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SECOND–ORDER FUNCTIONAL PROBLEMS WITH A RESONANCE OF DIMENSION ONE

NICKOLAI KOSMATOV AND WEIHUA JIANG

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Abstract. We obtain, using the coincidence degree theory, solvability conditions for all possible resonance scenarios $Lu = u'' = f(t,u,u') = Nu$, with linear functional conditions $B_i u = 0$, $i = 1,2$ with $\dim\ker L = 1$. Our work generalizes and improves the results of Zhao and Liang [18] and Cui [3] in several directions. We also construct a meaningful example of a nonlinear functional problem for a pendulum equation which not only satisfies the assumptions of an existence theorem but also has a closed-form solution.

1. Introduction

Resonant boundary value problems have been studied by a broad range of techniques [1, 2, 3, 4, 5, 6, 10, 11, 12, 15, 17, 18]. To this day, Mawhin's coincidence degree theory [14] continues to plays an important role in this active field. Recently, the attention has shifted to problems with integral boundary conditions and, more generally, to problems with linear functional conditions [18] and resonance scenarios [3] that have been completely overlooked in the past. Notably, some interesting results for systems of equations have also been obtained in [16] for a resonant problem that does not allow "uncoupling" in the sense that it cannot be treated as a scalar problem at resonance. We also mention [13], where a higher-order nonlocal problem at one-dimensional resonance was studied by reduction to a first order vector equation.

One of most general studies of a resonant problem for the differential operator $L : C^1[0,1] \to L_1[0,1]$, $Lx = x''$ known to us is done by Zhao and J. Liang [18], where the authors considered the functional differential problem

$$x''(t) = f(t,x(t),x'(t)), \quad t \in (0,1), \tag{1}$$

$$\Gamma_1(x) = 0, \quad \Gamma_2(x) = 0, \tag{2}$$

where Γ_1, Γ_2 are linear functionals on $C^1[0,1]$ satisfying the general resonance condition $\Gamma_1(t)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(t)$. (The non-resonant scenario subject to the condition (A_1): $\Gamma_1(t)\Gamma_2(1) \neq \Gamma_1(1)\Gamma_2(t)$ was also studied.) Specifically, the authors investigated the following resonant cases:

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(A_2) $\Gamma_1(t), \Gamma_1(1), \Gamma_2(1) = 0$, $\Gamma_2(t) \neq 0$;

(A_3) $\Gamma_1(t), \Gamma_1(1), \Gamma_2(t) = 0$, $\Gamma_2(1) \neq 0$;

(A_4) $\Gamma_1(1), \Gamma_2(t), \Gamma_2(1) = 0$, $\Gamma_1(t) \neq 0$;

(A_5) $\Gamma_1(t), \Gamma_2(1), \Gamma_2(t) = 0$, $\Gamma_1(1) \neq 0$;

(A_6) $\Gamma_1(1), \Gamma_1(t), \Gamma_2(1), \Gamma_2(t) = 0$.

The cases (A_2) and (A_4) result in $\ker L = \{c : c \in \mathbb{R}\}$ and (A_3) and (A_5) correspond to $\ker L = \{ct : c \in \mathbb{R}\}$. The case (A_6) describes a resonance with $\ker L = \{c_1 t + c_2 : c_1, c_2 \in \mathbb{R}\}$. The cases (A_2), (A_3), and (A_6) were investigated in full detail.

Although [18] generalizes and extends many results for nonlocal second-order problems at resonance, it does not contain a complete analysis of (1), (2). To see this, let $a, b, \alpha \in \mathbb{R}$ and $a, b \neq 0$ and set, for example, $\Gamma_1(t) = \alpha b$, $\Gamma_1(1) = \alpha a$, $\Gamma_2(t) = b$, and $\Gamma_2(1) = a$. Then $\Gamma_1(t)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(t)$ with

$$\ker L = \{c\,(at - b) : c \in \mathbb{R}\}, \quad \dim \ker L = 1.$$

This case cannot be derived from the results of [18] pertaining to the cases of (A_2) - (A_6). In [3], Cui considered such "slanted" kernels, which are also the main motivation of the present paper. To be exact, [3] studied

$$x''(t) = f(t, x(t), x'(t)), \quad t \in (0,1), \tag{3}$$

$$x(0) = \int_0^1 x(s)d\alpha(s), \quad x(1) = \int_0^1 x(s)d\beta(s), \tag{4}$$

where $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$; α and β are right continuous on $[0,1)$, left continuous at $t = 1$, and $\int_0^1 u(s)d\alpha(s)$ and $\int_0^1 u(s)d\beta(s)$ are Riemann-Stieltjes integrals. The resonance condition is $\kappa_1\kappa_4 - \kappa_2\kappa_3 = 0$, where

$$\kappa_1 = 1 - \int_0^1 (1-t)d\alpha(t), \quad \kappa_2 = \int_0^1 t d\alpha(t),$$
$$\kappa_3 = \int_0^1 (1-t)d\beta(t), \quad \kappa_4 = 1 - \int_0^1 t d\beta(t).$$

We can interpret $\Gamma_i(x) = 0$, $i = 1,2$ in [18], as

$$x(0) - \int_0^1 x(s)\,d\alpha(s) = 0, \quad x(1) - \int_0^1 x(s)\,d\beta(s) = 0,$$

respectively. In [18], the authors also interpret $\Gamma_i(u)$ as a linear combination of the Riemann-Stiltjes integrals of x and x' defined in terms of measures of bounded variation. Subsequently, the authors use these representations to obtain uniqueness theorems and to compare their results to those of [12]. We do not believe that it is necessary to rely on such a representation of the functionals Γ_i.

