R})$ is a space of piecewise functions defined on $[-r,0]$ to be specified later (see Section 2).

For each function y defined on $[-r,T]$ and for any $t \in J$, we denote by y_t the element of $PC([-r,0],\mathbb{R})$ defined by:

$$y_t(\theta) = y(t+\theta), \ \ \theta \in [-r,0],$$

$y_t(\cdot)$ represent the history of the state from time $t-r$ up to time t. Here $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where

$$y(t_k^+) = \lim_{h\to 0^+} y(t_k+h) \text{ and } y(t_k^-) = \lim_{h\to 0^-} y(t_k+h)$$

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represent the right and left limits of y_t at $t = t_k$, respectively.

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, biosciences, bioengineering, etc. See for example [1, 6, 7, 15, 20, 26], and references therein. On the other hand, impulsive differential equations have received much attention, we refer the reader to the books [2, 10, 16, 22, 23, 25], and the papers [13, 19, 28], and the references therein. Very recently, fractional differential equations have received considerable attention because they occur in the mathematical modeling of a variety of physical processes; See for example [3, 4, 8, 9, 14, 27, 31]. In [11, 12], the authors gave some existence and uniqueness results for some classes of implicit fractional order differential equations.

Motivated by the works mentioned above, we present in this work some existence, uniqueness and Ulam stability results for a class of problem for implicit fractional differential equations. The present paper is organized as follows. In Section 2, some notations are introduced and we recall some concepts of preliminaries about fractional calculus and auxiliary results. In Section 3, two results for the problem (1.1)-(1.3) are presented: the first one is based on the Banach contraction principle, the second one on Schaefer's fixed point theorem. In Section 4, we present Ulam-Hyers stability result for the problem (1.1)-(1.2). Finally, in the last Section, we give two examples to illustrate the applicability of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $T > 0, J = [0,T]$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

Let $J_0 = [t_0,t_1]$ and $J_k = (t_k,t_{k+1}]$ where $k = 1,\dots,m$.

Consider the set of functions

$$PC([-r,0],\mathbb{R}) = \{y : [-r,0] \to \mathbb{R} : y \in C((\tau_k,\tau_{k+1}],\mathbb{R}),\ k = 0,\dots,m' \text{ and there exist}$$
$$y(\tau_k^-) \text{ and } y(\tau_k^+),\ k = 1,\dots,m \text{ with } y(\tau_k^-) = y(\tau_k)\}.$$

$PC([-r,0],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in[-r,0]} |y(t)|.$$

$PC([-r,T],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC_1} = \sup_{t\in[-r,T]} |y(t)|.$$

$L^1(J,\mathbb{R})$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}$ with the norm

$$\|w\|_1 = \int_0^T |w(s)|ds.$$

$AC^n(J) = \{h : J \to \mathbb{R} : h, h', \dots h^{(n-1)} \in C(J,\mathbb{R})$ and $h^{(n-1)}$ is absolutely continuous$\}$.

In what follows $\alpha > 0$.

DEFINITION 1. ([21, 24]). The fractional (arbitrary) order integral of the function $h \in L^1([0,T],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0.$

DEFINITION 2. ([21, 24]). For a function $h \in AC^n(J)$, the Caputo fractional-order derivative of order α of h, is defined by

$$({}^cD_0^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

DEFINITION 3. ([21, 24]). Let $a \in [0,T], \delta > 0, a + \delta \leqslant T$, for a function $h \in AC^n[a,T]$, the Caputo fractional-order derivative of order α of h, is defined by

$$({}^cD_a^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

LEMMA 1. *([21, 24]) Let $\alpha \geqslant 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^cD_0^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

We need the following auxiliary lemmas.

LEMMA 2. *([21]) Let $\alpha > 0$, then the differential equation*

$${}^cD_0^\alpha k(t) = 0$$

has solutions $k(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, c_i \in \mathbb{R},\ i = 0,1,2,\dots,n-1,\ n = [\alpha] + 1$.

LEMMA 3. *([21]) Let* $\alpha > 0$*, then*

$$I^{\alpha c} D_0^{\alpha} k(t) = k(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1,\ n = [\alpha] + 1.$

LEMMA 4. [30] *Let* $\upsilon : [0,T] \longrightarrow [0,+\infty)$ *be a real function and* $\omega(\cdot)$ *is a non-negative, locally integrable function on* $[0,T]$ *and there are constants* $a > 0$ *and* $0 < \alpha \leqslant 1$ *such that*

$$\upsilon(t) \leqslant \omega(t) + a \int_0^t (t-s)^{-\alpha} \upsilon(s) ds.$$

Then, there exists a constant $K = K(\alpha)$ *such that*

$$\upsilon(t) \leqslant \omega(t) + Ka \int_0^t (t-s)^{-\alpha} \omega(s) ds, \ \textit{for every } t \in [0,T].$$

Bainov and Hristova [5] introduced the following integral inequality of Gronwall type for piecewise continuous functions which can be used in the sequel.

