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ON NONLINEAR PERTURBATIONS OF STURM–LIOUVILLE PROBLEMS IN DISCRETE AND CONTINUOUS SETTINGS

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Abstract. In this paper we provide sufficient conditions for the existence of solutions to certain classes of second-order discrete and continuous systems. In particular, we examine problems that can be posed as nonlinear perturbations of Sturm-Liouville problems. We first provide a lemma on the invertibility of a nonlinearly-perturbed invertible linear operator, and apply this result to extend previous work on these topics.

1. Introduction

The purpose of this paper is to provide sufficient conditions for the existence of solutions to nonlinearly-perturbed Sturm–Liouville problems, both in the differential equations and difference equations settings. The perturbations that we consider also include modifications to the boundary conditions. We extend the results of [28, 27] by removing differentiability requirements in both cases, and, in the differential case, we also work in a different normed space, which leads to benefits in a subset of problems. For previous work in the discrete case, see also [26].

The main lemma in Section 2 can be seen as extending the so-called global inverse function theorems of [7, 8], where the differentiability conditions are removed. The relationship between the eigenvalues of the original linear problem and the allowable nonlinearities is similar to [11], which studies closely related Hammerstein integral equations.

There has been much work done on similar problems or using similar approaches to the current paper. Graef and Kong [16] study multiple solutions to boundary value problems including nonlinear Sturm–Liouville problems. For analyses of nonlinear discrete systems with linear boundary conditions see [13, 14, 24, 23, 31, 30, 1]. For the differential equations setting, also with linear boundary conditions, see [32]. For the use of projection methods in more general nonlinear problems, see [29]. For the use of Galerkin methods in differential problems, see [22, 25]. In [18], nonlinear partial differential boundary value problems are considered. Three point boundary conditions in the context of nonlinear second-order differential equations are studied in [6], and [20] considered similar nonlinear equations under two-point boundary conditions. Periodic

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perturbations of second order equations were considered in [21], and then generalized to functions on $\mathbb{R}^n$ in [19]. Integral boundary conditions are studied in [5, 15, 17, 33]. Fixed point theorems applied to fractional differential equations can be found in [2, 9]. Problems in elliptic boundary values problems can be found in [4]. For boundary conditions that can be expressed as continuous linear functionals, see [34].

In Section 2 we cover the unified framework for the problems considered later. Section 3 deals with the continuous setting, and Section 4 examines the discrete setting.

2. Generalities

In this paper we consider two related problems, both of which fall into the general framework of searching for solutions to equations of the form

$$Lx = H(x), \tag{1}$$

where L is a linear operator, and H is a nonlinear operator, both defined on some Banach space. Suppose that L is known to have an inverse, and that $H = \Psi + G$. The strategy we shall employ is then to first find conditions under which $L - \Psi$ is guaranteed to also have an inverse. This will uniquely solve the related equation

$$Lx - \Psi(x) = y, \tag{2}$$

for any point y in the Banach space. Given a result of this type, we then study conditions under which (1) has a (possibly non-unique) solution by studying the operator $(L - \Psi)^{-1}G$. For this part we shall rely upon Schauder's fixed point theorem and degree theory arguments. To study conditions for which (2) can be solved for any y, we will make use of the following lemma, which considers when a nonlinear perturbation to an invertible linear operator preserves the invertibility.

LEMMA 1. *Let X be a Banach space, Y be a normed linear space, $D \subseteq X$ be any subspace, $T : X \to Y$, and $L : D \subseteq X \to Y$, where L is linear, L^{-1} exists and is Lipschitz continuous with constant K_1, $L^{-1} \circ T$ is Lipschitz continuous with constant $K_2 < 1$. Then, the map $L - T : D \subseteq X \to Y$ is invertible; furthermore, $(L - T)^{-1} : Y \to D$ is Lipschitz continuous with constant $K_1(1 - K_2)^{-1}$. In addition, if L^{-1} is compact and T is continuous, then $(L - T)^{-1}$ is also compact.*

Proof. Consider the map $S : X \times Y \to X$ defined by $S(x,y) \equiv L^{-1}(T(x) + y)$, and define the families of maps, $S_x : y \mapsto S(x,y)$ and $S_y : x \mapsto S(x,y)$. Let $x_1, x_2 \in X$ and $y \in Y$. Then,

$$\|S_y(x_1) - S_y(x_2)\| = \|L^{-1}T(x_1) - L^{-1}T(x_2)\| \leqslant K_2\|x_1 - x_2\|.$$

So, S_y is a uniform (over Y) contraction on X. Now, let $y_1, y_2 \in Y$ and $x \in X$. Then,

$$\|S_x(y_1) - S_x(y_2)\| = \|L^{-1}y_1 - L^{-1}y_2\| \leqslant K_1\|y_1 - y_2\|.$$

So, S_x is uniformly (over X) Lipschitz continuous with constant K_1. By the contraction mapping theorem with parameters [12, Corollary 2.3.2], there exists a Lipschitz continuous function, $g : Y \to X$, with constant $K_1(1-K_2)^{-1}$, such that

$$S(x,y) = x \iff x = g(y).$$

Since $L^{-1}(Y) = D$, $g : Y \to D$. For $x \in D$,

$$S(x,y) = x \iff Lx - T(x) = y,$$

which means that $g = (L-T)^{-1}$.

To see that g is compact if L^{-1} is compact and T is continuous, note that the fixed point relations means that

$$g = L^{-1} \circ ((T \circ g) + I),$$

which is the composition of a compact operator with a continuous operator, since g is continuous from above. □

We consider modifications to classical Sturm-Liouville problems, both in the differential equations setting and in the difference equation setting. Due to the infinite dimensional nature of the spaces considered when studying the differential equations setting, more issues arise that require special attention. For the difference equations that are considered, the finite dimensionality simplifies many of the problems encountered in the following section.

