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MULTIPLE SOLUTIONS TO THE NONHOMOGENEOUS KIRCHHOFF TYPE PROBLEM INVOLVING A NONLOCAL OPERATOR

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Abstract. This paper examines the nonhomogenous Kirchhoff type equation that involves a nonlocal operator. Using Ekeland's variational principle and the Mountain pass theorem, the existence of multiple solutions is established.

1. Introduction

In this article, we investigate the multiplicity of solutions to the boundary problem

$$\begin{cases} -M\left(\|u\|_{X_0}^2\right)\mathscr{L}_K u(x) = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where Ω is the complement of a smooth bounded domain D in $\mathbb{R}^n$, $n > 2s$ with $s \in (0,1)$, that is, $\Omega = \mathbb{R}^n \backslash D$, and $\lambda > 0$. Moreover,

(A_1) $M(t) = at^m + b$ with $a,b > 0$, $0 \leqslant m < \frac{2}{n-2s}$;

(A_2) $1 < q < 2 < 2(m+1) < p < 2_s^*$ with $2_s^* = \frac{2n}{n-2s}$ for $n > 2s$;

(A_3) f,g,h are continuous functions which may change sign on Ω, and

$$f(x) \in L^{q_0}(\Omega) \cap L^\infty(\Omega),\ g(x) \in L^{p_0}(\Omega) \cap L^\infty(\Omega),\ h(x) \in L^r(\Omega)$$

with

$$q_0 = \frac{2_s^*}{2_s^* - q},\ p_0 = \frac{2_s^*}{2_s^* - p},\ r = \frac{2_s^*}{2_s^* - 1},$$

and there exists a nonempty open domain $\tilde{\Omega} \subset \Omega$ such that $g(x) > 0$ in $\tilde{\Omega}$.

Furthermore, $\|\cdot\|_{X_0}$ is a functional norm which is defined in (2.2) and $\mathscr{L}_K$ is a nonlocal operator defined as follows:

$$\mathscr{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^n.$$

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Here $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ is a measurable function which satisfies

$$\begin{cases} \gamma K(x)\in L^1(\mathbb{R}^n) \ \text{ with } \ \gamma(x)=\min\{|x|^2,1\}; \\ \text{there exists } \theta>0 \text{ such that } K(x)\geqslant \theta|x|^{-(n+2s)} \text{ for any } x\in\mathbb{R}^n\setminus\{0\}; \\ K(x)=K(-x) \text{ for any } x\in\mathbb{R}^n\setminus\{0\}. \end{cases} \tag{1.2}$$

A typical example for K is given by $K(x)=|x|^{-(n+2s)}$. In this case

$$\mathscr{L}_K u(x)=-(-\triangle)^s u(x)$$

is the fractional Laplace operator which (up to normalization factors) can be defined as

$$-(-\Delta)^s u(x)=\frac{1}{2}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy, \quad \text{for} \quad x\in\mathbb{R}^n. \tag{1.3}$$

Recently, Fiscella and Valdinoci [8] have investigated the existence of a nontrivial solution to the following problem

$$\begin{cases} -M\left(\|u\|_{X_0}^2\right)\mathscr{L}_K u(x)=\lambda f(x,u)+|u|^{2_s^*-2}u & \text{in } \Omega, \\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{1.4}$$

where $\Omega\subset\mathbb{R}^n$ is an open bounded set, M and f are continuous functions. In particular, the authors gave some motivations for studying fractional Kirchhoff equations.

We remark that, in (1.1) and (1.4), the standard Dirichlet condition $u=0$ on $\partial\Omega$ is replaced by the condition that the function u vanishes outside Ω, consistently with the nonlocal characterization of the operator $\mathscr{L}_K$. Problems (1.1) and (1.4) have variational structures, thus we can construct solutions by finding critical points of the associated energy functional on some appropriate space. It turns out to work on the homogeneous fractional Sobolev space $H_0^s(\Omega)$ (see [6]). In order to study problems (1.1) and (1.4), it is important to encode the boundary condition $u=0$ in $\mathbb{R}^n\setminus\Omega$ in the weak formulation, by considering also that in the norm $\|u\|_{H^s(\mathbb{R}^n)}$ the interaction between Ω and $\mathbb{R}^n\setminus\Omega$ gives positive contribution. We will introduce the functional space in the next section.