LEMMA 5. *Let for* $t \geqslant t_0 \geqslant 0$ *the following inequality hold*

$$x(t) \leqslant a(t) + \int_{t_0}^t g(t,s) x(s) ds + \sum_{t_0 < t_k < t} \beta_k(t) x(t_k),$$

where $\beta_k(t) (k \in \mathbb{N})$ *are nondecreasing functions for* $t \geqslant t_0$*,* $a \in PC([t_0,\infty),\mathbb{R}_+)$*,* a *is nondecreasing and* $g(t,s)$ *is a continuous nonnegative function for* $t,s \geqslant t_0$ *and nondecreasing with respect to* t *for any fixed* $s \geqslant t_0$*. Then, for* $t \geqslant t_0$*, the following inequality is valid:*

$$x(t) \leqslant a(t) \prod_{t_0 < t_k < t} (1 + \beta_k(t)) exp\left(\int_{t_0}^t g(t,s) ds \right).$$

Here, we adopt the concepts in Wang *et al.* [29] and introduce Ulam's type stability concepts for the problem (1.1)-(1.2).
Let $z \in PC(J,\mathbb{R}), \varepsilon > 0, \psi > 0$ and $\omega \in PC(J,\mathbb{R}_+)$ is nondecreasing. We consider the set of inequalities

$$\begin{cases} |{}^c D^{\alpha} z(t) - f(t, z_t, {}^c D^{\alpha} z(t))| \leqslant \varepsilon, \ t \in (t_k, t_{k+1}], k = 1, \ldots, m \\ |\Delta z|_{t_k} - I_k(z_{t_k^-})| \leqslant \varepsilon, \ k = 1, \ldots, m; \end{cases} \tag{2.1}$$

the set of inequalities

$$\begin{cases} |{}^c D^{\alpha} z(t) - f(t, z_t, {}^c D^{\alpha} z(t))| \leqslant \omega(t), \ t \in (t_k, t_{k+1}], k = 1, \ldots, m \\ |\Delta z|_{t_k} - I_k(z_{t_k^-})| \leqslant \psi, \ k = 1, \ldots, m; \end{cases} \tag{2.2}$$

and the set of inequalities

$$\begin{cases} |{}^c D^{\alpha} z(t) - f(t, z_t, {}^c D^{\alpha} z(t))| \leqslant \varepsilon\omega(t), \ t \in (t_k, t_{k+1}], k = 1, \ldots, m \\ |\Delta z|_{t_k} - I_k(z_{t_k^-})| \leqslant \varepsilon\psi, \ k = 1, \ldots, m. \end{cases} \tag{2.3}$$

DEFINITION 4. The problem (1.1)-(1.2) is Ulam-Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in PC(J,\mathbb{R})$ of the inequality (2.1) there exists a solution $y \in PC(J,\mathbb{R})$ of the problem (1.1)-(1.2) with

$$|z(t) - y(t)| \leqslant c_{f,m}\varepsilon,\ t \in J.$$

DEFINITION 5. The problem (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}_+,\mathbb{R}_+), \theta_{f,m}(0) = 0$ such that for each solution $z \in PC(J,\mathbb{R})$ of the inequality (2.1) there exists a solution $y \in PC(J,\mathbb{R})$ of the problem (1.1)-(1.2) with

$$|z(t) - y(t)| \leqslant \theta_{f,m}(\varepsilon),\ t \in J.$$

DEFINITION 6. The problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to (ω,ψ) if there exists $c_{f,m,\omega} > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in PC(J,\mathbb{R})$ of the inequality (2.3) there exists a solution $y \in PC(J,\mathbb{R})$ of the problem (1.1)-(1.2) with

$$|z(t) - y(t)| \leqslant c_{f,m,\omega}\varepsilon(\omega(t) + \psi),\ t \in J.$$

DEFINITION 7. The problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to (ω,ψ) if there exists $c_{f,m,\omega} > 0$ such that for each solution $z \in PC(J,\mathbb{R})$ of the inequality (2.2) there exists a solution $y \in PC(J,\mathbb{R})$ of the problem (1.1)-(1.2) with

$$|z(t) - y(t)| \leqslant c_{f,m,\omega}(\omega(t) + \psi),\ t \in J.$$

REMARK 1. It is clear that: (i) Definition 4 implies Definition 5; (ii) Definition 6 implies Definition 7; (iii) Definition 6 for $\omega(t) = \psi = 1$ implies Definition 4.

