

SECOND-ORDER FUNCTIONAL PROBLEMS WITH A RESONANCE OF DIMENSION ONE

NICKOLAI KOSMATOV AND WEIHUA JIANG

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Abstract. We obtain, using the coincidence degree theory, solvability conditions for all possible resonance scenarios $Lu = u'' = f(t, u, u') = Nu$, with linear functional conditions $B_i u = 0$, $i = 1, 2$ with $\dim \ker L = 1$. Our work generalizes and improves the results of Zhao and Liang [18] and Cui [3] in several directions. We also construct a meaningful example of a nonlinear functional problem for a pendulum equation which not only satisfies the assumptions of an existence theorem but also has a closed-form solution.

1. Introduction

Resonant boundary value problems have been studied by a broad range of techniques [1, 2, 3, 4, 5, 6, 10, 11, 12, 15, 17, 18]. To this day, Mawhin's coincidence degree theory [14] continues to play an important role in this active field. Recently, the attention has shifted to problems with integral boundary conditions and, more generally, to problems with linear functional conditions [18] and resonance scenarios [3] that have been completely overlooked in the past. Notably, some interesting results for systems of equations have also been obtained in [16] for a resonant problem that does not allow “uncoupling” in the sense that it cannot be treated as a scalar problem at resonance. We also mention [13], where a higher-order nonlocal problem at one-dimensional resonance was studied by reduction to a first order vector equation.

One of most general studies of a resonant problem for the differential operator $L : C^1[0, 1] \rightarrow L_1[0, 1]$, $Lx = x''$ known to us is done by Zhao and J. Liang [18], where the authors considered the functional differential problem

$$x''(t) = f(t, x(t), x'(t)), \quad t \in (0, 1), \quad (1)$$

$$\Gamma_1(x) = 0, \quad \Gamma_2(x) = 0, \quad (2)$$

where Γ_1, Γ_2 are linear functionals on $C^1[0, 1]$ satisfying the general resonance condition $\Gamma_1(t)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(t)$. (The non-resonant scenario subject to the condition $(A_1) : \Gamma_1(t)\Gamma_2(1) \neq \Gamma_1(1)\Gamma_2(t)$ was also studied.) Specifically, the authors investigated the following resonant cases:

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- (A₂) $\Gamma_1(t), \Gamma_1(1), \Gamma_2(1) = 0, \Gamma_2(t) \neq 0;$
- (A₃) $\Gamma_1(t), \Gamma_1(1), \Gamma_2(t) = 0, \Gamma_2(1) \neq 0;$
- (A₄) $\Gamma_1(1), \Gamma_2(t), \Gamma_2(1) = 0, \Gamma_1(t) \neq 0;$
- (A₅) $\Gamma_1(t), \Gamma_2(1), \Gamma_2(t) = 0, \Gamma_1(1) \neq 0;$
- (A₆) $\Gamma_1(1), \Gamma_1(t), \Gamma_2(1), \Gamma_2(t) = 0.$

The cases (A₂) and (A₄) result in $\ker L = \{c : c \in \mathbb{R}\}$ and (A₃) and (A₅) correspond to $\ker L = \{ct : c \in \mathbb{R}\}$. The case (A₆) describes a resonance with $\ker L = \{c_1t + c_2 : c_1, c_2 \in \mathbb{R}\}$. The cases (A₂), (A₃), and (A₆) were investigated in full detail.

Although [18] generalizes and extends many results for nonlocal second-order problems at resonance, it does not contain a complete analysis of (1), (2). To see this, let $a, b, \alpha \in \mathbb{R}$ and $a, b \neq 0$ and set, for example, $\Gamma_1(t) = \alpha b, \Gamma_1(1) = \alpha a, \Gamma_2(t) = b,$ and $\Gamma_2(1) = a$. Then $\Gamma_1(t)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(t)$ with

$$\ker L = \{c(at - b) : c \in \mathbb{R}\}, \quad \dim \ker L = 1.$$

This case cannot be derived from the results of [18] pertaining to the cases of (A₂) - (A₆). In [3], Cui considered such “slanted” kernels, which are also the main motivation of the present paper. To be exact, [3] studied

$$x''(t) = f(t, x(t), x'(t)), \quad t \in (0, 1), \tag{3}$$

$$x(0) = \int_0^1 x(s) d\alpha(s), \quad x(1) = \int_0^1 x(s) d\beta(s), \tag{4}$$

where $f \in C([0, 1] \times \mathbb{R}^2, \mathbb{R}); \alpha$ and β are right continuous on $[0, 1)$, left continuous at $t = 1$, and $\int_0^1 u(s) d\alpha(s)$ and $\int_0^1 u(s) d\beta(s)$ are Riemann-Stieltjes integrals. The resonance condition is $\kappa_1 \kappa_4 - \kappa_2 \kappa_3 = 0$, where

$$\begin{aligned} \kappa_1 &= 1 - \int_0^1 (1-t) d\alpha(t), & \kappa_2 &= \int_0^1 t d\alpha(t), \\ \kappa_3 &= \int_0^1 (1-t) d\beta(t), & \kappa_4 &= 1 - \int_0^1 t d\beta(t). \end{aligned}$$

We can interpret $\Gamma_i(x) = 0, i = 1, 2$ in [18], as

$$x(0) - \int_0^1 x(s) d\alpha(s) = 0, \quad x(1) - \int_0^1 x(s) d\beta(s) = 0,$$

respectively. In [18], the authors also interpret $\Gamma_i(u)$ as a linear combination of the Riemann-Stieltjes integrals of x and x' defined in terms of measures of bounded variation. Subsequently, the authors use these representations to obtain uniqueness theorems and to compare their results to those of [12]. We do not believe that it is necessary to rely on such a representation of the functionals Γ_i .

