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ZEROS' DISTRIBUTION OF THE FIRST KIND BESSEL FUNCTIONS

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Abstract. The aim of this paper is to investigate the zeros' distribution of the first kind Bessel functions $J_\nu(z)$ of order $\nu \geqslant 1$. The problem arises from the conjecture given by the work [8] which considered the existence of smooth solutions for one-dimensional compressible Euler equation with gravity. In this article we show that $J_\nu(L\theta) \neq 0$ for any integer $L \geqslant 2$ provided that $J_\nu(\theta) = 0$, $\nu \geqslant 1$ and θ is sufficiently large. Moreover, if ν is half of an odd integer, we can remove the restriction of large θ and show that $J_\nu(L\theta) \neq 0$ for any integer $L \geqslant 2$.

1. Introduction

The aim of this work is to investigate the zeros' distribution of the first kind Bessel functions. The Bessel functions were first defined by Bernoulli and generalized by Bessel, which are canonical solutions of the following Bessel's differential equation:

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0, \tag{1.1}$$

where $z, \nu \in \mathbb{R}$. The parameter ν is called the order of the Bessel function. In the past years, Bessel function plays an important role in many problems, for example, the study of wave propagation and static potentials, etc.. According to the book of Watson [11], the Bessel functions can be classified into two classes: the first kind $J_\nu(z)$ and the second kind $Y_\nu(z)$. In the past, many properties of the Bessel functions have been widely investigated. For example, $J_0(z)$ has an infinity of real zeros and, obtained by Poisson, the formal expansion of $J_0(z)$ for large positive z was

$$\begin{aligned} J_0(z) = \Big(\frac{2}{\pi z}\Big)^{1/2}\Big[\cos\big(z - \frac{1}{4}\pi\big)\cdot\Big\{1 - \frac{1^2\cdot 3^2}{2!(8z)^2} + \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{4!(8z)^4} - \cdots\Big\} + \\ \sin\big(z - \frac{1}{4}\pi\big)\cdot\Big\{\frac{1^2}{1!8z} - \frac{1^2\cdot 3^2\cdot 5^2}{3!(8z)^3} + \cdots\Big\}\Big]. \end{aligned}$$

For elementary properties of the Bessel functions, we refer the readers to the book [11]. Some recent results on the zeros of Bessel functions can also be found in the literature [1, 3, 4, 9, 10].

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In recent work [8], Hsu et al considered the existence of smooth solutions for one-dimensional motions of polytropic gas governed by the compressible Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = -g\rho, \end{cases} \quad \text{with } \rho u|_{x=0} = 0, \tag{1.2}$$

for $t, x \geqslant 0$, here ρ, u, P and $g > 0$ are density, velocity, pressure and gravitational acceleration constant respectively. By expanding the solution of (1.2) as a power series of parameters, they found that the first-order term of the series is exactly the solution of their linearized problem, which can be represented by the first kind Bessel function $J_\nu(z)$ with index $\nu = (N-2)/2$ for some $N > 0$ (see Section 2 for more details). Moreover, in order to seek the formulas of the higher order terms of the series solution, they provided the following conjecture:

Conjecture. (see [8, p.719]) *Let $\nu \geqslant 1$ and θ be a positive zero of the first kind Bessel function $J_\nu(z)$. Then $J_\nu(L\theta) \neq 0$ for any integer $L \geqslant 2$.*

The conjecture provides an interesting property for the zeros of Bessel function. To the best of our knowledge, it was not considered before. Therefore, in this article we will verify the authenticity of this conjecture. Here we remark that the statement of the conjecture does not hold when $\nu < 1$. For example, when $\nu = 1/2$, we have

$$J_{\frac{1}{2}}(z) = (\sqrt{2}\sin z)/\sqrt{\pi z},$$

which has positive zeros $j_{\frac{1}{2},n} := n\pi$ for any $n \in \mathbb{Z}^+$. So that, for any integer L, we have $J_{\frac{1}{2}}(Lj_{\frac{1}{2},n}) = J_{\frac{1}{2}}(j_{\frac{1}{2},Ln}) = 0$.