Some interesting results were obtained by variational methods, which can be found in [1, 3, 4, 9, 11, 16, 17, 18] for Kirchhoff type problems involving the classical Laplacian operator, and [2, 5, 10] for the p-Laplacian case.

Motivated by these results, we are interested in the multiplicity of solutions to problem (1.1). Our main result can be stated as follows.

THEOREM 1. *Let $(A_1)-(A_3)$ hold. Then there exist $\lambda_0,c_0>0$ such that for all $\lambda\in(0,\lambda_0)$, problem (1.1) admits at least two nontrivial weak solutions when $\|h\|_{L^r(\Omega)}\leqslant c_0$.*

This paper is organized as follows. In Section 2, we give some notations and preliminaries. We prove Theorem 1 in Section 3.

2. Notations and preliminaries

Let us introduce the functional space that we will use in the following, which was introduced in [14]. For fixed $s \in (0,1)$, $n > 2s$, $\Omega \subset \mathbb{R}^n$ is an open bounded set, let X be the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2(\mathbb{R}^{2n} \backslash (\mathscr{C}\Omega \times \mathscr{C}\Omega), dxdy)$$

where $\mathscr{C}\Omega = \mathbb{R}^n \backslash \Omega$. Moreover, set

$$X_0 = \{g \in X \ : \ g = 0 \; a.e. \text{ in } \mathbb{R}^n \backslash \Omega\}.$$

According to the conditions of K, by Lemma 11 in [13], we know that $C_0^2(\Omega) \subseteq X_0$, so X and X_0 are nonempty. The spaces X and X_0 are endowed, respectively, with the norms defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2 K(x-y) dxdy\Big)^{1/2}, \tag{2.1}$$

and

$$\|g\|_{X_0} = \left(\int_Q |g(x)-g(y)|^2 K(x-y)\, dxdy\right)^{1/2}, \tag{2.2}$$

where $Q = \mathbb{R}^{2n} \setminus ((\mathscr{C}\Omega) \times (\mathscr{C}\Omega)) \subset \mathbb{R}^{2n}$. Since $g \in X_0$, then the integral in (2.2) can be extended to all $\mathbb{R}^{2n}$. Moreover, the norm on X_0 given in (2.2) is equivalent to the usual one defined in (2.1), see [14, Lemmas 6 and 7].

With the norm given in (2.2), X_0 is a Hilbert space with scalar product defined as

$$\langle u,v \rangle_{X_0} = \int_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, K(x-y)\, dxdy, \tag{2.3}$$

see [14, Lemma 7]. For further details on X and X_0 and also for their properties we refer to [14, 15].

In the following, $H^s(\Omega)$ denotes the usual fractional Sobolev space endowed with the norm (so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\, dxdy\Big)^{1/2}. \tag{2.4}$$

We remark that, even in the model case $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.4) are not the same, because $\Omega \times \Omega$ is strictly contained in Q. From [14] we have that the embedding $X_0 \hookrightarrow L^{2_s^*}(\Omega)$ is continuous where $2_s^* = \frac{2n}{n-2s}$. Let $K : \mathbb{R}^n \backslash \{0\} \to (0,\infty)$ satisfy assumptions (1.2), if $u \in X_0$, then $u \in H^s(\mathbb{R}^n)$. Moreover $\|u\|_{H^s(\Omega)} \leqslant \|u\|_{H^s(\mathbb{R}^n)} \leqslant c(\theta)\|u\|_{X_0}$. Using this fact and Sobolev inequality, there is a constant $S > 0$ such that for every $u \in X_0$,

$$S\left(\int_\Omega |u|^{2_s^*}\right)^{\frac{1}{2_s^*}} \leqslant \|u\|_{X_0}. \tag{2.5}$$