The authors of [18] make unnecessary respective assumptions $\Gamma_1(t^2) \neq 0$ and $\Gamma_1(t^3) \neq 0$ in Theorems 3.2 and 3.3, which yield existence criteria for the respective cases (A_2) and (A_3). In [3], such artificial conditions were also deemed necessary. In particular, for (3), (4) it is assumed that

$$\kappa_3 \int_0^1 t(1-t)d\alpha(t) + \kappa_1 \int_0^1 t(1-t)d\beta(t) \neq 0.$$

Conditions of this type are only needed to ensure that $Q: Z \to Z$ is well-defined and in our work we propose an approach that allows us to bypass this minor technical difficulty (see (H) below). Thus, we improve the results of [3] and [18] in that respect as well. In addition, due to the simplicity of our method, it is clearly preferred to that devised in [4], and can also be used for higher order problems with functional conditions.

Moreover, in many recent papers devoted to the second order problems (see, e. g., [12, 18]) the following assumption is imposed: There exists a constant $M_0 > 0$, such that if $|u(t)| + |u'(t)| > M_0$, $t \in [0,1]$, then $QNu \neq 0$. It is used to show that $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L : Lu = \lambda Nu, \lambda \in (0,1)\}$ is bounded. A similar comment can be made about [3]. In the present paper we also show that the boundedness of Ω_1 can be shown if $|u(t)| + |u'(t)| > M_0$ is replaced with just $|u(t)| > M_0$ or $|u'(t)| > M_0$ (see Theorems 4 and 5 below.) In Section 2 we will further discuss the methodology of á priori estimates in comparison with [3, 18].

We consider the differential equation

$$u''(t) = f(t, u(t), u'(t)), \quad t \in (0,1), \tag{5}$$

together with the functional conditions

$$B_1(u) = B_2(u) = 0. \tag{6}$$

We assume the following:

(B_1) The linear functionals $B_1, B_2 : X \to \mathbb{R}$ satisfy $B_1(t) = \alpha b$, $B_1(1) = \alpha a$, $B_2(t) = b$, $B_2(1) = a$, where $\alpha, a, b \in \mathbb{R}$.

(B_2) The functionals $B_1, B_2 : X \to \mathbb{R}$ are continuous with the respective norms β_1, β_2, that is, $|B_i(u)| \leqslant \beta_i \|u\|_X$.

Here $X = C^1[0,1]$ with the norm $\|u\|_X = \max\{\|u\|_0, \|u'\|_0\}$, where $\|\cdot\|_0$ is the max-norm, and $Z = L_1[0,1]$ with the usual norm $\|\cdot\|_1$.

Let us assume, without loss of generality, that

(H) the function $h \in Z$ satisfies

$$(B_1 - \alpha B_2)\left(\int_0^t (t-s)h(s)\,ds\right) = 1.$$

The following lemma shows that the assumption (H) is merely a matter of choice of such a function.

LEMMA 1. *Assume that* (B_1) *and* (B_2) *hold. Then there exists* $h \in Z$ *such that*

$$(B_1 - \alpha B_2)\left(\int_0^t (t-s)h(s)\,ds\right) \neq 0.$$

Proof. For convenience, set $B = B_1 - \alpha B_2$. Assume, by way of contradiction, that

$$B\left(\int_0^t (t-s)h(s)ds\right) = 0$$

for all $h \in Z$. In particular, for every integer $n \geqslant 0$,

$$(n+1)(n+2)B\left(\int_0^t (t-s)s^n\,ds\right) = B(t^{n+2}) = 0.$$

By (B_2), $B(1) = B(t) = 0$. Thus, $B(p) = 0$ for every polynomial p.

Since $B \neq 0$ on all of X, there exists $v_0 \in X$ such that $B(v_0) \neq 0$. Choose a sequence of polynomials $\{p_m\}$ such that $\|v_0 - p_m\|_X < \frac{1}{m}$. Then

$$0 \neq |B(v_0)| = |B(v_0 - p_m) + B(p_m)| = |B(v_0 - p_m)| \leqslant ||B|| \|v_0 - p_m\|_X < (\beta_1 + |\alpha|\beta_2)\frac{1}{m}$$

for all $m \in \mathbb{N}$, which is a contradiction. $\square$

Define the mapping $L : \operatorname{dom} L \subset X \to Z$ by

$$Lu = u'',$$

where

$$\operatorname{dom} L = \{u \in X : u'' \in Z \text{ and } B_1(u) = B_2(u) = 0\}.$$

We assume that the function f satisfies the Carathéodory conditions and define $N : X \to Z$ by

$$Nu(t) = f(t, u(t), u'(t)).$$

The functional differential problem (5), (6) is now equivalent to the abstract equation $Lu = Nu$.

DEFINITION 1. Let X and Z be real normed spaces. A linear mapping $L : \operatorname{dom} L \subset X \to Z$ is called a Fredholm mapping if $\ker L$ has a finite dimension and $\operatorname{Im} L$ is closed and has a finite co-dimension.

If L is a Fredholm mapping, its (Fredholm) *index* is the integer $\operatorname{Ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L$.

Define continuous projectors $P : X \to X$ and $Q : Z \to Z$ so that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

For a Fredholm mapping L of index zero, the inverse of the map

$$L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is denoted by $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$.