This work is organized as follows. In Section 2, we recall some results of [8] which provides the derivation of the conjecture. Then, in Section 3, we prove the conjecture when $\nu \geqslant 1$ and θ is sufficiently large. In the last section, we can remove the restriction of large θ and prove the conjecture when $\nu = m + \frac{1}{2}$ with $m \in \mathbb{N}$.

2. Derivation of the conjecture

To illustrate the significance of the conjecture, we briefly describe its derivation in this section. For more details, see the literature [6, 8].

Let's consider system (1.2) under the assumption that the pressure satisfying the gamma-law (see [2]), i.e. $P = P(\rho) = A\rho^\gamma$ for some constants A and γ with $A > 0$ and $1 < \gamma \leqslant 2$. One can easily verify that equilibria of (1.2) are of the form

$$\bar{\rho} = \begin{cases} A_1(x_+ - x)^{\frac{1}{\gamma-1}}, \text{ if } 0 \leqslant x \leqslant x_+, \\ 0, \text{ if } x_+ < x, \end{cases} \quad \text{where } A_1 := ((\gamma-1)g/\gamma A)^{1/(\gamma-1)} \tag{2.1}$$

and x_+ is an arbitrary positive value. Without loss of generality, we may assume $x_+ = 1$, $A_1 = 1$ and $A = 1/\gamma$. Using the Lagrangian variable $m = \int_0^x \rho dx$ as the independent variable instead of x, we can write equations (1.2) by

$$x_{tt} + P_m = -g, \tag{2.2}$$

where $P = \gamma^{-1}(x_m)^{-\gamma}$. Let us fix an equilibrium $x = \bar{x}(m) = 1 - A_2(m_+ - m)^{\frac{\gamma-1}{\gamma}}$, where

$$A_2 := \big(\gamma/(\gamma-1)\big)^{\gamma/(\gamma-1)},\ m_+ := (\gamma-1)/\gamma \text{ and } 0 \leqslant m \leqslant m_+.$$

Putting $x(t,m) = \bar{x}(m) + y$ and taking $\bar{x}(m)$ as the independent variable instead of m (still write it as x), then equation (2.2) is reduced to

$$y_{tt} - \frac{1}{\bar{\rho}(x)}(\gamma \bar{P}(x) G(y_x))_x = 0 \ \text{ with } y|_{x=0} = 0, \text{ for } 0 < x < 1, \tag{2.3}$$

where

$$G(v) := \gamma^{-1}(1-(1+v)^{-\gamma}),\ \bar{\rho}(x) = (1-x)^{\frac{1}{\gamma-1}} \text{ and } \bar{P}(x) = \gamma^{-1}(1-x)^{\frac{\gamma}{\gamma-1}}.$$

In addition, using new variables $z := 1-x$ and $N := 2\gamma/(\gamma-1)$, equation (2.3) can be written by

$$y_{tt} - \Delta y = \bar{G}(v)\Delta y + \hat{G}(v), \tag{2.4}$$

where $v := -\partial y/\partial z$,

$$\Delta y := z y_{zz} + N y_z/2,\ \bar{G}(v) := DG(v) - 1 \text{ and } \hat{G}(v) := N(vDG(v) - G(v))/2.$$

For the linearized equation of (2.4), i.e.

$$y_{tt} - \Delta y = 0 \quad \text{and} \quad y|_{z=1} = 0, \tag{2.5}$$

Hsu et al [6] showed that (2.5) admits a time periodic solution

$$y = y_1(t,z) := (\lambda_n z)^{\frac{2-N}{4}} \sin(\sqrt{\lambda_n} t + \theta) J_{\frac{N-2}{2}}(2\sqrt{\lambda_n z}), \tag{2.6}$$