3. Differential Equations

We consider differential equations on the interval $[0,1]$ of the form

$$(p(t)x'(t))' + q(t)x(t) + \psi(x)(t) = G(x)(t), \tag{3}$$

subject to boundary conditions of the form

$$\alpha x(0) + \beta x'(0) + \eta_1(x) = \phi_1(x) \tag{4a}$$
$$\gamma x(1) + \delta x'(1) + \eta_2(x) = \phi_2(x). \tag{4b}$$

In the above equations, ψ and G are function-valued operators, and η_1, η_2, ϕ_1, and ϕ_2 are real-valued functions. These represent the nonlinear perturbations of the classical linear problem, but conditions on the right-hand and left-hand side perturbations that guarantee the existence of a solution will be qualitatively different. We make the usual assumptions on the linear portions of the problem, namely that $p, p\prime$, and q are continuous, $p > 0$, and $\alpha^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$.

The strategy for analyzing (3)-(4) will be to first determine conditions under which we can uniquely solve the following problem:

$$(p(t)x'(t))' + q(t)x(t) + \psi(x)(t) = h(t), \tag{5}$$

subject to

$$\alpha x(0) + \beta x'(0) + \eta_1(x) = v_1 \tag{6a}$$

$$\gamma x(1) + \delta x'(1) + \eta_2(x) = v_2. \tag{6b}$$

This problem corresponds to finding an inverse for the operator representing the left-hand side of (5)-(6), and is dealt with in Theorem 1.

In general, we use capital Roman letters to denote sets, and capital script letters to denote the corresponding spaces equipped with a specified norm. On any Euclidean space, $\mathbb{R}^n$, we denote the usual norm by $|\cdot|$, regardless of dimension. Let L^2 denote the set of square Lebesgue integrable functions on $[0,1]$, and let $\mathscr{L}^2 = (L^2, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the usual norm defined by the inner product $\langle x,y\rangle = \int_0^1 x(t)y(t)dt$ for $x,y \in L^2$. Let $D = \{x \in L^2 | x''(\text{weak}) \in L^2\}$, and let $\mathscr{D}_2 = (D, \|\cdot\|_2)$. On this space we define the operators representing the linear part of the problem, $\mathbb{A} : D \to L^2$ and $\mathbb{B} : D \to \mathbb{R}^2$, where

$$\mathbb{A}(x) \equiv (px')' + qx,$$

$$\mathbb{B}(x) \equiv \begin{pmatrix} \alpha x(0) + \beta x'(0) \\ \gamma x(1) + \delta x'(1) \end{pmatrix},$$

and let

$$\mathbb{L}(x) \equiv \begin{pmatrix} \mathbb{A}(x) \\ \mathbb{B}(x) \end{pmatrix}.$$

For the nonlinear portion, let $\eta = (\eta_1, \eta_2)$ and $\phi = (\phi_1, \phi_2)$. Then define $\Psi, \mathscr{G} : D \to L^2 \times \mathbb{R}^2$ as

$$\Psi(x) \equiv \begin{pmatrix} -\psi(x) \\ -\eta(x) \end{pmatrix}, \mathscr{G}(x) \equiv \begin{pmatrix} G(x) \\ \phi(x) \end{pmatrix}.$$

On $L^2 \times \mathbb{R}^2$ we will use the sum of the the two usual component norms, and denote this space $\mathscr{L}^2 \times \mathbb{R}^2$. The properties of these components guaranteeing the existence of solutions will be the content of the main results of this paper. For now, simply view them as nonlinear maps. With this notation, we rewrite (3)-(4) as

$$\mathbb{L}x - \Psi(x) = \mathscr{G}(x),$$

for $x \in D$. The norm with which we choose to pair D will have a major impact on the conditions for existence of solutions.

Classical Sturm-Liouville theory deals with equations of the form $\mathbb{A}x = h$ for $x \in \mathbb{B}^{-1}(\{0\})$, and we will require well-known properties from this case (see, for example, [3]). First, on $\mathbb{B}^{-1}(\{0\}) \cap \mathscr{D}_2$, $\mathbb{A}$ has countably many simple eigenvalues, $\{-\lambda_k\}_{k=1}^{\infty}$, which we can assume are in decreasing order, such that $\lambda_k \to \infty$ as $k \to \infty$. For each $-\lambda_k$, there is an eigenfunction, u_k, with unit norm, which spans the corresponding eigenspace. In general, $\mathbb{A}$ might fail to be injective. Because of this, we take $\mu \in \mathbb{R}$, which is not an eigenvalue, and let

$$\mathbb{L}_\mu x \equiv \begin{pmatrix} \mathbb{A}x + \mu x \\ \mathbb{B}x \end{pmatrix}, \Psi_\mu(x) \equiv \begin{pmatrix} -\psi(x) + \mu x \\ -\eta(x) \end{pmatrix}.$$

This implies that on the set $\mathbb{B}^{-1}(\{0\}) \cap \mathscr{D}_2$, the operator $\mathbb{A} + \mu I$ has eigenvalues $\{\mu - \lambda_k\}_{k=1}^{\infty}$ corresponding to the same eigenfunctions, but where it is now guaranteed that 0 is not an eigenvalue.

From the theory of second order differential equations, it is known that $(\mathbb{A} + \mu I)x = 0$ has a two dimensional solution space. We can choose a basis, $\{w_1, w_2\}$, for this space such that $\|w_1\|_2 + \|w_2\|_2 \leqslant 1$. Then let $w = (w_1, w_2)^t$ and define the 2×2 matrix $B = [\mathbb{B}w_1 | \mathbb{B}w_2]$. With these definitions we can now give the form of the inverse of $\mathbb{L}_\mu$. The following result is cited from [28], but we provide a short proof.

LEMMA 2. (from [28]) *$\mathbb{L}_\mu : D \to L^2 \times \mathbb{R}^2$ is bijective and*

$$\mathbb{L}_\mu^{-1}(h, v) = \sum_{k=1}^{\infty} \frac{\langle h, u_k \rangle}{\mu - \lambda_k} u_k + w^t B^{-1} v, \tag{7}$$

where the limit is in the sense of uniform convergence.