We say that u is a weak solution of problem (1.1), if u satisfies

$$\left(a\|u\|_{X_0}^{2m}+b\right)\langle u,\phi\rangle_{X_0}=\int_\Omega\left(\lambda f(x)|u|^{q-2}u+g(x)|u|^{p-2}u+h(x)\right)\phi(x)dx \tag{2.6}$$

for all $\phi\in X_0$. u is a weak solution, which is equivalent to be a critical point of the functional $J_\lambda : X_0\to\mathbb{R}$ defined as

$$\begin{aligned}J_\lambda(u)=\frac{a}{2(m+1)}&\|u\|_{X_0}^{2(m+1)}+\frac{b}{2}\|u\|_{X_0}^2\\&-\frac{\lambda}{q}\int_\Omega f(x)|u|^q dx-\frac{1}{p}\int_\Omega g(x)|u|^p dx-\int_\Omega h(x)udx.\end{aligned} \tag{2.7}$$

We can see that $J_\lambda\in C^1(X_0,\mathbb{R})$ and for any $\phi\in X_0$, there holds

$$\begin{aligned}\langle J_\lambda'(u),\phi\rangle_{X_0}=&\left(a\|u\|_{X_0}^{2m}+b\right)\langle u,\phi\rangle_{X_0}\\&-\int_\Omega\left(\lambda f(x)|u|^{q-2}u+g(x)|u|^{p-2}u+h(x)\right)\phi(x)dx.\end{aligned} \tag{2.8}$$

3. Proof of Theorem 1

We first prove that J_λ satisfies the geometric conditions of the Mountain pass Lemma.

LEMMA 1. *Suppose that $(A_1),(A_2)$ and (A_3) hold. Then*
(i) there exist some constants $\rho,\alpha,\lambda_0,c_0>0$ such that for $\lambda\in(0,\lambda_0)$ and $\|h\|_r\leqslant c_0$, $J_\lambda(u)\geqslant\alpha$ when $\|u\|_{X_0}=\rho$.
(ii) there exists a function $v\in X_0$ with $\|v\|_{X_0}>\rho$ such that $J_\lambda(v)<0$.

Proof. (i) It follows from Hölder's inequality and (2.5) that

$$\int_\Omega|f(x)||u|^q dx\leqslant\|f\|_{q_0}\|u\|_{2_s^*}^q\leqslant S^{-q}\|f\|_{q_0}\|u\|_{X_0}^q\quad\text{with } q_0=\frac{2_s^*}{2_s^*-q}. \tag{3.1}$$

Similarly,

$$\int_\Omega|g(x)||u|^p dx\leqslant\|g\|_{p_0}\|u\|_{2_s^*}^p\leqslant S^{-p}\|g\|_{p_0}\|u\|_{X_0}^p\quad\text{with } p_0=\frac{2_s^*}{2_s^*-p}, \tag{3.2}$$

and by Young inequality, we have

$$\int_\Omega|h(x)||u|dx\leqslant\|h\|_r\|u\|_{2_s^*}\leqslant S^{-1}\|h\|_r\|u\|_{X_0}\leqslant\varepsilon\|u\|_{X_0}^2+C_\varepsilon\|h\|_r^2 \tag{3.3}$$

with $r=\frac{2_s^*}{2_s^*-1}$, $\varepsilon>0, C_\varepsilon>0$. Thus

$$J_\lambda(u)\geqslant\frac{a}{2(m+1)}\|u\|_{X_0}^{2(m+1)}+\frac{b}{2}\|u\|_{X_0}^2-\lambda C_1\|u\|_{X_0}^q-C_2\|u\|_{X_0}^p-\varepsilon\|u\|_{X_0}^2$$

$$
\begin{aligned}
&-C_\varepsilon\|h\|_r^2\\
\geqslant\ &\|u\|_{X_0}^{2(m+1)}\left(\frac{a}{2(m+1)}-\lambda C_1\|u\|_{X_0}^{q-2(m+1)}-C_2\|u\|_{X_0}^{p-2(m+1)}\right)\\
&-C_\varepsilon\|h\|_r^2 \qquad (3.4)
\end{aligned}
$$

with $0<\varepsilon<\frac{b}{2}$ and $C_1=\frac{1}{q}S^{-q}\|f\|_{q_0}$, $C_2=\frac{1}{p}S^{-p}\|g\|_{p_0}$. Let

$$\psi(t)=\lambda C_1 t^{q-2(m+1)}+C_2 t^{p-2(m+1)},\quad t>0. \qquad (3.5)$$

To complete the proof of (i), it is sufficient to show that $\psi(t_0)<\frac{a}{4}$ for some $t_0=\|u\|_{X_0}>0$. Indeed, note that $\psi(t)\to+\infty$ whenever $t\to 0^+$ and $t\to+\infty$, then $\psi(t)$ has a minimum at

$$t_0=\lambda^{\frac{1}{p-q}}\left(\frac{C_1(2(m+1)-q)}{C_2(p-2(m+1))}\right)^{\frac{1}{p-q}}.$$