DEFINITION 2. Let $L : \mathrm{dom} L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N : E \to Z$ be a mapping. We say that N is L-compact on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are continuous and compact on E. In addition, we say, that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

The following is the Kolmogorov-Riesz criterion (see, for example, [7]):

THEOREM 1. *For* $1 \leqslant p < \infty$, $E \subset L_p[0,1]$ *is compact if*

(a) E *is bounded;*

(b) the limit

$$\lim_{\varepsilon \to 0} \int_0^1 |g(s+\varepsilon) - g(s)|^p \, ds = 0$$

is uniform in E.

The compactness of $K_{P,Q}N : E \to X$ and $QN : E \to Z$ will follow from the Arzela-Ascoli theorem and the Kolmogorov-Riesz criterion, respectively. However, we will omit the corresponding details as straightforward.

The equation $Lu = Nu$ will be shown to have a solution by means of Theorem IV.13 [14]:

THEOREM 2. *Let* $\Omega \subset X$ *be open and bounded,* L *be a Fredholm mapping of index zero and* N *be* L*-compact on* $\overline{\Omega}$. *Assume that the following conditions are satisfied:*

(i) $Lu \neq \lambda Nu$ *for every* $(u,\lambda) \in ((\mathrm{dom} L \setminus \ker L) \cap \partial\Omega) \times (0,1)$*;*

(ii) $Nu \notin \mathrm{Im} L$ *for every* $u \in \ker L \cap \partial\Omega$*;*

(iii) $\deg(QN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, *with* $Q : Z \to Z$ *a continuous projector such that* $\ker Q = \mathrm{Im} L$.

Then the equation $Lu = Nu$ *has at least one solution in* $\mathrm{dom} L \cap \overline{\Omega}$.

REMARK 1. The condition (B_1) incorporates the cases $\Gamma_1(t) = \Gamma_2(t) = 0$ and $\Gamma_1(1) = \Gamma_2(1) = 0$ considered in [18]. In this form, it is needed for Theorem 3. Furthermore, in order to prove Theorems 4 and 5 we will assume, in addition, that $a \neq 0$ and $\frac{b}{a} \notin (0,1)$, respectively.

We will demonstrate now that (B_1) is a critical condition, that is, the functional problem (5), (6) is at resonance and, moreover, $\dim \ker L = 1$.

LEMMA 2. *Let* (B_1) *with* $a^2 + b^2 \neq 0$, (B_2) *and* (H) *hold. Then the mapping* $L : \mathrm{dom} L \subset X \to Z$ *is a Fredholm mapping of index zero.*

Proof. If $u \in \operatorname{dom} L$ and $Lu = 0$, we have $u = c_1 t + c_2$ and

$$B_1(c_1 t + c_2) = c_1 B_1(t) + c_2 B_1(1) = \alpha b c_1 + \alpha a c_2 = 0,$$

$$B_2(c_1 t + c_2) = c_1 B_2(t) + c_2 B_2(1) = b c_1 + a c_2 = 0.$$

Therefore,

$$\ker L = \{c\,(at - b) : c \in \mathbb{R}\}, \quad \dim\ker L = 1.$$

Now we verify

$$\operatorname{Im} L = \left\{ g \in Z : (B_1 - \alpha B_2)\left(\int_0^t (t-s) g(s)\, ds\right) = 0 \right\}. \tag{7}$$

Let $g \in \operatorname{Im} L$, then there exists $u \in \operatorname{dom} L$ such that $g = Lu$, that is,

$$u = \int_0^t (t-s) g(s)\, ds + u'(0)t + u(0)$$

and $B_1 u = B_2 u = 0$. Hence,

$$0 = B_i u = B_i\left(\int_0^t (t-s) g(s)\, ds\right) + u'(0) B_i(t) + u(0) B_i(1), \quad i = 1, 2,$$

and, using the resonance condition (B_1),

$$\begin{aligned} &B_1\left(\int_0^t (t-s) g(s)\, ds\right) + u'(0) B_1(t) + u(0) B_1(1) \\ &= B_1\left(\int_0^t (t-s) g(s)\, ds\right) + \alpha b u'(0) + \alpha a u(0) = 0, \\ &B_2\left(\int_0^t (t-s) g(s)\, ds\right) + u'(0) B_2(t) + u(0) B_2(1) \\ &= B_2\left(\int_0^t (t-s) g(s)\, ds\right) + b u'(0) + a u(0) = 0. \end{aligned}$$

Hence $g \in \left\{ g \in Z : (B_1 - \alpha B_2)\left(\int_0^t (t-s) g(s)\, ds\right) = 0 \right\}$. That is,

$$\operatorname{Im} L \subseteq \left\{ g \in Z : (B_1 - \alpha B_2)\left(\int_0^t (t-s) g(s)\, ds\right) = 0 \right\}.$$

If $g \in \left\{ g \in Z : (B_1 - \alpha B_2)\left(\int_0^t (t-s) g(s)\, ds\right) = 0 \right\}$, take

$$u(t) = -\frac{bt + a}{a^2 + b^2} B_2\left(\int_0^t (t-s) g(s)\, ds\right) + \int_0^t (t-s) g(s) ds.$$

It is clear that $Lu = g$ and $B_1 u = B_2 u = 0$. That is,

$$\left\{ g \in Z : (B_1 - \alpha B_2)\left(\int_0^t (t-s) g(s)\, ds\right) = 0 \right\} \subseteq \operatorname{Im} L.$$

Combining the above we obtain (7).