Proof. It is well known that the map, $h \mapsto \sum_{k=1}^{\infty} u_k \langle h, u_k \rangle / (\mu - \lambda_k)$ is the eigenfunction expansion of the integral operator

$$\mathbb{G} : h \to \int_0^1 g(\cdot, s) h(s) ds,$$

where g is the so-called Green's function for the problem

$$(\mathbb{A} + \mu I)x(t) = (p(t)x'(t))' + q(t)x(t) = h(t)$$
$$\mathbb{B}x = \begin{pmatrix} \alpha x(0) + \beta x'(0) \\ \gamma x(1) + \delta x'(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The eigenfunction expansion is known to converge in $\mathscr{L}^2$, and furthermore this convergence can also be shown to be uniform. Furthermore, the kernel of $\mathbb{G}$ is known to be $\{0\}$. This fact, along with the definitions of w and B, imply that

$$(\mathbb{A} + \mu I)\mathbb{L}_\mu^{-1}(h, v) = h + (\mathbb{A} + \mu I) w^t B^{-1} v = h,$$
$$\mathbb{B}\mathbb{L}_\mu^{-1}(h, v) = 0 + BB^{-1} v = v,$$

which proves the result. $\square$

LEMMA 3. *$\mathbb{L}_\mu^{-1}$ is a compact operator from $\mathscr{L}^2 \times \mathbb{R}^2$ onto $\mathscr{D}_2$.*

Proof. The map $h \mapsto \int_0^1 g(\cdot,s)h(s)ds$ is known to be a compact map from $\mathscr{L}^2$ to $\mathscr{L}^2$, hence it is compact on the product domain. The map $v \mapsto w^t B^{-1} v$ is also compact on the domain since its range is finite dimensional. Thus, the sum is compact. □

The next corollary provides estimates on norms of interest related to $\mathbb{L}_\mu^{-1}$. The operator norm is given the same notation as the norm on the corresponding domain of the operator. The definition of the graph norm, $\|\cdot\|_{gr}$, can be found in proposition 1.

COROLLARY 1. *Let* $A_0 = \sup_k |\mu - \lambda_k|^{-1}$, $B_0 = \|B^{-1}\|$, $C_0 = \sup_k |\lambda_k/(\mu-\lambda_k)|$, *and* $D_0 = \max\{|\mu|, \|B^{-1}\|\}$.

$$\begin{aligned}
\|\mathbb{L}_\mu^{-1}(h,v)\|_2 &\leqslant A_0\|h\|_2 + B_0|v|,\\
\|\mathbb{L}_\mu^{-1}\|_2 &\leqslant \max\{A_0, B_0\} \equiv K^*,\\
\|\mathbb{L}_\mu^{-1}(h,v)\|_{gr} &\leqslant (A_0+C_0)\|h\|_2 + (B_0+D_0)|v|,\\
\|\mathbb{L}_\mu^{-1}\|_{gr} &\leqslant \max\{A_0+C_0, B_0+D_0\} \equiv N^*.
\end{aligned}$$

Proof.

$$\begin{aligned}
\|\mathbb{L}_\mu^{-1}(h,v)\|_2 &\leqslant \sum_{k=1}^{\infty}\left|\frac{\langle h,u_k\rangle}{\mu-\lambda_k}\right| + \|w^t B^{-1} v\|_2 \leqslant A_0 \sum_{k=1}^{\infty} |\langle h,u_k\rangle|^2 + |B^{-1}v|\\
&\leqslant A_0\|h\|_2 + B_0|v|,\\
\|\mathbb{A}\mathbb{L}_\mu^{-1}(h,v)\|_2 &= \left\| h - \sum_{k=1}^{\infty} \frac{\mu\langle h,u_k\rangle}{\mu-\lambda_k} u_k - \mu w^t B^{-1} v\right\|_2\\
&= \left\| \sum_{k=1}^{\infty} \frac{-\lambda_k}{\mu-\lambda_k}\langle h,u_k\rangle u_k - \mu w^t B^{-1} v\right\|_2\\
&\leqslant C_0\|h\|_2 + |\mu|\|B^{-1}\||v|.\\
|\mathbb{B}\mathbb{L}_\mu^{-1}(h,v)| &= |v|.
\end{aligned}$$

So we have that

$$\begin{aligned}
\|\mathbb{L}_\mu^{-1}(h,v)\|_{gr} &\leqslant A_0\|h\|_2 + B_0|v| + \max\{C_0\|h\|_2 + |\mu|\|B^{-1}\||v|, |v|\}\\
&\leqslant (A_0+C_0)\|h\|_2 + (B_0+D_0)|v|.
\end{aligned}$$

The bounds on operator norms easily follow from these inequalities. □

PROPOSITION 1. $\mathbb{L} : \mathscr{D}_2 \to \mathscr{L}^2 \times \mathscr{R}^2$ *is a closed operator. Hence,* $\mathscr{D}_{gr} \equiv (D, \|\cdot\|_{gr})$ *is a Banach space, where the norm is the graph norm defined by* $\|x\|_{gr} = \|x\|_2 + \max\{\|\mathbb{A}x\|_2, |\mathbb{B}x|\}$.

Proof. $\mathbb{L}_\mu$ is a closed operator since it has a continuous inverse. The operator μI is clearly a bounded linear operator. Thus $\mathbb{L} = \mathbb{L}_\mu - \mu I$ is a closed operator. It follows that $\mathscr{D}_{gr}$ is a Banach space by the closed graph theorem.

REMARK 1. The fact that $\mathbb{L}$ is closed can be used to show that the convergence of the sum in (7) is in the stronger sense of the graph norm.

As opposed to relying on a global inverse function theorem, as in [28], we instead use Lemma 1. One significant benefit of this approach is the removal of any differentiability requirements, which are then replaced by requirements of Lipschitz continuity. As we shall see later on, another advantage is the ability to work on an incomplete space, which allows certain constraints on constants to be relaxed.

REMARK 2. In our case, $\mathscr{D}_2$ is incomplete, but we can view it as a subset of its completion, $\mathscr{L}^2$. To apply the lemma, we will then need to only consider differences with operators defined on all of $\mathscr{L}^2$. Since $\mathscr{D}_{gr}$ is already a Banach space, we do not need to consider operators defined on any larger set.

THEOREM 1.