where θ is a constant and $\lambda_n := (j_{\frac{N-2}{2},n})^2/4$. Note that $j_{\frac{N-2}{2},n}$ is the n-th positive zero of $J_{\frac{N-2}{2}}(z)$. Recently, by using the Nash-Moser Theorem (cf. [5]), the authors of [8] established long time existence of true solution, for which the time-periodic function (2.6) around an equilibrium is the first approximation. Their main idea is to find a formal solution of (2.4) of the form

$$y(t,z) = \sum_{k=1}^{\infty} y_k(t,z)\varepsilon^k,$$

where ε stands for a small parameter. Substituting the series solution into (2.4) and starting from $y_1(t,z)$, we can solve $y_k(t,z)$ with the boundary condition $y_k(1) = 0$ successively. For simplicity, we only consider the case $k = 2$. One can verify that $y_2(t,z)$ satisfies

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big) y_2 = -\frac{2(N-1)}{N-2}(\Delta y_1 + \frac{N}{4} v_1) v_1, \tag{2.7}$$

where $v_1 = -\partial y_1/\partial z$. Since $y_1(t,z)$ is an entire function, we can write the right-hand side of (2.7) by $f_0(z)+(\cos 2\Theta)f_1(z)$, where $\Theta := \sqrt{\lambda_{n_0}}t+\theta_0$ for some n_0, $f_0(z)$ and $f_1(z)$ are entire functions of z. To solve equation (2.7), we first consider the problem

$$-\Delta w = f_0(z), \quad w|_{z=1} = 0. \tag{2.8}$$

It's easy to see that $w(z)$ can be represented by

$$w(z) = -\frac{2}{N-2}\int_0^z \Big(1-\Big(\frac{\zeta}{z}\Big)^{\frac{N}{2}-1}\Big)f_0(\zeta)d\zeta + \frac{2}{N-2}\int_0^1 (1-\zeta^{\frac{N}{2}-1})f_0(\zeta)d\zeta. \tag{2.9}$$

Next, we consider the problem

$$w_{tt}-\Delta w = (\cos 2\Theta)f_1(z), \quad w|_{z=1} = 0. \tag{2.10}$$

Note that $\lambda_n = (j_{\frac{N-2}{2},n})^2/4$ is the eigenvalues of the operator $-\Delta$ with the Dirichlet boundary condition. To solve the problem (2.10), we need to consider the following two cases: (i) $4\lambda_{n_0}$ is not an eigenvalue and (ii) there is an eigenvalue $\lambda_q = 4\lambda_{n_0}$.

For case (i), (2.10) have a solution of the form $w(t,z) = (\cos 2\Theta)W(z)$, where

$$\big(-4\lambda_{n_0}-\Delta\big)W = f_1(z), \quad W|_{z=1} = 0. \tag{2.11}$$

According to Proposition 3 of [8], the first equation of (2.11) has a solution $W_0(z)$, which is an entire function of z such that $W_0(0) = 1$. Then, for any constant C,

$$W(z) = W_0(z) + C(4\lambda_{n_0}z)^{\frac{2-N}{4}} J_{\frac{N-2}{2}}(4\sqrt{\lambda_{n_0}z})$$

is a solution of (2.11), too. Since $4\lambda_{n_0}$ is not an eigenvalue, we have $J_{\frac{N-2}{2}}(4\sqrt{\lambda_{n_0}}) \neq 0$. Therefore, $W(z)$ satisfies the boundary value condition provided C satisfies

$$W(1) = W_0(1) + C(4\lambda_{n_0})^{\frac{2-N}{4}} J_{\frac{N-2}{2}}(4\sqrt{\lambda_{n_0}}) = 0.$$

On the other hand, the case (ii) implies $j_{\frac{N-2}{2},q} = 2j_{\frac{N-2}{2},n_0}$, i.e. $J_{\frac{N-2}{2}}(2j_{\frac{N-2}{2},n_0}) = 0$. Similarly, for any case $k \geqslant 3$, we also need to verify $j_{\frac{N-2}{2},q} = kj_{\frac{N-2}{2},n_0}$ or not. This gives us the reason why we have to investigate the correctness of the conjecture as stated in Section 1.