The authors of [18] make unnecessary respective assumptions $\Gamma_1(t^2) \neq 0$ and $\Gamma_1(t^3) \neq 0$ in Theorems 3.2 and 3.3, which yield existence criteria for the respective

cases (A_2) and (A_3) . In [3], such artificial conditions were also deemed necessary. In particular, for (3), (4) it is assumed that

$$\kappa_3 \int_0^1 t(1-t)d\alpha(t) + \kappa_1 \int_0^1 t(1-t)d\beta(t) \neq 0.$$

Conditions of this type are only needed to ensure that $Q : Z \rightarrow Z$ is well-defined and in our work we propose an approach that allows us to bypass this minor technical difficulty (see (H) below). Thus, we improve the results of [3] and [18] in that respect as well. In addition, due to the simplicity of our method, it is clearly preferred to that devised in [4], and can also be used for higher order problems with functional conditions.

Moreover, in many recent papers devoted to the second order problems (see, e. g., [12, 18]) the following assumption is imposed: There exists a constant $M_0 > 0$, such that if $|u(t)| + |u'(t)| > M_0, t \in [0, 1]$, then $QNu \neq 0$. It is used to show that $\Omega_1 = \{u \in \text{dom}L \setminus \ker L : Lu = \lambda Nu, \lambda \in (0, 1)\}$ is bounded. A similar comment can be made about [3]. In the present paper we also show that the boundedness of Ω_1 can be shown if $|u(t)| + |u'(t)| > M_0$ is replaced with just $|u(t)| > M_0$ or $|u'(t)| > M_0$ (see Theorems 4 and 5 below.) In Section 2 we will further discuss the methodology of *a priori* estimates in comparison with [3, 18].

We consider the differential equation

$$u''(t) = f(t, u(t), u'(t)), \quad t \in (0, 1), \tag{5}$$

together with the functional conditions

$$B_1(u) = B_2(u) = 0. \tag{6}$$

We assume the following:

(B_1) The linear functionals $B_1, B_2 : X \rightarrow \mathbb{R}$ satisfy $B_1(t) = \alpha b, B_1(1) = \alpha a, B_2(t) = b, B_2(1) = a$, where $\alpha, a, b \in \mathbb{R}$.

(B_2) The functionals $B_1, B_2 : X \rightarrow \mathbb{R}$ are continuous with the respective norms β_1, β_2 , that is, $|B_i(u)| \leq \beta_i \|u\|_X$.

Here $X = C^1[0, 1]$ with the norm $\|u\|_X = \max\{\|u\|_0, \|u'\|_0\}$, where $\|\cdot\|_0$ is the max-norm, and $Z = L_1[0, 1]$ with the usual norm $\|\cdot\|_1$.

Let us assume, without loss of generality, that

(H) the function $h \in Z$ satisfies

$$(B_1 - \alpha B_2) \left(\int_0^t (t-s)h(s) ds \right) = 1.$$

The following lemma shows that the assumption (H) is merely a matter of choice of such a function.

LEMMA 1. Assume that (B_1) and (B_2) hold. Then there exists $h \in Z$ such that

$$(B_1 - \alpha B_2) \left(\int_0^t (t-s)h(s) ds \right) \neq 0.$$

Proof. For convenience, set $B = B_1 - \alpha B_2$. Assume, by way of contradiction, that

$$B \left(\int_0^t (t-s)h(s) ds \right) = 0$$

for all $h \in Z$. In particular, for every integer $n \geq 0$,

$$(n+1)(n+2)B \left(\int_0^t (t-s)s^n ds \right) = B(t^{n+2}) = 0.$$

By (B_2) , $B(1) = B(t) = 0$. Thus, $B(p) = 0$ for every polynomial p .

Since $B \neq 0$ on all of X , there exists $v_0 \in X$ such that $B(v_0) \neq 0$. Choose a sequence of polynomials $\{p_m\}$ such that $\|v_0 - p_m\|_X < \frac{1}{m}$. Then

$$0 \neq |B(v_0)| = |B(v_0 - p_m) + B(p_m)| = |B(v_0 - p_m)| \leq \|B\| \|v_0 - p_m\|_X < (\beta_1 + |\alpha|\beta_2) \frac{1}{m}$$

for all $m \in \mathbb{N}$, which is a contradiction. \square

Define the mapping $L : \text{dom}L \subset X \rightarrow Z$ by

$$Lu = u'',$$

where

$$\text{dom}L = \{u \in X : u'' \in Z \text{ and } B_1(u) = B_2(u) = 0\}.$$

We assume that the function f satisfies the Carathéodory conditions and define $N : X \rightarrow Z$ by

$$Nu(t) = f(t, u(t), u'(t)).$$

The functional differential problem (5), (6) is now equivalent to the abstract equation $Lu = Nu$.

DEFINITION 1. Let X and Z be real normed spaces. A linear mapping $L : \text{dom}L \subset X \rightarrow Z$ is called a Fredholm mapping if $\ker L$ has a finite dimension and $\text{Im}L$ is closed and has a finite co-dimension.

If L is a Fredholm mapping, its (Fredholm) index is the integer $\text{Ind}L = \dim \ker L - \text{codim Im}L$.