REMARK 2. A function $z \in PC(J,\mathbb{R})$ is a solution of the inequality (2.3) if and only if there is $\sigma \in PC(J,\mathbb{R})$ and a sequence $\sigma_k, k = 1,\dots,m$ (which depend on z) such that

i) $|\sigma(t)| \leqslant \varepsilon\omega(t),\ t \in (t_k,t_{k+1}], k = 1,\dots,m$ and $|\sigma_k| \leqslant \varepsilon\psi, k = 1,\dots,m$;

ii) ${}^cD^\alpha z(t) = f(t,z_t,{}^cD^\alpha z(t)) + \sigma(t),\ t \in (t_k,t_{k+1}], k = 1,\dots,m$;

iii) $\Delta z|_{t_k} = I_k(z_{t_k^-}) + \sigma_k,\ k = 1,\dots,m.$

One can have similar remarks for inequalities 2.2 and 2.1.

THEOREM 1. [18] *(theorem of Ascoli-Arzela). Let $A \subset C(J,\mathbb{R})$, A is relatively compact (i.e $\overline{A}$ is compact) if:*

1. *A is uniformly bounded i.e, there exists $M > 0$ such that*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in J.$$

2. *A is equicontinuous i.e, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x,\overline{x} \in J$, $|x - \overline{x}| \leqslant \delta$ implies $|f(x) - f(\overline{x})| \leqslant \varepsilon$, for every $f \in A$.*

THEOREM 2. *([17]) (**Banach's** fixed point theorem). Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

THEOREM 3. *([17]) (**Schaefer's** fixed point theorem) Let X be a Banach space, and $N : X \longrightarrow X$ completely continuous operator.*
If the set $\mathscr{E} = \{y \in X : \; y = \lambda Ny, \; for some \; \lambda \in (0,1)\}$ is bounded, then N has fixed points.

3. Existence of Solutions

DEFINITION 8. A function $y \in PC([-r,T],\mathbb{R})$ whose α-derivative exists on J_k is said to be a solution of (1.1)-(1.3) if y satisfies the equation ${}^cD^{\alpha}_{t_k}y(t) = f(t,y_t,{}^cD^{\alpha}_{t_k}y(t))$ on J_k, and satisfy the conditions

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \; k = 1,\dots,m,$$
$$y(t) = \varphi(t), \;\; t \in [-r,0].$$

To prove the existence of solutions to (1.1)-(1.3), we need the following auxiliary Lemma.

LEMMA 6. *Let $0 < \alpha \leqslant 1$ and let $\sigma : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\sigma(s)ds & if\, t \in [0,t_1] \\ \\ \varphi(0) + \displaystyle\sum_{i=1}^{k} I_i(y_{t_i^-}) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)ds & \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds, & if\, t \in (t_k,t_{k+1}], \\ \\ \varphi(t), \; t \in [-r,0], \end{cases} \tag{3.1}$$

where $k = 1,\dots,m$, if and only if y is a solution of the following fractional problem

$$ {}^cD^{\alpha}y(t) = \sigma(t), \quad t \in J_k, \tag{3.2}$$
$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \quad k = 1,\dots,m, \tag{3.3}$$
$$y(t) = \varphi(t), \; t \in [-r,0]. \tag{3.4}$$

Proof. Assume y satisfies (3.2)-(3.4). If $t \in [0,t_1]$ then

$${}^cD^{\alpha}y(t) = \sigma(t).$$

Lemma 3 implies

$$y(t) = \varphi(0) + I^{\alpha}\sigma(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)ds.$$

If $t \in (t_1, t_2]$ then Lemma 3 implies

$$\begin{aligned}
y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\
&= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\
&= I_1(y_{t_1^-}) + \left[\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds. \\
&= \varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds.
\end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 3, we get

$$\begin{aligned}
y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}\sigma(s)ds \\
&= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}\sigma(s)ds \\
&= I_2(y_{t_2^-}) + \left[\varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds\right] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}\sigma(s)ds. \\
&= \varphi(0) + \left[I_1(y_{t_1^-}) + I_2(y_{t_2^-})\right] + \left[\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds\right] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}\sigma(s)ds.
\end{aligned}$$

Repeating the process in this ways, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$ where $k = 1, \dots, m$, can be written as

$$\begin{aligned}
y(t) = \varphi(0) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)ds \\
+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}\sigma(s)ds.
\end{aligned}$$

Conversely, assume that y satisfies the impulsive fractional integral equation (3.1). If $t \in [0,t_1]$ then $y(0) = \varphi(0)$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$ {}^cD^\alpha y(t) = \sigma(t) \quad \text{for each } t \in [0,t_1]. $$

If $t \in (t_k,t_{k+1}]$, $k = 1,\dots,m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$ {}^cD^\alpha y(t) = \sigma(t) \text{ for each } t \in (t_k,t_{k+1}]. $$

Also, we can easily show that

$$ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), \quad k = 1,\dots,m. $$

We are now in a position to state and prove our existence result for the problem $(1.1)-(1.3)$ based on Banach's fixed point.