We consider

$$Pu(t)=\frac{1}{a^2+b^2}(au'(0)-bu(0))(at-b)$$

It is easy to check that $P^2u=Pu$, $u\in X$. It is also elementary to confirm the identity $\operatorname{Im}P=\ker L$. Moreover, choose $u=c(at-b)\in\operatorname{Im}P$. If, in addition, $c(at-b)\in\ker P$, then $0=au'(0)-bu(0)=c(a^2+b^2)$. Now, since $a^2+b^2\neq 0$, we must have $c=0$. That is, $\ker L\oplus\ker P=X$.

Define an operator $Q:Z\to Z$ as

$$Qg(t)=(B_1-\alpha B_2)\left(\int_0^t (t-s)g(s)ds\right)h(t).$$

Then, by (B_2) and (H), $Q:Z\to Z$ is a continuous linear projector such that $\operatorname{Im}L=\ker Q$ and $\operatorname{Im}Q=\{ch(t):c\in\mathbb{R}\}$. It is clear that $Z=\operatorname{Im}L\oplus\operatorname{Im}Q$ and $\dim\ker L=\operatorname{codim}\operatorname{Im}L$, that is, L is a Fredholm mapping of index zero. □

LEMMA 3. *The map $K_P:Z\to\operatorname{dom}L\cap\ker P$ defined by*

$$K_Pg(t)=-\frac{bt+a}{a^2+b^2}B_2\left(\int_0^t (t-s)g(s)\,ds\right)+\int_0^t (t-s)g(s)\,ds$$

is the inverse of L.

Proof. Obviously, $LK_pg=g$ for all $g\in Z$. For $u\in\operatorname{dom}L\cap\ker P$,

$$\begin{aligned}(K_PLu)(t)=(K_Pu'')(t)&=-\frac{bt+a}{a^2+b^2}B_2\left(\int_0^t (t-s)u''(s)\,ds\right)+\int_0^t (t-s)u''(s)\,ds\\&=-\frac{bt+a}{a^2+b^2}B_2(u-u'(0)t-u(0))+u(t)-u'(0)t-u(0)\\&=\frac{bt+a}{a^2+b^2}(bu'(0)+au(0))+u(t)-u'(0)t-u(0)\\&=u(t)-\frac{at-b}{a^2+b^2}(au'(0)-bu(0))\\&=u(t)-(Pu)(t)\\&=u(t).\end{aligned}$$

Thus,

$$K_P=(L|_{\operatorname{dom}L\cap\ker P})^{-1}.\quad\square$$

The next lemma provides norm-estimates needed for the main results.

LEMMA 4. *For $g\in Z$,*

$$\|(K_pg)'\|_0\leqslant A\|g\|_1,\quad A=1+\frac{|b|}{a^2+b^2}\beta_2, \tag{8}$$

and

$$\|K_p g\|_0 \leqslant B\|g\|_1, \quad B = 1 + \frac{||bt+a||_0}{a^2+b^2}\beta_2. \tag{9}$$

Moreover,

$$\|K_P g||_X \leqslant \|K_P\|\|g\|_1, \quad \|K_P\| = 1 + \frac{\beta_2}{a^2+b^2}||bt+a||_X. \tag{10}$$

Proof. Observe that due to $|B_2(u)| \leqslant \beta_2\|u\|_X$,

$$\begin{aligned} |K_P g(t)| &\leqslant \frac{|bt+a|}{a^2+b^2}\beta_2 \left\| \int_0^t (t-s)g(s)\,ds \right\|_X + \left\| \int_0^t (t-s)g(s)\,ds \right\|_0 \\ &\leqslant \left(\frac{|bt+a|}{a^2+b^2}\beta_2 + 1 \right) \|g\|_1 \\ &= \left(\frac{\beta_2}{a^2+b^2}||bt+a\|_0 + 1 \right) \|g\|_1, \end{aligned}$$

and we arrive at (9). Similarly,

$$\begin{aligned} |(K_P g)'(t)| &\leqslant \frac{|b|}{a^2+b^2}\beta_2 \left\| \int_0^t (t-s)g(s)\,ds \right\|_X + \left\| \int_0^t g(s)\,ds \right\|_0 \\ &\leqslant \left(\frac{|b|}{a^2+b^2}\beta_2 + 1 \right) \|g\|_1, \end{aligned}$$

which implies (8). Finally, (10) follows from (9) and (8) since $\|bt+a\|_X = \max\{\|bt+a\|_0, |b|\}$. $\square$

2. Main results

THEOREM 3. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function. Assume that (B_1) with $a^2+b^2 \neq 0$, (B_2), (H) and the following conditions hold:*

(A_1) *There exists a constant $M_0 > 0$, such that if $|u(t)| + |u'(t)| > M_0$, then*

$$(B_1 - \alpha B_2)\left(\int_0^t (t-s)f(s,u(s),u'(s))\,ds \right) \neq 0.$$

(A_2) *There exist nonnegative functions $\alpha, \beta, \gamma \in L_1[0,1]$ with*

$$\|\alpha\|_1 + \|\beta\|_1 < 1 \tag{11}$$

such that

$$|f(t,u,v)| \leqslant \gamma(t) + \alpha(t)|u| + \beta(t)|v|, \quad \text{a.e.} \, t \in [0,1], \quad u,v \in \mathbb{R}.$$

(A_3) *There exists a constant $M_1 > 0$, such that if $|c| > M_1$, then*

$$c(B_1 - \alpha B_2)\left(\int_0^t (t-s)f(s,c(as-b),ca)\,ds \right) > 0. \tag{12}$$

Then the boundary value problem (5), (6) has at least one solution.