Define continuous projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ so that

$$\text{Im}P = \ker L, \quad \ker Q = \text{Im}L, \quad X = \ker L \oplus \ker P, \quad Z = \text{Im}L \oplus \text{Im}Q.$$

For a Fredholm mapping L of index zero, the inverse of the map

$$L|_{\text{dom}L \cap \ker P} : \text{dom}L \cap \ker P \rightarrow \text{Im}L$$

is denoted by $K_P : \text{Im}L \rightarrow \text{dom}L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \rightarrow \text{dom}L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$.

DEFINITION 2. Let $L : \text{dom}L \subset X \rightarrow Z$ be a Fredholm mapping, E be a metric space, and $N : E \rightarrow Z$ be a mapping. We say that N is L -compact on E if $QN : E \rightarrow Z$ and $K_{P,Q}N : E \rightarrow X$ are continuous and compact on E . In addition, we say, that N is L -completely continuous if it is L -compact on every bounded $E \subset X$.

The following is the Kolmogorov-Riesz criterion (see, for example, [7]):

THEOREM 1. For $1 \leq p < \infty$, $E \subset L_p[0, 1]$ is compact if

(a) E is bounded;

(b) the limit

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 |g(s + \varepsilon) - g(s)|^p ds = 0$$

is uniform in E .

The compactness of $K_{P,Q}N : E \rightarrow X$ and $QN : E \rightarrow Z$ will follow from the Arzela-Ascoli theorem and the Kolmogorov-Riesz criterion, respectively. However, we will omit the corresponding details as straightforward.

The equation $Lu = Nu$ will be shown to have a solution by means of Theorem IV.13 [14]:

THEOREM 2. Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L -compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:

- (i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in ((\text{dom}L \setminus \ker L) \cap \partial\Omega) \times (0, 1)$;
- (ii) $Nu \notin \text{Im}L$ for every $u \in \ker L \cap \partial\Omega$;
- (iii) $\text{deg}(QN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q : Z \rightarrow Z$ a continuous projector such that $\ker Q = \text{Im}L$.

Then the equation $Lu = Nu$ has at least one solution in $\text{dom}L \cap \overline{\Omega}$.

REMARK 1. The condition (B_1) incorporates the cases $\Gamma_1(t) = \Gamma_2(t) = 0$ and $\Gamma_1(1) = \Gamma_2(1) = 0$ considered in [18]. In this form, it is needed for Theorem 3. Furthermore, in order to prove Theorems 4 and 5 we will assume, in addition, that $a \neq 0$ and $\frac{b}{a} \notin (0, 1)$, respectively.

We will demonstrate now that (B_1) is a critical condition, that is, the functional problem (5), (6) is at resonance and, moreover, $\dim \ker L = 1$.

LEMMA 2. Let (B_1) with $a^2 + b^2 \neq 0$, (B_2) and (H) hold. Then the mapping $L : \text{dom}L \subset X \rightarrow Z$ is a Fredholm mapping of index zero.

Proof. If $u \in \text{dom}L$ and $Lu = 0$, we have $u = c_1t + c_2$ and

$$B_1(c_1t + c_2) = c_1B_1(t) + c_2B_1(1) = \alpha bc_1 + \alpha ac_2 = 0,$$

$$B_2(c_1t + c_2) = c_1B_2(t) + c_2B_2(1) = bc_1 + ac_2 = 0.$$

Therefore,

$$\ker L = \{c(at - b) : c \in \mathbb{R}\}, \quad \dim \ker L = 1.$$

Now we verify

$$\text{Im}L = \left\{ g \in Z : (B_1 - \alpha B_2) \left(\int_0^t (t-s)g(s) ds \right) = 0 \right\}. \quad (7)$$

Let $g \in \text{Im}L$, then there exists $u \in \text{dom}L$ such that $g = Lu$, that is,

$$u = \int_0^t (t-s)g(s) ds + u'(0)t + u(0)$$

and $B_1u = B_2u = 0$. Hence,

$$0 = B_iu = B_i \left(\int_0^t (t-s)g(s) ds \right) + u'(0)B_i(t) + u(0)B_i(1), \quad i = 1, 2,$$

and, using the resonance condition (B_1) ,

$$\begin{aligned} & B_1 \left(\int_0^t (t-s)g(s) ds \right) + u'(0)B_1(t) + u(0)B_1(1) \\ &= B_1 \left(\int_0^t (t-s)g(s) ds \right) + \alpha bu'(0) + \alpha au(0) = 0, \\ & B_2 \left(\int_0^t (t-s)g(s) ds \right) + u'(0)B_2(t) + u(0)B_2(1) \\ &= B_2 \left(\int_0^t (t-s)g(s) ds \right) + bu'(0) + au(0) = 0. \end{aligned}$$

Hence $g \in \{g \in Z : (B_1 - \alpha B_2) \left(\int_0^t (t-s)g(s) ds \right) = 0\}$. That is,

$$\text{Im}L \subseteq \left\{ g \in Z : (B_1 - \alpha B_2) \left(\int_0^t (t-s)g(s) ds \right) = 0 \right\}.$$

If $g \in \{g \in Z : (B_1 - \alpha B_2) \left(\int_0^t (t-s)g(s) ds \right) = 0\}$, take

$$u(t) = -\frac{bt+a}{a^2+b^2} B_2 \left(\int_0^t (t-s)g(s) ds \right) + \int_0^t (t-s)g(s) ds.$$

It is clear that $Lu = g$ and $B_1u = B_2u = 0$. That is,

$$\left\{ g \in Z : (B_1 - \alpha B_2) \left(\int_0^t (t-s)g(s) ds \right) = 0 \right\} \subseteq \text{Im}L.$$

Combining the above we obtain (7).