Moreover $\psi(t_0)<\frac{a}{2(m+1)}$ if and only if

$$\psi(t_0)=C_1\frac{p-q}{p-2(m+1)}\left(\frac{C_1(2(m+1)-q)}{C_2(p-2(m+1))}\right)^{\frac{q-2(m+1)}{p-q}}\lambda^{\frac{p-2(m+1)}{p-q}}<\frac{a}{2(m+1)}.$$

Therefore, we obtain that there exist $\lambda_0,c_0,\alpha>0$ such that $J_\lambda(u)\geqslant\alpha$ with $\lambda\in(0,\lambda_0)$, $\rho=t_0=\|u\|_{X_0}$ and $\|h\|_r\leqslant c_0$ for each $h\in L^r(\Omega)$.

(ii) Let $\Omega_0\subset\tilde{\Omega}$ be a bounded domain, where $\tilde{\Omega}$ is given in (A_3). Choose $\phi\in C_0^\infty(\Omega_0)$, $\phi\geqslant 0$, $\phi\not\equiv 0$ in Ω_0, let $\phi=0$ for $x\in\Omega\backslash\Omega_0$, then

$$
\begin{aligned}
J_\lambda(t\phi)=\ &\frac{at^{2(m+1)}}{2(m+1)}\|\phi\|_{X_0}^{2(m+1)}+\frac{bt^2}{2}\|\phi\|_{X_0}^2-\frac{\lambda t^q}{q}\int_\Omega f(x)|\phi|^q dx\\
&-\frac{t^p}{p}\int_\Omega g(x)|\phi|^p dx-t\int_\Omega h(x)\phi dx,
\end{aligned}
$$

and $J_\lambda(t\phi_1)\to-\infty$ as $t\to+\infty$, since $q<2(m+1)<p$. Therefore there exists $t_1>0$ large enough such that $J_\lambda(t\phi)<0$. Then we take $v=t_1\phi\in X_0$ and $J_\lambda(v)<0$.

Next, let us prove J_λ satisfies (PS) condition in X_0.

LEMMA 2. *Suppose $(A_1),(A_2)$ and (A_3) hold. Then J_λ satisfies (PS) condition in X_0.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of J_λ in X_0, that is

$$J_\lambda(u_n)\ \text{is bounded},\qquad J'_\lambda(u_n)\to 0\ \text{in}\ X_0^*\quad \text{as}\ n\to+\infty. \qquad (3.6)$$

We claim that $\{u_n\}$ is bounded in X_0. Indeed, using (3.1), for n large enough, we find

$$C+\|u_n\|_{X_0}\geqslant J_\lambda(u_n)-\frac{1}{p}\langle J'_\lambda(u_n),u_n\rangle_{X_0}$$

$$
\begin{aligned}
&= \left(\frac{1}{2(m+1)} - \frac{1}{p}\right) a\|u_n\|_{X_0}^{2(m+1)} + \left(\frac{1}{2} - \frac{1}{p}\right) b\|u_n\|_{X_0}^2 \\
&\quad + \lambda \left(\frac{1}{p} - \frac{1}{q}\right) \int_\Omega f(x)|u_n|^q dx + \left(\frac{1}{p} - 1\right) \int_\Omega h(x)|u_n| dx \\
&\geqslant \left(\frac{1}{2(m+1)} - \frac{1}{p}\right) a\|u_n\|_{X_0}^{2(m+1)} + \left(\frac{1}{2} - \frac{1}{p}\right) b\|u_n\|_{X_0}^2 \\
&\quad - \lambda D_1 \|u_n\|_{X_0}^q - D_2\|u_n\|_{X_0} \qquad (3.7)
\end{aligned}
$$

with

$$D_1 = \left(\frac{1}{q} - \frac{1}{p}\right) S^{-q}\|f\|_{q_0} \quad \text{and} \quad D_2 = \left(1 - \frac{1}{p}\right) S^{-1}\|h\|_r.$$