REMARK 2. The inequality (12) may be replaced by

$$c(B_1-\alpha B_2)\left(\int_0^t (t-s)f(s,c(as-b),ca)\,ds\right)<0. \tag{13}$$

The proof of Theorem 3 will be based on the next three lemmas.

LEMMA 5. *Assume that (B_1) with $a^2+b^2\neq 0$, (B_2), (H), (A_1), and (A_2) hold. Then*

$$\Omega_1=\{u\in \operatorname{dom}L\setminus \ker L:\ Lu=\lambda Nu,\ \lambda\in(0,1)\}$$

is bounded.

Proof. If $u\in\Omega_1$, then by (A_1), there exists a constant $t_0\in[0,1]$ such that

$$|u(t_0)|,|u'(t_0)|\leqslant M_0.$$

Since

$$u(t)=\int_{t_0}^t u'(s)\,ds+u(t_0),$$

we get

$$|u(t)|\leqslant M_0+\|u'\|_0,\quad t\in[0,1]. \tag{14}$$

By $Lu=\lambda Nu$, we obtain

$$u'(t)=\lambda\int_{t_0}^t f(s,u(s),u'(s))\,ds+u'(t_0)$$

and thus

$$|u'(t)|<\|Nu\|_1+M_0.$$

By (A_2) and (14), we have

$$\begin{aligned}|u'(t)|&<\|\gamma\|_1+\|\alpha\|_1\|u\|_0+\|\beta\|_1\|u'\|_0+M_0\\&<M_0+\|\alpha\|_1M_0+\|\gamma\|_1+(\|\alpha\|_1+\|\beta\|_1)\|u'\|_0.\end{aligned}$$

So,

$$\|u'\|_0<\frac{\|\gamma\|_1+M_0(\|\alpha\|_1+1)}{1-\|\alpha\|_1-\|\beta\|_1}<\infty.$$

This, together with (14), shows that Ω_1 is bounded. $\square$

LEMMA 6. *Assume (B_1) with $a^2+b^2\neq 0$, (B_2), (H), and (A_3) hold. Then*

$$\Omega_2=\{u\in\ker L: Nu\in \operatorname{Im}L\}$$

is bounded.

Proof. Let $u \in \Omega_2$. Then $u = c(at-b)$ for some $c \in \mathbb{R}$ and

$$QNu(t) = (B_1 - \alpha B_2)\left(\int_0^t (t-s)f(s,c(as-b),ca)\,ds\right)h(t) = 0.$$

By (A_3), we have $|c| \leqslant M_1$. So, $\|u\|_X \leqslant M_1\|at-b\|_X$, that is, Ω_2 is bounded. $\square$

Since L is a Fredholm map of index zero, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$. For example, define

$$Jg(t) = (B_1 - \alpha B_2)\left(\int_0^t (t-s)g(s)\,ds\right)(at-b).$$

Obviously, $J : Z \to \ker L$. If $g \in \operatorname{Im} Q$, then $g = ch$, where h is introduced in (H), and

$$\begin{aligned} Jg(t) &= J(ch)(t) \\ &= (B_1 - \alpha B_2)\left(\int_0^t (t-s)ch(s)\,ds\right)(at-b) \\ &= c(at-b) \end{aligned}$$

by the property of h in (H). Thus, $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.

LEMMA 7. *Assume (B_1) with $a^2+b^2 \neq 0$, (B_2), (H), and (A_3) hold. Then*

$$\Omega_3 = \{u : \rho\lambda u + (1-\lambda)JQNu = 0,\, u \in \ker L,\, \lambda \in [0,1]\}$$

is bounded, where $J : \operatorname{Im} Q \to \ker L$ is defined above and

$$\rho = \begin{cases} 1, & \text{if (12) holds,} \\ -1, & \text{if (13) holds.} \end{cases}$$

Proof. Suppose that (12) holds and let $u \in \Omega_3$. Then $u(t) = c(at-b)$, $c \in \mathbb{R}$, and $\lambda u + (1-\lambda)JQNu = 0$. If $\lambda = 1$, then $u = 0$, that is, $c = 0$. If $\lambda = 0$, then $JQNu = JQN[c(at-b)] = 0$. In this case, by Lemma 6, $\|u\|_X \leqslant M_1\|at-b\|_X$. For $\lambda \in (0,1)$,

$$\lambda c(at-b) = -(1-\lambda)(B_1 - \alpha B_2)\left(\int_0^t (t-s)f(s,c(as-b),ac)\,ds\right)(at-b).$$

Hence

$$\lambda c^2 = -(1-\lambda)c(B_1 - \alpha B_2)\left(\int_0^t (t-s)f(s,c(as-b),ac)\,ds\right).$$

If $|c| > M_1$, by the definition of ρ and (12), we get $c^2 < 0$, which is a contradiction. The treatment of the case $\rho = -1$ subject to (13) is similar. $\square$

Now we are in position to prove Theorem 3.

Proof. Lemma 2 establishes that L is a Fredholm mapping of index zero. Let Ω be open and bounded such that $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$, where Ω_i, $i=1,2,3$, are as in Lemmas 5, 6, 7, respectively. Then the assumptions (i) and (ii) of Theorem 2 are fulfilled. The compactness of $K_{P,Q}N : E \to X$ and $QN : E \to Z$ follows from the Arzela-Ascoli theorem and the Kolmogorov-Riesz criterion, respectively. Hence, N is L-compact on $\overline{\Omega}$.