THEOREM 4. *Assume*

(H1) The function $f : J \times PC([-r,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(H2) There exist constants $K > 0$ *and* $0 < L < 1$ *such that*

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant K\|u-\bar{u}\|_{PC} + L|v-\bar{v}| $$

for any $u,\bar{u} \in PC([-r,0],\mathbb{R})$, $v,\bar{v} \in \mathbb{R}$ *and* $t \in J$.

(H3) There exists a constant $\widetilde{l} > 0$ *such that*

$$ |I_k(u) - I_k(\overline{u})| \leqslant \widetilde{l}\|u-\overline{u}\|_{PC}, $$

for each $u,\overline{u} \in PC([-r,0],\mathbb{R})$ *and* $k = 1,\dots,m$.

If

$$ m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} < 1, \tag{3.5} $$

then there exists a unique solution for the problem (1.1)-(1.3) *on* J.

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $N : PC([-r,T],\mathbb{R}) \to PC([-r,T],\mathbb{R})$ defined by

$$ Ny(t) = \begin{cases} \varphi(0) + \sum\limits_{0<t_k<t} I_k(y_{t_i^-}) + \dfrac{1}{\Gamma(\alpha)} \sum\limits_{0<t_k<t} \displaystyle\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds, & t \in [0,T], \\ \\ \varphi(t), & t \in [-r,0], \end{cases} \tag{3.6} $$

where $g \in C(J,\mathbb{R})$ be such that

$$ g(t) = f(t,y_t,g(t)). $$

Clearly, the fixed points of operator N are solutions of problem (1.1)-(1.3). Let $u,w \in PC([-r,T],\mathbb{R})$. If $t \in [-r,0]$, then

$$|N(u)(t)-N(w)(t)| = 0.$$

For $t \in J$, we have

$$\begin{aligned}|N(u)(t)-N(w)(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g(s)-h(s)|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g(s)-h(s)|ds\\ &+\sum_{0<t_k<t}|I_k(u_{t_k^-})-I_k(w_{t_k^-})|,\end{aligned}$$

where $g,h \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t,u_t,g(t)),$$

and

$$h(t) = f(t,w_t,h(t)).$$

By $(H2)$ we have

$$\begin{aligned}|g(t)-h(t)| &= |f(t,u_t,g(t))-f(t,w_t,h(t))|\\ &\leqslant K\|u_t-w_t\|_{PC}+L|g(t)-h(t)|.\end{aligned}$$

Then

$$|g(t)-h(t)| \leqslant \frac{K}{1-L}\|u_t-w_t\|_{PC}.$$

Therefore, for each $t \in J$

$$\begin{aligned}|N(u)(t)-N(w)(t)| &\leqslant \frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|u_s-w_s\|_{PC}ds\\ &+\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|u_s-w_s\|_{PC}ds\\ &+\sum_{k=1}^{m}\widetilde{l}\|u_{t_k^-}-w_{t_k^-}\|_{PC}.\\ &\leqslant \left[m\widetilde{l}+\frac{mKT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\right.\\ &\quad\left.+\frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_{PC_1}.\end{aligned}$$

Thus

$$\|N(u)-N(w)\|_{PC_1} \leqslant \left[m\widetilde{l}+\frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_{PC_1}.$$

By (3.5), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (1.1)-(1.3).

Our second result is based on Schaefer's fixed point theorem.

THEOREM 5. *Assume (H1), (H2) and*

(H4) There exist $p,q,r \in C(J,\mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$|f(t,u,w)| \leqslant p(t)+q(t)\|u\|_{PC}+r(t)|w| \text{ for } t \in J,\ u \in PC([-r,0],\mathbb{R}) \text{ and } w \in \mathbb{R}.$$

(H5) The functions $I_k : PC([-r,0],\mathbb{R}) \to \mathbb{R}$ *are continuous and there exist constants* $M^*,N^* > 0$ *with* $mM^* < 1$ *such that*

$$|I_k(u)| \leqslant M^*\|u\|_{PC}+N^* \text{ for each } u \in PC([-r,0],\mathbb{R}),\ k=1,\dots,m.$$