(i) *Assume* $\psi-\mu I:\mathscr{L}^2\to\mathscr{L}^2$ *is Lipschitz continuous with constant* K_1, *and* $\eta:\mathscr{L}^2\to\mathbb{R}^2$ *is Lipschitz continuous with constant* K_2. *Then, if* $A_0K_1+B_0K_2<1$, $\mathbb{L}-\Psi:\mathscr{D}_2\to\mathscr{L}^2\times\mathbb{R}^2$ *is invertible; moreover* $(\mathbb{L}-\Psi)^{-1}$ *is compact and Lipschitz continuous with constant* $K\equiv K^*(1-A_0K_1-B_0K_2)^{-1}$.

(ii) *Assume* $\psi-\mu I:\mathscr{D}_{gr}\to\mathscr{L}^2$ *is Lipschitz continuous with constant* K_1, *and* $\eta:\mathscr{D}_{gr}\to\mathbb{R}^2$ *is Lipschitz continuous with constant* K_2. *Then, if* $(A_0+C_0)K_1+(B_0+D_0)K_2<1$, $\mathbb{L}-\Psi:\mathscr{D}_{gr}\to\mathscr{L}^2\times\mathbb{R}^2$ *is invertible; moreover* $(\mathbb{L}-\Psi)^{-1}$ *is Lipschitz continuous with constant* $N\equiv N^*(1-(A_0+C_0)K_1-(B_0+D_0)K_2)^{-1}$.

Proof. First, notice that $\mathbb{L}-\Psi=\mathbb{L}_\mu-\Psi_\mu$.

(i) We check the conditions of lemma (1). Let $x_1,x_2\in D$. Then, using corollary (1),

$$\begin{aligned}\|\mathbb{L}_\mu^{-1}\Psi_\mu(x_1)-\mathbb{L}_\mu^{-1}\Psi_\mu(x_2)\|_2 &\leqslant A_0\|(\psi-\mu I)(x_2)-(\psi-\mu I)(x_1)\|_2\\ &\quad+B_0\|\eta(x_2)-\eta(x_1)\|_2\\ &\leqslant (A_0K_1+B_0K_2)\|x_2-x_1\|.\end{aligned}$$

Lemma (1) now implies the desire result since $\mathbb{L}_\mu^{-1}$ is compact and Lipschitz continuous with constant K^* by corollary (1).

(ii) We check the conditions of lemma (1). Let $x_1,x_2\in D$. Then, using corollary (1),

$$\begin{aligned}\|\mathbb{L}_\mu^{-1}\Psi_\mu(x_1)-\mathbb{L}_\mu^{-1}\Psi_\mu(x_2)\|_{gr} &\leqslant(A_0+C_0)\|(\psi-\mu I)(x_2)-(\psi-\mu I)(x_1)\|_2\\ &\quad+(B_0+D_0)\|\eta(x_2)-\eta(x_1)\|_2\\ &\leqslant((A_0+C_0)K_1+(B_0+D_0)K_2)\|x_2-x_1\|.\end{aligned}$$

Lemma (1) now implies the desired result since $\mathbb{L}_\mu^{-1}$ is Lipschitz continuous with constant N^* by corollary (1). $\square$

In part (2), it can be assumed that $\psi-\mu I$ and η are only defined on D, since $\mathscr{D}_{gr}$ is a Banach space whereas $\mathscr{D}_2$ is not. This conclusion was a result from [28]; however, it was proved under different conditions, notably, the assumption of differentiability replaced our Lipschitz conditions. It should be noted that the important constant for the result to hold is strictly larger in part (2) than in part (1). This shows that there is a trade-off between the size of the domain of the functions of interest and the restrictions of their corresponding Lipschitz constants. For example, consider the operator η_1 in (4). Theorem 1 part (2) can be used for the case of

$$\eta_1(x) = \sum_{i=1}^{n} f(x(t_i)),$$

where $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, and $t_i \in [0,1]$. This is a so-called multipoint boundary condition. Theorem 1 part (1) does not cover this case. However, the case of

$$\eta_1(x) = \int_0^1 f(x(t))dt,$$

where $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, can be handled by part (1), and the constraints on the constants are less stringent than part (2) would require.

Before stating the next theorem, we must recall some basic facts about the spaces of interest. In [28], it was shown that, on D, $\|\cdot\|_{gr}$ is equivalent to the Sobolev norm, $\|\cdot\|_{2,2}$. It is well known that the Sobolev space $\mathscr{H}^2$ has compact embeddings into both $(C^1[0,1],\|\cdot\|_{C^1})$ and $\mathscr{L}^2$, and hence, so does $\mathscr{D}_{gr}$. Here we have used $C^1[0,1]$ to denote the functions on $[0,1]$ which have a continuous first derivative, and for $x\in C^1[0,1]$, $\|x\|_{C^1} = \sup|x(t)| + \sup|x'(t)|$. Denote these embeddings by $j_1 : \mathscr{D}_{gr} \hookrightarrow C^1[0,1]$ and $j_2 : \mathscr{D}_{gr} \hookrightarrow \mathscr{L}^2$, and their embedding constants by C_1 and C_2, respectively.

THEOREM 2. *Assume the conditions of Theorem 1 (1) for part 1 below; assume the conditions of Theorem 1 (2) for parts 2-3 below.*

(i) *Let $\mathscr{G} : \mathscr{L}^2 \to \mathscr{L}^2 \times \mathbb{R}^2$ be continuous and assume that there exists an $M \in \mathbb{N}$ such that, for $\|x\|_2 \leqslant M$, $\|\mathscr{G}(x)\|_2 \leqslant K^{-1}(M - \|(\mathbb{L}-\Psi)^{-1}(0)\|_2)$. Then there exists at least one point, $x_0 \in D$ such that $\mathbb{L}x_0 - \Psi(x_0) = \mathscr{G}(x_0)$.*

(ii) *Let $\mathscr{G} : \mathscr{D}_{gr} \to \mathscr{L}^2 \times \mathbb{R}^2$ be compact and assume that there exists an $M \in \mathbb{N}$ such that, for $\|x\|_{gr} \leqslant M$, $\|\mathscr{G}(x)\|_2 \leqslant N^{-1}(M - \|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr})$. Then there exists at least one point, $x_0 \in D$ such that $\mathbb{L}x_0 - \Psi(x_0) = \mathscr{G}(x_0)$.*

(iii) *Let $\mathscr{G} : C^1[0,1] \to \mathscr{L}^2 \times \mathbb{R}^2$ be continuous, and map bounded sets into bounded sets. Assume that there exists an $M \in \mathbb{N}$ such that, for $\|x\|_{C^1} \leqslant M$,*

$$\|\mathscr{G}(x)\|_2 \leqslant N^{-1}(C_1^{-1}M - \|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr}).$$

Then there exists at least one point, $x_0 \in D$ such that $\mathbb{L}x_0 - \Psi(x_0) = \mathscr{G}(x_0)$.