We consider

$$Pu(t) = \frac{1}{a^2 + b^2}(au'(0) - bu(0))(at - b)$$

It is easy to check that $P^2u = Pu$, $u \in X$. It is also elementary to confirm the identity $\text{Im}P = \ker L$. Moreover, choose $u = c(at - b) \in \text{Im}P$. If, in addition, $c(at - b) \in \ker P$, then $0 = au'(0) - bu(0) = c(a^2 + b^2)$. Now, since $a^2 + b^2 \neq 0$, we must have $c = 0$. That is, $\ker L \oplus \ker P = X$.

Define an operator $Q : Z \rightarrow Z$ as

$$Qg(t) = (B_1 - \alpha B_2) \left(\int_0^t (t - s)g(s)ds \right) h(t).$$

Then, by (B_2) and (H) , $Q : Z \rightarrow Z$ is a continuous linear projector such that $\text{Im}L = \ker Q$ and $\text{Im}Q = \{ch(t) : c \in \mathbb{R}\}$. It is clear that $Z = \text{Im}L \oplus \text{Im}Q$ and $\dim \ker L = \text{codim Im}L$, that is, L is a Fredholm mapping of index zero. \square

LEMMA 3. The map $K_P : Z \rightarrow \overline{\text{dom}L \cap \ker P}$ defined by

$$K_P g(t) = -\frac{bt + a}{a^2 + b^2} B_2 \left(\int_0^t (t - s)g(s) ds \right) + \int_0^t (t - s)g(s) ds$$

is the inverse of L .

Proof. Obviously, $LK_P g = g$ for all $g \in Z$. For $u \in \text{dom}L \cap \ker P$,

$$\begin{aligned} (K_P Lu)(t) &= (K_P u'')(t) = -\frac{bt + a}{a^2 + b^2} B_2 \left(\int_0^t (t - s)u''(s) ds \right) + \int_0^t (t - s)u''(s) ds \\ &= -\frac{bt + a}{a^2 + b^2} B_2(u - u'(0)t - u(0)) + u(t) - u'(0)t - u(0) \\ &= \frac{bt + a}{a^2 + b^2} (bu'(0) + au(0)) + u(t) - u'(0)t - u(0) \\ &= u(t) - \frac{at - b}{a^2 + b^2} (au'(0) - bu(0)) \\ &= u(t) - (Pu)(t) \\ &= u(t). \end{aligned}$$

Thus,

$$K_P = (L|_{\text{dom}L \cap \ker P})^{-1}. \quad \square$$

The next lemma provides norm-estimates needed for the main results.

LEMMA 4. For $g \in Z$,

$$\|(K_P g)'\|_0 \leq A \|g\|_1, \quad A = 1 + \frac{|b|}{a^2 + b^2} \beta_2, \tag{8}$$

and

$$\|K_P g\|_0 \leq B \|g\|_1, \quad B = 1 + \frac{\|bt + a\|_0}{a^2 + b^2} \beta_2. \tag{9}$$

Moreover,

$$\|K_P g\|_X \leq \|K_P\| \|g\|_1, \quad \|K_P\| = 1 + \frac{\beta_2}{a^2 + b^2} \|bt + a\|_X. \tag{10}$$

Proof. Observe that due to $|B_2(u)| \leq \beta_2 \|u\|_X$,

$$\begin{aligned} |K_P g(t)| &\leq \frac{|bt + a|}{a^2 + b^2} \beta_2 \left\| \int_0^t (t-s)g(s) ds \right\|_X + \left\| \int_0^t (t-s)g(s) ds \right\|_0 \\ &\leq \left(\frac{|bt + a|}{a^2 + b^2} \beta_2 + 1 \right) \|g\|_1 \\ &= \left(\frac{\beta_2}{a^2 + b^2} \|bt + a\|_0 + 1 \right) \|g\|_1, \end{aligned}$$

and we arrive at (9). Similarly,

$$\begin{aligned} |(K_P g)'(t)| &\leq \frac{|b|}{a^2 + b^2} \beta_2 \left\| \int_0^t (t-s)g(s) ds \right\|_X + \left\| \int_0^t g(s) ds \right\|_0 \\ &\leq \left(\frac{|b|}{a^2 + b^2} \beta_2 + 1 \right) \|g\|_1, \end{aligned}$$

which implies (8). Finally, (10) follows from (9) and (8) since $\|bt + a\|_X = \max\{\|bt + a\|_0, |b|\}$. \square

2. Main results

THEOREM 3. *Let $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Carathéodory function. Assume that (B_1) with $a^2 + b^2 \neq 0$, (B_2) , (H) and the following conditions hold:*

(A_1) *There exists a constant $M_0 > 0$, such that if $|u(t)| + |u'(t)| > M_0$, then*

$$(B_1 - \alpha B_2) \left(\int_0^t (t-s)f(s, u(s), u'(s)) ds \right) \neq 0.$$

(A_2) *There exist nonnegative functions $\alpha, \beta, \gamma \in L_1[0, 1]$ with*

$$\|\alpha\|_1 + \|\beta\|_1 < 1 \tag{11}$$

such that

$$|f(t, u, v)| \leq \gamma(t) + \alpha(t)|u| + \beta(t)|v|, \quad \text{a.e. } t \in [0, 1], \quad u, v \in \mathbb{R}.$$

(A_3) *There exists a constant $M_1 > 0$, such that if $|c| > M_1$, then*

$$c(B_1 - \alpha B_2) \left(\int_0^t (t-s)f(s, c(as-b), ca) ds \right) > 0. \tag{12}$$

Then the boundary value problem (5), (6) has at least one solution.