Then the problem (1.1)-(1.3) *has at least one solution.*

Proof. Let the operator N defined in (3.6). We shall use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC([-r,T],\mathbb{R})$. If $t \in [-r,0]$, then

$$|N(u_n)(t)-N(u)(t)| = 0.$$

For $t \in J$, we have

$$\begin{aligned}|N(u_n)(t)-N(u)(t)| \leqslant{}& \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g_n(s)-g(s)|ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g_n(s)-g(s)|ds\\&+\sum_{0<t_k<t}|I_k(u_{nt_k^-})-I_k(u_{t_k^-})|,\end{aligned}\tag{3.7}$$

where $g_n,g \in C(J,\mathbb{R})$ such that

$$g_n(t) = f(t,u_{nt},g_n(t)),$$

and

$$g(t) = f(t,u_t,g(t)).$$

By $(H2)$, we have

$$\begin{aligned}|g_n(t)-g(t)| &= |f(t,u_{nt},g_n(t))-f(t,u_t,g(t))|\\&\leqslant K\|u_{nt}-u_t\|_{PC}+L|g_n(t)-g(t)|.\end{aligned}$$

Then

$$|g_n(t)-g(t)| \leqslant \frac{K}{1-L}\|u_{nt}-u_t\|_{PC}.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leqslant \eta$ and $|g(t)| \leqslant \eta$. Then, we have

$$\begin{aligned}(t-s)^{\alpha-1}|g_n(s)-g(s)| &\leqslant (t-s)^{\alpha-1}[|g_n(s)|+|g(s)|]\\ &\leqslant 2\eta(t-s)^{\alpha-1},\end{aligned}$$

and

$$\begin{aligned}(t_k-s)^{\alpha-1}|g_n(s)-g(s)| &\leqslant (t_k-s)^{\alpha-1}[|g_n(s)|+|g(s)|]\\ &\leqslant 2\eta(t_k-s)^{\alpha-1}.\end{aligned}$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k-s)^{\alpha-1}$ are integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (3.7) imply that

$$|N(u_n)(t)-N(u)(t)| \to 0 \; as \; n \to \infty,$$

and hence

$$\|N(u_n)-N(u)\|_{PC_1} \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Step 2: *N maps bounded sets into bounded sets in $PC([-r,T],\mathbb{R})$.*
Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in PC([-r,T],\mathbb{R}) : ||u||_{PC_1} \leqslant \eta^*\}$, we have $\|N(u)\|_{PC_1} \leqslant \ell$. We have for each $t \in J$,

$$\begin{aligned}N(u)(t) &= \varphi(0)+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds+\sum_{0<t_k<t}I_k(u_{t_k^-}),\end{aligned} \tag{3.8}$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t,u_t,g(t)).$$

By (H4), we have for each $t \in J$,

$$\begin{aligned}|g(t)| &= |f(t,u_t,g(t))|\\ &\leqslant p(t)+q(t)\|u_t\|_{PC}+r(t)|g(t)|\\ &\leqslant p(t)+q(t)\|u\|_{PC_1}+r(t)|g(t)|\\ &\leqslant p(t)+q(t)\eta^*+r(t)|g(t)|\\ &\leqslant p^*+q^*\eta^*+r^*|g(t)|,\end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t\in J} q(t)$.
Then

$$|g(t)| \leqslant \frac{p^* + q^*\eta^*}{1-r^*} := M.$$

Thus (3.8) implies

$$\begin{aligned}
|N(u)(t)| &\leqslant |\varphi(0)| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u_{t_k^-}\|_{PC} + N^*) \\
&\leqslant |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u\|_{PC_1} + N^*) \\
&\leqslant |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\eta^* + N^*) := R.
\end{aligned}$$

And if $t \in [-r, 0]$, then

$$|N(u)(t)| \leqslant \|\varphi\|_{PC},$$

thus

$$\|N(u)\|_{PC_1} \leqslant \max\ \{R, \|\varphi\|_{PC}\} := \ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of $PC([-r,T],\mathbb{R})$.*
Let $\tau_1, \tau_2 \in (0,T]$, $\tau_1 < \tau_2$, B_{η^*} be a bounded set of $PC([-r,T],\mathbb{R})$ as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}
&|N(u)(\tau_2) - N(u)(\tau_1)| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}||g(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}||g(s)|ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(u_{t_k^-})| \\
&\leqslant \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)(M^*\|u_{t_k^-}\|_{PC} + N^*) \\
&\leqslant \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)(M^*\|u\|_{PC_1} + N^*) \\
&\leqslant \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)(M^*\eta^* + N^*).
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $N : PC([-r,T],\mathbb{R}) \to PC([-r,T],\mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.* Now it remains to show that the set

$$E = \{u \in PC([-r,T],\mathbb{R}) : u = \lambda N(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in E$, then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$u(t) = \lambda \varphi(0) + \frac{\lambda}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds + \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds + \lambda \sum_{0<t_k<t} I_k(u_{t_k^-}). \tag{3.9}$$

And, by (H4), we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t, u_t, g(t))| \\ &\leqslant p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ &\leqslant p^* + q^*\|u_t\|_{PC} + r^*|g(t)|. \end{aligned}$$