Thus $\{u_n\}$ is bounded in X_0. Then we can take a subsequence(still denote by u_n) such that $u_n \rightharpoonup u$ in X_0 as $n \to \infty$. By (2.8), we have

$$
\begin{aligned}
&\langle J'_\lambda(u_n), u_n - u\rangle_{X_0} \\
&= \left(a\|u_n\|_{X_0}^{2(m+1)} + b\right) \langle u_n, u_n - u\rangle_{X_0} \\
&\quad - \int_\Omega \left(\lambda f(x)|u_n|^{q-2}u_n + g(x)|u_n|^{p-2}u_n + h(x)\right)(u_n - u)(x) dx. \qquad (3.8)
\end{aligned}
$$

First, the left side of (3.8) goes to zero as $n \to +\infty$, because $J'_\lambda(u_n) \to 0$ in X_0^* as $n \to +\infty$. Moreover, from Hölder inequality and using the facts that u_n is bounded by some constant times $\|u_n\|_{X_0}$, and $f \in L^{q_0}(\Omega)$, we have

$$
\begin{aligned}
\int_\Omega |f(x)||u_n|^{q-1}|u_n - u| dx &\leqslant \left(\int_\Omega |f(x)||u_n|^q dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |f(x)||u_n - u|^q dx\right)^{\frac{1}{q}} \\
&\leqslant C\left(\int_\Omega |f(x)||u_n - u|^q dx\right)^{\frac{1}{q}}. \qquad (3.9)
\end{aligned}
$$

Since $f(x) \in L^{q_0}(\Omega)$, then for every $\varepsilon > 0$, there is $\rho_0 > 0$, such that

$$\int_{\Omega_\rho^c} |f(x)|^{q_0} dx < \varepsilon \quad \text{for } \rho \geqslant \rho_0, \qquad (3.10)$$

where $\Omega_\rho = B_\rho \backslash D$, $\Omega_\rho^c = \Omega \backslash \Omega_\rho$, and B_ρ is an open ball in X_0 centered at the origin with radius ρ. Let ρ be so large that $D \subset B_\rho$ for any $\rho \geqslant \rho_0$. By the Sobolev compact embedding theorem in the bounded domain Ω_ρ, u_n has a subsequence, still denoted by u_n, which converges v in $L^q(\Omega_\rho)$. Note that f is bounded in Ω_ρ and u_n is also bounded in $L^{2_s^*}(\Omega)$. Thus, Hölder inequality implies that

$$\int_\Omega |f(x)||u_n - u|^q dx \leqslant \|f\|_\infty \int_{\Omega_\rho} |u_n - u|^q dx + \left(\int_{\Omega_\rho^c} |f|^{q_0}\right)^{\frac{1}{q_0}} \left(\int_{\Omega_\rho^c} |u_n - u|^{2_s^*}\right)^{\frac{q}{2_s^*}}.$$

Since $u_n \to u$ in $L^q(\Omega_\rho)$ as $n \to \infty$, this together with (3.10) gives that

$$\int_\Omega |f(x)||u_n - u|^q dx \to 0 \quad \text{as } n \to +\infty. \qquad (3.11)$$

From (3.9) and (3.11), we get as $n \to \infty$,

$$\int_\Omega |f(x)||u_n|^{q-1}|u_n-u|dx$$

$$\leqslant \left(\int_\Omega |f(x)||u_n|^q dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |f(x)||u_n-u|^q dx\right)^{\frac{1}{q}} \to 0. \quad (3.12)$$

Similarly,

$$\int_\Omega |g(x)||u_n|^{p-1}|u_n-u|dx \to 0. \quad (3.13)$$

From (3.8), (3.12), (3.13), we obtain that

$$\left(a\|u_n\|_{X_0}^{2(m+1)}+b\right)\langle u_n, u_n-u\rangle_{X_0} \to 0 \quad \text{as } n\to\infty. \quad (3.14)$$

On the other hand, by the fact $u_n \rightharpoonup u$ in X_0 as $n \to \infty$, we find that

$$\left(a\|u_n\|_{X_0}^{2(m+1)}+b\right)\langle u, u_n-u\rangle_{X_0} \to 0 \quad \text{as } n\to\infty. \quad (3.15)$$

Combining (3.14) with (3.15), we have that $\|u_n-u\|_{X_0} \to 0$ as $n\to\infty$. Thus J_λ satisfies (PS) condition on X_0.