Proof.

(i) $H \equiv (\mathbb{L}-\Psi)^{-1}\mathscr{G} : \mathscr{L}^2 \to \mathscr{L}^2$ is compact since it is the composition of a compact operator with a continuous operator. Now, let $B = \{z \in L^2 | \|z\|_2 \leqslant M\}$. Let $x \in B$, then

$$\|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x)\|_2 \leqslant \|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x) - (\mathbb{L}-\Psi)^{-1}(0)\|_2 + \|(\mathbb{L}-\Psi)^{-1}(0)\|_2$$

$$\leqslant KK^{-1}(M-\|(\mathbb{L}-\Psi)^{-1}(0)\|_2)+\|(\mathbb{L}-\Psi)^{-1}(0)\|_2$$
$$\leqslant M.$$

Therefore, $H(B)\subseteq B$, which is clearly closed, bounded, and convex, so by Schauder's fixed point theorem [10, Theorem 8.8], there exists an $x_0\in L^2$ such that

$$(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x_0)=x_0.$$

Since $(\mathbb{L}-\Psi)^{-1}(L^2\times\mathbb{R}^2)=D$, $x_0\in D$, the result is proven.
(ii) $H\equiv(\mathbb{L}-\Psi)^{-1}\mathscr{G}:\mathscr{D}_{gr}\to\mathscr{D}_{gr}$ is compact since it is the composition of a continuous operator with a compact operator. Now, let $B=\{z\in D|\|z\|_{gr}\leqslant M\}$. Let $x\in B$, then

$$\begin{aligned}\|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x)\|_{gr}&\leqslant\|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x)-(\mathbb{L}-\Psi)^{-1}(0)\|_{gr}+\|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr}\\&\leqslant NN^{-1}(M-\|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr})+\|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr}\\&\leqslant M.\end{aligned}$$

Therefore, $H(B)\subseteq B$, which is clearly closed, bounded, and convex, so by Schauder's fixed point theorem, the result is proven.
(iii) $H\equiv(\mathbb{L}-\Psi)^{-1}\mathscr{G}\circ j_1:\mathscr{D}_{gr}\to\mathscr{D}_{gr}$ is compact since it is the composition of two continuous operators with a compact operator. Now, let $B=\{z\in D|\|z\|_{gr}\leqslant C_1^{-1}M\}$, and notice that $\|z\|_{gr}\leqslant C_1^{-1}M\implies\|z\|_{C^1}\leqslant M$. Let $x\in B$, then

$$\begin{aligned}\|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(j_1x)\|_{gr}&\leqslant\|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(j_1x)-(\mathbb{L}-\Psi)^{-1}(0)\|_{gr}+\|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr}\\&\leqslant NN^{-1}(M-\|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr})+\|(\mathbb{L}-\Psi)^{-1}(0)\|_{gr}\\&\leqslant M.\end{aligned}$$

Therefore, $H(B)\subseteq B$, which is clearly closed, bounded, and convex, so by Schauder's fixed point theorem, the result is proven. □

REMARK 3. In the cases that require the compactness of $\mathscr{G}$, this requirement can be replaced by the assumption that $\mathscr{G}$ is α-Lipschitz with constant less than K^{-1}, where α is a measure of noncompactness. The generalized form of Schauder's theorem would then be used.

If the norm conditions in Theorem 2 can be shown not to be sharp, in the sense that for $\|x\|=M$, the bound for the norm of $\|\mathscr{G}\|$ can be reduced by some $\varepsilon>0$, then the result can be extended slightly. We state the extension only for the first case; however, it applies analogously to the others.

COROLLARY 2. *Let $\mathscr{G}:\mathscr{L}^2\to\mathscr{L}^2\times\mathbb{R}^2$ be continuous and assume that there exists an $M\in\mathbb{N}$ such that, for $\|x\|_2\leqslant M$, $\|\mathscr{G}(x)\|_2\leqslant K^{-1}(M-\|(\mathbb{L}-\Psi)^{-1}(0)\|_2)-\delta$, where $\delta>0$. Consider $\mathscr{G}_\varepsilon\equiv\mathscr{G}+\varepsilon\mathscr{F}$, where $\mathscr{F}:\mathscr{L}^2\to\mathscr{L}^2\times\mathbb{R}^2$ is continuous such that $\sup_{x\in B}\|\mathscr{F}(x)\|=F<\infty$. Then, for every $\varepsilon<K\delta/F$, there exists at least one point, $x_0\in D$ such that $\mathbb{L}x_0-\Psi(x_0)=\mathscr{G}_\varepsilon(x_0)$.*

Proof. In Theorem 2, it is shown that the operator $H \equiv (\mathbb{L}-\Psi)^{-1}\mathscr{G} : \mathscr{L}^2 \to \mathscr{L}^2$ has a fixed point on the set $B = \{z \in L^2 | \|z\|_2 \leqslant M\}$, which means that $d(I-H,\mathring{B},0) \neq 0$, where d is the Leray-Schauder degree. By the assumed condition,

$$\rho(0,(I-H)(\partial B)) \geqslant K\delta$$

since for $\|x\|_2 = M$, $\|H(x)\| \leqslant M - K\delta$, where ρ is the $\mathscr{L}^2$ distance. By a well-known property of the degree [10, Theorem 8.2], any compact operator, H_2 within an $K\delta$-ball (sup norm) of H will satisfy $d(I-H,\mathring{B},0) = d(I-H_2,\mathring{B},0)$. The operator $H_2 \equiv (\mathbb{L}-\Psi)^{-1}(\mathscr{G}+\varepsilon\mathscr{F})$ clearly satisfies this with the given assumption on $\mathscr{F}$. □

We now provide examples for each of the four cases of Theorem 2. We focus only on the first component of $\mathscr{G}$, G, which maps into $\mathscr{L}^2$. Assume $k : \mathbb{R} \times [0,1] \to \mathbb{R}$ is continuous.