Thus

$$|g(t)| \leqslant \frac{1}{1-r^*}(p^* + q^*\|u_t\|_{PC}).$$

This implies, by (3.9) and (H5), that for each $t \in J$ we have

$$\begin{aligned} |u(t)| \leqslant |\varphi(0)| &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ m(M^*\|u_{t_k^-}\|_{PC} + N^*). \end{aligned}$$

Consider the function ν defined by

$$\nu(t) = \sup\{|u(s)| : -r \leqslant s \leqslant t\}, 0 \leqslant t \leqslant T,$$

then, there exists $t^* \in [-r, T]$ such that $\nu(t) = |u(t^*)|$. If $t^* \in [0, T]$,then by the previous inequality, we have for $t \in J$

$$\begin{aligned} \nu(t) \leqslant |\varphi(0)| &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\nu(s))ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\nu(s))ds \\ &+ mM^*\nu(t) + mN^*. \end{aligned}$$

Thus

$$\begin{aligned} \nu(t) \leqslant \frac{|\varphi(0)| + mN^*}{1 - mM^*} &+ \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\nu(s))ds \\ &+ \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\nu(s))ds \end{aligned}$$

$$\leqslant \frac{|\varphi(0)|+mN^*}{1-mM^*}+\frac{(m+1)p^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}$$
$$+\frac{(m+1)q^*}{(1-mM^*)(1-r^*)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds.$$

Applying Lemma 4, we get

$$v(t) \leqslant \left[\frac{|\varphi(0)|+mN^*}{1-mM^*}+\frac{(m+1)p^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right]$$
$$\times\left[1+\frac{\delta(m+1)q^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right] := A,$$

where $\delta=\delta(\alpha)$ a constant. If $t^*\in[-r,0]$, then $v(t)=\|\varphi\|_{PC}$, thus for any $t\in[-r,T], \|u\|_{PC_1}\leqslant v(t)$, we have

$$\|u\|_{PC_1}\leqslant \max\{\|\varphi\|_{PC}, A\}$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (1.1)-(1.3).

4. Ulam-Hyers Rassias Stability

Now, we state the following Ulam-Hyers-Rassias stable result.

THEOREM 6. *Assume* $(H1)$-$(H3)$, *(3.5) and*

$(H6)$ *there exists a nondecreasing function* $\omega\in PC(J,\mathbb{R}_+)$ *and there exists* $\lambda_\omega>0$ *such that for any* $t\in J$:
$$I^\alpha\omega(t)\leqslant\lambda_\omega\omega(t)$$
are satisfied, then the problem (1.1)-(1.2) *is Ulam-Hyers-Rassias stable with respect to* (ω,ψ).

Proof. Let $z\in PC([-r,T],\mathbb{R})$ be a solution of the inequality (2.3). Denote by y the unique solution of the following problem

$$\begin{cases} {}^cD^\alpha_{t_k}y(t)=f(t,y_t,{}^cD^\alpha_{t_k}y(t)),\ t\in(t_k,t_{k+1}], k=1,\dots,m\,;\\ \Delta y|_{t=t_k}=I_k(y_{t_k^-}),\ k=1,\dots,m\,;\\ y(t)=z(t)=\varphi(t),\ t\in[-r,0]. \end{cases}$$

Using Lemma 6, we obtain for each $t\in(t_k,t_{k+1}]$

$$y(t)=\varphi(0)+\sum_{i=1}^k I_i(y_{t_i^-})+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}g(s)ds,\ t\in(t_k,t_{k+1}],$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, y_t, g(t)).$$

Since z solution of the inequality (2.3) and by Remark 2, we have

$$\begin{cases} {}^cD^{\alpha}_{t_k} z(t) = f(t, z_t, {}^cD^{\alpha}_{t_k} z(t)) + \sigma(t),\ t \in (t_k, t_{k+1}], k = 1, \ldots, m\,; \\ \Delta z|_{t=t_k} = I_k(z_{t_k^-}) + \sigma_k,\ k = 1, \ldots, m\,. \end{cases} \tag{4.1}$$

Clearly, the solution of (4.1) is given by

$$\begin{aligned} z(t) &= \varphi(0) + \sum_{i=1}^{k} I_i(z_{t_i^-}) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \sigma(s)ds,\ t \in (t_k, t_{k+1}], \end{aligned}$$

where $h \in C(J,\mathbb{R})$ be such that

$$h(t) = f(t, z_t, h(t)).$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows that

$$\begin{aligned} |z(t) - y(t)| \leqslant & \sum_{i=1}^{k} |\sigma_i| + \sum_{i=1}^{k} |I_i(z_{t_i^-}) - I_i(y_{t_i^-})| \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)|ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s) - g(s)|ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |\sigma(s)|. \end{aligned}$$