3. Result for the case $\nu \geqslant 1$

According to the book [11] (see Section 7.3, p. 205), we have

$$J_\nu(z) = \sqrt{2}(\pi z)^{-1/2}\big(p_\nu(z)\cos(z-\frac{\nu}{2}\pi-\frac{\pi}{4}) - q_\nu(z)\sin(z-\frac{\nu}{2}\pi-\frac{\pi}{4})\big), \tag{3.1}$$

where

$$\begin{cases} p_\nu(z) := \dfrac{1}{2}\dfrac{1}{\Gamma(\nu+\frac{1}{2})}\displaystyle\int_0^{+\infty} e^{-u}u^{\nu-\frac{1}{2}}\Big(\Big(1+\frac{iu}{2z}\Big)^{\nu-\frac{1}{2}}+\Big(1-\frac{iu}{2z}\Big)^{\nu-\frac{1}{2}}\Big)du, \\ q_\nu(z) := \dfrac{1}{2i}\dfrac{1}{\Gamma(\nu+\frac{1}{2})}\displaystyle\int_0^{+\infty} e^{-u}u^{\nu-\frac{1}{2}}\Big(\Big(1+\frac{iu}{2z}\Big)^{\nu-\frac{1}{2}}-\Big(1-\frac{iu}{2z}\Big)^{\nu-\frac{1}{2}}\Big)du \end{cases} \tag{3.2}$$

are real-valued for $z>0$, $i=\sqrt{-1}$. By the formulae of (3.2), if $|z|\to\infty$ and $|\arg z|<\pi$, we can write the asymptotic expansions of $p_\nu(z)$ and $q_\nu(z)$ more precisely by the form:

$$p_\nu(z)\sim\sum_{k=0}^{\infty}\frac{(-1)^k(\nu,2k)}{(2z)^{2k}} \quad\text{and}\quad q_\nu(z)\sim\sum_{k=0}^{\infty}\frac{(-1)^k(\nu,2k+1)}{(2z)^{2k+1}},$$

$$\text{where }\ (\nu,k):=\frac{\Gamma(\nu+k+\frac{1}{2})}{k!\Gamma(\nu-k+\frac{1}{2})}=\frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-(2k-1)^2)}{2^{2k}k!}.$$

Hence, it follows that

$$p_\nu(z)=1+O(z^{-2}) \quad\text{and}\quad q_\nu(z)=\Big(\frac{\nu^2}{2}-\frac{1}{8}\Big)\frac{1}{z}(1+O(z^{-2})) \ \text{ as } z\to+\infty. \tag{3.3}$$

Using the asymptotic expansions of $J_\nu(z)$, we can obtain the following result.

THEOREM 1. *Let $\nu\geqslant 1$ and θ be a positive zero of the Bessel function J_ν. If θ is sufficiently large, then $J_\nu(L\theta)\neq 0$ for any integer $L\geqslant 2$.*

Proof. Suppose $J_\nu(\theta)=0$ and θ is sufficiently large, by (3.1) and (3.3), we have

$$\tan\Big(\theta-\frac{\nu}{2}\pi-\frac{\pi}{4}\Big)=A(\theta):=\frac{p_\nu(\theta)}{q_\nu(\theta)}=\frac{8}{4\nu^2-1}\theta(1+O(\theta^{-2})).$$

If $J_\nu(L\theta)=0$ for $L\geqslant 2$ and θ being sufficiently large, then

$$\tan\Big(L\theta-\frac{\nu}{2}\pi-\frac{\pi}{4}\Big)=A(L\theta) \quad\text{and}\quad \frac{\tan(L\theta-\frac{\nu}{2}\pi-\frac{\pi}{4})}{\tan(\theta-\frac{\nu}{2}\pi-\frac{\pi}{4})}\to L, \quad\text{as } \theta\to\infty. \tag{3.4}$$