REMARK 2. The inequality (12) may be replaced by

$$c(B_1 - \alpha B_2) \left(\int_0^t (t-s)f(s, c(as-b), ca) ds \right) < 0. \tag{13}$$

The proof of Theorem 3 will be based on the next three lemmas.

LEMMA 5. Assume that (B_1) with $a^2 + b^2 \neq 0$, (B_2) , (H) , (A_1) , and (A_2) hold. Then

$$\Omega_1 = \{u \in \text{dom}L \setminus \ker L : Lu = \lambda Nu, \lambda \in (0, 1)\}$$

is bounded.

Proof. If $u \in \Omega_1$, then by (A_1) , there exists a constant $t_0 \in [0, 1]$ such that

$$|u(t_0)|, |u'(t_0)| \leq M_0.$$

Since

$$u(t) = \int_{t_0}^t u'(s) ds + u(t_0),$$

we get

$$|u(t)| \leq M_0 + \|u'\|_0, \quad t \in [0, 1]. \tag{14}$$

By $Lu = \lambda Nu$, we obtain

$$u'(t) = \lambda \int_{t_0}^t f(s, u(s), u'(s)) ds + u'(t_0)$$

and thus

$$|u'(t)| < \|Nu\|_1 + M_0.$$

By (A_2) and (14), we have

$$\begin{aligned} |u'(t)| &< \|\gamma\|_1 + \|\alpha\|_1 \|u\|_0 + \|\beta\|_1 \|u'\|_0 + M_0 \\ &< M_0 + \|\alpha\|_1 M_0 + \|\gamma\|_1 + (\|\alpha\|_1 + \|\beta\|_1) \|u'\|_0. \end{aligned}$$

So,

$$\|u'\|_0 < \frac{\|\gamma\|_1 + M_0(\|\alpha\|_1 + 1)}{1 - \|\alpha\|_1 - \|\beta\|_1} < \infty.$$

This, together with (14), shows that Ω_1 is bounded. \square

LEMMA 6. Assume (B_1) with $a^2 + b^2 \neq 0$, (B_2) , (H) , and (A_3) hold. Then

$$\Omega_2 = \{u \in \ker L : Nu \in \text{Im}L\}$$

is bounded.

Proof. Let $u \in \Omega_2$. Then $u = c(at - b)$ for some $c \in \mathbb{R}$ and

$$QNu(t) = (B_1 - \alpha B_2) \left(\int_0^t (t-s)f(s, c(as-b), ca) ds \right) h(t) = 0.$$

By (A₃), we have $|c| \leq M_1$. So, $\|u\|_X \leq M_1 \|at - b\|_X$, that is, Ω_2 is bounded. \square

Since L is a Fredholm map of index zero, there exists an isomorphism $J : \text{Im } Q \rightarrow \ker L$. For example, define

$$Jg(t) = (B_1 - \alpha B_2) \left(\int_0^t (t-s)g(s) ds \right) (at - b).$$

Obviously, $J : Z \rightarrow \ker L$. If $g \in \text{Im } Q$, then $g = ch$, where h is introduced in (H), and

$$\begin{aligned} Jg(t) &= J(ch)(t) \\ &= (B_1 - \alpha B_2) \left(\int_0^t (t-s)ch(s) ds \right) (at - b) \\ &= c(at - b) \end{aligned}$$

by the property of h in (H). Thus, $J : \text{Im } Q \rightarrow \ker L$ is an isomorphism.

LEMMA 7. Assume (B₁) with $a^2 + b^2 \neq 0$, (B₂), (H), and (A₃) hold. Then

$$\Omega_3 = \{u : \rho \lambda u + (1 - \lambda)JQNu = 0, u \in \ker L, \lambda \in [0, 1]\}$$

is bounded, where $J : \text{Im } Q \rightarrow \ker L$ is defined above and

$$\rho = \begin{cases} 1, & \text{if (12) holds,} \\ -1, & \text{if (13) holds.} \end{cases}$$

Proof. Suppose that (12) holds and let $u \in \Omega_3$. Then $u(t) = c(at - b)$, $c \in \mathbb{R}$, and $\lambda u + (1 - \lambda)JQNu = 0$. If $\lambda = 1$, then $u = 0$, that is, $c = 0$. If $\lambda = 0$, then $JQNu = JQN[c(at - b)] = 0$. In this case, by Lemma 6, $\|u\|_X \leq M_1 \|at - b\|_X$. For $\lambda \in (0, 1)$,

$$\lambda c(at - b) = -(1 - \lambda)(B_1 - \alpha B_2) \left(\int_0^t (t-s)f(s, c(as-b), ac) ds \right) (at - b).$$

Hence

$$\lambda c^2 = -(1 - \lambda)c(B_1 - \alpha B_2) \left(\int_0^t (t-s)f(s, c(as-b), ac) ds \right).$$

If $|c| > M_1$, by the definition of ρ and (12), we get $c^2 < 0$, which is a contradiction. The treatment of the case $\rho = -1$ subject to (13) is similar. \square

Now we are in position to prove Theorem 3.

Proof. Lemma 2 establishes that L is a Fredholm mapping of index zero. Let Ω be open and bounded such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$, where Ω_i , $i = 1, 2, 3$, are as in Lemmas 5, 6, 7, respectively. Then the assumptions (i) and (ii) of Theorem 2 are fulfilled. The compactness of $K_{P,Q}N : E \rightarrow X$ and $QN : E \rightarrow Z$ follows from the Arzela-Ascoli theorem and the Kolmogorov-Riesz criterion, respectively. Hence, N is L -compact on $\overline{\Omega}$.