EXAMPLE 1. The operator $G(x) = k(x(\cdot),\cdot)$ is continuous from $\mathscr{L}^2$.

EXAMPLE 2. The operator $G(x) = \int_0^1 k(s,\cdot)x''(s)ds$ is compact from $\mathscr{D}_{gr}$.

EXAMPLE 3. The operator $G(x) = k(x'(\cdot),\cdot)$ is continuous from $C^1[0,1]$.

The next corollary is an immediate consequence of Theorem 2, and will apply to all parts of Theorem 2. The norms will be left ambiguous, but correspond to those in the respective parts of the theorem.

COROLLARY 3. *If $\mathscr{G}$ is sublinear in the sense that $\|\mathscr{G}(x)\| \leqslant a + b\|x\|^{\zeta}$, with $\zeta \in [0,1)$, then Theorem 2 holds.*

As an application of Theorem 2, we state the following two corollaries, where we use the notation from above for the linear part of the differential equation.

COROLLARY 4. *Consider the multipoint boundary value problem on the interval $[0,1]$*

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t)), \tag{8a}$$

subject to the boundary conditions

$$\alpha x(0) + \beta x'(0) + \sum_{i=1}^{N} h_{1,i}(x(t_i)) = 0 \tag{8b}$$

$$\gamma x(1) + \delta x'(1) + \sum_{i=1}^{M} h_{2,i}(x(s_i)) = 0, \tag{8c}$$

where $\psi, G, h_{j,i} : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions. Let $t \mapsto \psi(t) - \mu t$ be Lipschitz with constant K_1, and let $h_{j,i}$ be Lipschitz with constant $k_{j,i}$. Let $K_2 = \sum_{i=1}^{N} k_{1,i} + \sum_{i=1}^{M} k_{2,i}$. Then if $\exists\, a,b \in \mathbb{R}, \zeta \in [0,1)$ such that $|G(t)| \leqslant a + b|t|^{\zeta}$ for all $t \in \mathbb{R}$, and, in addition, $(A_0+C_0)K_1 + (B_0+D_0)K_2 < 1$, then there exists a solution to the above problem. If $G = 0$, then this solution is unique.

COROLLARY 5. *Consider the boundary value problem on the interval* $[0,1]$

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t)), \tag{9a}$$

subject to the boundary conditions

$$\alpha x(0) + \beta x'(0) + \int_0^1 h_1(x(t))dt = 0 \tag{9b}$$

$$\gamma x(1) + \delta x'(1) + \int_0^1 h_2(x(t))dt = 0, \tag{9c}$$

where $\psi, G, h_j : \mathbb{R} \to \mathbb{R}$ *are continuous functions. Let* $t \mapsto \psi(t) - \mu t$ *be Lipschitz with constant* K_1*, and let* h_j *be Lipschitz with constant* k_j*. Let* $K_2 = k_1 + k_2$*. Then if* $\exists\ a, b \in \mathbb{R}, \zeta \in [0,1)$ *such that* $|G(t)| \leqslant a + b|t|^{\zeta}$ *for all* $t \in \mathbb{R}$*, and, in addition,* $A_0K_1 + B_0K_2 < 1$*, then there exists a solution to the above problem. If* $G = 0$*, then this solution is unique.*

This last corollary shows the trade-off that using the weaker norm in the domain has on a typical result. If the problem only contains operators which can be defined on all of L^2, then the condition on the Lipschitz constants can be weakened. The case of point evaluations can, however, be approximated using this last corollary, by taking the integrals in the boundary conditions to be the convolution with a mollifier. The level of approximation to the point evaluation will also determine the Lipschitz constant, which will increase along with the accuracy.

4. Difference Equations

We consider difference equations of the form

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x)(t) = G(x)(t) \tag{10}$$

for $t = a+1, \ldots, b+1$, subject to boundary conditions of the form

$$\alpha x(a) + \beta \Delta x(a) + \eta_1(x) = \phi_1(x) \tag{11a}$$

$$\gamma x(b+1) + \delta \Delta x(b+1) + \eta_2(x) = \phi_2(x). \tag{11b}$$

Since we will work over the integers, we will drop the intersection in the notation and simply view $[a,b]$ as $[a,b] \cap \mathbb{Z}$. We assume that $x \in \mathbb{R}^{[a,b+2]}$, $p \in \mathbb{R}^{[a,b+1]}$, and $q \in \mathbb{R}^{[a+1,b+1]}$. Analogous to the previous section, ψ and G are function-valued operators, and η_1, η_2, ϕ_1, and ϕ_2 are real-valued functions. These represent the nonlinear perturbations of the classical linear problem, but conditions on the right-hand and left-hand side perturbations that guarantee the existence of a solution will be qualitatively different. We make the usual assumptions on the linear portions of the problem, namely that $p, p\prime$, and q are continuous, $p > 0$, $\alpha^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$, $\alpha \neq \beta$, and $\gamma \neq \delta$.

The strategy for analyzing (10)-(11) will be to first determine conditions under which we can uniquely solve the following problem:

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x)(t) = h(t), \tag{12}$$

subject to

$$\alpha x(a) + \beta \Delta x(a) + \eta_1(x) = v_1 \tag{13a}$$
$$\gamma x(b+1) + \delta \Delta x(b+1) + \eta_2(x) = v_2, \tag{13b}$$

where $h \in \mathbb{R}^{[a+1,b+1]}$ and $v_1, v_2 \in \mathbb{R}$. This problem corresponds to finding an inverse for the operator representing the left-hand side of (12)-(13), and is dealt with in Theorem 3.