Denote $\Theta:=\theta-\frac{\nu}{2}\pi-\frac{\pi}{4}$ and $\beta:=\tan((L-1)(\frac{\nu}{2}\pi+\frac{\pi}{4}))$, we have

$$\begin{aligned} &\tan\Theta\ \sim 2\Theta/\Big(\nu^2-\frac{1}{4}\Big)\to\infty \text{ as } \theta\to\infty, \text{ and} \\ &\tan\Big(L\theta-\frac{\nu}{2}\pi-\frac{\pi}{4}\Big)=\tan\Big(L\Theta+(L-1)\Big(\frac{\nu}{2}\pi+\frac{\pi}{4}\Big)\Big) \end{aligned} \tag{3.5}$$

$$= \begin{cases} \dfrac{\beta + \tan(L\Theta)}{1 - \beta\tan(L\Theta)}, \text{ if } \beta \neq \pm\infty, \\ \quad -\cot(L\Theta), \text{ if } \beta = \pm\infty. \end{cases} \tag{3.6}$$

Moreover, by induction, we obtain that

$$\tan(L\theta) = \begin{cases} \dfrac{(-1)^{\ell+1}L(\tan\theta)^{2\ell-1} + \cdots + L\tan\theta}{(-1)^{\ell}(\tan\theta)^{2\ell} + \cdots + 1}, \text{ if } L = 2\ell,\ \ell \in \mathbb{Z}^+, \\ \dfrac{(-1)^{\ell}(\tan\theta)^{2\ell+1} + \cdots + L\tan\theta}{(-1)^{\ell}L(\tan\theta)^{2\ell} + \cdots + 1}, \text{ if } L = 2\ell+1,\ \ell \in \mathbb{Z}^+. \end{cases} \tag{3.7}$$

Now we consider the following two cases.

(1) Assume $L = 2\ell$, $\ell \in \mathbb{Z}$. By (3.5) and (3.7), as $\theta \to \infty$, we have $\tan(L\Theta) \sim -L/\tan\Theta \to 0$ as $\theta \to \infty$. Then, from (3.6), we obtain that

$$\frac{\tan(L\theta - \frac{\nu}{2}\pi - \frac{\pi}{4})}{\tan(\theta - \frac{\nu}{2}\pi - \frac{\pi}{4})} \sim \begin{cases} \dfrac{\beta}{\tan\Theta}, \text{ if } \beta \neq \pm\infty, \\ -\dfrac{\cot(L\Theta)}{\tan\Theta}, \text{ if } \beta = \pm\infty \end{cases} \to \begin{cases} 0, \text{ if } \beta \neq \pm\infty, \\ \dfrac{1}{L}, \text{ if } \beta = \pm\infty, \end{cases} \tag{3.8}$$

as $\theta \to \infty$. However, by (3.4) and (3.8), we have either $L = 0$ or $L = 1/L$ and which gives a contradiction.

(2) Assume $L = 2\ell + 1$, $\ell \in \mathbb{Z}$. By (3.5) and (3.7), we have

$$\tan(L\Theta) \sim \tan\Theta/L \to \infty, \text{ as } \theta \to \infty.$$

Then, from (3.6), we obtain that

$$\frac{\tan(L\theta - \frac{\nu}{2}\pi - \frac{\pi}{4})}{\tan(\theta - \frac{\nu}{2}\pi - \frac{\pi}{4})} \sim \begin{cases} -\dfrac{1}{\beta\tan\Theta}, \text{ if } \beta \neq 0, \pm\infty, \\ \dfrac{\tan(L\Theta)}{\tan\Theta}, \text{ if } \beta = 0, \\ -\dfrac{\cot(L\Theta)}{\tan\Theta}, \text{ if } \beta = \pm\infty \end{cases} \to \begin{cases} 0, \text{ if } \beta \neq 0, \pm\infty, \\ \dfrac{1}{L}, \text{ if } \beta = 0, \\ 0, \text{ if } \beta = \pm\infty, \end{cases} \tag{3.9}$$

as $\theta \to \infty$. However, by (3.4) and (3.9), we have either $L = 0$ or $L = 1/L$ and which gives a contradiction. Hence $J_\nu(\theta)$ and $J_\nu(L\theta)$ can not vanish simultaneously provided that θ is sufficiently large and $L \geqslant 2$. The proof is complete.