Thus

$$\begin{aligned} |z(t) - y(t)| \leqslant & \; m\varepsilon\psi + (m+1)\varepsilon\lambda_{\omega}\omega(t) + \sum_{i=1}^{k} \widetilde{l} \|z_{t_i^-} - y_{t_i^-}\|_{PC} \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s) - g(s)|ds. \end{aligned}$$

By (H2), we have

$$\begin{aligned}|h(t)-g(t)| &= |f(t,z_t,h(t))-f(t,y_t,g(t))|\\ &\leqslant K\|z_t-y_t\|_{PC}+L|g(t)-h(t)|.\end{aligned}$$

Then

$$|h(t)-g(t)| \leqslant \frac{K}{1-L}\|z_t-y_t\|_{PC}.$$

Therefore, for each $t\in J$

$$\begin{aligned}|z(t)-y(t)| \leqslant\ & m\varepsilon\psi+(m+1)\varepsilon\lambda_\omega\omega(t)+\sum_{i=1}^{k}\widetilde{l}\|z_{t_i^-}-y_{t_i^-}\|_{PC}\\ &+\frac{K}{(1-L)\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds\\ &+\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds.\end{aligned}$$

Thus

$$\begin{aligned}|z(t)-y(t)| \leqslant\ & \sum_{0<t_i<t}\widetilde{l}\|z_{t_i^-}-y_{t_i^-}\|_{PC}+\varepsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\ &+\frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds.\end{aligned}$$

We consider the function v_1 defined by

$$v_1(t)=\sup\{|z(s)-y(s)| : -r\leqslant s\leqslant t\},0\leqslant t\leqslant T,$$

then, there exists $t^*\in[-r,T]$ such that $v_1(t)=|z(t^*)-y(t^*)|$.
If $t^*\in[-r,0]$, then $v_1(t)=0$.
If $t^*\in[0,T]$, then by the previous inequality, we have

$$\begin{aligned}v_1(t) \leqslant\ & \sum_{0<t_i<t}\widetilde{l}v_1(t_i^-)+\varepsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\ &+\frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v_1(s)ds.\end{aligned}$$

Applying Lemma 5, we get

$$\begin{aligned}v_1(t) &\leqslant \varepsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\ &\quad\times\left[\prod_{0<t_i<t}(1+\widetilde{l})\exp\left(\int_0^t\frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]\\ &\leqslant c_\omega\varepsilon(\psi+\omega(t)),\end{aligned}$$

where

$$c_\omega=(m+(m+1)\lambda_\omega)\left[\prod_{i=1}^{m}(1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]$$

$$= (m+(m+1)\lambda_\omega)\left[(1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m.$$

Thus, the problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to (ω,ψ). The proof is complete.

Next, we present the following Ulam-Hyers stable result.

THEOREM 7. *Assume that (H1)-(H3) and* (3.5) *are satisfied, then, the problem* (1.1)-(1.2) *is Ulam-Hyers stable.*

Proof. Let $z \in PC([-r,T],\mathbb{R})$ be a solution of the inequality (2.1). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD_{t_k}^\alpha y(t) = f(t,y_t,{}^cD_{t_k}^\alpha y(t)),\ t\in(t_k,t_{k+1}], k=1,\dots,m\,; \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}),\ k=1,\dots,m\,; \\ y(t)=z(t)=\varphi(t),\ t\in[-r,0]. \end{cases}$$

From the proof of Theorem 6, we get the inequality

$$v_1(t) \leqslant \sum_{0<t_i<t} \widetilde{l} v_1(t_i^-) + m\varepsilon + \frac{T^\alpha \varepsilon(m+1)}{\Gamma(\alpha+1)} + \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_1(s)ds.$$

Applying Lemma 5, we get

$$\begin{aligned} v_1(t) &\leqslant \varepsilon\left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\right) \\ &\quad\times\left[\prod_{0<t_i<t}(1+\widetilde{l})\exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right] \\ &\leqslant c_\omega\varepsilon, \end{aligned}$$

where

$$\begin{aligned} c_\omega &= \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[\prod_{i=1}^m (1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right] \\ &= \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[(1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m. \end{aligned}$$

Which completes the proof of the Theorem.

Moreover, if we set $\gamma(\varepsilon)=c_\omega\varepsilon; \gamma(0)=0$, then, the problem (1.1)-(1.2) is generalized Ulam-Hyers stable.

5. Examples

Example 1. Consider the following impulsive problem

$$ {}^cD_{t_k}^{\frac{1}{2}}y(t) = \frac{e^{-t}}{(11+e^t)}\left[\frac{y_t}{1+y_t} - \frac{{}^cD_{t_k}^{\frac{1}{2}}y(t)}{1+{}^cD_{t_k}^{\frac{1}{2}}y(t)}\right], \text{ for each, } t \in J_0 \cup J_1. \tag{5.1} $$

$$ \Delta y|_{t=\frac{1}{2}} = \frac{y(\frac{1}{2}^-)}{10+y(\frac{1}{2}^-)}. \tag{5.2} $$

$$ y(t) = \varphi(t),\ t \in [-r,0], r > 0, \tag{5.3} $$

where $\varphi \in PC([-r,0],\mathbb{R})$, $J_0 = [0,\frac{1}{2}]$, $J_1 = (\frac{1}{2},1]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.