Let $X = \mathbb{R}^{[a,b+2]}$, $Y = \mathbb{R}^{[a+1,b+1]}$, and $N = b - a + 1$. On any Euclidean space, $\mathbb{R}^n$, we denote the usual norm by $|\cdot|$, regardless of dimension. On X and Y we use the inner product $\langle x, y \rangle = \sum_t x(t)y(t)$, where the sum ranges over the domain of the functions. Along with this inner product, we use the typical norm of $\|x\| = \langle x, x \rangle$.

With this notation, we define the operators representing the linear part of the problem, $\mathbb{A} : X \to Y$ and $\mathbb{B} : X \to \mathbb{R}^2$, where

$$\mathbb{A}(x) \equiv \Delta(p(t-1)\Delta x(t-1)) + q(t)x(t),$$
$$\mathbb{B}(x) \equiv \begin{pmatrix} \alpha x(a) + \Delta \beta x(a) \\ \gamma x(b+1) + \delta \Delta x(b+1) \end{pmatrix},$$

and let

$$\mathbb{L}(x) \equiv \begin{pmatrix} \mathbb{A}(x) \\ \mathbb{B}(x) \end{pmatrix}.$$

For the nonlinear portion, let $\eta = (\eta_1, \eta_2)$ and $\phi = (\phi_1, \phi_2)$. Then define $\Psi, \mathscr{G} : X \to Y \times \mathbb{R}^2$ as

$$\Psi(x) \equiv \begin{pmatrix} -\psi(x) \\ -\eta(x) \end{pmatrix}, \mathscr{G}(x) \equiv \begin{pmatrix} G(x) \\ \phi(x) \end{pmatrix}.$$

On $Y \times \mathbb{R}^2$ we will use the sum of the the two usual component norms. The properties of these components guaranteeing the existence of solutions will be the content of the main results of this paper. For now, simply view them as nonlinear maps. With this notation, we rewrite (10)-(11) as

$$\mathbb{L}x - \Psi(x) = \mathscr{G}(x),$$

for $x \in X$.

Classical Sturm-Liouville theory deals with equations of the form $\mathbb{A}x = h$ for $x \in \mathbb{B}^{-1}(\{0\})$, and we will require well-known properties from this case. First, $\mathbb{A}$ has N simple eigenvalues, $\{-\lambda_k\}_{k=1}^N$, which we can assume are in decreasing order. For each $-\lambda_k$, there is an eigenfunction, u_k, with unit norm, which spans the corresponding eigenspace. It is known that for $k \neq j$, $\langle u_k, u_j \rangle = 0$. In general, $\mathbb{A}$ might fail to be injective. Because of this we take $\mu \in \mathbb{R}$, which is not an eigenvalue, and let

$$\mathbb{L}_\mu x \equiv \begin{pmatrix} \mathbb{A}x + \mu x \\ \mathbb{B}x \end{pmatrix}, \Psi_\mu(x) \equiv \begin{pmatrix} -\psi(x) + \mu x \\ -\eta(x) \end{pmatrix}.$$

This implies that the operator $\mathbb{A}+\mu I$ has eigenvalues $\{\mu-\lambda_k\}_{k=1}^N$ corresponding to the same eigenfunctions, but where it is now guaranteed that 0 is not an eigenvalue.

From the theory of second order difference equations, it is known that $(\mathbb{A}+\mu I)x = 0$ has a two dimensional solution space. We can choose a basis, $\{w_1, w_2\}$, for this space such that $\|w_1\|+\|w_2\| \leqslant 1$. Then let $w=(w_1,w_2)^t$ and define the 2×2 matrix $B=[\mathbb{B}w_1|\mathbb{B}w_2]$. With these definitions we can now give the form of the inverse of $\mathbb{L}_\mu$. The following result is cited from [27].

LEMMA 4. (from [27]) $\mathbb{L}_\mu : X \to Y\times\mathbb{R}^2$ *is bijective and*

$$\mathbb{L}_\mu^{-1}(h,v) = \sum_{k=1}^N \frac{\langle h,u_k\rangle}{\mu-\lambda_k}u_k + w^t B^{-1} v.$$

The next corollary provides estimates on norms of interest related to $\mathbb{L}_\mu^{-1}$. The operator norm is given the same notation as the norm on the corresponding domain of the operator.

COROLLARY 6. *Let* $A_0 = \sup_k |\mu-\lambda_k|^{-1}$, $B_0 = \|B^{-1}\|$. *Then,*

$$\|\mathbb{L}_\mu^{-1}(h,v)\|_2 \leqslant A_0\|h\|_2 + B_0|v| \quad \text{and} \quad \|\mathbb{L}_\mu^{-1}\|_2 \leqslant \max\{A_0,B_0\} \equiv K^*.$$

Proof.

$$\begin{aligned}\|\mathbb{L}_\mu^{-1}(h,v)\|_2 &\leqslant \sum_{k=1}^N \left|\frac{\langle h,u_k\rangle}{\mu-\lambda_k}\right| + \|w^t B^{-1} v\|_2 \\ &\leqslant A_0 \sum_{k=1}^N |\langle h,u_k\rangle|^2 + |B^{-1}v| \\ &\leqslant A_0\|h\|_2 + B_0|v|. \quad \square\end{aligned}$$

As opposed to relying on a global inverse function theorem, as in [27], we instead employ Lemma 1. As in the previous section, the significant benefit of this approach is the removal of any differentiability requirements, which are then replaced by requirements of Lipschitz continuity.

THEOREM 3. *Assume* $\psi-\mu I : X\to X$ *is Lipschitz continuous with constant* K_1, *and* $\eta : X\to\mathbb{R}^2$ *is Lipschitz continuous with constant* K_2. *Then, if* $A_0K_1 + B_0K_2 < 1$, $\mathbb{L}-\Psi : X\to Y\times\mathbb{R}^2$ *is invertible; moreover* $(\mathbb{L}-\Psi)^{-1}$ *is Lipschitz continuous with constant* $K \equiv K^*(1-A_0K_1-B_0K_2)^{-1}$.