Using the identity map $I : \ker L \rightarrow \ker L$, we define (in the appropriate case) the homotopy

$$H(u, \lambda) = \pm \lambda Iu + (1 - \lambda)JQN u.$$

By the degree property of invariance under a homotopy, if $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned} \deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \ker L, 0) \\ &= \deg(\pm I, \Omega \cap \ker L, 0) \neq 0. \end{aligned}$$

Finally, the assumption (iii) of Theorem 2 is fulfilled and the proof is completed. \square

REMARK 3. Here we compare our result to Theorem 3.2 [18]. Specifically, we discuss the part that deals with the boundedness of Ω_1 . The authors rely on the hypotheses that are very similar to ours (with $\gamma_1, \gamma_2 \equiv 0$):

(H_1) There exist functions $\rho, \alpha, \beta, \gamma_1, \gamma_2 \in L_1[0, 1]$, $\theta_1, \theta_2 \in [0, 1]$ such that for all $(x_1, x_2) \in \mathbb{R}^2$, $t \in [0, 1]$,

$$f(t, x_1, x_2) \leq \rho(t) + \alpha(t)|x_1| + \beta(t)|x_2| + \gamma_1(t)|x_1|^{\theta_1} + \gamma_2(t)|x_2|^{\theta_2}.$$

(H_2) There exists a constant $A > 0$ such that for $x \in \text{dom } L$, if $|x(t)| + |x'(t)| > A$ for all $t \in [0, 1]$, then

$$\Gamma_1 \left(\int_0^t (t-s)f(s, x(s), x'(s)) ds \right) \neq 0,$$

(H_3) is identical to our (A_3).

Then it is shown that (1), (2) has at least one solution provided

$$\|\alpha\|_1 + \|\beta\|_1 < \frac{|\Gamma_2(t)|}{|\Gamma_2(t^2)| + 2|\Gamma_2(t)|}.$$

Note that

$$\frac{|\Gamma_2(t)|}{|\Gamma_2(t^2)| + 2|\Gamma_2(t)|} < \frac{1}{2}.$$

In our method of proof of Lemma 5, we have achieved an improved upper bound given by (11). This is due to the fact that if we rely (H_1), then $u \in \Omega_1$ immediately implies that there exists $t_0 \in [0, 1]$ such that $|x(t_0)|, |x'(t_0)| \leq A$ (or M_0 in our notation). There is no need, in this case, to separately consider Pu and $(I - P)u$ as it is originally done in

[12] whose method [18] reproduces. However, the decomposition of $u \in \Omega_1$ as the sum of Pu and $(I - P)u$ could not be avoided if $|u(t)| + |u'(t)| > M_0$ is replaced with either $|u(t)| > M_0$ or $|u'(t)| > M_0$. Also, in [3] using the notations of [18], this assumption is phrased as follows: there exists a constant $A > 0$ such that for $x \in \text{dom } L$, if $|x(t)| > A$ or $|x'(t)| > A$ for all $t \in [0, 1]$, then

$$\Gamma_1 \left(\int_0^t (t - s)f(s, x(s), x'(s)) ds \right) \neq 0.$$

Again, this directly leads to $t_0 \in [0, 1]$ such that $|x(t_0)| \leq A$ and $|x'(t_0)| \leq A$, which makes the use of $x = Px + (I - P)x$ unnecessary in showing that Ω_1 is bounded.

THEOREM 4. *Let $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Carathéodory function. Assume that (B_1) with $a \neq 0$, (B_2) , (H) , and (A_3) (of Theorem 3) and the following conditions hold:*

(A_4) *There exists a constant $M_0 > 0$ such that if $|u'(t)| > M_0$, then*

$$(B_1 - \alpha B_2) \left(\int_0^t (t - s)f(s, u(s), u'(s)) ds \right) \neq 0.$$

(A_5) *There exist nonnegative functions $\alpha, \beta, \gamma \in L_1[0, 1]$ with*

$$|f(t, u, v)| \leq \gamma(t) + \alpha(t)|u| + \beta(t)|v|, \quad t \in [0, 1], \quad u, v \in \mathbb{R},$$

where

$$(\|K_P\| + \|t - b/a\|_X (A + 1)) (\|\alpha\|_1 + \|\beta\|_1) < 1$$

and A and $\|K_P\|$ are given by (8) and (10), respectively.

Then the boundary value problem (5), (6) has at least one solution.

Proof. As in the proof of Lemma 5, $u \in \Omega_1$ implies, by (A_4) , there exist a constant $t_0 \in [0, 1]$ such that $|u'(t_0)| \leq M_0$.

REMARK 4. Note that we do not readily have $|u(t_0)| \leq M_0$, which follows directly from (A_1) of Theorem 3.