Set

$$ f(t,u,v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right],\ t \in [0,1],\ u \in PC([-r,0],\mathbb{R}) \text{ and } v \in \mathbb{R}. $$

Clearly, the function f is jointly continuous.

For each $u,\bar{u} \in PC([-r,0],\mathbb{R}), v,\bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$ \begin{aligned} |f(t,u,v) - f(t,\bar{u},\bar{v})| &\leqslant \frac{e^{-t}}{(11+e^t)}(\|u-\bar{u}\|_{PC} + |v-\bar{v}|) \\ &\leqslant \frac{1}{12}\|u-\bar{u}\|_{PC} + \frac{1}{12}|v-\bar{v}|. \end{aligned} $$

Hence condition (H2) is satisfied with $K = L = \frac{1}{12}$.

And let

$$ I_1(u) = \frac{u}{10+u},\quad u \in PC([-r,0],\mathbb{R}). $$

Let $u,v \in PC([-r,0],\mathbb{R})$. Then we have

$$ |I_1(u) - I_1(v)| = \left|\frac{u}{10+u} - \frac{v}{10+v}\right| \leqslant \frac{1}{10}\|u-v\|_{PC}. $$

Thus condition

$$ \begin{aligned} m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} &= \left[\frac{1}{10} + \frac{\frac{2}{12}}{(1-\frac{1}{12})\Gamma(\frac{3}{2})}\right] \\ &= \frac{4}{11\sqrt{\pi}} + \frac{1}{10} < 1, \end{aligned} $$

is satisfied with $T = 1, m = 1$ and $\widetilde{l} = \frac{1}{10}$. It follows from Theorem 4 that the problem (5.1)–(5.3) has a unique solution on J.

Set for any $t \in [0,1]$, $\omega(t) = t, \psi = 1$. Since

$$ I^{\frac{1}{2}}\omega(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{\frac{1}{2}-1}s ds \leqslant \frac{2t}{\sqrt{\pi}}, $$

then, condition (H6) is satisfied with $\lambda_\omega = \frac{2}{\sqrt{\pi}}$. It follows that the problem (5.1)-(5.2) is Ulam-Hyers-Rassias stable with respect to (ω, ψ).

Example 2. Consider the following impulsive problem

$$ {}^cD_{t_k}^{\frac{1}{2}} y(t) = \frac{2 + |y_t| + |{}^cD_{t_k}^{\frac{1}{2}} y(t)|}{108e^{t+3}(1 + |y_t| + |{}^cD_{t_k}^{\frac{1}{2}} y(t)|)}, \text{ for each, } t \in J_0 \cup J_1. \quad (5.4)$$

$$ \Delta y|_{t=\frac{1}{3}} = \frac{|y(\frac{1}{3}^-)|}{6 + |y(\frac{1}{3}^-)|}, \quad (5.5)$$

$$ y(t) = \varphi(t),\ t \in [-r, 0], r > 0, \quad (5.6)$$

where $\varphi \in PC([-r,0],\mathbb{R})$ $J_0 = [0, \frac{1}{3}]$, $J_1 = (\frac{1}{3}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$.
Set

$$ f(t,u,v) = \frac{2 + |u| + |v|}{108e^{t+3}(1 + |u| + |v|)},\ t \in [0,1],\ u \in PC([-r,0],\mathbb{R}), v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.

For any $u, \bar{u} \in PC([-r,0],\mathbb{R}), v, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant \frac{1}{108e^3}(\|u - \bar{u}\|_{PC} + |v - \bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{108e^3}$.

We have, for each $t \in [0,1]$,

$$ |f(t,u,v)| \leqslant \frac{1}{108e^{t+3}}(2 + \|u\|_{PC} + |v|).$$

Thus condition (H4) is satisfied with

$$ p(t) = \tfrac{1}{54e^{t+3}} \text{ and } q(t) = r(t) = \tfrac{1}{108e^{t+3}}.$$

Let

$$ I_1(u) = \frac{|u|}{6 + |u|},\ u \in PC([-r,0],\mathbb{R}).$$

We have, for each $u \in PC([-r,0],\mathbb{R})$,

$$ |I_1(u)| \leqslant \frac{1}{6}\|u\|_{PC} + 1$$

Thus condition (H5) is satisfied with $M^* = \frac{1}{6}$ and $N^* = 1$. It follows from Theorem 5 that the problem (5.4)–(5.6) has at least one solution on J.

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