Using the identity map $I : \ker L \to \ker L$, we define (in the appropriate case) the homotopy

$$H(u,\lambda) = \pm\lambda I u + (1-\lambda)JQNu.$$

By the degree property of invariance under a homotopy, if $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned}\deg(JQN|_{\ker L\cap\partial\Omega}, \Omega\cap\ker L, 0) &= \deg(H(\cdot,0), \Omega\cap\ker L, 0)\\ &= \deg(H(\cdot,1), \Omega\cap\ker L, 0)\\ &= \deg(\pm I, \Omega\cap\ker L, 0) \neq 0.\end{aligned}$$

Finally, the assumption (iii) of Theorem 2 is fulfilled and the proof is completed. □

REMARK 3. Here we compare our result to Theorem 3.2 [18]. Specifically, we discuss the part that deals with the boundedness of Ω_1. The authors rely on the hypotheses that are very similar to ours (with $\gamma_1, \gamma_2 \equiv 0$):

(H_1) There exist functions $\rho, \alpha, \beta, \gamma_1, \gamma_2 \in L_1[0,1]$, $\theta_1, \theta_2 \in [0,1)$ such that for all $(x_1,x_2) \in \mathbb{R}^2$, $t \in [0,1]$,

$$f(t,x_1,x_2) \leqslant \rho(t) + \alpha(t)|x_1| + \beta(t)|x_2| + \gamma_1(t)|x_1|^{\theta_1} + \gamma_2(t)|x_2|^{\theta_2}.$$

(H_2) There exists a constant $A > 0$ such that for $x \in \operatorname{dom} L$, if $|x(t)| + |x'(t)| > A$ for all $t \in [0,1]$, then

$$\Gamma_1\left(\int_0^t (t-s)f(s,x(s),x'(s))\,ds\right) \neq 0,$$

(H_3) is identical to our (A_3).

Then it is shown that (1), (2) has at least one solution provided

$$\|\alpha\|_1 + \|\beta\|_1 < \frac{|\Gamma_2(t)|}{|\Gamma_2(t^2)| + 2|\Gamma_2(t)|}.$$

Note that

$$\frac{|\Gamma_2(t)|}{|\Gamma_2(t^2)| + 2|\Gamma_2(t)|} < \frac{1}{2}.$$

In our method of proof of Lemma 5, we have achieved an improved upper bound given by (11). This is due to the fact that if we rely (H_1), then $u \in \Omega_1$ immediately implies that there exists $t_0 \in [0,1]$ such that $|x(t_0)|, |x'(t_0)| \leqslant A$ (or M_0 in our notation). There is no need, in this case, to separately consider Pu and $(I-P)u$ as it is originally done in

[12] whose method [18] reproduces. However, the decomposition of $u \in \Omega_1$ as the sum of Pu and $(I-P)u$ could not be avoided if $|u(t)|+|u'(t)| > M_0$ is replaced with either $|u(t)| > M_0$ or $|u'(t)| > M_0$. Also, in [3] using the notations of [18], this assumption is phrased as follows: there exists a constant $A > 0$ such that for $x \in \operatorname{dom} L$, if $|x(t)| > A$ or $|x'(t)| > A$ for all $t \in [0,1]$, then

$$\Gamma_1\left(\int_0^t (t-s)f(s,x(s),x'(s))\,ds\right) \neq 0.$$

Again, this directly leads to $t_0 \in [0,1]$ such that $|x(t_0)| \leqslant A$ and $|x'(t_0)| \leqslant A$, which makes the use of $x = Px + (I-P)x$ unnecessary in showing that Ω_1 is bounded.

THEOREM 4. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function. Assume that (B_1) with $a \neq 0$, (B_2), (H), and (A_3) (of Theorem 3) and the following conditions hold:*

(A_4) *There exists a constant $M_0 > 0$ such that if $|u'(t)| > M_0$, then*

$$(B_1 - \alpha B_2)\left(\int_0^t (t-s)f(s,u(s),u'(s))\,ds\right) \neq 0.$$

(A_5) *There exist nonnegative functions $\alpha, \beta, \gamma \in L_1[0,1]$ with*

$$|f(t,u,v)| \leqslant \gamma(t) + \alpha(t)|u| + \beta(t)|v|, \quad t \in [0,1], \quad u,v \in \mathbb{R},$$

where

$$(||K_P|| + \|t - b/a\|_X (A+1))(\|\alpha\|_1 + \|\beta\|_1) < 1$$

and A and $||K_P||$ are given by (8) and (10), respectively.

Then the boundary value problem (5), (6) has at least one solution.

Proof. As in the proof of Lemma 5, $u \in \Omega_1$ implies, by (A_4), there exist a constant $t_0 \in [0,1]$ such that $|u'(t_0)| \leqslant M_0$.

REMARK 4. Note that we do not readily have $|u(t_0)| \leqslant M_0$, which follows directly from (A_1) of Theorem 3.