Since

$$u'(t) = \int_{t_0}^t u''(s) ds + u'(t_0),$$

we have

$$|u'(t)| \leq M_0 + \|Lu\|_1 < M_0 + \|Nu\|_1, \quad t \in [0, 1]. \tag{15}$$

Write $u = u_1 + u_2$, where $u_1 = (I - P)u \in \text{dom } L \cap \ker P$ and $u_2 = Pu \in \text{Im } P$. Then since $u_1 = (I - P)u \in \text{dom } L \cap \ker P$, $u_1 = K_P L u_1 = K_P L (I - P)u = K_P L u = \lambda K_P N u$. As in the proof of (8),

$$|u'_1(t)| < |(K_P N u)'(t)| \leq A \|Nu\|_1 \tag{16}$$

and, as in (10),

$$\|u_1\|_X < \|K_P\| \|Nu\|_1. \tag{17}$$

Now, $u_2 = u - u_1$, so $u'_2 = u' - u'_1$ and

$$|u'_2(t)| \leq |u'(t)| + |u'_1(t)| < M_0 + \|Nu\|_1 + A\|Nu\|_1,$$

by (15), (16). Recall that $u_2(t) = Pu(t) = c(u)(at - b)$, where

$$c(u) = \frac{1}{a^2 + b^2} (au'(0) - bu(0))$$

is introduced for the sake of brevity. Hence

$$|u'_2(t)| = |c(u)a| < M_0 + \|Nu\|_1 + A\|Nu\|_1.$$

That is,

$$|c(u)| \leq \frac{1}{|a|} (M_0 + (A + 1)\|Nu\|_1).$$

Thus,

$$\|u_2\|_X = |c(u)| \|at - b\|_X < \|t - b/a\|_X (M_0 + (A + 1)\|Nu\|_1), \tag{18}$$

for $u \in \text{dom } L \setminus \ker L$.

By (17) and (18),

$$\|u\|_X \leq \|u_1\|_X + \|u_2\|_X < C_1 + C_2 \|Nu\|_1 < C_1 + C_2 \|\gamma\|_1 + C_2 (\|\alpha\|_1 + \|\beta\|_1) \|u\|_X,$$

where

$$C_2 = \|K_P\| + \|t - b/a\|_X (A + 1).$$

By (A_5) , Ω_1 is bounded. The rest of the proof repeats that of Theorem 3. \square

We now provide an example that satisfies the assumptions of Theorem 4. Consider a kind of pendulum equation

$$u''(t) = \gamma_0(t) - \frac{1}{16} \sin u(t) + \frac{1}{16} u'(t), \quad t \in (0, 1), \tag{19}$$

where

$$\gamma_0(t) = \frac{1}{16} (\sin(10^3 - 9t^2 - 12t + 13) - 30t^2 + 978t - 276),$$

satisfying

$$B_1(u) = u'(0) + 2u\left(\frac{1}{2}\right) = 0, \quad B_2(u) = u(0) - 2 \int_0^1 u(s) ds = 0. \tag{20}$$

It is easy to see that $B_1(t) = 2$, $B_1(1) = 2$, $B_2(t) = -1$, $B_2(1) = -1$, so that $\alpha = -2$, $a = b = -1$ and $\ker L = \{c(t - 1) : c \in \mathbb{R}\}$. It is not difficult to verify that $h \equiv -\frac{12}{5}$ satisfies (H).

Also,

$$|B_2(u)| \leq |u(0)| + 2 \int_0^1 |u(s)| ds \leq 3\|u\|_X,$$

that is, $\beta_2 = 3$. With $a = b = -1$, $A = \frac{5}{2}$ and $\|K_P\| = 4$, and $\|t - b/a\|_X = 1$.

The right side of the differential equation satisfies the inequality

$$|f(t, u, v)| \leq \gamma(t) + \alpha(t)|u| + \beta(t)|v|, \quad t \in (0, 1),$$

where $\gamma(t) = |\gamma_0(t)|$, $\|\alpha\|_1 = \|\beta\|_1 = \frac{1}{16}$, and

$$(\|K_P\| + \|t - b/a\|_X(A + 1))(\|\alpha\|_1 + \|\beta\|_1) = \frac{15}{16} < 1,$$

which verifies (A_4) .

Note that $\gamma'_0(t) > 0$ is $[0, 1]$, and thus

$$-18 < \frac{1}{16} \sin 13 - \frac{69}{4} \leq \gamma_0(t) \leq \frac{1}{16} \sin 2 + 42 < 43.$$

Let $M_0 = 689$. If $u'(t) > 689$, then

$$\begin{aligned} Nu(t) &= \gamma_0(t) - \frac{1}{16} \sin u(t) + \frac{1}{16} u'(t) \\ &> -18 - \frac{1}{16} + \frac{1}{16} M_0 \\ &> 0, \end{aligned}$$

and, if $u'(t) < -689$, then

$$\begin{aligned} Nu(t) &= \gamma_0(t) - \frac{1}{16} \sin u(t) + \frac{1}{16} u'(t) \\ &< 43 + \frac{1}{16} - \frac{1}{16} M_0 \\ &< 0. \end{aligned}$$

Observe that

$$\begin{aligned} &(B_1 - \alpha B_2) \left(\int_0^t (t-s) Nu(s) ds \right) \\ &= 2 \int_0^{\frac{1}{2}} \left(\frac{1}{2} - s \right) Nu(s) ds - 2 \int_0^1 (1-s)^2 Nu(s) ds \\ &= - \int_0^{\frac{1}{2}} (1-2s+2s^2) Nu(s) ds - 2 \int_{\frac{1}{2}}^1 (1-s)^2 Nu(s) ds \\ &= \int_0^1 \mathcal{K}(s) Nu(s) ds, \end{aligned}$$

where

$$\mathcal{K}(s) = \begin{cases} -1 + 2s - 2s^2, & 0 \leq s \leq \frac{1}{2}, \\ -2 + 4s - 2s^2, & \frac{1}{2} < s \leq 1. \end{cases}$$

Obviously, $\mathcal{K}(s) \leq 0$ in $[0, 1]$. Therefore,

$$(B_1 - \alpha B_2) \left(\int_0^t (t-s)Nu(s) ds \right) \neq 0$$

provided $u \in \text{dom}L \setminus \ker L$ satisfies $|u'(t)| > M_0 = 689$. This shows that (A_5) holds.