4. Result for the case $\nu = m + \frac{1}{2}$

In this section, we further improve the result of Theorem 1 to the case $\nu = m + \frac{1}{2}$ for some $m \in \mathbb{N}$. If ν is half of an odd integer, $J_\nu(z)$ can also be written by (see [11])

$$J_\nu(z) = (-1)^{\nu - \frac{1}{2}} \sqrt{\frac{2}{\pi}} z^\nu \left(\frac{1}{z}\frac{d}{dz}\right)^{\nu - \frac{1}{2}} \frac{\sin z}{z}. \tag{4.1}$$

By (4.1), we can remove the restriction of large θ in Theorem 1.

THEOREM 2. *Suppose* $\nu = m+\frac{1}{2}$ *for some integer* $m \geqslant 1$. *If* θ *is a positive zero of* J_ν, *then* $J_\nu(L\theta) \neq 0$ *for any integer* $L \geqslant 2$.

Proof. Since $\nu = m+\frac{1}{2}$ for some integer $m \geqslant 1$, by (4.1) and mathematical induction on m, there are polynomials $P_m(z), Q_m(z)$ of z with coefficients in $\mathbb{Z}$ such that

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \frac{\sin z}{z} = \frac{1}{z^{2m}}\left(P_m(z)\cos z - Q_m(z)\frac{\sin z}{z}\right). \tag{4.2}$$

Moreover, we can show that these polynomials are of the form

$$\begin{aligned} P_{2k} &= (-1)^k k(2k+1)z^{2k-2} + [\text{lower order terms}], \\ P_{2k+1} &= (-1)^k z^{2k} + [\text{lower order terms}], \\ Q_{2k} &= (-1)^{k+1} z^{2k} + [\text{lower order terms}], \\ Q_{2k+1} &= (-1)^k (k+1)(2k+1)z^{2k} + [\text{lower order terms}]. \end{aligned}$$

Suppose that $J_\nu(\theta) = J_\nu(L\theta) = 0$ for some $\theta > 0$ and an integer $L \geqslant 2$. Then it follows from (4.2) that

$$P_m(\theta)\cos\theta - Q_m(\theta)\frac{\sin\theta}{\theta} = 0 \quad \text{and} \quad P_m(L\theta)\cos L\theta - Q_m(L\theta)\frac{\sin L\theta}{L\theta} = 0. \tag{4.3}$$

By Siegel's theorem (cf. [11], p. 485), which claims that if θ is a non-zero algebraic number then $J_\nu(\theta)$ is transcendental, we know that θ is transcendental. Then $Q_m(\theta) \neq 0, Q_m(L\theta) \neq 0$ and which imply that $\cos\theta \neq 0, \cos L\theta \neq 0$,

$$\tan\theta = \frac{\theta P_m(\theta)}{Q_m(\theta)} \quad \text{and} \quad \tan L\theta = \frac{L\theta P_m(L\theta)}{Q_m(L\theta)}. \tag{4.4}$$

On the other hand, by (3.7), we may write $\tan(L\theta) = R_L(\tan\theta)/S_L(\tan\theta)$ where $R_L(T)$ and $S_L(T)$ are polynomials of T with coefficients in $\mathbb{Z}$ which consist of odd powers of T and even powers of T respectively. Moreover, $S_L \not\equiv 1$ for $L \geqslant 2$. Then it follows from (4.4) that

$$\frac{R_L(\frac{P_m(\theta)}{Q_m(\theta)}\theta)}{S_L(\frac{P_m(\theta)}{Q_m(\theta)}\theta)} = \frac{P_m(L\theta)}{Q_m(L\theta)}L\theta \tag{4.5}$$

and which implies that θ is algebraic, a contradiction. Therefore, $J_\nu(\theta)$ and $J_\nu(L\theta)$ can not vanish simultaneously for any $L \geqslant 2$. The proof is complete.

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