Since

$$u'(t) = \int_{t_0}^t u''(s)\,ds + u'(t_0),$$

we have

$$|u'(t)| \leqslant M_0 + \|Lu\|_1 < M_0 + \|Nu\|_1, \quad t \in [0,1]. \tag{15}$$

Write $u = u_1 + u_2$, where $u_1 = (I-P)u \in \operatorname{dom} L \cap \ker P$ and $u_2 = Pu \in \operatorname{Im} P$. Then since $u_1 = (I-P)u \in \operatorname{dom} L \cap \ker P$, $u_1 = K_P L u_1 = K_P L(I-P)u = K_P L u = \lambda K_P N u$. As in the proof of (8),

$$|u_1'(t)| < |(K_P N u)'(t)| \leqslant A\|Nu\|_1 \tag{16}$$

and, as in (10),

$$||u_1\|_X < ||K_P||||Nu||_1. \tag{17}$$

Now, $u_2 = u - u_1$, so $u_2' = u' - u_1'$ and

$$|u_2'(t)| \leqslant |u'(t)| + |u_1'(t)| < M_0 + \|Nu\|_1 + A||Nu||_1,$$

by (15), (16). Recall that $u_2(t) = Pu(t) = c(u)(at - b)$, where

$$c(u) = \frac{1}{a^2 + b^2}(au'(0) - bu(0))$$

is introduced for the sake of brevity. Hence

$$|u_2'(t)| = |c(u)a| < M_0 + \|Nu\|_1 + A||Nu||_1.$$

That is,

$$|c(u)| \leqslant \frac{1}{|a|}(M_0 + (A+1)||Nu||_1).$$

Thus,

$$\|u_2\|_X = |c(u)|||at - b||_X < \|t - b/a\|_X \,(M_0 + (A+1)||Nu||_1), \tag{18}$$

for $u \in \operatorname{dom} L \backslash \ker L$.

By (17) and (18),

$$\|u\|_X \leqslant ||u_1||_X + ||u_2||_X < C_1 + C_2||Nu||_1 < C_1 + C_2||\gamma||_1 + C_2\,(||\alpha||_1 + ||\beta||_1)\,\|u\|_X,$$

where

$$C_2 = ||K_P|| + \|t - b/a\|_X\,(A+1).$$

By (A_5), Ω_1 is bounded. The rest of the proof repeats that of Theorem 3. □

We now provide an example that satisfies the assumptions of Theorem 4. Consider a kind of pendulum equation

$$u''(t) = \gamma_0(t) - \frac{1}{16}\sin u(t) + \frac{1}{16}u'(t), \quad t \in (0,1), \tag{19}$$

where

$$\gamma_0(t) = \frac{1}{16}\left(\sin\left(10^3 - 9t^2 - 12t + 13\right) - 30t^2 + 978t - 276\right),$$

satisfying

$$B_1(u) = u'(0) + 2u\left(\frac{1}{2}\right) = 0, \quad B_2(u) = u(0) - 2\int_0^1 u(s)\,ds = 0. \tag{20}$$

It is easy to see that $B_1(t) = 2,\ B_1(1) = 2,\ B_2(t) = -1,\ B_1(1) = -1$, so that $\alpha = -2,\ a = b = -1$ and $\ker L = \{c(t-1) : c \in \mathbb{R}\}$. It is not difficult to verify that $h \equiv -\frac{12}{5}$ satisfies (H).

Also,

$$|B_2(u)| \leqslant |u(0)| + 2\int_0^1 |u(s)|\,ds \leqslant 3\|u\|_X,$$

that is, $\beta_2 = 3$. With $a = b = -1$, $A = \frac{5}{2}$ and $\|K_P\| = 4$, and $\|t - b/a\|_X = 1$.

The right side of the differential equation satisfies the inequality

$$|f(t,u,v)| \leqslant \gamma(t) + \alpha(t)|u| + \beta(t)|v|, \quad t \in (0,1),$$

where $\gamma(t) = |\gamma_0(t)|$, $\|\alpha\|_1 = \|\beta\|_1 = \frac{1}{16}$, and

$$(||K_P|| + \|t - b/a\|_X\,(A+1))\,(\|\alpha\|_1 + \|\beta\|_1) = \frac{15}{16} < 1,$$

which verifies (A_4).

Note that $\gamma_0'(t) > 0$ is $[0,1]$, and thus

$$-18 < \frac{1}{16}\sin 13 - \frac{69}{4} \leqslant \gamma_0(t) \leqslant \frac{1}{16}\sin 2 + 42 < 43.$$

Let $M_0 = 689$. If $u'(t) > 689$, then

$$\begin{aligned} Nu(t) &= \gamma_0(t) - \frac{1}{16}\sin u(t) + \frac{1}{16}u'(t) \\ &> -18 - \frac{1}{16} + \frac{1}{16}M_0 \\ &> 0, \end{aligned}$$

and, if $u'(t) < -689$, then

$$\begin{aligned} Nu(t) &= \gamma_0(t) - \frac{1}{16}\sin u(t) + \frac{1}{16}u'(t) \\ &< 43 + \frac{1}{16} - \frac{1}{16}M_0 \\ &< 0. \end{aligned}$$

Observe that

$$\begin{aligned} &(B_1 - \alpha B_2)\left(\int_0^t (t-s)Nu(s)\,ds\right) \\ &= 2\int_0^{\frac{1}{2}} \left(\frac{1}{2} - s\right) Nu(s)\,ds - 2\int_0^1 (1-s)^2 Nu(s)\,ds \\ &= -\int_0^{\frac{1}{2}} (1 - 2s + 2s^2)Nu(s)\,ds - 2\int_{\frac{1}{2}}^1 (1-s)^2 Nu(s)\,ds \\ &= \int_0^1 \mathscr{K}(s)Nu(s)\,ds, \end{aligned}$$

where

$$\mathscr{K}(s)=\begin{cases}-1+2s-2s^2, & 0\leqslant s\leqslant \frac{1}{2},\\ -2+4s-2s^2, & \frac{1}{2}<s\leqslant 1.\end{cases}$$

Obviously, $\mathscr{K}(s)\leqslant 0$ in $[0,1]$. Therefore,

$$(B_1-\alpha B_2)\left(\int_0^t (t-s)Nu(s)\,ds\right)\neq 0$$

provided $u\in \operatorname{dom}L\setminus\ker L$ satisfies $|u'(t)|>M_0=689$. This shows that (A_5) holds.