Finally, for $u \in \ker L$, $u_c = c(1-t)$,

$$\begin{aligned} Nu(t) &= \gamma_0(t) - \frac{1}{16} \sin u_c(t) + \frac{1}{16} u'_c(t) \\ &= \gamma_0(t) - \frac{1}{16} \sin(c(1-t)) - \frac{1}{16} c. \end{aligned}$$

Consequently,

$$\begin{aligned} &c(B_1 - \alpha B_2) \left(\int_0^t (t-s)Nu_c(s) ds \right) \\ &= \int_0^1 \mathcal{K}(s)cNu_c(s) ds \\ &= \int_0^1 \mathcal{K}(s) \left(c\gamma_0(t) - \frac{c}{16} \sin(c(1-t)) - \frac{1}{16}c^2 \right) ds \\ &> 0 \end{aligned}$$

since $\mathcal{K}(s) \leq 0$ in $[0, 1]$ and

$$c\gamma_0(t) - \frac{c}{16} \sin(c(1-t)) - \frac{1}{16}c^2 \leq 43|c| + \frac{1}{16}|c| - \frac{1}{16}c^2 < 0$$

provided $|c| > 689$. That is, (A_3) of Theorem 3, which carries over to the assumption set of Theorem 4, is also fulfilled.

At last, notice that

$$u(t) = 10t^3 - 9t^2 - 12t + 13$$

is a solution of (19) and (20).

THEOREM 5. Let $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Carathéodory function. Assume that (B_1) with $\frac{b}{a} < 0$ or $\frac{b}{a} > 1$, (B_2) , (H) , and (A_3) (of Theorem 3) and the following conditions hold:

(A_6) There exists a constant $M_0 > 0$ such that if $|u(t)| > M_0$, then

$$(B_1 - \alpha B_2) \left(\int_0^t (t-s)f(s, u(s), u'(s)) ds \right) \neq 0.$$

(A7) *There exist nonnegative functions $\alpha, \beta, \gamma \in L_1[0, 1]$ such that*

$$|f(t, u, v)| \leq \gamma(t) + \alpha(t)|u| + \beta(t)|v|, \quad t \in [0, 1], \quad u, v \in \mathbb{R},$$

where

$$\left(\|K_P\| + \frac{\max\{|a-b|, |b|\}}{\min\{|a-b|, |b|\}} B \right) (\|\alpha\|_1 + \|\beta\|_1) < 1$$

and B and $\|K_P\|$ are given by (9) and (10), respectively.

Then the boundary value problem (5), (6) has at least one solution.

Proof. As in the proof of Lemma 5, $u \in \Omega_1$ means $QNu = 0$. By (A6), there exist a constant $t_0 \in [0, 1]$ such that $|u(t_0)| \leq M_0$.

REMARK 5. Similar to Remark 4, in this case $|u'(t_0)| \leq M_0$ does not come for free.

As in the proof of Theorem 4, $u = u_1 + u_2$, where $u_1 = (I - P)u \in \text{dom}L \cap \ker P$ and $u_2 = Pu \in \text{Im}P$. Similarly, $u_1 = (I - P)u \in \text{dom}L \cap \ker P$, so that $u_1 = K_P L u_1 = K_P L (I - P)u = K_P L u = \lambda K_P N u$. As in the proof of (9),

$$|u_1(t)| < |(K_P N u)(t)| \leq B \|Nu\|_1. \tag{21}$$

Again, (17) holds.

Now, $u_2 = u - u_1$, so, by (21),

$$|u_2(t_0)| \leq |u(t_0)| + |u_1(t_0)| < M_0 + B \|Nu\|_1.$$

As in the proof of Theorem 4, $u_2(t) = Pu(t) = c(u)(at - b)$. Hence

$$|u_2(t_0)| = |c(u)|at_0 - b| < M_0 + B \|Nu\|_1.$$

Since $at_0 - b \neq 0$ in $[0, 1]$,

$$|c(u)| \leq \frac{1}{\min_{t \in [0,1]} |at - b|} (M_0 + B \|Nu\|_1) = \frac{1}{\min\{|a-b|, |b|\}} (M_0 + B \|Nu\|_1).$$

Thus,

$$\|u_2\|_X = |c(u)| \|at - b\|_X < \frac{\|at - b\|_X}{\min\{|a-b|, |b|\}} (M_0 + B \|Nu\|_1), \tag{22}$$

for $u \in \text{dom}L \setminus \ker L$.

By (17) and (22),

$$\|u\|_X \leq \|u_1\|_X + \|u_2\|_X < C_1 + C_2 \|Nu\|_1 < C_1 + C_2 \|\gamma\|_1 + C_2 (\|\alpha\|_1 + \|\beta\|_1) \|u\|_X,$$

where

$$C_2 = \|K_P\| + \frac{\|at - b\|_X}{\min\{|a-b|, |b|\}} B.$$

By (A7), Ω_1 is bounded and the rest of proof is identical to those above. \square

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Nikolai Kosmatov
 Department of Mathematics and Statistics
 University of Arkansas at Little Rock
 Little Rock, AR 72204-1099, USA
 e-mail: nxkosmatov@ualr.edu

Weihua Jiang
 College of Sciences
 Hebei University of Science and Technology
 Shijiazhuang, 050018, Hebei, PR China
 e-mail: weihuajiang@hebust.edu.cn