Proof. First, notice that $\mathbb{L}-\Psi = \mathbb{L}_\mu - \Psi_\mu$. We check the conditions of lemma (1). Let $x_1, x_2 \in X$. Then, using corollary (1),

$$\|\mathbb{L}_\mu\Psi_\mu(x_1) - \mathbb{L}_\mu\Psi_\mu(x_2)\|_2 \leqslant A_0\|(\psi-\mu I)(x_2) - (\psi-\mu I)(x_1)\|_2$$

$$+B_0\|\eta(x_2)-\eta(x_1)\|_2$$
$$\leqslant (A_0K_1+B_0K_2)\|x_2-x_1\|.$$

Lemma (1) now implies the desired result since $\mathbb{L}_\mu^{-1}$ is Lipschitz continuous with constant K^* by Corollary (6). □

The conclusion was a result from [27]; however, it was proved under different conditions, notably the assumption of differentiability replaced our Lipschitz conditions.

THEOREM 4. *Assume the conditions of Theorem 1. Let $\mathscr{G}: X \to Y\times\mathbb{R}^2$ be continuous and assume that there exists an $M\in\mathbb{N}$ such that, for $\|x\|\leqslant M$, $\|\mathscr{G}(x)\|\leqslant K^{-1}(M-\|(\mathbb{L}-\Psi)^{-1}(0)\|)$. Then there exists at least one point, $x_0\in X$ such that $\mathbb{L}x_0-\Psi(x_0)=\mathscr{G}(x_0)$.*

Proof. $H\equiv(\mathbb{L}-\Psi)^{-1}\mathscr{G}: X\to X$ is continuous since it is the composition of two continuous operators. Now, let $B=\{z\in X\,|\,\|z\|_2\leqslant M\}$, and let $x\in B'$, then

$$\begin{aligned}\|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x)\|_2 &\leqslant \|(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x)-(\mathbb{L}-\Psi)^{-1}(0)\|_2+\|(\mathbb{L}-\Psi)^{-1}(0)\|_2\\ &\leqslant KK^{-1}(M-\|(\mathbb{L}-\Psi)^{-1}(0)\|_2)+\|(\mathbb{L}-\Psi)^{-1}(0)\|_2\leqslant M.\end{aligned}$$

Therefore, $H(B)\subseteq B$, which is clearly closed, bounded, and convex, so by Brouwer's fixed point theorem [10, Theorem 3.2], there exists an $x_0\in X$ such that

$$(\mathbb{L}-\Psi)^{-1}\mathscr{G}(x_0)=x_0. \quad \square$$

COROLLARY 7. *Let $\mathscr{G}: X\to Y\times\mathbb{R}^2$ be continuous and assume that there exists an $M\in\mathbb{N}$ such that, for $\|x\|_2\leqslant M$, $\|\mathscr{G}(x)\|_2\leqslant K^{-1}(M-\|(\mathbb{L}-\Psi)^{-1}(0)\|_2)$. Assume that for any $x\in X$ such that $\mathbb{L}x-\Psi(x)=\mathscr{G}(x)$, $\|x\|<M$, and consider $\mathscr{G}_\varepsilon\equiv\mathscr{G}+\varepsilon\mathscr{F}$, where $\mathscr{F}: X\to Y\times\mathbb{R}^2$ is continuous. Then, for every ε small enough, there exists at least one point, $x_0\in X$ such that $\mathbb{L}x_0-\Psi(x_0)=\mathscr{G}_\varepsilon(x_0)$.*

Proof. In Theorem 4, it is shown that the operator $H\equiv(\mathbb{L}-\Psi)^{-1}\mathscr{G}: X\to X$ has a fixed point on the set $B=\{z\in X\,|\,\|z\|\leqslant M\}$, which means that $d(I-H,\mathring{B},0)\neq 0$, where d is the Brouwer degree. Since ∂B is compact, so is $(I-H)(\partial B)$, and hence $\rho(0,(I-H)(\partial B))=\delta$, for some $\delta>0$, where ρ is the Euclidean distance. This is because, by assumption, $I-H$ has no zeroes on the boundary of B. By a well-known property of the degree [10, Theorem 3.1], any continuous operator, H_2, within an δ-ball (sup norm) of H will satisfy $d(I-H,\mathring{B},0)=d(I-H_2,\mathring{B},0)$. Let $\sup_{x\in B}\|\mathscr{F}(x)\|=F$. Then, the operator $H_2\equiv(\mathbb{L}-\Psi)^{-1}(\mathscr{G}+\varepsilon\mathscr{F})$ clearly satisfies this whenever $\varepsilon<K\delta/F$. □

The next corollary is an immediate consequence of Theorem 4.

COROLLARY 8. *If $\mathscr{G}$ is sublinear in the sense that $\|\mathscr{G}(x)\|\leqslant a+b\|x\|^\zeta$, with $\zeta\in[0,1)$, then Theorem 4 holds.*

As an application of Theorem 4, we state the following two corollaries, where we assume the notation as above for the linear part of the differential equation.

COROLLARY 9. *Consider the multipoint boundary value problem on the interval* $[0,1]$,

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x(t)) = G(x(t)), \tag{14a}$$

subject to the boundary conditions

$$\alpha x(a) + \beta\Delta x(a) + \sum_{i=1}^{M_1} h_{1,i}(x(t_i)) = 0 \tag{14b}$$

$$\gamma x(b+1) + \delta\Delta x(b+1) + \sum_{i=1}^{M_2} h_{2,i}(x(s_i)) = 0, \tag{14c}$$

where $\psi, G, h_{j,i} : \mathbb{R} \to \mathbb{R}$ *are Lipschitz continuous functions. Let* $t \mapsto \psi(t) - \mu t$ *be Lipschitz with constant* K_1, *and let* $h_{j,i}$ *be Lipschitz with constant* $k_{j,i}$. *Let* $K_2 = \sum_{i=1}^{M_1} k_{1,i} + \sum_{i=1}^{M_2} k_{2,i}$. *Then if* $\exists\, a,b \in \mathbb{R}, \zeta \in [0,1)$ *such that* $|G(t)| \leqslant a + b|t|^{\zeta}$ *for all* $t \in \mathbb{R}$, *and, in addition,* $A_0K_1 + B_0K_2 < 1$, *then there exists a solution to the above problem. If* $G = 0$, *then this solution is unique.*

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