Finally, for $u\in\ker L$, $u_c=c(1-t)$,

$$\begin{aligned}Nu(t)&=\gamma_0(t)-\frac{1}{16}\sin u_c(t)+\frac{1}{16}u_c'(t)\\ &=\gamma_0(t)-\frac{1}{16}\sin(c(1-t))-\frac{1}{16}c.\end{aligned}$$

Consequently,

$$\begin{aligned}&c(B_1-\alpha B_2)\left(\int_0^t (t-s)Nu_c(s)\,ds\right)\\ &=\int_0^1 \mathscr{K}(s)cNu_c(s)\,ds\\ &=\int_0^1 \mathscr{K}(s)\left(c\gamma_0(t)-\frac{c}{16}\sin(c(1-t))-\frac{1}{16}c^2\right)ds\\ &>0\end{aligned}$$

since $\mathscr{K}(s)\leqslant 0$ in $[0,1]$ and

$$c\gamma_0(t)-\frac{c}{16}\sin(c(1-t))-\frac{1}{16}c^2\leqslant 43|c|+\frac{1}{16}|c|-\frac{1}{16}c^2<0$$

provided $|c|>689$. That is, (A_3) of Theorem 3, which carries over to the assumption set of Theorem 4, is also fulfilled.

At last, notice that

$$u(t)=10t^3-9t^2-12t+13$$

is a solution of (19) and (20).

THEOREM 5. *Let $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a Carathéodory function. Assume that (B_1) with $\frac{b}{a}<0$ or $\frac{b}{a}>1$, (B_2), (H), and (A_3) (of Theorem 3) and the following conditions hold:*

(A_6) *There exists a constant $M_0>0$ such that if $|u(t)|>M_0$, then*

$$(B_1-\alpha B_2)\left(\int_0^t (t-s)f(s,u(s),u'(s))ds\right)\neq 0.$$

(A_7) *There exist nonegative functions* $\alpha,\beta,\gamma\in L_1[0,1]$ *such that*

$$|f(t,u,v)|\leqslant\gamma(t)+\alpha(t)|u|+\beta(t)|v|,\quad t\in[0,1],\quad u,v\in\mathbb{R},$$

where

$$\left(||K_P||+\frac{\max\{|a-b|,|b|\}}{\min\{|a-b|,|b|\}}B\right)(\|\alpha\|_1+\|\beta\|_1)<1$$

and B *and* $||K_P||$ *are given by (9) and (10), respectively.*

Then the boundary value problem (5), (6) has at least one solution.

Proof. As in the proof of Lemma 5, $u\in\Omega_1$ means $QNu=0$. By (A_6), there exist a constant $t_0\in[0,1]$ such that $|u(t_0)|\leqslant M_0$.

REMARK 5. Similar to Remark 4, in this case $|u'(t_0)|\leqslant M_0$ does not come for free.

As in the proof of Theorem 4, $u=u_1+u_2$, where $u_1=(I-P)u\in\operatorname{dom}L\cap\ker P$ and $u_2=Pu\in\operatorname{Im}P$. Similarly, $u_1=(I-P)u\in\operatorname{dom}L\cap\ker P$, so that $u_1=K_PLu_1=K_PL(I-P)u=K_PLu=\lambda K_PNu$. As in the proof of (9),

$$|u_1(t)|<|(K_PNu)(t)|\leqslant B\|Nu\|_1. \tag{21}$$

Again, (17) holds.

Now, $u_2=u-u_1$, so, by (21),

$$|u_2(t_0)|\leqslant|u(t_0)|+|u_1(t_0)|<M_0+B||Nu||_1.$$

As in the proof of Theorem 4, $u_2(t)=Pu(t)=c(u)(at-b)$. Hence

$$|u_2(t_0)|=|c(u)|at_0-b|<M_0+B||Nu||_1.$$

Since $at_0-b\neq0$ in $[0,1]$,

$$|c(u)|\leqslant\frac{1}{\min_{t\in[0,1]}|at-b|}(M_0+B||Nu||_1)=\frac{1}{\min\{|a-b|,|b|\}}(M_0+B||Nu||_1).$$

Thus,

$$\|u_2\|_X=|c(u)|||at-b||_X<\frac{\|at-b\|_X}{\min\{|a-b|,|b|\}}(M_0+B||Nu||_1), \tag{22}$$

for $u\in\operatorname{dom}L\setminus\ker L$.

By (17) and (22),

$$\|u\|_X\leqslant||u_1||_X+||u_2||_X<C_1+C_2||Nu||_1<C_1+C_2||\gamma||_1+C_2(||\alpha||_1+||\beta||_1)\|u\|_X,$$

where

$$C_2=||K_P||+\frac{\|at-b\|_X}{\min\{|a-b|,|b|\}}B.$$

By (A_7), Ω_1 is bounded and the rest of proof is identical to those above. □

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Nickolai Kosmatov
Department of Mathematics and Statistics
University of Arkansas at Little Rock
Little Rock, AR 72204-1099, USA
e-mail: nxkosmatov@ualr.edu

Weihua Jiang
College of Sciences
Hebei University of Science and Technology
Shijiazhuang, 050018, Hebei, PR China
e-mail: weihuajiang@hebust.edu.cn