Proof of Theorem 1*:* From Lemmas 1 and 2, J_λ satisfies the Mountain pass theorem [12]. Then there exists $u_1 \in X_0$ such that u_1 is a solution of (1.1). Moreover, $J_\lambda(u_1) \geqslant \alpha > 0$.

We look for the second solution u_2 in the following. Choosing $\varphi \in C_0^\infty(\Omega)$ such that $\int_\Omega h(x)\varphi(x)dx > 0$, then we have

$$\begin{aligned} J_\lambda(t\varphi) = {} & \frac{at^{2(m+1)}}{2(m+1)}\|\varphi\|_{X_0}^{2(m+1)} + \frac{bt^2}{2}\|\varphi\|_{X_0}^2 - \frac{\lambda t^q}{q}\int_\Omega f(x)|\varphi|^q dx \\ & -\frac{t^p}{p}\int_\Omega g(x)|\varphi|^p dx - t\int_\Omega h(x)\varphi dx < 0, \end{aligned}$$

for small $t>0$, and for any open ball $B_\tau \subset X_0$, we have

$$-\infty < c_\tau = \inf_{u\in \overline{B}_\tau} J_\lambda(u). \quad (3.16)$$

Thus,

$$c_\rho = \inf_{u\in \overline{B}_\rho} J_\lambda(u) < 0, \quad \text{and} \quad \inf_{u\in\partial B_\rho} J_\lambda(u) > 0,$$

where ρ is given in Lemma 1. Let $\varepsilon_k \downarrow 0$ be such that

$$0 < \varepsilon_k < \inf_{u\in\partial B_\rho} J_\lambda(u) - \inf_{u\in B_\rho} J_\lambda(u).$$

By Ekeland's variational principle in [7], there exists $u_k \in \overline{B}_\rho$ such that

$$c_\rho \leqslant J_\lambda(u_k) \leqslant c_\rho + \varepsilon_k, \tag{3.17}$$

and

$$J(u_k) < J(u) + \varepsilon_k \|u_k - u\|_{X_0}, \quad \forall\, u \in \bar{B}_\rho,\ \ u \neq u_k. \tag{3.18}$$

Then we have

$$J(u_k) < c_\rho + \varepsilon_k \leqslant \inf_{u \in B_\rho} J(u) + \varepsilon_k < \inf_{u \in \partial B_\rho} J(u). \tag{3.19}$$

Thus $u_k \in B_\rho$.

Next we show that $J'_\lambda(u_k) \to 0$ in X_0^*. Indeed, for any $u \in X_0$ with $\|u\|_{X_0} = 1$, let $w_k = u_k + tu$ and for a fixed $k \geqslant 1$, we have $\|w_k\|_{X_0} \leqslant \|u_k\|_{X_0} + t < \rho$ if $t > 0$ small enough. It follows from (3.18) that $J_\lambda(u_k + tu) \geqslant J_\lambda(u_k) - t\varepsilon_k\|u\|_{X_0}$, that is,

$$\frac{J_\lambda(u_k + tu) - J_\lambda(u_k)}{t} \geqslant -\varepsilon_k \|u\|_{X_0} = -\varepsilon_k.$$

Letting $t \to 0$, we see that $\langle J'_\lambda(u_k), u\rangle \geqslant -\varepsilon_k$. It yields $|\langle J'_\lambda(u_k), u\rangle_{X_0}| < \varepsilon_k$, for any $u \in X_0$ with $\|u\|_{X_0} = 1$. Then $J'_\lambda(u_k) \to 0$ in X_0^*. Therefore, there exists a subsequence $\{u_k\} \subset B_\rho$ such that $J_\lambda(u_k) \to c_\rho$ and $J'_\lambda(u_k) \to 0$ in X_0^* as $k \to \infty$. By Lemma 2, $\{u_k\}$ has a convergent subsequence in X_0, still denoted by $\{u_k\}$, such that $u_k \to u_2$ in X_0. Thus u_2 is a solution of (1.1) with $J_\lambda(u_2) < 0$. We complete the proof of Theorem 1.

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